Lecture #4:

Integration Algorithms for Rate-independent Plasticity (1D)

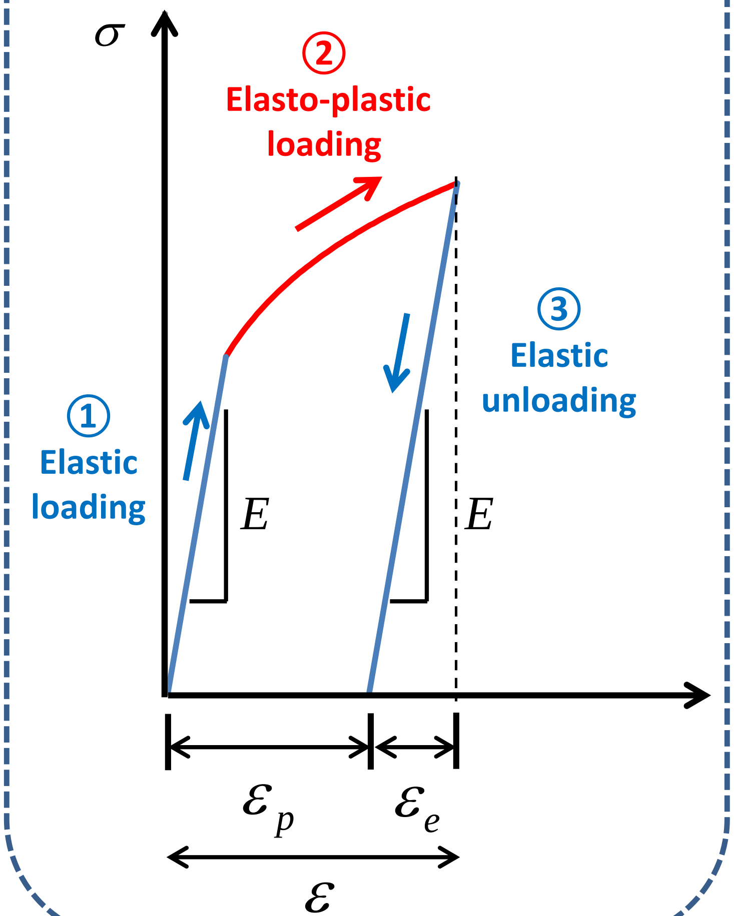
by Dirk Mohr

ETH Zurich,
Department of Mechanical and Process Engineering,
Chair of Computational Modeling of Materials in Manufacturing

Recall: Important difference

ELASTO-PLASTIC

(e.g. metals, concrete, thermoplastics)



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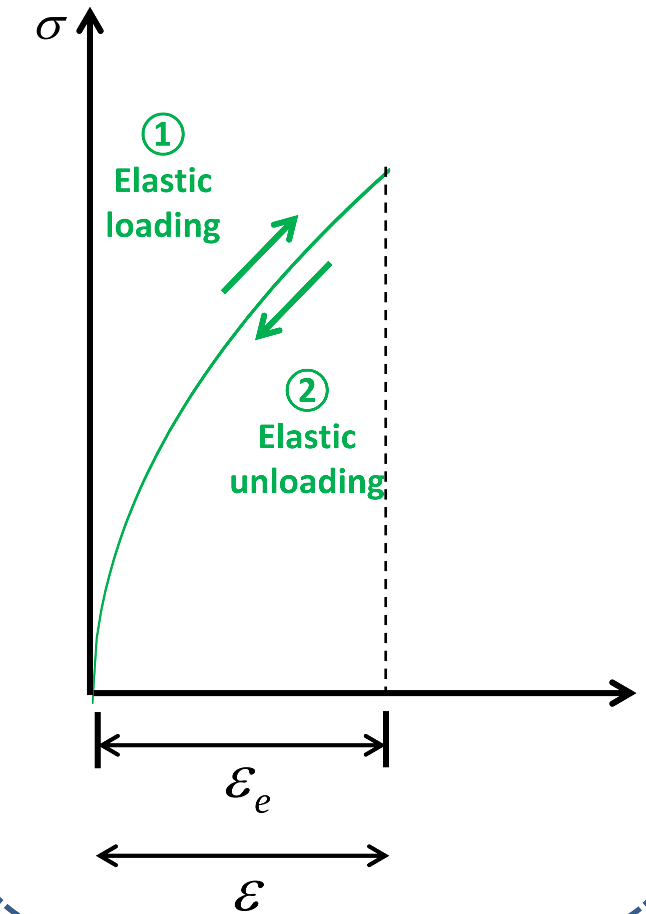
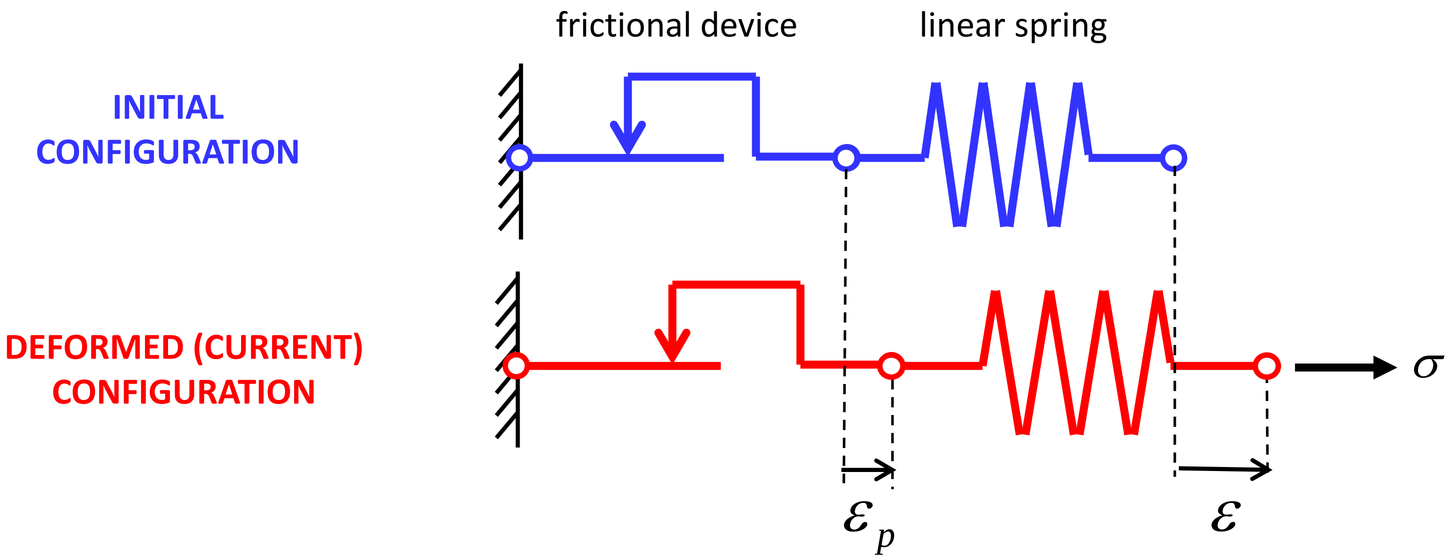
NON-LINEAR ELASTIC

(e.g. rubbers, foams)



Rate-independent perfect plasticity

- **Simplified rheological model:**



The strain is split into an elastic and a plastic part

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

i.e. the elastic strain is

$$\varepsilon_e = \varepsilon - \varepsilon_p$$

Rate-independent perfect plasticity - Summary

i. Constitutive equation for stress

$$\sigma = E(\varepsilon - \varepsilon_p)$$

ii. Yield function

$$f[\sigma, k] = |\sigma| - k$$

iii. Flow rule

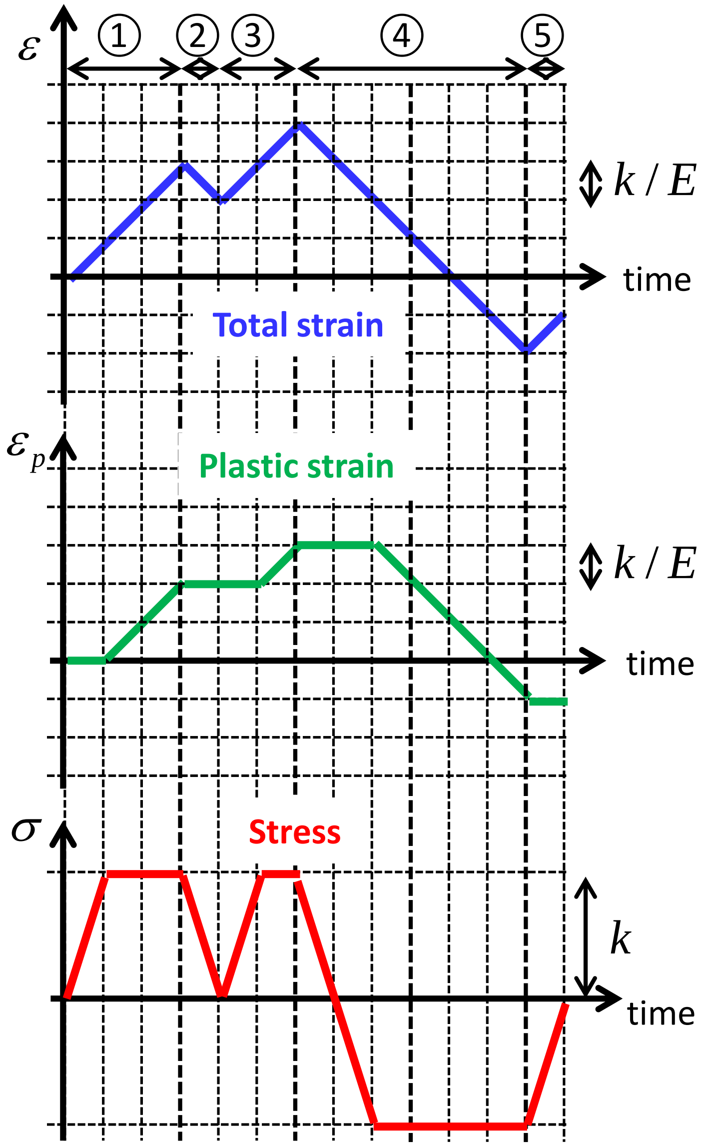
$$\dot{\varepsilon}_p = \dot{\gamma}\,\mathrm{sign}[\sigma]$$

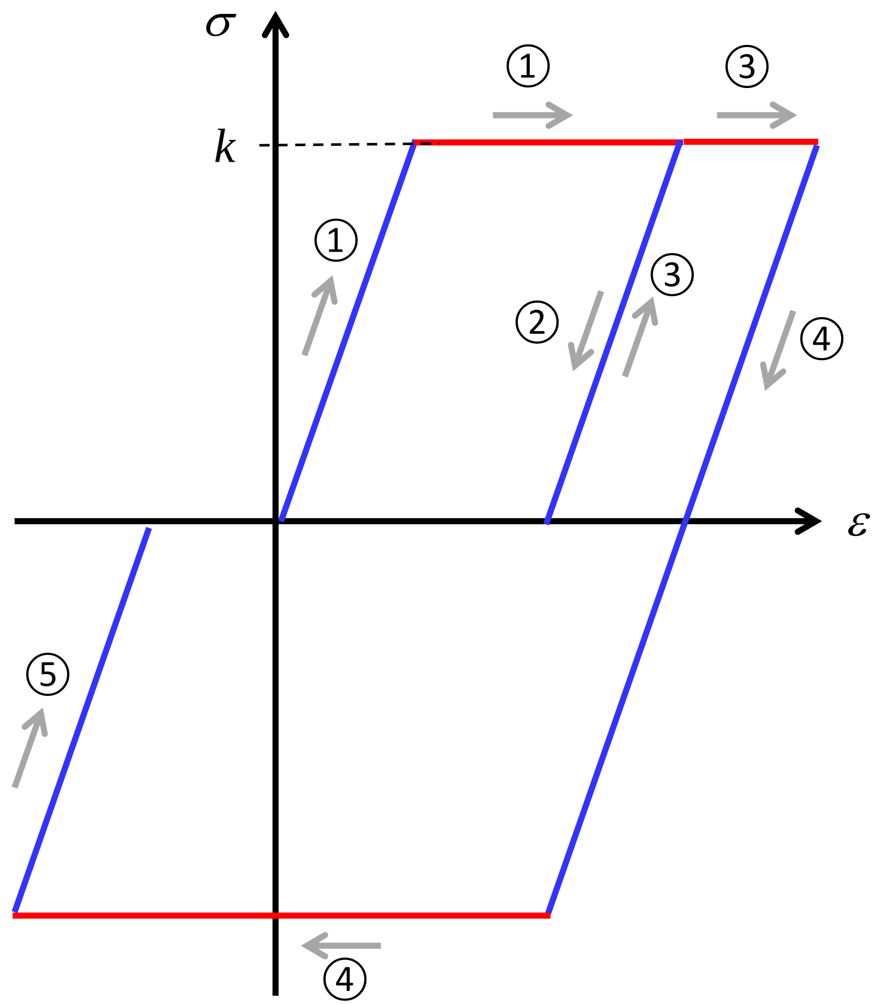
iv. Loading/unloading conditions

$$\dot{\gamma} = \begin{cases} 0 & \text{if } f < 0 \\ > 0 & \text{if } f = 0 \text{ and } \dot{f} = 0 \\ 0 & \text{if } f = 0 \text{ and } \dot{f} < 0 \end{cases}$$

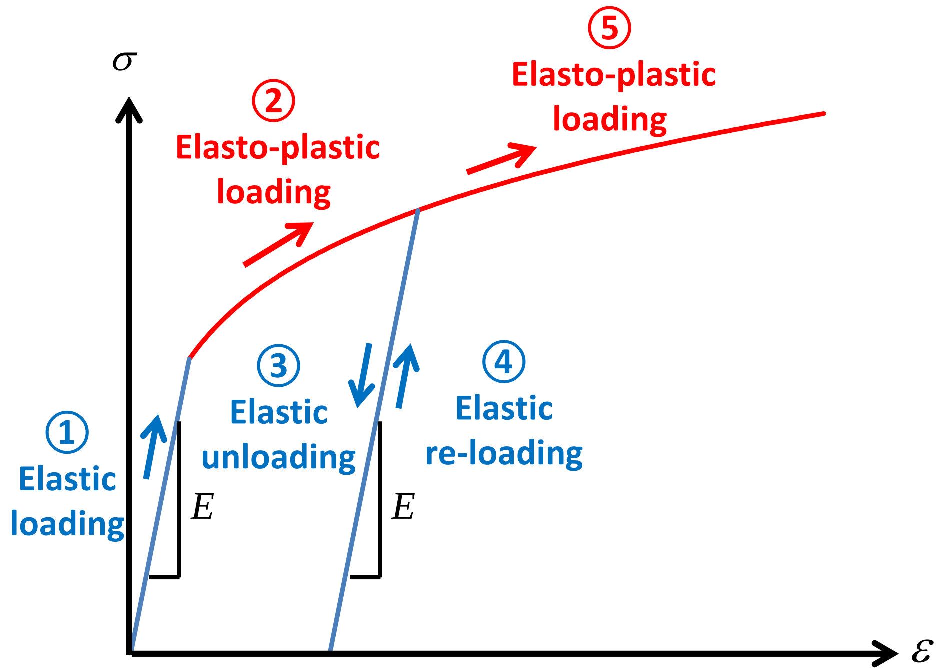
Material model parameters: (1) Young's modulus E, and (2) flow stress k.

Rate-independent perfect plasticity - Application





Rate-independent isotropic hardening plasticity



The magnitude of the **stress increases due to strain hardening** when the material is deformed in the elasto-plastic range. For isotropic hardening materials, it is described through an evolution equation for the flow stress k.

Rate-independent isotropic hardening plasticity

Firstly, we introduce a scalar valued non-negative function

$$\bar{\varepsilon}_p = \int \dot{\gamma} dt$$

to measure the amount of plastic flow (slip). This measure is often called **equivalent plastic strain**. Unlike the plastic strain, the magnitude of the equivalent plastic strain can only increase!

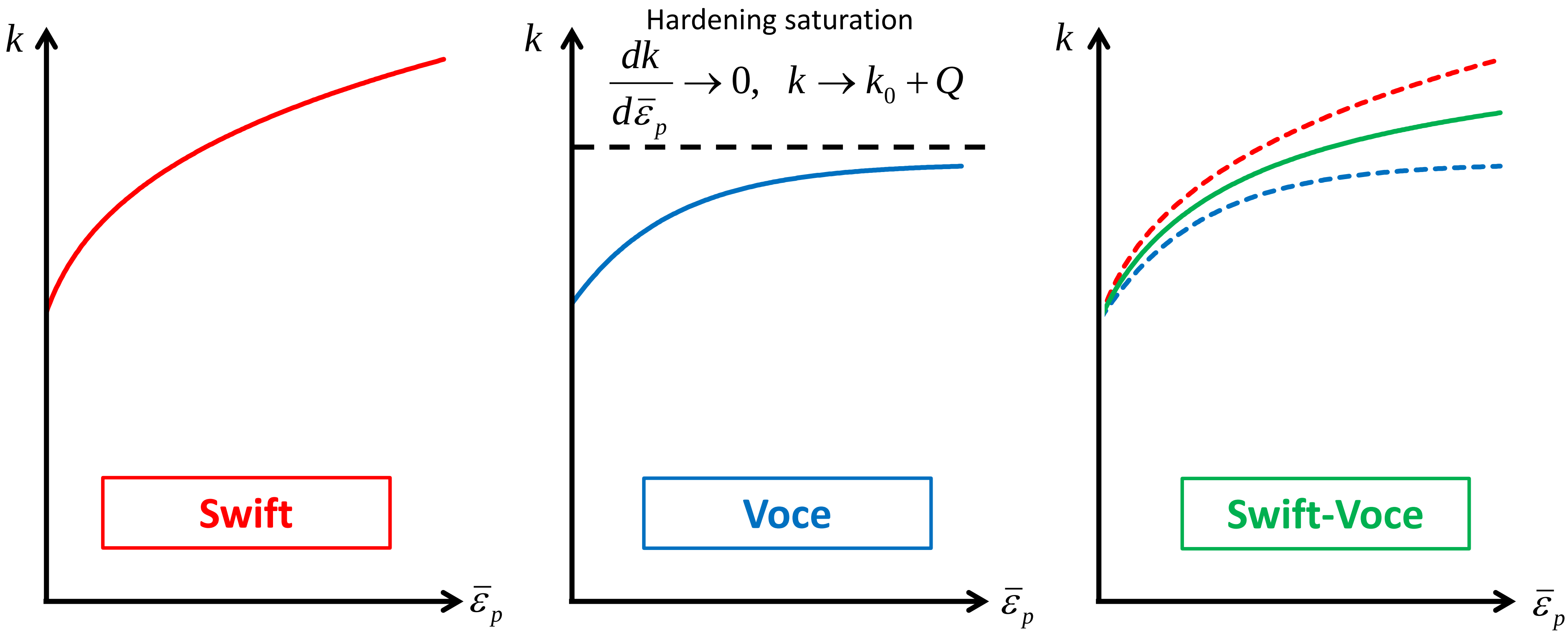
It is then assumed that the flow stress is a monotonically increasing smooth differentiable function of the equivalent plastic strain

$$k = k[\bar{\varepsilon}_p]$$

This equation describes the **isotropic hardening law**.

Rate-independent isotropic hardening plasticity

Frequently used parametric forms of the function $k = k[\bar{\varepsilon}_p]$ are the Swift and Voce laws:



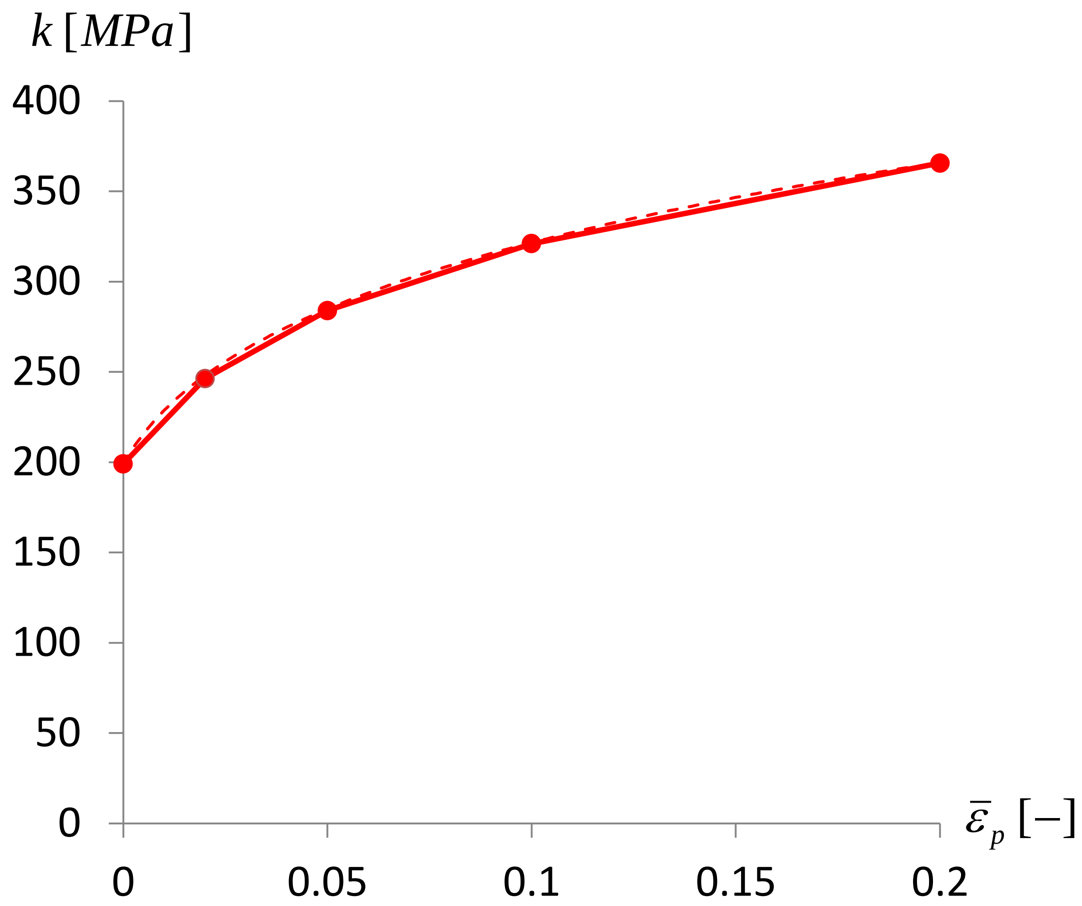
$$k_S = A(\bar{\varepsilon}_p + \varepsilon_0)^n$$

$$k_V = k_0 + Q\left(1 - \exp[-\beta\bar{\varepsilon}_p]\right)$$

$$k = (1-\alpha)k_V + \alpha k_S$$

Rate-independent isotropic hardening plasticity

In engineering practice, the isotropic hardening function is often represented by a piece-wise linear function



PEEQ	k
0.000	199.1
0.020	246.3
0.050	283.9
0.100	321.0
0.200	365.6

Isotropic hardening plasticity - Summary

i. Constitutive equation for stress

$$\sigma = E(\varepsilon - \varepsilon_p)$$

ii. Yield function

$$f[\sigma, \bar{\varepsilon}_p] = |\sigma| - k[\bar{\varepsilon}_p]$$

iii. Flow rule

$$\dot{\varepsilon}_p = \dot{\gamma}\,\mathrm{sign}[\sigma]$$

iv. Loading/unloading conditions

$$\dot{\gamma} = \begin{cases} 0 & \text{if } f < 0 \\ > 0 & \text{if } f = 0 \text{ and } \dot{f} = 0 \\ 0 & \text{if } f = 0 \text{ and } \dot{f} < 0 \end{cases}$$

v. Isotropic hardening law

$$k = k[\bar{\varepsilon}_p] \quad \text{with} \quad \bar{\varepsilon}_p = \int \dot{\gamma}\, dt$$

Differential equation to be solved

- First-order ordinary differential equation

$$\left.\begin{aligned}\sigma &= E(\varepsilon - \varepsilon_p) \\ \dot{\varepsilon}_p &= \dot{\gamma}\,\mathrm{sign}[\sigma]\end{aligned}\right\} \quad \dot{\varepsilon}_p = \dot{\gamma}\,\mathrm{sign}\left[\varepsilon[t] - \varepsilon_p[t]\right]E$$

- Initial condition

$$\varepsilon_p[t=0] = 0$$

- Prescribed loading

$$\varepsilon = \varepsilon[t]$$

- Multiplier $\dot{\gamma}$ to satisfy the constraints

$$\dot{\gamma} = \begin{cases} 0 & \text{if } f < 0 \\ >0 & \text{if } f = 0 \text{ and } \dot{f} = 0 \\ 0 & \text{if } f = 0 \text{ and } \dot{f} < 0 \end{cases}$$

with $f[\sigma, \bar{\varepsilon}_p] = |\sigma| - k[\bar{\varepsilon}_p]$, $\sigma = E(\varepsilon - \varepsilon_p)$ and $\bar{\varepsilon}_p = \int \dot{\gamma}\,dt$

Numerical solution of Differential Equations

Without the loading and unloading conditions, the plasticity problem reduces to solving an **ordinary first-order differential equation** for the plastic strain, considering time as the only independent variable:

- D.E. $\dot{\varepsilon}_p = \dot{\gamma}\,\text{sign}\lfloor \varepsilon[t] - \varepsilon_p[t] \rfloor E \quad \Leftrightarrow \quad \frac{dy}{dt} = g[y]$
- I.C. $\varepsilon_p[t=0] = 0 \quad \Leftrightarrow \quad y[t=0] = y_0$

Such equations are solved numerically using **integration algorithms**. Instead of the calculating the exact analytical solution, we limit our attention to calculating the approximated solution

$$y_n \cong y[t_n]$$

at equally-spaced instants t_n, $n=1,\ldots,N$ with the **time step** Δt,

$$\Delta t = t_{n+1} - t_n$$

Numerical solution of Differential Equations

A first popular method is the so-called **forward (explicit) Euler algorithm:**

$$y_0$$

$$y_1 = y_0 + \Delta t g[y_0]$$

$$y_2 = y_1 + \Delta t g[y_1]$$

$$\vdots$$

$$y_{n+1} = y_n + \Delta t g[y_n]$$

Starting with the initial condition, the approximations can be progressively calculated.

Numerical solution of Differential Equations

Recall that $y' = g[y]$ and thus the forward (explicit) Euler algorithm may also be written as

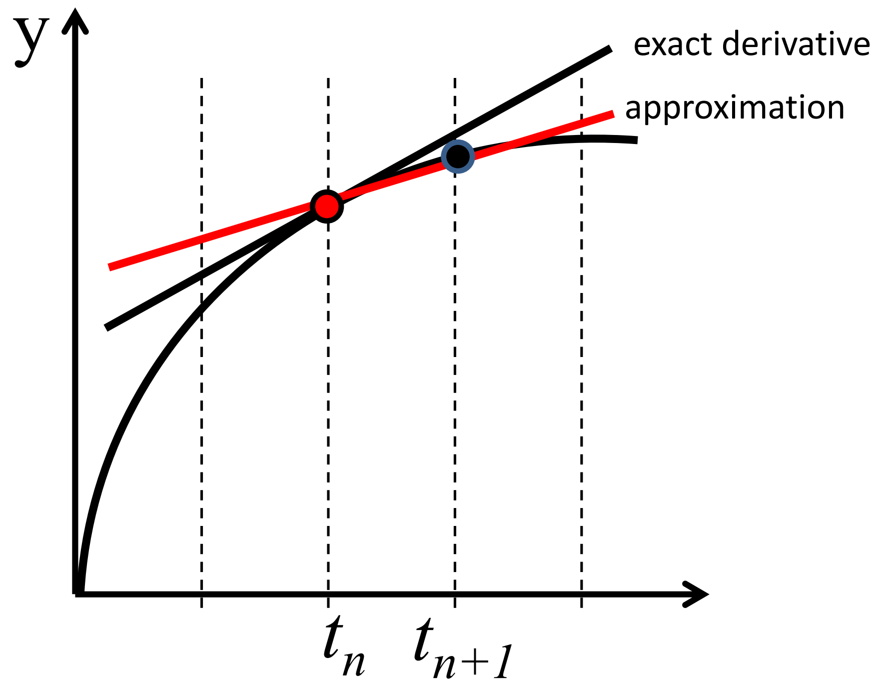
$$y_0$$

$$y_1 = y_0 + \Delta t \dot{y}_0$$

$$y_2 = y_1 + \Delta t \dot{y}_1$$

$$\vdots$$

$$y_{n+1} = y_n + \Delta t \dot{y}_n$$



In other words, the time derivative at time t_n is given by the approximation

$$\dot{y}[t_n] \cong \frac{y_{n+1} - y_n}{\Delta t}$$

Numerical solution of Differential Equations

A second popular method is the so-called **backward (implicit) Euler** algorithm:

y_0

y_1 which is obtained from solving the implicit equation $y_1 = y_0 + \Delta t g[y_1]$

y_2 which is obtained from solving the implicit equation $y_2 = y_1 + \Delta t g[y_2]$

.
.
.

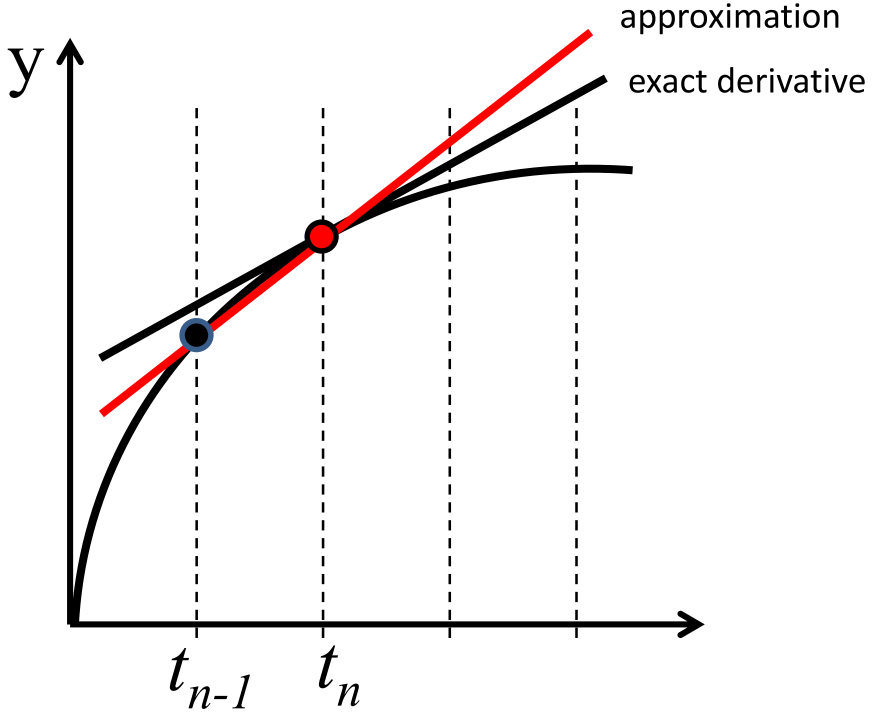
y_{n+1} which is obtained from solving the implicit equation $y_{n+1} = y_n + \Delta t g[y_{n+1}]$

Starting with the initial condition, the approximations can be progressively calculated. However, at each time step t_i, an often implicit equation needs to be solved.

Numerical solution of Differential Equations

According to the **backward (implicit) Euler** algorithm, the time derivative at time t_n is given by the approximation

$$\left.\begin{array}{l} y_n = y_{n-1} + \Delta t g[y_n] \\ \dot{y} = g[y] \end{array}\right\} \Rightarrow \dot{y}[t_n] \cong \frac{y_n - y_{n-1}}{\Delta t}$$



Illustration

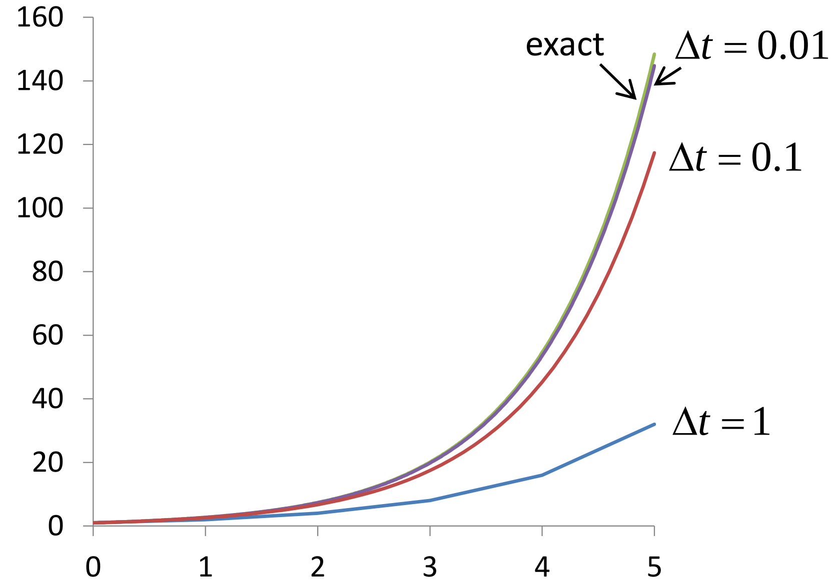
Example: $\dfrac{dy}{dt} = y$ (differential equation)

$y[t=0]=1$ (initial condition)

$\Rightarrow$ $y = \exp[t]$ (exact solution)

The approximate solution with **forward (explicit) Euler** algorithm for a time step of $\Delta t=1$ reads (we have $g[y]=y$ and $y_0=1$):

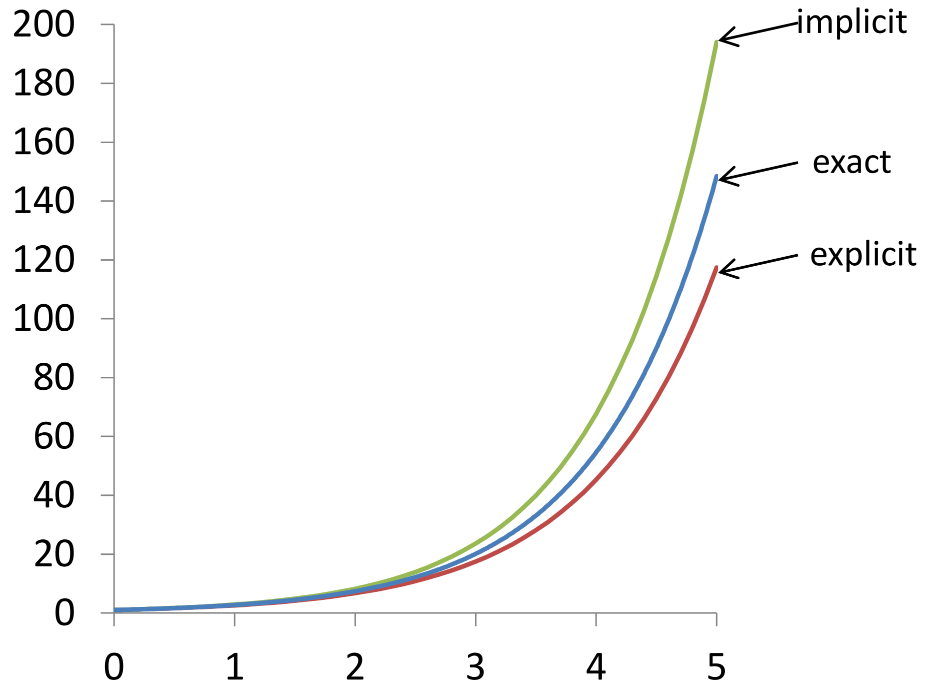
$$y_0 = 1$$
$$y_1 = y_0 + \Delta t g[y_0] = 1 + 1\cdot 1 = 2$$
$$y_2 = y_1 + \Delta t g[y_1] = 2 + 1\cdot 2 = 4$$
$$y_3 = y_2 + \Delta t g[y_2] = 4 + 1\cdot 4 = 8$$
$$y_4 = y_3 + \Delta t g[y_3] = 8 + 1\cdot 8 = 16$$
$$y_5 = y_4 + \Delta t g[y_4] = 16 + 1\cdot 16 = 32$$



Observe from the graph that the method converges for $\Delta t \rightarrow \infty$

Comparison implicit vs. explicit

The approximate solutions with **forward (explicit) Euler** and **backward (implicit) Euler** algorithms for a time step of $\Delta t = 0.1$



… back to the plasticity problem

Differential equation:

$$\dot{\varepsilon}_p = \dot{\gamma}\,\mathrm{sign}\left[\varepsilon[t]-\varepsilon_p[t]\right]E \quad \Rightarrow \quad \varepsilon^p_{n+1} = \varepsilon^p_n + \Delta t g[t_{n+1}]$$

$$= \varepsilon^p_n + \underbrace{\Delta t \dot{\gamma}}_{\Delta\gamma}\,\mathrm{sign}\left[\varepsilon_{n+1} - \varepsilon^p_{n+1}\right]E$$

$$= \varepsilon^p_n + \Delta\gamma\,\mathrm{sign}\left[\sigma_{n+1}\right]$$

Initial condition:

$$\varepsilon^p_0 = 0$$

State variable:

$$\bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \Delta\gamma$$

$$\bar{\varepsilon}^p_0 = 0$$

Dependent variables:

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_{n+1})$$

$$k_{n+1} = k[\bar{\varepsilon}^p_{n+1}]$$

$$f_{n+1} = \left|\sigma_{n+1}\right| - k_{n+1}$$

plus "discrete" evolution constraints:

$$f_{n+1} \le 0$$

$$\Delta\gamma \ge 0$$

$$(\Delta\gamma) f_{n+1} = 0 \quad \Leftrightarrow \quad \begin{cases} \text{if } \Delta\gamma > 0 \text{ then } f_{n+1} = 0 \\ \text{if } f_{n+1} < 0 \text{ then } \Delta\gamma = 0 \end{cases}$$

… back to the plasticity problem

Differential equation:

$$\dot{\varepsilon}_p = \dot{\gamma}\,\mathrm{sign}\lfloor \varepsilon[t] - \varepsilon_p[t] \rfloor E \quad \Rightarrow \quad \varepsilon^p_{n+1} = \varepsilon^p_n + \Delta t g[t_{n+1}]$$

$$= \varepsilon^p_n + \underbrace{\Delta t \dot{\gamma}}_{\Delta\gamma}\,\mathrm{sign}\left[\varepsilon_{n+1} - \varepsilon^p_{n+1}\right]E$$

$$= \varepsilon^p_n + \Delta\gamma\,\mathrm{sign}\left[\sigma_{n+1}\right]$$

Initial condition:

$$\varepsilon^p_0 = 0$$

State variable:

$$\bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \Delta\gamma$$

$$\bar{\varepsilon}^p_0 = 0$$

Dependent variables:

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_{n+1})$$

$$k_{n+1} = k[\bar{\varepsilon}^p_{n+1}]$$

$$f_{n+1} = |\sigma_{n+1}| - k_{n+1}$$

plus "discrete" evolution constraints:

$$f_{n+1} \le 0$$

$$\Delta\gamma \ge 0$$

$$(\Delta\gamma) f_{n+1} = 0 \quad \Leftrightarrow \quad \begin{cases} \text{if } \Delta\gamma > 0 \text{ then } f_{n+1} = 0 \\ \text{if } f_{n+1} < 0 \text{ then } \Delta\gamma = 0 \end{cases}$$

Main unknown:

Plastic multiplier $\Delta\gamma$

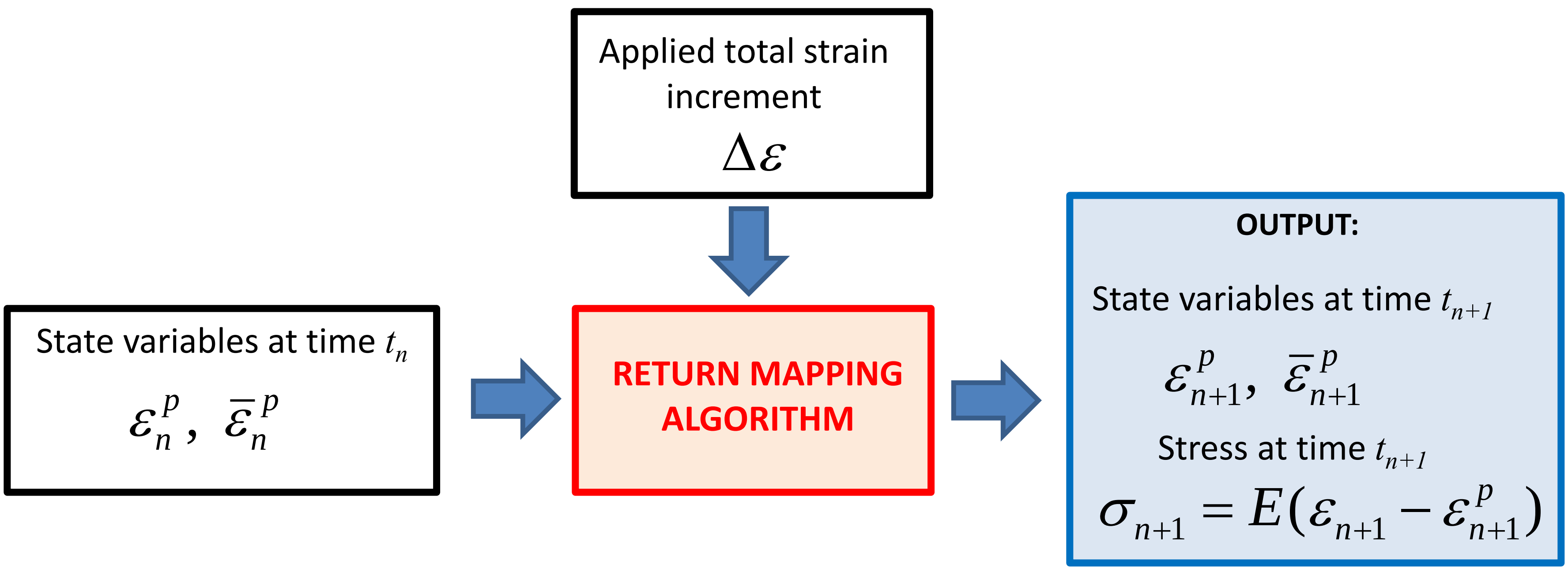
… which makes our problem more complex than solving an ordinary first order differential equation!

Return Mapping Algorithm

We solve the plasticity problem assuming a **strain-driven process**, i.e. for a given increment in the applied total strain,

$$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n$$

we determine numerical approximations of the corresponding stress and state variables at time t_{n+1} based on their values at time t_n.



Return mapping procedure

When computing the solution at time t_{n+1}, we first compute the **trial elastic state** by assuming that the material response is purely elastic (no plastic evolution) when applying $\Delta\varepsilon$:

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p) = E(\Delta\varepsilon + \varepsilon_n - \varepsilon_n^p) = \sigma_n + E(\Delta\varepsilon)$$

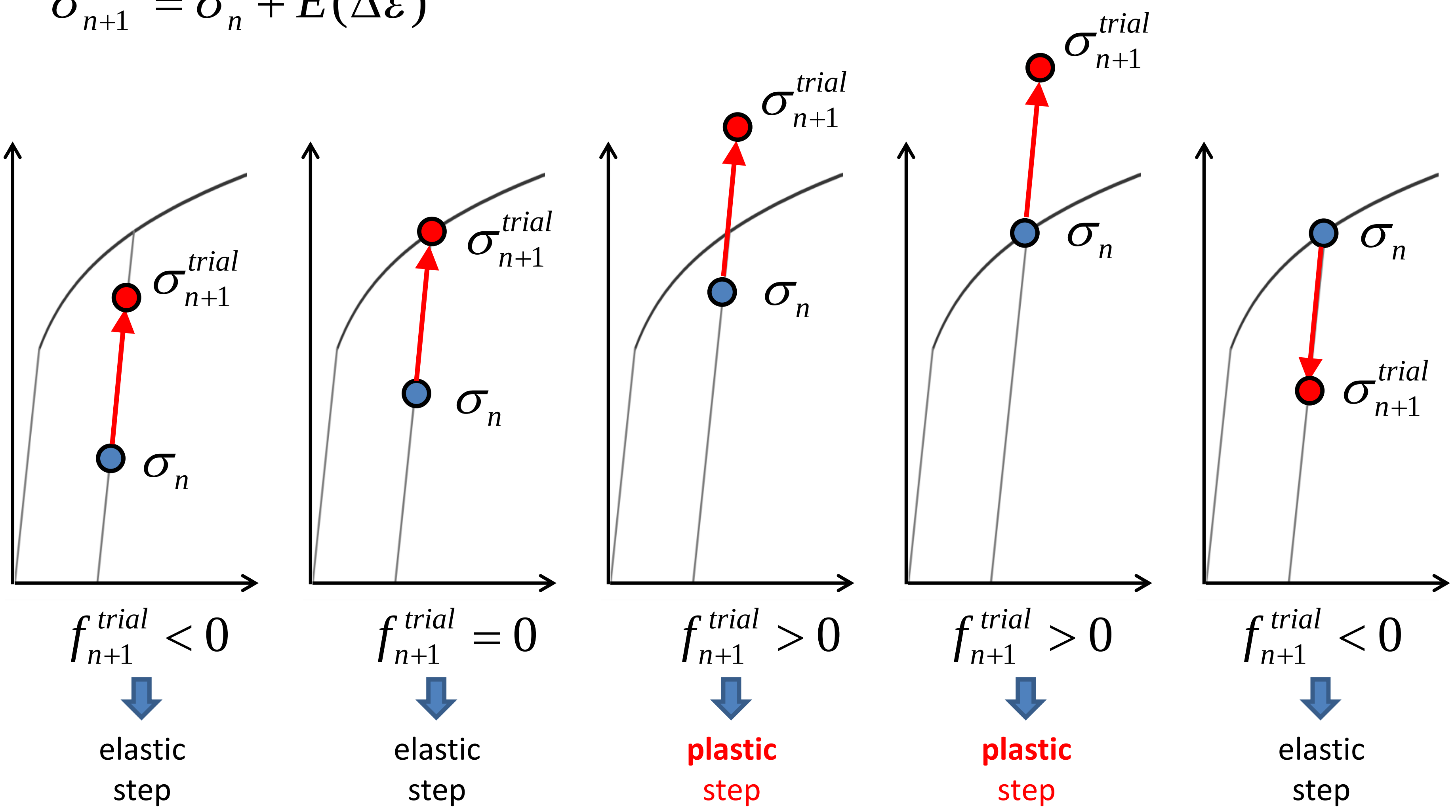
$$\varepsilon_{n+1}^{p,trial} = \varepsilon_n^p$$

$$\bar{\varepsilon}_{n+1}^{p,trial} = \bar{\varepsilon}_n^p$$

$$f_{n+1}^{trial} = \left|\sigma_{n+1}^{trial}\right| - k[\bar{\varepsilon}_n^p] \Rightarrow \begin{cases} \text{if } f_{n+1}^{trial} \leq 0 & \text{then elastic loading step} \\ \text{if } f_{n+1}^{trial} > 0 & \text{then plastic loading step} \end{cases}$$

Return mapping procedure

$$\sigma_{n+1}^{trial} = \sigma_n + E(\Delta\varepsilon)$$



Return mapping procedure

Elastic loading step:

$$\Delta\gamma = 0$$

$$f_{n+1} = f_{n+1}^{trial} \le 0$$

State variables at time t_n

$$\varepsilon_n^p, \ \bar{\varepsilon}_n^p$$

Applied total strain increment

$$\Delta\varepsilon$$

Calculate Trial State

$$\sigma_{n+1}^{trial}, \ f_{n+1}^{trial}$$

$$f_{n+1}^{trial} \le 0$$

$$\Delta\gamma = 0$$

OUTPUT:

State variables at time t_{n+1}

$$\varepsilon_{n+1}^p = \varepsilon_n^p$$

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p$$

Stress at time t_{n+1}

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon_{n+1}^p)$$

Return mapping procedure

Plastic loading step: $f_{n+1}^{trial} > 0 \Rightarrow \Delta\gamma > 0$

In a plastic loading step, the plastic multiplier $\Delta\gamma > 0$ must be determined such that the yield condition at time t_{n+1} is full filled.

$$f_{n+1} = |\sigma_{n+1}| - k_{n+1} \quad (1)$$

Firstly, we express the absolute value of the stress σ_{n+1} as a function of the unknown plastic multiplier:

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon_{n+1}^p) = E\left(\varepsilon_{n+1} - \varepsilon_n^p - \overbrace{(\varepsilon_{n+1}^p - \varepsilon_n^p)}^{\Delta\varepsilon_p}\right) = \sigma_{n+1}^{trial} - E\Delta\varepsilon_p$$

while

$$\Delta\varepsilon_p = \varepsilon_{n+1}^p - \varepsilon_n^p = \Delta\gamma\,\mathrm{sign}[\sigma_{n+1}]$$

And hence

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon_{n+1}^p) = \sigma_{n+1}^{trial} - E(\Delta\gamma)\mathrm{sign}[\sigma_{n+1}]$$

Return mapping procedure

$$\left|\sigma_{n+1}\right| = \text{sign}[\sigma_{n+1}]\sigma_{n+1}$$
$$= \text{sign}[\sigma_{n+1}]\left(\sigma_{n+1}^{trial} - E(\Delta\gamma)\text{sign}[\sigma_{n+1}]\right)$$
$$= \text{sign}[\sigma_{n+1}]\sigma_{n+1}^{trial} - E(\Delta\gamma)$$

observe that

$$\text{sign}[\sigma_{n+1}] = \text{sign}[\sigma_{n+1}^{trial}] \Rightarrow \boxed{\left|\sigma_{n+1}\right| = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma} \quad (2)$$

Secondly, we express the flow stress k_{n+1} as a function of the unknown plastic multiplier:

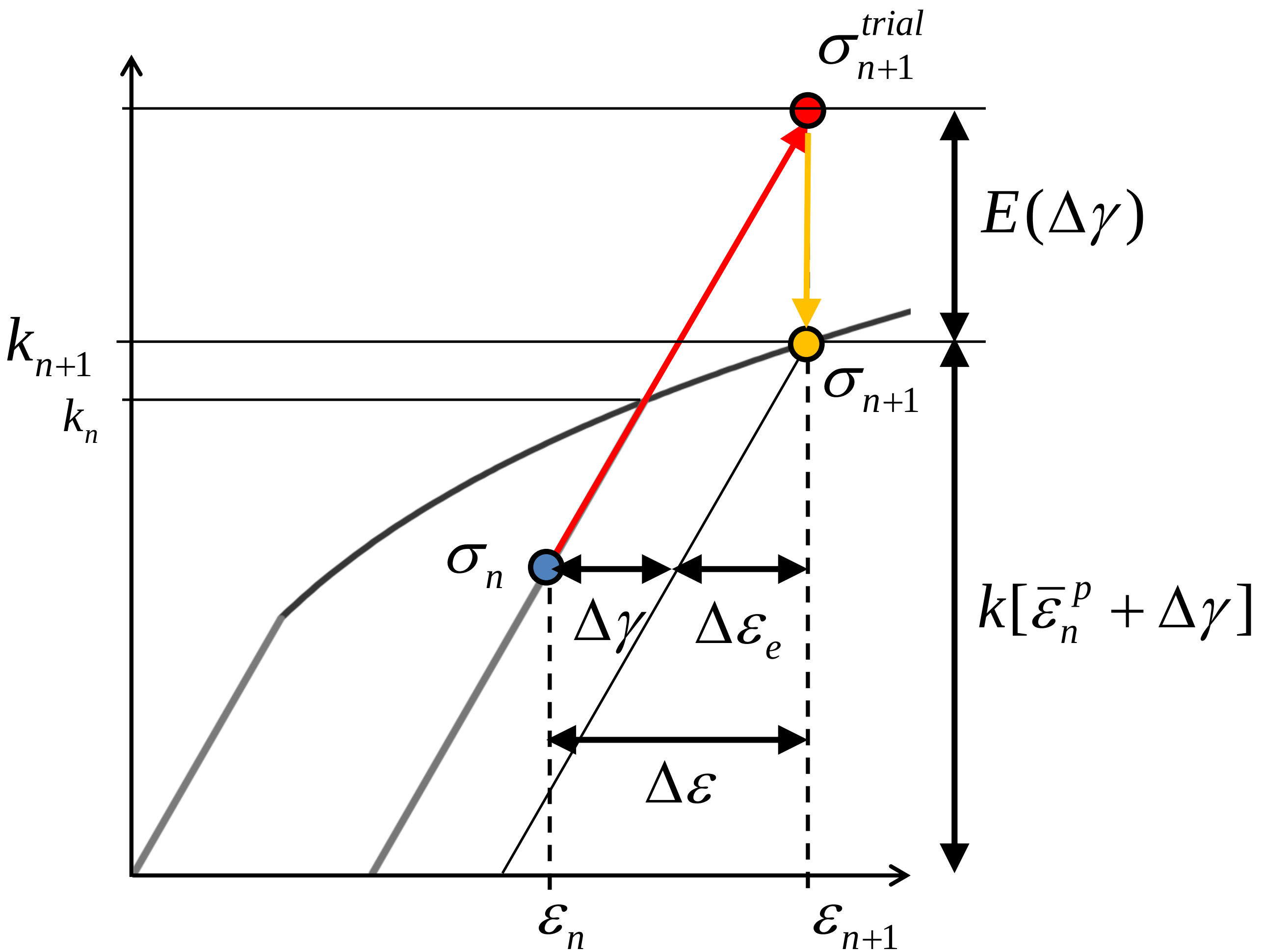
$$\bar{\varepsilon}_{n+1}^{p} = \bar{\varepsilon}_{n}^{p} + \Delta\gamma \Rightarrow \boxed{k_{n+1} = k[\bar{\varepsilon}_{n}^{p} + \Delta\gamma]} \quad (3)$$

Then, using the results (2) and (3) in (1), we obtain the so-called **discrete consistency condition**:

$$\boxed{f_{n+1} = \left|\sigma_{n+1}\right| - k_{n+1} = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - k[\bar{\varepsilon}_{n}^{p} + \Delta\gamma] = 0}$$

Return mapping procedure

$$f_{n+1} = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - k[\bar{\varepsilon}_n^p + \Delta\gamma] = 0$$



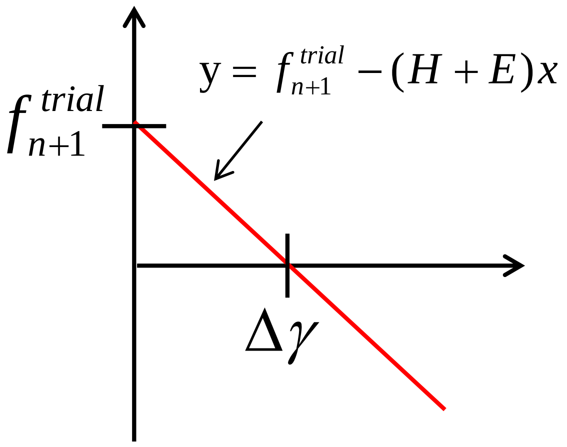
Solving the discrete consistency condition

Example #1: Linear hardening law

$$k[\bar{\varepsilon}_p] = k_0 + H\bar{\varepsilon}_p$$ with constant hardening modulus H

The discrete consistency condition then reads

$$f_{n+1} = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - \left(k_0 + H(\bar{\varepsilon}_n^p + \Delta\gamma)\right)$$
$$= \left|\sigma_{n+1}^{trial}\right| - (k_0 + H\bar{\varepsilon}_n^p) - H\Delta\gamma - E\Delta\gamma$$
$$= f_{n+1}^{trial} - (H+E)\Delta\gamma = 0$$

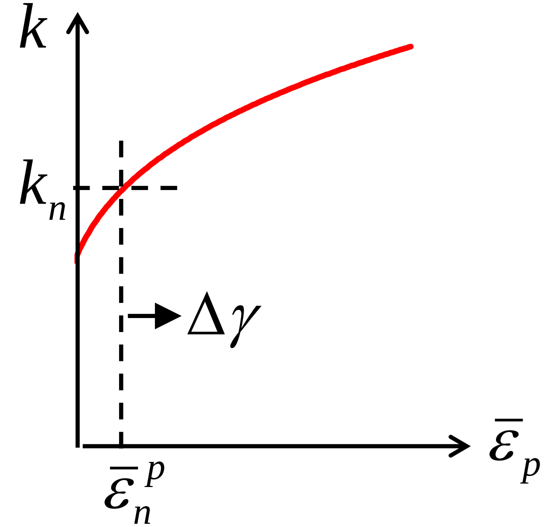


from which we determine the plastic multiplier

$$\Delta\gamma = \frac{f_{n+1}^{trial}}{H+E}$$

Solving the discrete consistency condition

Example #2: General non-linear concave hardening law $k[\bar{\varepsilon}_p]$

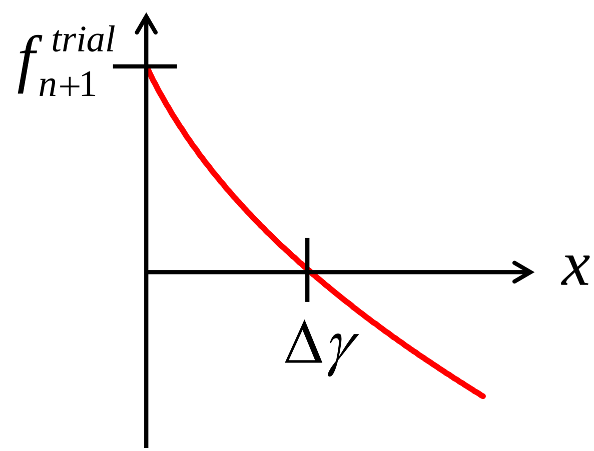


The discrete consistency condition then reads

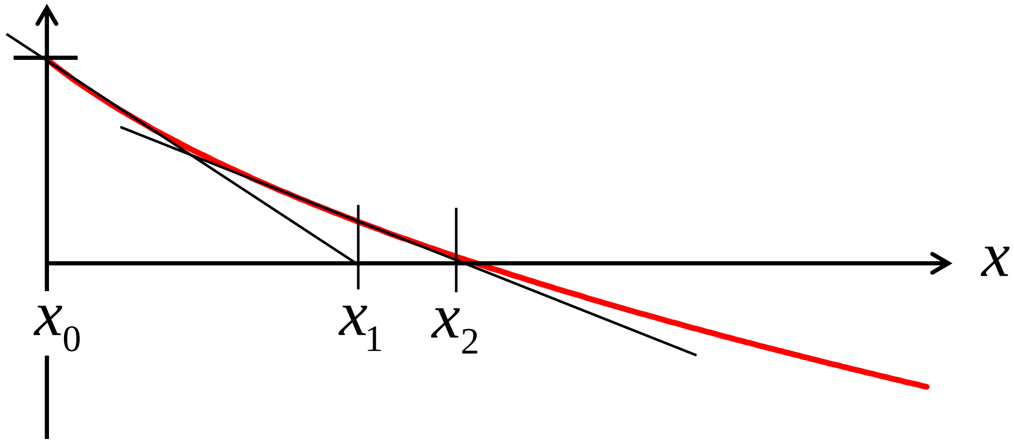
$$f_{n+1} = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - k[\bar{\varepsilon}_n^p + \Delta\gamma]$$
$$= \left|\sigma_{n+1}^{trial}\right| - k_n - E\Delta\gamma - \left(k[\bar{\varepsilon}_n^p + \Delta\gamma] - k_n\right)$$
$$= f_{n+1}^{trial} - E\Delta\gamma - \left(k[\bar{\varepsilon}_n^p + \Delta\gamma] - k_n\right) = 0$$

Which corresponds to seeking the root of the convex function

$$\mathrm{y}[\mathrm{x}] = f_{n+1}^{trial} - Ex - \left(k[\bar{\varepsilon}_n^p + x] - k_n\right)$$



Solving the discrete consistency condition



Seeking the root of a C[1]-continuous function is a standard problem in applied mathematics. For example, it can be found using a Newton-Raphson scheme:

$$x_0 = 0 \Rightarrow x_1 = x_0 - \frac{y[x_0]}{y'[x_0]} \Rightarrow x_2 = x_1 - \frac{y[x_1]}{y'[x_1]} \Rightarrow \ldots \Rightarrow x_{n+1} = x_n - \frac{y[x_n]}{y'[x_n]}$$

iterate until $\left| y[x_{n+1}] \right| < TOL$ then $\Delta\gamma \cong x_{n+1}$

Elasto-plastic Tangent Modulus

The derivative $\mathrm{d}\sigma/\mathrm{d}\varepsilon$ is called **elasto-plastic tangent modulus**.
During plastic tensile loading $(d\varepsilon > 0, d\sigma > 0, d\varepsilon_p > 0)$, we have

$$df = d\sigma - dk = E(d\varepsilon - d\gamma) - \left(\frac{dk}{d\bar{\varepsilon}_p}\right) d\gamma = 0 \Rightarrow d\gamma = \frac{E}{E + \left(\frac{dk}{d\bar{\varepsilon}_p}\right)} d\varepsilon$$

and thus

$$d\sigma = E(d\varepsilon - d\gamma) = \frac{E\left(\frac{dk}{d\bar{\varepsilon}_p}\right)}{E + \left(\frac{dk}{d\bar{\varepsilon}_p}\right)} d\varepsilon \Rightarrow \boxed{\frac{d\sigma}{d\varepsilon} = \frac{E\left(\frac{dk}{d\bar{\varepsilon}_p}\right)}{E + \left(\frac{dk}{d\bar{\varepsilon}_p}\right)}}$$

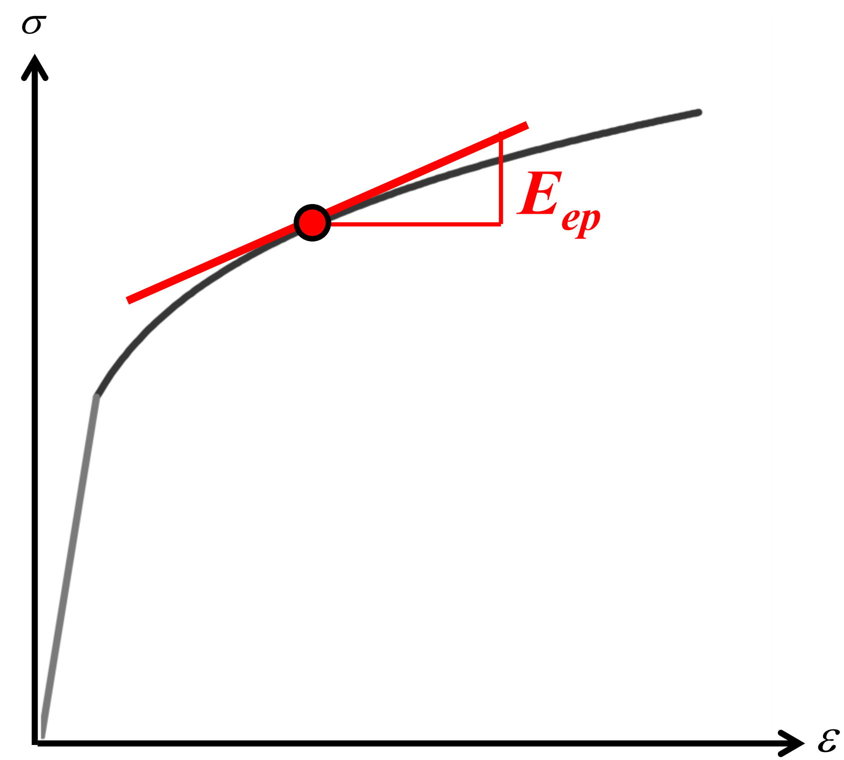
Elasto-plastic Tangent Modulus

Formally, we note the incremental stress-strain response as

$$d\sigma = E_{ep}(d\varepsilon)$$

with

$$E_{ep} = \begin{cases} E & \text{if} \quad \dot{\gamma} = 0 \\ \dfrac{E\left(\dfrac{dk}{d\bar{\varepsilon}_p}\right)}{E + \left(\dfrac{dk}{d\bar{\varepsilon}_p}\right)} & \text{if} \quad \dot{\gamma} > 0 \end{cases}$$



Summary: Return Mapping Algorithm

Applied total strain increment

$$\Delta\varepsilon$$

State variables at time t_n

$$\varepsilon_n^p, \ \bar{\varepsilon}_n^p$$

Calculate Trial State

$$\sigma_{n+1}^{trial}, \ f_{n+1}^{trial}$$

$$f_{n+1}^{trial} \le 0$$

$$\Delta\gamma = 0$$

OUTPUT:

State variables at time t_{n+1}

$$\varepsilon_{n+1}^p = \varepsilon_n^p$$

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p$$

Stress at time t_{n+1}

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon_{n+1}^p)$$

$$f_{n+1}^{trial} > 0$$

Solve: $\left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - k[\bar{\varepsilon}_n^p + \Delta\gamma] = 0$

$$\Rightarrow \Delta\gamma > 0$$

OUTPUT:

State variables at time t_{n+1}

$$\varepsilon_{n+1}^p = \varepsilon_n^p + (\Delta\gamma)\mathrm{sign}[\sigma_{n+1}^{trial}]$$

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p + \Delta\gamma$$

Stress at time t_{n+1}

$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon_{n+1}^p)$$

Reading Materials for Lecture #4

- J.C. Simo and T.J.R. Hughes, “Computational Inelasticity” (first chapter): http://link.springer.com/book/10.1007%2Fb98904
- M.E. Gurtin, E. Fried, L. Anand, “The Mechanics and Thermodynamics of Continua”, Cambridge University Press, 2010.