2.10 Floating Base Kinematics

Free-floating robots like the quadruped and humanoid depicted in Fig. 2.15 are described by $n_b$ un-actuated base coordinates $q_b$ and $n_j$ actuated joint coordinates $q_j$:

$$ q = \begin{pmatrix} q_b \\ q_j \end{pmatrix} $$

(2.233)

The unactuated base is free in translation and rotation

$$ q_b = \begin{pmatrix} q_{b_p} \\ q_{b_R} \end{pmatrix} \in \mathbb{R}^3 \times SO(3), $$

(2.234)

whereby the position $q_{b_p}$ and rotation $q_{b_R}$ can be parameterized using different representations as seen in sections 2.2.1 and 2.4.3. Hence, the dimension of the generalized coordinate vector of a floating base system $n_b + n_j$ depends on the parameterization of the rotation, whereby the minimal number of generalized coordinates for the base is $n_{b0} = 6$.

There are several challenges linked with floating base systems. First, there are typically no (onboard) sensors that allow to directly measure the base position and orientation. To cope with this, some research areas use motion tracking systems, i.e. external cameras and markers on the robot that allows measuring the pose of the base. Otherwise, it is necessary to use sensor fusion algorithms in order to estimate the pose from different other sensor information. Second, since the base is not directly actuated, the motion of the system of bodies (i.e. the total linear and angular impulse) can only be changed through additional external forces resulting from contacts (see also section 2.10.4).

2.10.1 Generalized Velocity and Acceleration

Since differentiation in $SO(3)$ is different from $\mathbb{R}^3$, people often introduce generalized velocity and acceleration velocity vectors

$$ \dot{u} = \begin{pmatrix} I \dot{\mathbf{V}}_B \\ B \dot{\mathbf{V}}_B^{\mathbf{B}} \end{pmatrix} \in \mathbb{R}^{n_u} = \mathbb{R}^{6+n_j} \quad \ddot{u} = \begin{pmatrix} I \ddot{\mathbf{V}}_B \\ B \ddot{\mathbf{V}}_B^{\mathbf{B}} \end{pmatrix} \in \mathbb{R}^{n_u} $$

(2.235)
As shown in section 2.5.1, there exists a direct mapping from rotational velocity \( \omega \) to time derivatives of end-effector rotation coordinates \( \dot{q}_{bR} \) such that

\[
\mathbf{u} = \mathbf{E}_{fb} \cdot \dot{q}, \text{ with } \mathbf{E}_{fb} = \begin{bmatrix}
I_{3 \times 3} & 0 & 0 \\
0 & \mathbf{E}_{X,n} & 0 \\
0 & 0 & I_{n_j \times n_j}
\end{bmatrix}
\]

(2.236)

whereby the rotation parameterization dependent matrices \( \mathbf{E}_{X,n} \) and its inverse \( \mathbf{E}_{X,n}^{-1} \) are given in

- Euler Angles (2.77) and (2.78)
- ZYX: (2.75) and (2.76)
- ZYZ: (2.79) and (2.80)
- Quaternions (2.87) and (2.88)
- Angle Axis (2.91) and (2.92)
- Rotation Vector (2.95) and (2.96)

Please note that many textbooks write \( \dot{q} \), but implicitly mean \( u \) and not the time-derivative of the parameterization. However, however working with floating base systems, you must be aware of the difference. In these lecture notes we try to be consistent in this context and hence always use \( u \) to become applicable for floating base systems as well.

### 2.10.2 Forward Kinematics

We wish to derive the relationship between the generalized velocities \( u \) and the operational space velocities \( \dot{\mathbf{r}}_Q \) of a point \( Q \), which is fixed at the end of a kinematic chain that stems from a floating base \( B \). The position vector \( \dot{\mathbf{r}}_B = \dot{\mathbf{r}}_{BQ}(q) \) of a point w.r.t. the inertial frame \( I \) is given by:

\[
\dot{\mathbf{r}}_{BQ}(q) = \dot{\mathbf{r}}_{IB}(q) + \mathbf{C}_{IB}(q) \cdot \dot{\mathbf{r}}_{BQ}(q),
\]

(2.237)

where the rotation matrix \( \mathbf{C}_{IB}(q) \) describes the orientation of the floating base \( B \) w.r.t. the inertial frame \( I \). \( \dot{\mathbf{r}}_{IB}(q) \) represents the position of the floating base \( B \) w.r.t. the inertial frame \( I \) expressed in the inertial frame, \( \dot{\mathbf{r}}_{BQ}(q) \) represents the position of \( Q \) w.r.t. the floating Base \( B \) expressed in frame \( B \), and \( \mathbf{q} = \mathbf{q}(t) \) is a function of time \( t \).

### 2.10.3 Differential Kinematics of Floating Base Systems

Time differentiation of the position vector (2.237) yields:

\[
\dot{\mathbf{r}}_Q = \dot{\mathbf{r}}_B + \dot{\mathbf{C}}_{IB} \cdot \dot{\mathbf{r}}_{BQ} + \mathbf{C}_{IB} \cdot \dot{\mathbf{r}}_{BQ} = \dot{\mathbf{r}}_B + \dot{\mathbf{C}}_{IB} \cdot [\mathbf{B} \omega_{IB}] \times \cdot \dot{\mathbf{r}}_{BQ} + \mathbf{C}_{IB} \cdot \dot{\mathbf{r}}_{BQ}
\]

(2.238)
If we attach a frame at $r_Q$, we can derive a similar mapping for angular velocities. The orientation of frame $Q$ w.r.t. the inertial frame $I$ is described by:

$$C_{IQ} = C_{IB} \cdot C_{BQ}$$  \hspace{1cm} (2.239)

Time differentiation of both sides of (2.239) yields:

$$[\dot{\omega}_{IB}]_x \cdot C_{IQ} = [\dot{\omega}_{IB}]_x \cdot C_{IB} \cdot C_{BQ} + C_{IB} \cdot [\dot{\omega}_{BQ}]_x \cdot C_{BQ}$$

$$= [\dot{\omega}_{IQ}]_x \cdot C_{IQ} + [\dot{\omega}_{BQ}]_x \cdot C_{IQ}$$  \hspace{1cm} (2.240)

which gives finally:

$$\dot{\omega}_{IQ} = \dot{\omega}_{IB} + \dot{\omega}_{BQ}$$

Hence, the mapping from generalized velocities $u$ to the operational space twist $[T \dot{v}_Q \ T \omega_{TQ}]^T$ of frame $Q$ is realized by the spatial Jacobian:

$$\begin{bmatrix} T \dot{J}_P \ \\ T \dot{J}_R \end{bmatrix} = \begin{bmatrix} \dot{J}_C \\ 0_{3 \times 3} \end{bmatrix} - C_{IB} \cdot [\dot{b}r_{BQ}]_x + C_{IB} \cdot BJ_{R_{q_j}}(q_j)$$  \hspace{1cm} (2.242)

### 2.10.4 Contacts and Constraints

In kinematics, contacts between the robot and its environment can be modeled as kinematic constraints. Every point $C_i$ that is in contact with the environment (attached to coordinate frame $I$) imposes three constraints

$$T r_{IC_i} = const, \quad T \dot{r}_{IC_i} = T \dot{r}_{IC_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (2.243)

These contact constraints can be expressed as a function of the generalized velocities and accelerations using the contact point Jacobian

$$T J_{C_i} u = 0, \quad T J_{C_i} \dot{u} + T \dot{J}_{C_i} u = 0$$  \hspace{1cm} (2.244)

In case there are $n_c$ active contacts, the constraints are simply stacked to

$$J_c = \begin{bmatrix} J_{C_1} \\ \vdots \\ J_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_u}.$$  \hspace{1cm} (2.245)

The rank($J_c$) indicates the number of independent contact constraints. This stacked Jacobian can be split into

$$J_c = [J_{c,b} \ J_{c,j}] = \begin{bmatrix} \frac{\partial r}{\partial q_b} \\ \frac{\partial r}{\partial q_j} \end{bmatrix},$$  \hspace{1cm} (2.246)
whereby $J_{c,b}$ indicates the relation between base motion and contact constraints. In case the rank of $J_{c,b}$ (i.e. number of base constraints) is full ($\text{rank}(J_{c,b}) = 6$), the system features enough constraints such that the base motion can be controlled from joint motion. The difference between the number of independent contact constraints $\text{rank}(J_c)$ and the number of base constraints $\text{rank}(J_{c,b})$ is the number of internal kinematic constraints that must be fulfilled.

Point Contacts - Quadruped

![Figure 2.16](image.png)

Figure 2.16: Depending on the number and arrangement of point contacts, a quadruped is fully-constrained (a) or under-constrained (b).

The point feet of a quadrupedal robot impose three (independent) constraints each. In case of two point contacts, the stacked contact Jacobian has $\text{rank}(J_c) = 6$ but the Jacobian w.r.t. base coordinates has only $\text{rank}(J_{c,b}) = 5$. This implies that the system is under-actuated and the base can not be arbitrarily moved by the joints. This becomes intuitively clear when looking at Fig. 2.16(b) as the robot cannot change the orientation around the line of support.

In contrast thereto, three point contacts as illustrated in Fig. 2.16(a) imply $\text{rank}(J_c) = 9$ and $\text{rank}(J_{c,b}) = 6$. This means that the body position and orientation is fully controllable through the joints. At the same time, there are three internal constraints that can be interpreted by the fact that the three legs cannot be moved one with respect to another.

Extended Contacts - Humanoid

For systems with extended feet, additional constraints are required in order to limit the foot rotation. One possible option is to introduce a rotational Jacobian. Much more common is to assign multiple contact points on the same link. A single contact point (Fig. 2.18, left) imposes three constraints as already seen in the previous section. In case of two contact points, the rank of the constraints is $\text{rank}(J_c) = \text{rank}(J_{c,b}) = 5$ and in case of three points assigned to the same element we get $\text{rank}(J_c) = \text{rank}(J_{c,b}) = 6$ although $J_c \in \mathbb{R}^{9 \times n_j}$.

2.10.5 Support Consistent Inverse Kinematics

Applying inverse kinematics to floating base systems answers the question how to move individual joints in order to achieve certain task-space motion without violating contact constraints. This seeks for the application of a multi-task approach with prioritization, whereby the contact constraints are considered to have higher priority.
Figure 2.17: Depending on the number and arrangement of point contacts, a humanoid is fully-constrained (a) or under-constrained (b).

Figure 2.18: Multiple contact points attached to a single foot.

than the task-space motion. As introduced above, contact constraints of the $n_c$ legs in ground contact are given by

$$
\mathbf{J}_c \dot{\mathbf{q}} = 0
$$

(2.247)

with

$$
\mathbf{J}_c = 
\begin{bmatrix}
\frac{\partial r_{c1}(\mathbf{q})}{\partial \mathbf{q}} \\
\vdots \\
\frac{\partial r_{cnc}(\mathbf{q})}{\partial \mathbf{q}}
\end{bmatrix},
$$

(2.248)

which implies that the motion of the system in contact is given by

$$
\dot{\mathbf{q}} = \mathbf{J}_c^+ \mathbf{0} + \mathbf{N}(\mathbf{J}_c) \dot{\mathbf{q}}_0 = \mathbf{N}_c \dot{\mathbf{q}}_0.
$$

(2.249)

Hence, given a demanded task space motion

$$
\mathbf{w}_t = \mathbf{J}_t \dot{\mathbf{q}}_t
$$

(2.250)

the joint velocity required to achieve this is

$$
\dot{\mathbf{q}} = \mathbf{N}_c (\mathbf{J}_t \mathbf{N}_c)^+ \mathbf{w}_t
$$

(2.251)