CER-ETH – Center of Economic Research at ETH Zurich

Open Rule Legislative Bargaining

V. Britz, H. Gersbach

Working Paper 20/346
November 2020

Economics Working Paper Series

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Open Rule Legislative Bargaining

Volker Britz*　　Hans Gersbach*

This version: October 8, 2020

Abstract

We consider non–cooperative bargaining on the division of a surplus under simple majority rule. We use the "open rule" bargaining protocol as originally suggested by Baron and Ferejohn (1989): Proposals can be amended before they are voted on. It is widely known that there are significant gaps in our understanding of open rule bargaining. In order to address these gaps, we provide a fresh analysis of a particularly simple class of equilibria. Our results shed new light on the efficiency and fairness implications of using an open vs. closed rule in bargaining. In particular, our results on the open rule model suggest that equilibrium delays tend to be longer, and surplus allocations tend to be less egalitarian than originally predicted by Baron and Ferejohn. Understanding the efficiency and fairness properties of different bargaining protocols is crucial for institutional design.

Keywords: Bargaining, Legislatures, Open Rules, Baron and Ferejohn, Stationary Equilibrium

JEL Codes: C72, C78, D72

**CER–ETH – Center of Economic Research at ETH Zurich, Zürichbergstrasse 18, 8092 Zurich, Switzerland. E–mail addresses: vbritz@ethz.ch (Volker Britz) and hgersbach@ethz.ch (Hans Gersbach).* *The authors would like to thank Tilman Börgers, Georgy Egorov, Jürgen Eichberger, Hans Haller, Volker Hahn, Matthew Jackson, Roger Myerson, Bernhard Pachl, and seminar participants at ETH Zürich, Heidelberg University, and Princeton University for helpful comments. Tettje Halbertsma has provided excellent research assistance.*

1 Introduction

We consider open rule bargaining on the division of a surplus under a simple majority rule. There is a vast game-theoretic literature on the resolution of surplus division problems through bargaining, see for instance the seminal papers by Rubinstein (1982) as well as Baron and Ferejohn (1989). The bargaining approach to surplus division problems has found important applications in political economy: The classical example is a parliament that negotiates about the allocation of funds from the government budget, while each member of the legislature wishes to obtain funds for projects in their own district. Such *legislative bargaining* models have been studied, among others, by Banks and Duggan (2000), Eraslan (2002), Battaglini and Coate (2007), and Battaglini et al. (2014). On the notion and current state of legislative bargaining, see Eraslan and Evdokimov (2019).

The game-theoretic bargaining literature emphasizes the importance of formal rules that structure the bargaining process. In particular, it assumes that negotiations consist of several rounds, where the rejection of a proposal triggers the next round. In these models, equilibrium outcomes depend crucially on who has the right to make a proposal at what point in time, and on how costly the delay in moving from one round to the next is.

Moreover, it matters how each round is structured. One important distinction can be made between "open rule" and "closed rule" bargaining. Under an *open rule,* a proposal may be amended one or several times before it is put to a vote. In contrast, under a *closed rule,* each proposal is immediately voted upon, allowing only Yes-or-No approval. A new proposal is only made when the previous one has been rejected. The seminal paper by Baron and Ferejohn (1989) offers a comparison of open and closed rule bargaining.

While immediate agreement is a standard result for closed rule bargaining, equilibrium delays typically occur under open rules. The tendency to delay agreement under an open rule can be understood as follows: Ideally, a proposer would want to ensure that his proposal is not amended before a vote. To this end, he would have to propose a surplus allocation that is sufficiently attractive for other players to discourage them from making any amendment. However, the proposer is uncertain about *which* player will get a chance to make an amendment. Hence, he has to weigh the risk of an amendment against the cost of discouraging other players from amendments. Typically, the optimal proposal gives sufficiently generous offers to some, but not all, players. As a result, there remains positive probability of an amendment, leading to delay.

In practice, open rules are very common in legislative decision-making. For instance, open rule procedures are an important part of legislative bargaining in the U.S. Congress (Oleszek (2011)). In particular, the House of Representatives makes use of a variety of rules, including open rules, structured rules, and closed rules.[1] Open rules are not only prevalent at the federal level in the United States, but also at the state level (Primo (2003)).

However, most of the bargaining-theoretic literature has focused on closed rules. In particular, this is true for most of the contributions following Baron and Ferejohn's seminal paper. The theoretical understanding of closed rule bargaining has been furthered considerably by the work of Ansolabehere et al. (2005), Banks and Duggan (2000, 2006), Diermeier and Feddersen (1998), Eraslan (2002), and McCarty (2000), among others.

Compared to bargaining games with a closed rule, Baron and Ferejohn's model of open rule legislative bargaining has received less attention. Primo (2007) shows how a proposer can randomize between different coalitions in open rule legislative bargaining. His findings indicate that the equilibrium found by Baron and Ferejohn cannot be unique. Fréchette et al. (2003) provide a theoretical and experimental investigation comparing closed and open rules. Falconieri (2004) gives a comparative analysis of open and closed rules in the context of lobbying and delegation.

Arguably, one reason for the lack of literature on open rule bargaining is that the model proposed by Baron and Ferejohn (1989) is not easily tractable. Neither

[1]Detailed information on the legislative processes in the House of Representatives can be found on the website of its Committee on Rules: https://rules.house.gov (retrieved on July 12th, 2019).

is it obvious what the right equilibrium concept for this model would be, nor is there a sharp equilibrium prediction as in the closed rule bargaining game. In the present paper, we address this gap in the literature: We propose a new approach to Baron and Ferejohn's model of open rule bargaining game. We introduce a suitable equilibrium refinement, which makes the analysis more tractable, and we obtain a sharp equilibrium prediction in the limit as the discount factor goes to one.

More specifically, we make the following contributions:

- We provide a rigorous definition of stationary strategies in the open rule legislative bargaining model. We define a class of equilibrium candidates that consists of relatively simple, and thus tractable, stationary strategies. We derive necessary and sufficient conditions under which such an equilibrium candidate is indeed a stationary subgame–perfect Nash equilibrium of the game.

- For the limit case, as the discount factor converges to one, we compute an explicit equilibrium prediction of the proposals, the payoffs, and the expected length of delay before an agreement.

- We compare the equilibrium outcome of open rule bargaining to that of the canonical closed rule bargaining process. While closed rule bargaining leads to immediate agreement, open rule bargaining typically leads to delays on the equilibrium path of play. We find that these equilibrium delays can be much longer than predicted by Baron and Ferejohn. We also show that the inefficiency inherent to an open rule can be so large that *all* players would be ex ante better off with a closed rule.

Our work is complementary to a stream of literature that analyzes bargaining with an endogenous status quo, see Anesi (2010), Diermeier and Fong (2011, 2012), and Bowen and Zahran (2012). In that class of bargaining models, negotiations continue even after an agreement has been reached. In each round, the status quo is given by the most recent agreement. This is different from bargaining under an open rule, where any agreement ends the game.

The rest of the paper is organized as follows: Section 2 contains the formal description of the open rule legislative bargaining game. In Section 3, we provide a rigorous definition of stationary strategies which is suitable for the analysis of this game. We provide a more detailed account of the relation between our present paper and Baron and Ferejohn's work in Section 4. In Sections 5 and 6, we conduct an equilibrium analysis of a slightly simplified version of the open rule bargaining game. In Section 7, we consider the limit as players are sufficiently patient and explicitly compute the equilibrium predictions. Afterwards, we use some numerical examples to illustrate our findings in Section 8. In Section 9, we argue that the main results and conclusions we have obtained for the simplified open rule bargaining game carry over to the original game. We offer some concluding remarks in Section 10. Most of the proofs are relegated to the appendix.

2 The open rule legislative bargaining game

We consider open rule bargaining with an odd[2] number of players, $n \geq 3$. The set of players is denoted by N, and we will frequently use i or j to index its members. There is a surplus of unit size to be divided among the players. Thus, the space of possible agreements is $\Delta^n = \{\theta \in \mathbb{R}^n_+ \,|\, \sum_{i \in N} \theta_i \leq 1\}$.[3] The decision to implement a particular surplus division is taken by simple majority voting, that is, it requires the approval of at least $(n+1)/2$ players. The bargaining process is structured in rounds $t = 0, 1, \ldots$. The number of rounds is potentially infinite.

Baron and Ferejohn's open rule bargaining process involves a complex chain of events in which players can make proposals, suggest amendments, choose between a proposal and an amendment, and eventually vote on the implementation of a

[2] We follow Baron and Ferejohn (1989) in assuming an odd number of players in order to avoid ties.

[3] Without loss of generality, we will focus on feasible agreements that satisfy $\sum_{i \in N} \theta_i = 1$. In particular, a proposer never finds it optimal to make a proposal that does not fully divide the available surplus.

proposal. In order to make this open rule bargaining process clear, we divide the description into the following three steps:

Step 1: Proposal on the floor

Consider any bargaining round t. Two cases must be distinguished: Either, bargaining round t begins with a *proposal on the floor,* or it begins *without a proposal on the floor.*

- *Proposal on the floor:* If round t begins with a proposal on the floor, then round t of the game directly proceeds to Step 2 below.

- *No proposal on the floor:* If round t begins without a proposal on the floor, then a *proposer* is randomly chosen from N with equal probability. Let us say that player i is chosen as the proposer. Then, player i chooses some proposal $\overline{\theta} \in \Delta^n$, which thereby becomes the proposal on the floor.[4] Now the game proceeds to Step 2.

Bargaining round t can only begin with a proposal on the floor if that proposal has been made in a previous round. Therefore, the initial bargaining round $t = 0$ begins without a proposal on the floor.

Step 2: Amendment or endorsement

Suppose that the proposal $\overline{\theta} \in \Delta^n$ made by some player $i \in N$ is on the floor in round t. Now, a new proposer is randomly chosen with equal probability from $N \setminus \{i\}$. Let us say that player j has been chosen. Then, player j decides whether to *endorse* or *amend* the proposal on the floor.[5]

- *Endorsement:* If player j *endorses* the proposal on the floor $\overline{\theta}$, then round t of the game proceeds directly to Step 3.

[4] Here, we renounce indexing $\overline{\theta}$ by i to ease presentation.

[5] The assumption here is that the original proposer cannot be chosen to endorse or amend his own proposal. The underlying idea is that a non-trivial endorsement is required for one's proposal before it can be voted on.

- *Amendment:* Suppose that player j chooses to amend the proposal $\overline{\theta}$ on the floor. He does so by announcing an *amendment* $\theta' \in \Delta^n$. Then, all players simultaneously cast votes in favor of the proposal on the floor $\bar{\theta}$ or in favor of the amendment θ'. If at least $(n+1)/2$ players vote in favor of θ', then bargaining round $t+1$ begins with the amendment θ' as the new proposal on the floor. If at least $(n+1)/2$ players vote in favor of $\overline{\theta}$, then bargaining round $t+1$ begins with the proposal $\overline{\theta}$ on the floor. Note that players can keep making amendments, and thus repeating Step 2, indefinitely. However, a new bargaining round begins every time a new amendment is made.

Step 3: Voting on an endorsed proposal

Now suppose that in some bargaining round t, a proposal on the floor $\overline{\theta}$ is endorsed by player j. Then, all players simultaneously cast votes in favor or against the endorsed proposal $\overline{\theta}$. Again, there are two cases:

- *Majority approval:* If at least $(n+1)/2$ players accept the endorsed proposal $\overline{\theta}$, then the game ends and $\overline{\theta}$ is implemented.

- *No majority approval:* If strictly less than $(n+1)/2$ players accept $\overline{\theta}$, then the game moves to round $t+1$. That bargaining round begins again in Step 1, without a proposal on the floor.

Note that a new bargaining round begins whenever either *(i)* an amendment is made (in Step 2), or *(ii)* an endorsed proposal is not approved by the majority (in Step 3). Every time a new bargaining round starts, a discount factor $\delta \in (0,1)$ is applied. This discount factor can be suitably interpreted as a measure for the players' impatience. Furthermore, we assume that players are risk-neutral. Thus, if a proposal $\theta \in \Delta^n$ is implemented in bargaining round t, then player i receives a payoff of $\delta^t \theta_i$. If no proposal is ever endorsed, or if no endorsed proposal is ever approved by the majority, then bargaining is trapped in *perpetual disagreement,* which gives all players zero payoffs. This completes the formal description of the open rule legislative bargaining game (henceforth ORBG) $G(\delta, n)$. It corresponds to the open rule bargaining game originally proposed by Baron and Ferejohn (1989).

Throughout most of this paper, we are going to analyze a slightly abridged version of the ORBG, which we call the *simplified ORBG,* and denote by $\widehat{G}(\delta, n)$. In Section 9, we will argue that the main results and conclusions derived for the simplified ORBG also hold in the ORBG itself. The simplified ORBG differs from the ORBG as follows: Whenever a player makes an amendment to a proposal on the floor in Step 2, the amendment immediately replaces the proposal on the floor, without a vote being held.

We make two remarks for a better understanding of bargaining power in this game:

First, consider a history of this game where players are in Step 3 and thus vote on an endorsed proposal. Their choice is either to stop bargaining and implement the proposal now, or to move back to Step 1 and start bargaining from scratch in the next round. This is similar to the choice that players make when responding to proposals in a closed rule bargaining game. In such games, the prospect of discounting discourages players from rejecting a proposal, which leads to a bargaining advantage for the proposer, often called the *proposer premium.* With an open rule, it seems intuitive that this proposer premium is shared between the player who has originally made the proposal, and the one who has endorsed it. One crucial question in this paper will be how many players share the proposer premium, and how it is divided.

Second, consider a history of the game where players vote between a proposal on the floor and an amendment. They decide whether the current proposal on the floor remains the proposal on the floor in the next round, or whether the amendment becomes the new proposal on the floor. Regardless of the outcome of such a vote, another round of bargaining is required to reach an agreement, and thus another round of discounting will occur either way. Loosely speaking, while players are under "time pressure" when they vote on an endorsed proposal, this time pressure does not affect them when choosing between a proposal on the floor and an amendment.

3 Stationary strategies

It is well-known that non-cooperative bargaining games with more than two players admit a wide multiplicity of subgame-perfect equilibrium allocations. The bargaining literature has focused on analyzing subgame-perfect equilibria in stationary strategies (SSPE). In closed rule bargaining games in the tradition of Rubinstein (1982), the meaning of a "stationary" strategy is straightforward: Each player makes the same proposal every time he is the proposer. Moreover, each player uses the same acceptance rule whenever he is a responder. In this paper, however, we depart from most of the bargaining literature by considering an open rule. In the open rule bargaining game, the appropriate definition of a "stationary" strategy is less obvious. The purpose of the present section is to define the stationary strategies that we use in our analysis. These strategies are "mixed" in the following sense: They allow players to choose lotteries over different actions only when they are making proposals or amendments. The purpose of these lotteries is to ensure that the proposer can treat all other players equally. That is, a proposer can choose a configuration of payoffs that he wants to offer to the other players, while leaving it to chance which payoff is offered to which player.

In order to make this idea more precise, let us define an *anonymous proposal* as a vector $\eta \in \Delta^n$ such that $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. Furthermore, let $\mathcal{M}(i)$ be the collection of $(n \times n)$-permutation matrices M such that $m_{1,i} = 1$. For any anonymous proposal $\eta \in \Delta^n$, let $\Theta^i(\eta) = \{\theta \in \Delta^n | \, \theta = M^\top \eta \text{ for some } M \in \mathcal{M}(i)\}$. Each of the proposals in $\Theta^i(\eta)$ assigns η_1 to player i, and the payoffs $\eta_2, \ldots, \eta_n$ to the remaining players. Of course, the assignment of the payoffs to the individual players $N \setminus \{i\}$ differs across the different elements of $\Theta^i(\eta)$.[6]

[6]The requirement that player i randomize uniformly among all the elements of $\Theta^i(\eta)$ adds an anonymity requirement to the stationarity requirement. Strategies in which proposals are made in a "stationary but not anonymous" way play no role in our paper. They would complicate the analysis without offering new insights. Therefore, it seems convenient to include the anonymity requirement in the definition of stationarity.

Formally, in the open rule legislative bargaining game $G(\delta, n)$, a *stationary strategy* for player i consists of the following four elements:

1. An *anonymous proposal* $\eta^i \in \Delta^n$, such that at every history at which there is no proposal on the floor and player i is the proposer, he randomizes uniformly among all the elements of $\Theta^i(\eta^i)$.

2. Let $P(\Delta^n)$ denote the power set of Δ^n. An *amendment rule* is a map $\psi^i : \Delta^n \times N \setminus \{i\} \to P(\Delta^n)$ which prescribes how player i should behave when he is the proposer and proposal θ, made by player k, is on the floor. If $\psi^i(\theta, k) = \{\theta\}$, player i endorses player k's proposal. If $\psi^i(\theta, k) = \{\theta'\}$ for some $\theta' \neq \theta$, player i makes an amendment θ' when k's proposal θ is on the floor. If $\psi^i(\theta, k)$ consists of $m \geq 2$ elements and $\theta \in \psi^i(\theta, k)$, then player i endorses k's proposal θ with probability $1/m$, and chooses every element of $\psi^i(\theta, k) \setminus \{\theta\}$ as an amendment with probability $1/m$. If $\psi^i(\theta, k)$ consists of at least two elements, and $\theta \notin \psi^i(\theta, k)$, then player i amends player k's proposal θ. In particular, he chooses the amendment from $\psi^i(\theta, k)$ uniformly at random.

3. A *selection rule* $\chi^i : \Delta^n \times \Delta^n \times N \times N \to \{\text{Proposal, Amendment}\}$ indicates player i's behavior at histories where he votes between a proposal and an amendment. This voting decision can be conditioned on the proposal on the floor, on the amendment, and on the identities of the players who have made the proposal on the floor and the amendment.

4. An *acceptance rule* $A^i \subset \Delta^n$ describes player i's voting decisions at histories where he votes on an endorsed proposal. More precisely, player i votes in favor of an endorsed proposal θ if and only if $\theta \in A^i$. Of course, the set A^i is specified independently of the history of play.

We use $\sigma^i = (\eta^i, A^i, \psi^i, \chi^i)$ to describe the *stationary strategy* for player i and we write $\sigma = (\sigma^1, \ldots, \sigma^n)$ for a profile of stationary strategies. We note that our definition of stationary strategies implies the following: When there is no proposal on the floor, a proposer can only make an anonymous proposal. Amendments, however, need not be anonymous. The reason is that we want to allow an amendment to condition on the identity of the player who has made the proposal on the floor. More specifically, we will be interested in amendments which permute the amounts offered to the current and previous proposers, while leaving the remaining $n-2$ components of the proposal unchanged. Given this definition of a stationary strategy, the equilibrium concept is perfectly standard: Indeed, a *stationary subgame-perfect equilibrium (SSPE)* is a profile of stationary strategies that is a subgame-perfect equilibrium. In the simplified ORBG, a stationary strategy consists of an anonymous proposal, an amendment rule, and an acceptance rule, while a selection rule is redundant.[7]

4 Relation to Baron and Ferejohn

Although the present paper deals with the open rule bargaining model proposed by Baron and Ferejohn (1989), our analysis and results differ from theirs in several respects. In this section, we discuss the fundamental reasons for these differences:

1. Baron and Ferejohn's equilibrium analysis lacks a comprehensive description of the relevant strategy profiles. In particular, their equilibrium strategies are not fully specified off the path of play. This is problematic at histories where players vote on whether or not to replace the current proposal on the floor with an amendment. More specifically, Baron and Ferejohn impose that a player votes in favor of the amendment if he is "indifferent" between the proposal on the floor and the amendment. Unfortunately, it is not straightforward what it means to be indifferent between the proposal on the floor and the amendment: Player i's preferences over the proposal on the floor, say $\overline{\theta}$, and the amendment, say θ', do not only depend on the components $\overline{\theta}_i$ and θ'_i, but also on the probabilities with which either $\overline{\theta}$ or

[7] Mutatis mutandis, the definition of an SSPE in the simplified ORBG corresponds to that in the ORBG, to which we will return in Section 9.

θ' will be endorsed or amended in the future. For instance, even if $\bar{\theta}_i < \theta'_i$ player i may want to vote in favor of $\bar{\theta}$ because he believes that $\bar{\theta}$ will be endorsed with a higher probability than θ'. Along a path of play of Baron and Ferejohn's supposed SSPE, a proposal on the floor and an amendment always have the same probability of being endorsed. However, this is no longer true off the equilibrium path. In this paper, we work around this problem in two different ways: First, we analyze the simplified ORBG in which the problem is redundant. Second, in Section 9, we return to the original ORBG and show that players' best-responses to deviations from SSPE must have a certain recursive structure. Therefore, we can do an equilibrium analysis without explicitly determining the optimal voting behavior for each player, for each proposal on the floor, and a for each amendment. This analysis confirms that the results and conclusions obtained in the simplified ORBG carry over to the ORBG itself.

2. Baron and Ferejohn tacitly assume that a player who is willing to vote for a given proposal is also willing to endorse it. However, we will demonstrate that this need not be true: We will see that there may be players who would want to amend a proposal if they had the chance to do so, but who would nevertheless want to vote in favor of that same proposal once it had been endorsed.[8] Taking this possibility into account changes some of the analysis and conclusions. In particular, we find longer equilibrium delays and less egalitarian allocations than Baron and Ferejohn.

[8]This problem with Baron and Ferejohn's analysis was recognized earlier in a working paper by Fahrenberger and Gersbach (2007). In the present paper, we adopt a new approach that differs from the one in Baron and Ferejohn (1989) as well as from Fahrenberger and Gersbach (2007): We focus on a class of stationary equilibria that involves particularly simple amendment rules which we will call "simple swap." This makes the problem more tractable than in any previous work we are aware of. Based on our new approach, we can construct and test equilibrium candidates for any values of the model parameters. Within the class of SSPE we consider, we can explicitly compute the limit of equilibrium payoffs as $\delta \to 1$ and the equilibrium number of players who endorse a proposal.

3. In the present paper, we look at a class of stationary equilibria with the following property: On the equilibrium path, whenever a player amends a proposal, he does so while leaving unchanged the payoffs offered to all players other than himself and the player who has made the previous proposal. Put another way, an amendment merely permutes two components of a proposal. We will call this kind of amendment a *simple swap.* Our focus on this class of stationary equilibria differs from the approach taken by Baron and Ferejohn. It allows us to express the proposals and payoffs associated with the equilibrium candidates as solutions to a relatively simple and tractable system of linear equations.

The relation of the present paper to Baron and Ferejohn's work can be summarized as follows: We point out that the strategy profiles which they claim to be equilibria are not fully specified. We have found it impossible to write down equilibrium strategies which generally support Baron and Ferejohn's supposed equilibrium payoffs. Whether such equilibrium strategies exist at all remains an open question. Our analysis focuses on an alternative class of stationary equilibria that is more easily tractable. Based on this class of stationary equilibria, we obtain results that qualify some of Baron and Ferejohn's conclusions.

5 Equilibrium candidates for the simplified ORBG

In this section, we entirely focus on the simplified ORBG, and discuss a particular family of stationary strategy profiles that we call *k-candidates with simple swaps.* Such a stationary strategy profile has the following properties:

- On the path of play induced by a k-candidate with simple swaps, whenever a player amends a proposal on the floor, he does so by simply swapping his component of the proposal on the floor with that of the player who has made the proposal on the floor.
- Every proposal and every amendment made on a path of play of a k-candidate with simple swaps has the following structure: The proposer offers

k players a payoff that makes them willing to endorse the proposal, and to vote in its favor. If $k \leq \frac{n-1}{2}$, the proposer offers an additional $\frac{n-1}{2} - k$ players a payoff that makes them willing to vote in favor of the proposal once it has been endorsed, but not to endorse it themselves.

In the open rule legislative bargaining game, one obtains a multitude of equilibria by changing the way in which amendments reshuffle the surplus allocation relative to proposals, see Primo (2003). To the best of our knowledge, we are the first to use simple swaps to reshuffle allocations. This choice simplifies the equations which determine the equilibrium variables. A player who makes an amendment is not interested in the other players' payoff, so that it seems natural to assume that he would choose the simplest possible reshuffling. One way in which this idea could be formalized is to assume that writing an amendment requires a small amount of costly effort, and this cost depends on how different the new bill is from the old one. In such circumstances, reshuffling allocations with "simple swaps" would be chosen.

Take the number of players n and the discount factor δ as given. For any $k = 1, \ldots, n-1$, let the quadruple (V_k, W_k, X_k, Y_k) be defined as the solution to the following system of equations:

$$V_k = \left(\frac{k}{n-1}\right) X_k + \left(\frac{n-1-k}{n-1}\right) \delta W_k, \tag{1}$$

$$W_k = \frac{Y_k - V_k}{n-1}, \tag{2}$$

$$X_k = 1 - k\delta V_k - \max\left\{0, \frac{n-1}{2} - k\right\} \left(\frac{\delta}{n}\right) Y_k, \tag{3}$$

$$Y_k = \frac{k}{\delta k + (1-\delta)(n-1)}. \tag{4}$$

We will show below that a solution to this system exists. Before proceeding to the formal definition of the equilibrium candidates, let us intuitively describe how the variables (V_k, W_k, X_k, Y_k) enter the construction of a k-candidate with simple swaps: The proposer always offers the amount X_k to himself. If his proposal is endorsed, it will also be accepted, and so the proposer will receive X_k. If the proposal on the floor is not endorsed, then the current proposer will certainly not be the proposer again in the next round. He may, however, be chosen to amend the proposal in the next round.[9] In this case, he will receive the expected payoff of a responding player. Hence, anticipating one round of discounting, the current proposer expects δW_k in case his proposal is rejected. Hence, the proposer's expected payoff is a weighted average of X_k and δW_k.

The weight given to X_k is the probability that the proposal is endorsed. This probability equals $k/(n-1)$ because the proposer gives k out of the $n-1$ responding players an incentive to endorse the proposal. It will turn out that δV_k is the amount that makes a responding player exactly indifferent between endorsing the proposal on the floor and making an amendment. Indeed, it is the expected payoff from making an amendment: A player who makes an amendment triggers one round of discounting, and then takes the place of the proposer. These considerations explain Eqn. (1).

It will turn out that a player who is willing to endorse a proposal is also willing to vote in its favor once it has been endorsed. Therefore, if $k \geq (n-1)/2$, a majority of players votes in favor of the endorsed proposal. If $k \leq (n-3)/2$, however, the proposer and the k players willing to endorse the proposal on the floor do not form a majority. Hence, the proposer must convince $\frac{n-1}{2} - k$ additional players to vote for the proposal on the floor once it has been endorsed. The expected continuation payoff for any player after the rejection of an endorsed proposal is $\frac{\delta}{n}(V_k + (n-1)W_k)$. Writing Y_k for the quantity $V_k + (n-1)W_k$ as in Eqn. (2), this explains Eqn. (3) above. All that remains to be explained is Eqn. (4). The quantity Y_k is the sum of the expected payoffs to all players. Equivalently, it can be thought of as the value of the surplus discounted by the expected delay. If the very first proposal which is made is endorsed, the total surplus of size one is divided. If t amendments are made before the t^{th} amendment is endorsed, the surplus divided is of size δ^t. Since each proposal is endorsed with probability

[9] We stress that the player making the amendment is randomly chosen from the $n-1$ remaining players and that the amendment is a simple swap. No matter which player is selected, the current proposer's continuation payoff after a rejection of his proposal corresponds to the payoff of a current responder.

$k/(n-1)$, the probability that the t^{th} amendment is endorsed is $\left(1-\frac{k}{n-1}\right)^t \left(\frac{k}{n-1}\right)$. We note that $1-k/(n-1)<1$ since $1 \leq k \leq n-1$. Summing over t from zero to infinity and rewriting yields the expression in Eqn. (4).

For the analysis in the remainder of the paper, an important auxiliary result is that the variables (V_k, W_k, X_k, Y_k) as defined in Eqns. (1)-(4) are strictly positive. The formal claim is stated in Proposition 1 below. The proof is provided in the appendix.

Proposition 1. *For every $k = 1, \ldots, n-1$, the system of Eqns. (1)-(4) has a unique solution. Furthermore: (i) If $k = n-1$ and $\delta = 1$, then $V_k = W_k > 0$. (ii) For any other choices of $k = 1, \ldots, n-1$ and $\delta \in (0, 1]$, it holds that $V_k > W_k > 0$. (iii) For any $k = 1, \ldots, n-1$, all components of solutions (V_k, W_k, X_k, Y_k) to the system of equations (1)-(4) are strictly positive.*

To provide a formal definition of the k-candidate with simple swaps, we need the following notation: For any proposal $\theta \in \Delta^n$ and any two players $i, j \in N$, let $\pi_{i\leftrightarrow j}(\theta)$ be the permutation of θ which swaps components i and j, while leaving all other components unchanged. Let $H^{\emptyset}$ be the set of histories at which there is no proposal on the floor. Let H^{θ} be the set of histories at which the proposal θ is on the floor. In particular, let $H^{i,\theta} \subset H^{\theta}$ be the set of histories at which the proposal on the floor is θ, and the author of that proposal is player i. Finally, let $H^{i,\theta}_j \subset H^{i,\theta}$ be the set of histories at which the proposal on the floor is θ, the author of that proposal is player i, and the current proposer is player j.

We now provide a formal definition of the k-candidate with simple swaps:

Definition 1. *Consider the simplified ORBG $\widehat{G}(\delta, n)$. Let (V_k, W_k, X_k, Y_k) be defined as solutions to Eqns. (1)-(4). For every $k = 1, \ldots, n-1$, a profile of stationary strategies is a* k-candidate with simple swaps *if the following hold:*

1. *At every history $h \in H^{\emptyset}$, the proposer makes an anonymous proposal η which gives himself X_k, gives δV_k to k other players, gives $\frac{\delta}{n}((n-1)W_k + V_k)$ to $\max\{0, \frac{n-1}{2} - k\}$ more players, and zero to all remaining players.*

2. *Consider a history $h \in H_j^{i,\theta}$. At such a history, the proposer is $j \in N$, and the proposal on the floor θ was made by player $i \in N$. Suppose that $\theta \in \Theta^i(\eta)$, where η is some anonymous proposal. Player j endorses the proposal on the floor θ if and only if $\theta_j \geq \delta V_k$. Otherwise, he makes the amendment $\pi_{i \leftrightarrow j}(\theta)$. Now suppose that $\theta \notin \Theta^i(\eta)$. Player j endorses θ if and only if there are at least $(n-1)/2$ players l in $N \setminus \{i\}$ with $\theta_l \geq \frac{\delta}{n}((n-1)W_k + V_k)$ and, moreover, it holds that $\theta_j \geq \delta V_k$. Otherwise, player j randomly chooses an amendment from $\Theta^j(\eta)$ with equal probability.*

3. *Whenever player i votes on an endorsed proposal θ, he votes in favor if and only if $\theta_i \geq \frac{\delta}{n}((n-1)W_k + V_k)$.*

Note that the three points of Definition 1 above correspond to the elements of a stationary strategy as defined in Section 3: Point 1 describes the *anonymous proposal,* Point 2 specifies the *amendment rule,* and the *acceptance rule* is spelled out in Point 3. Recall that, since we are considering the simplified ORBG, it is redundant to specify a selection rule. Since $V_k \geq W_k$, according to Proposition 1, we have $\frac{\delta}{n} Y_k \leq \delta V_k$, and thus a player who is willing to endorse a proposal will also vote in its favor. Finally, we conclude from Proposition 1 above that a proposer offers the highest share of surplus to himself. Indeed, we obtain the following corollary. The proof can be found in the appendix.

Corollary 1. *(i) If $k = n - 1$, then $X_k = V_k$. (ii) For any other choices of $k = 1, \ldots, n-2$ and $\delta \leq 1$, it holds that $X_k > \delta V_k$.*

By restricting attention to the simplified ORBG and to k-candidates with simple swaps, the analysis of stationary strategy profiles becomes more tractable than in any previous work on open rule bargaining that we know of. There are two reasons for this:

- In a simplified ORBG, there is a strategic equivalence between subgames that start at a node where a proposal on the floor can be amended or endorsed, and subgames that start at a node where no proposal is on the floor.

- If a k-candidate with simple swaps is played, the actions taken after a history $H_j^{i,\theta}$ do not depend on whether the proposal θ was originally made as an amendment to some other proposal, or whether it was made at a history without a proposal on the floor.

Intuitively, in a k-candidate with simple swaps played in a simplified ORBG, a player who can make an amendment to a proposal on the floor can achieve the same payoff (up to discounting) that he could also achieve if he were the proposer at a history without a proposal on the floor.

6 Testing equilibrium candidates

In this section, we introduce a test to verify whether a k-candidate with simple swaps is an SSPE of a simplified ORBG.

Proposition 2. *A k-candidate with simple swaps is an SSPE of the simplified ORBG if and only if there is no profitable unilateral deviation from it at any history $h \in H^{\emptyset}$.*

Proof. It is easily verified that a profitable unilateral deviation from a k-candidate with simple swaps is impossible at histories where players vote on an endorsed proposal. Thus, we have to focus on the possibility of profitable unilateral deviations from k-candidates with simple swaps at histories where a proposal can be made. Recall that a proposal can be made at histories in $H^{\emptyset}$ or at histories in H^{θ} through an amendment. Consider a history $h \in H^{\theta}$ at which player i chooses to endorse or amend the proposal on the floor θ. Suppose that player i obtains an expected payoff of $\delta\widetilde{V}$ if he makes the amendment $\widetilde{\theta}$. Now consider a history in $H^{\emptyset}$ where player i is the proposer. At that history, player i can obtain a payoff of $\widetilde{V}$ by proposing $\widetilde{\theta}$. When player i proposes at a history in $H^{\emptyset}$, his expected payoff is V_k, and when he proposes at a history in H^{θ}, his expected payoff is δV_k. Thus, if player i has a profitable deviation at a history in H^{θ}, then he also has a profitable deviation at a history in $H^{\emptyset}$. $\square$

Proposition 2 shows that, in order to test whether a k-candidate with simple swaps is an SSPE in the simplified ORBG, we only have to consider profitable unilateral deviations at histories in $H^{\emptyset}$. In the simplified ORBG, all histories at which a particular player can make a proposal or an amendment are "equivalent" in the sense that the continuation game is the same.

Next, we consider deviations from the k-candidate with simple swaps. Suppose that player i makes a unilateral one-shot deviation from the k-candidate with simple swaps by proposing the amount δV_k to $k+1$ instead of k players, proposing $\left(\frac{\delta}{n}\right) Y_k$ to $\max\left\{0, \frac{n-1}{2} - (k+1)\right\}$ players, and proposing to take the remainder for himself. We denote the proposer's expected gain from such a deviation by λ_k^+. Similarly, let λ_k^- denote the proposer's expected gain from a unilateral one-shot deviation under which the proposer offers the amount δV_k only to $k-1$ instead of to k players, and offers $\left(\frac{\delta}{n}\right) Y_k$ to $\max\left\{0, \frac{n-1}{2} - (k-1)\right\}$ players. In order to understand the expressions below, recall that Point 2 in Definition 1 says that if after the deviation, the proposal is amended, then that amendment is again based on the anonymous proposal associated with the k-candidate with simple swaps.

$$\lambda_k^+ = \begin{cases} 0 & \text{if } k = n-1, \\ -\left(\frac{k+1}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{(n-1)/2, \ldots, n-2\}, \\ -\left(\frac{k+1}{n-1}\right)\left(\delta V_k - \delta\left(\frac{1}{n}\right) Y_k\right) + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{1, \ldots, (n-3)/2\}. \end{cases} \tag{5}$$

$$\lambda_k^- = \begin{cases} \left(\frac{k-1}{n-1}\right)\delta V_k - \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{(n+1)/2, \ldots, n-1\}, \\ \left(\frac{k-1}{n-1}\right)\left(\delta V_k - \delta\left(\frac{1}{n}\right) Y_k\right) - \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{2, \ldots, (n-1)/2\}, \\ 0 & \text{if } k = 1. \end{cases} \tag{6}$$

It is straightforward that the k-candidate with simple swaps can only be an SSPE if λ_k^+ and λ_k^- are non-positive. The next proposition implies the converse: If the proposer has any profitable deviation, then either λ_k^+ or λ_k^- must be strictly positive. In particular, if the proposer cannot gain by offering δV_k to one additional player, or to one player less, then he cannot gain either by offering δV_k to any number of players other than k.

Proposition 3. *If there exists $\widehat{\theta} \in \Delta^n$ such that proposing $\widehat{\theta}$ instead of the proposal prescribed by the k-candidate with simple swaps is a profitable deviation for the proposer, then either $\lambda_k^+ > 0$ or $\lambda_k^- > 0$.*

The proof is relegated to the appendix. Propositions 2 and 3 lead to the following:

Proposition 4. *A k-candidate with simple swaps is an SSPE of the simplified ORBG if and only if $\lambda_k^+ \leq 0$ and $\lambda_k^- \leq 0$.*

Baron and Ferejohn (1989) have already argued that a proposer should choose a large k when δ is small. The intuition is as follows: For small δ, any delay is very costly. Hence, it seems intuitive that the proposer finds it optimal to ensure immediate endorsement and acceptance of his proposal. In order to ensure that his proposal is endorsed immediately with probability one, he needs to make all other players willing to endorse it. The proposition below formalizes this argument. The proof is given in the appendix.

Proposition 5. *If δ is sufficiently small, then the $(n-1)$-candidate with simple swaps is an SSPE.*

In an $(n-1)$-candidate with simple swaps, immediate agreement is reached on an allocation which gives $\frac{1}{1+\delta(n-1)}$ to the proposer and $\frac{\delta}{1+\delta(n-1)}$ to each of the other players. This corresponds exactly to the payoff division that one would expect under closed rule unanimity bargaining.[10]

It is important to emphasize that our analysis so far does not yield results on the "existence" or "uniqueness" of k-candidates that are SSPE in the simplified ORBG. Without any restrictions on the parameters δ and n, we do not claim that there must be a k so that the k-candidate with simple swaps is an SSPE. We do

[10] A closed rule bargaining game with linear utility functions and equal recognition probabilities is a special case of the games studied in Laruelle and Valenciano (2008) and Britz et al. (2014).

not show either that there is at most one k so that the k-candidate with simple swaps is an SSPE. In Section 8, however, we do consider some numerical examples. In each of the examples, it does turn out that exactly one k-candidate with simple swaps is an SSPE.

In the next section, we consider k–candidates with simple swaps that are SSPE in the limit as $\delta \to 1$ and $n \geq 9$. In that case, we do obtain results which show that there is a unique k such that the k–candidate with simple swap is an SSPE.

7 Stationary equilibrium with patient players

So far, we have defined a family of equilibrium candidates in the simplified ORBG, and we have introduced a test to verify which of these candidates are indeed SSPE of the simplfied ORBG. In the present section, we will focus on the case where the discount factor is sufficiently close to one. In that case, we will explicitly compute the limit of SSPE payoffs.

As a first step, we show that for sufficiently large δ and n, a k-candidate with simple swaps can only be an SSPE if $k \leq (n-3)/2$. Consequently, for sufficiently large δ and n, a k-candidate with simple swaps can only be an SSPE if there are players who are willing to vote in favor of proposals that they would not be willing to endorse.

We say that a k-candidate involves *majority endorsement* if $k \geq (n-1)/2$, and it involves *super-majority endorsement* if $k \geq (n+1)/2$. Intuitively, (super-)majority endorsement means that the proposer and the players who are willing to endorse his proposal form a (super-)majority.

Proposition 6. *If a k-candidate with super-majority endorsement is an SSPE of the simplified ORBG, then it holds that $\delta(n+1) \leq 4$.*

The proof of Proposition 6 can be found in the appendix.

One implication is that a k-candidate with super-majority endorsement cannot be an SSPE if δ is sufficiently close to one.[11] Another implication is that, for any

[11] This follows from Proposition 6 for $n \geq 5$. For the special case with $n = 3$, it can be verified

given $\delta > 0$, a k-candidate with super-majority endorsement cannot be an SSPE if the number of players satisfies $n > \frac{4}{\delta} - 1$, and thus if the number of players is not too small.

Proposition 7. *For any $\delta \in (0,1)$, there exists an odd integer n_δ sufficiently large, so that a k-candidate with majority endorsement cannot be an SSPE of the simplified ORBG if $n \geq n_\delta$.*

The proof of Proposition 7 can be found in the appendix. Intuitively, the argument runs as follows: Consider an $((n-1)/2)$-candidate. Suppose that a proposer makes a unilateral deviation under which he offers one player only $\frac{\delta}{n}Y_k$ instead of δV_k. This player would no longer be willing to endorse the proposal. However, he would still be willing to vote in favor of the proposal once it was endorsed. In the formal proof of Proposition 7, we derive a parameter condition under which this deviation is profitable for the proposer, and we show that this condition boils down to an upper bound on n. In particular, the corollary below follows from the proof of Proposition 7.

Proposition 8. *1. Suppose that $n \leq 7$. If δ is sufficiently close to one, then the $((n-1)/2)$-candidate is an SSPE of the simplified ORBG.*

2. Suppose that $n \geq 9$. If δ is sufficiently close to one, a k-candidate with majority endorsement cannot be an SSPE of the simplified ORBG.

The proof of Proposition 8 can be found in the appendix.

The two propositions above reflect the gist of how our results differ from those in Baron and Ferejohn (1989): Their findings suggest that, for δ sufficiently high, equilibrium proposals are endorsed by $(n-1)/2$ players.[12] Hence, the probability

by direct computation: Plugging in $n = 3$ and $k = 2$ as well as $\delta = 1$ into Eqns. (1)-(4) yields $X_2 = V_2 = W_2 = 1/3$. Plugging into Eqns. (5)-(6), we see that $\lambda_2^+ = 0$ and $\lambda_2^- = 1/6 > 0$. Indeed, the 2-candidate is not an SSPE.

[12]To be more precise, when we cite the findings and conclusions of Baron and Ferejohn, we refer to the strategies that they claim to be equilibria, and to their numerical examples. Baron and Ferejohn do not claim equilibrium uniqueness, and nor do we.

of an endorsement is one half for each proposal on the equilibrium path. Since an endorsed proposal is implemented in equilibrium, this further implies that the game ends in bargaining round t with probability $\frac{1}{2^{t+1}}$, which corresponds to an expected equilibrium delay of length one. Baron and Ferejohn's supposed equilibrium is based on strategies in which amendments are made in a more complicated way than with the simple swaps used here. We conclude that k-candidates with simple swaps that involve majority endorsement are not SSPE when $n \geq 9$ and δ is close to one. In Section 8, we provide an example with $n = 51$ and δ close to one in which only 7 (rather than 25) of the 50 responding players endorse the proposal in equilibrium. In that example, the probability that any particular proposal is endorsed on the equilibrium path is only $7/50 = 0.14$ (instead of $1/2$). As a result, the expected length of equilibrium delay is more than six times as long as it would be with $k = 25$.[13]

So far, we have shown that for $n \geq 9$ and sufficiently large δ, a k-candidate with simple swaps can only be an SSPE if $k \leq (n-3)/2$. In that case, Eqns. (1)-(6) reduce to expressions that are continuous in δ. Hence, computing the limit behavior of the variables $V_k, W_k, X_k, Y_k, \lambda_k^+$, and λ_k^- when δ converges to one is equivalent to computing them while setting δ equal to one. Indeed, let us restate Eqns. (1)-(6) for $\delta = 1$ and $k \leq (n-3)/2$:

$$\overline{V}_k = \left(\frac{k}{n-1}\right)\overline{X}_k + \left(\frac{n-1-k}{n-1}\right)\overline{W}_k, \tag{7}$$

$$\overline{W}_k = (1-\overline{V}_k)/(n-1), \tag{8}$$

$$\overline{X}_k = 1 - k\overline{V}_k - \left(\frac{n-1-2k}{2n}\right), \tag{9}$$

$$\overline{\lambda}_k^+ = -\left(\frac{k+1}{n-1}\right)\left(\overline{V}_k - 1/n\right) + \left(\frac{1}{n-1}\right)(\overline{X}_k - \overline{W}_k), \tag{10}$$

$$\overline{\lambda}_n^- = \left(\frac{k-1}{n-1}\right)(\overline{V}_k - 1/n) - \left(\frac{1}{n-1}\right)(\overline{X}_k - \overline{W}_k). \tag{11}$$

[13] On the path of play of a k-candidate with simple swaps, the probability that the proposal on the floor is endorsed (and then approved by majority voting) is $\frac{k}{n-1}$ in every round. Thus, the expected length of equilibrium delay can be written as $\frac{k}{n-1}\sum_{t=0}^{\infty}(1-\frac{k}{n-1})^t\, t = \frac{n-1-k}{k}$. For any n, if $k = \frac{n-1}{2}$, the expected length of delay is always one. In our example with $n = 51$ and $k = 7$, however, it is $\frac{51-1-7}{7} = \frac{43}{7} \approx 6.14$.

Eqns. (7)-(9) are a system of three independent linear equations in three unknowns. We can solve this system for the variables $\overline{V}_k, \overline{W}_k$, and $\overline{X}_k$, and substitute the resulting expressions into Eqns. (10)-(11) to obtain:

$$\overline{\lambda}_k^+ = (n-k-k^2)\left(\frac{n-1}{2n}\right)\left(\frac{1}{k^2(n-1)-k+n(n-1)}\right), \tag{12}$$

$$\overline{\lambda}_k^- = (k^2-k-n)\left(\frac{n-1}{2n}\right)\left(\frac{1}{k^2(n-1)-k+n(n-1)}\right). \tag{13}$$

Recalling that $n \geq 5$ and $1 \leq k \leq n-1$, it is easily verified that

$$\left(\frac{n-1}{2n}\right)\left(\frac{1}{k^2(n-1)-k+n(n-1)}\right) > 0.$$

Therefore, $\overline{\lambda}_k^+ > 0$ if and only if $n-k-k^2 > 0$, and $\overline{\lambda}_k^- > 0$ if and only if $k^2-k-n > 0$. Combined with Proposition 4, this implies Theorem 1.

Theorem 1. *Suppose that $n \geq 9$ and δ is sufficiently close to one. A k-candidate with simple swaps is an SSPE of the simplified ORBG if and only if the inequalities $k \leq (n-3)/2$ and $k^2-k \leq n \leq k^2+k$ are satisfied.*[14]

It follows that, for δ sufficiently close to one, the k-candidate equilibrium with simple swaps is an SSPE if $k \leq (n-3)/2$ and

$$k \in \left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right].$$

Corollary 2. *Suppose that $n \geq 9$ and δ is sufficiently close to one. There exists a unique $k = 1, \ldots, (n-3)/2$ such that the k-candidate with simple swaps is an SSPE of the simplified ORBG. This k is the unique integer contained in the interval $\left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right]$.*

The proof of Corollary 2 can be found in the appendix.

Proposition 8 and Theorem 1 readily imply the following existence result:

Corollary 3. *Suppose that δ is sufficiently close to one. There exists $k = 1, \ldots, n-1$ such that the k-candidate with simple swaps is an SSPE of the simplified ORBG.*

[14] Recall that n is odd, and so the inequalities will always be strict.

8 Numerical illustration

8.1 Optimal choice of k

In this section, we give some numerical examples for our findings.[15]

For $n = 51$ and various values of δ, Table 1 shows the unique value of k such that the k-candidate with simple swaps is an SSPE of the simplified ORBG.[16] Recall that the payoffs induced by a k-candidate with simple swaps are given as the solutions to a system of equations which is continuous at $\delta = 1$. Therefore, we can find the limit values as δ converges to one by considering the relevant equations for $\delta = 1$.

For values of δ close to zero, we see that $k = n - 1 = 50$. This exemplifies the finding in Proposition 5 that a proposal is endorsed by all players in an SSPE when discounting is sufficiently severe. As δ increases, the equilibrium value of k decreases, which is in line with Baron and Ferejohn's findings. However, Baron and Ferejohn predict that the equilibrium value of k falls only until it reaches $(n-1)/2 = 25$. Again, this is because they do not take into account that players who do not endorse a proposal may still vote for it. In our model, however, the equilibrium value of k continues to fall. For δ close to one, it eventually reaches 7, which is indeed the integer close to $\sqrt{n} = \sqrt{51} \approx 7.1$.

δ	Equilibrium value of k
0.02	50
0.03	41
0.08	25
0.09	24
0.5	10
1	7

Table 1: Equilibrium values of k for $n = 51$ and various values of δ.

[15] The code used to simulate these examples is available from the authors upon request.

[16] We note that in all the numerical examples listed in the table, there is exactly one $k = 1, \ldots, n-1$, so that the k-candidate with simple swaps is an SSPE.

8.2 Efficiency and equity for varying discount factors

One purpose of the original analysis by Baron and Ferejohn was to compare closed rule and open rule bargaining procedures with regard to the efficiency and equity of equilibrium outcomes. While open rules tend to lead to a more egalitarian distribution of the surplus, they suffer from inefficiencies. The reason is that the option of making amendments tends to lead to equilibrium delays, while closed rule bargaining models always predict immediate agreement. One important question is how one could weigh the efficiency loss against the equity gain.

We consider the example with $n = 51$ for different values of the discount factor. Examples 1 and 2 suggest that there is a large gain in fairness and no loss in efficiency when δ is either close to zero or close to one. However, Example 3 shows that for intermediate values of δ, the efficiency loss from open rules can be so large that even the responding players are ex ante better off under a closed rule.

Example 1. Let us focus on the example where $n = 51$. First, we consider the case where δ is very small, say $\delta = 0.02$. If players were bargaining under a closed rule, agreement would be immediate and so the surplus divided would be of size one. The proposer would "buy" 25 players' votes by offering each of them the reservation payoff $\delta/n = 0.02/51 < 0.0004$. Hence, the proposer could secure a majority by offering less than $25 \times 0.0004 = 0.01$ to other players. Under closed rule, the proposer would be able to keep more than 99 percent of the surplus for himself.

Now we turn to the case of open rule bargaining. If $n = 51$ and $\delta = 0.02$, we have previously computed that $k = 50$. Since δ is so small, it is prohibitive for the proposer to risk bargaining delay. In equilibrium, agreement is reached immediately and the size of the surplus divided is one, just as it would be under closed rule bargaining. Substituting for n, δ, and k into Eqns. (1)-(4), we find that $V_{50} = X_{50} = 0.5$, while $W_{50} = 0.01$ and $Y_{50} = 1$. The equilibrium outcome under open rule can be described as follows: The proposer receives half of the surplus himself. He distributes the remaining half of the surplus equally to the other fifty players; hence, each of them receives $\delta V_{50} = 0.02 \times 0.5 = 0.01$, and is willing to endorse the proposal.

Clearly, the outcome under open rule is much more equitable than under closed rule, and equally efficient.

Example 2. Now we consider the example with $n = 51$ in the limit as $\delta \to 1$. First, suppose that bargaining takes place under the closed rule. In that case, the proposer needs to offer each of 25 players their reservation payoffs $\delta/n \to 1/51 \approx 0.0196$. The proposer can keep the remainder $1 - 25/51 \approx 0.5098$.

Now turn to the case of open rule bargaining. We have computed before that $n = 51$ and δ close to one give $k = 7$. Substituting into Eqns. (1)-(4) and solving the resulting system, we find $V_7 \approx 0.054$, $W_7 \approx 0.0189$, and $X_7 \approx 0.2693$.

Hence, under open rule, the proposer receives just over one fourth of the surplus (instead of more than a half under closed rule). Seven players receive 0.054 instead of just 0.0196. All other players obtain the same payoff under open rule bargaining as under closed rule bargaining. Indeed, the open rule bargaining leads to a more equitable, and equally efficient, outcome.

Example 3. As an example, suppose that $n = 51$ and $\delta = 0.5$. With closed rule bargaining, 25 players would get the reservation payoff $\delta/n = 0.5/51 \approx 0.0098$. The proposer would keep the remaining $1 - 25 \times 0.0098 = 0.755$.

Under open rule bargaining, we find $k = 10$ and the relevant system of equations becomes

$$\begin{aligned}
V_{10} &= 0.2X_{10} + 0.4W_{10}, \\
W_{10} &= (Y_{10} - V_{10})/50, \\
X_{10} &= 1 - 5V_{10} - \frac{15 \times 0.5}{51} Y_{10}, \\
Y_{10} &= 1/3.
\end{aligned}$$

Solving this system yields $X_{10} \approx 0.4707$ and $\delta V_{10} \approx 0.048$. Hence, under open rule bargaining, one would expect the proposer to receive about 0.4707 and ten players to receive 0.048. Another 15 players would receive $0.5/51 \approx 0.0098$ and the remaining 25 players would receive nothing.

However, with open rule, the expected delay is 4 and the expected surplus is 1/3. So while the outcome under open rule is certainly more equitable than under closed rule, it is much less efficient.

Recall that V_{10} and W_{10} are the ex ante expected payoffs of the proposer and any player other than the proposer, respectively. From the above equations, we can compute $V_{10} \approx 0.096$ and $W_{10} = 0.0047$. In a closed rule bargaining game, the analogous ex ante payoffs would be 0.755 for the proposer (since agreement is immediate) and $0.5\frac{\delta}{n} = 0.0049$ for any other player. Note that ex ante, all players are better off with closed rule bargaining than with open rule bargaining for $\delta = 0.5$. The efficiency loss from delay is so great that even the gain in fairness cannot compensate the responders for it.

9 Return to the ORBG

In previous sections, we focused on the simplified ORBG. In the present section, we return to the original ORBG, as formally defined in Section 2. Recall the crucial difference between both games: In the original ORBG, whenever a player makes an amendment to a proposal on the floor, a vote determines whether or not the amendment replaces the proposal on the floor. We will show in this section that the main results and conclusions from our analysis of the simplified ORBG carry over to the ORBG itself. To this end, we first define a set of equilibrium candidates which are analogous to the k–candidates with simple swaps in the simplified ORBG. For this definition, we first recall the system of Eqns. (1)-(4)

$$
\begin{aligned}
V_k &= \left(\frac{k}{n-1}\right) X_k + \left(\frac{n-1-k}{n-1}\right) \delta W_k, \\
W_k &= \frac{Y_k - V_k}{n-1}, \\
X_k &= 1 - k\delta V_k - \max\left\{0, \frac{n-1}{2} - k\right\} \left(\frac{\delta}{n}\right) Y_k, \\
Y_k &= \frac{k}{\delta k + (1-\delta)(n-1)}.
\end{aligned}
$$

Definition 2. *Consider the ORBG $G(\delta, n)$. For every $k = 1, \ldots, n-1$, a profile of stationary strategies is a* generalized k-candidate *if the following holds:*

1. *At every history $h \in H^{\emptyset}$, the proposer makes an anonymous proposal η which gives himself X_k, gives δV_k to k other players, gives $\frac{\delta}{n}((n-1)W_k + V_k)$ to* $\max\{0, \frac{n-1}{2} - k\}$ *more players, and zero to all remaining players.*

2. *At any history $h \in H_j^{i,\theta}$, player i expects the same payoff, say $p_{ij}(\theta, \sigma)$.*

3. *Let $p_i(\theta, \sigma) = \left(\frac{1}{n-1}\right) \sum_{j \in N \setminus \{i\}} p_{ij}(\theta, \sigma)$. For any $i_1, i_2 \in N$, it holds that $p_{i_2}(\pi_{i_1 \leftrightarrow i_2}(\theta), \sigma) = p_{i_1}(\theta, \sigma)$.*

4. *For every $\theta \in \Delta^n$, there is a set $T(\sigma, \theta) \subset \Delta^n$ such that the following holds: Whenever players vote between the proposal on the floor θ and some amendment θ', then the majority votes in favor of θ' if and only if $\theta' \in T(\sigma, \theta)$. Moreover, for every $\theta \in \Delta^n$, the set $T(\sigma, \theta)$ has the following properties: $\pi_{i \leftrightarrow j}(\theta) \in T(\sigma, \theta)$ and $\pi_{i \leftrightarrow j}(\widetilde{\theta}) \in T(\sigma, \pi_{i \leftrightarrow j}(\theta))$ if $\widetilde{\theta} \in T(\sigma, \theta)$ for any $i, j \in N$.*

5. *Whenever player i votes on an endorsed proposal θ, he votes in favor of θ if and only if $\theta_i \geq \frac{\delta}{n}((n-1)W_k + V_k)$.*

Let us compare the generalized k–candidates to the k–candidates with simple swaps in Definition 1:

Points 1 and 5 in Definition 2 are familiar from Definition 1. In contrast to that earlier definition, however, we now also have to specify a *selection rule.* We impose on that selection rule a number of stationarity and anonymity requirements spelled out in Points 2, 3, and 4 of Definition 2. We verbally discuss these points in turn:

- Point 2 in Definition 2 imposes a *stationarity requirement* for histories at which an amendment can be made: Whenever player j can endorse or amend the same proposal θ made by the same player i, he acts in the same way. One implication is that player i has the same expected payoff, independently of the amendment, whenever his proposal θ is on the floor.

- Point 3 in Definition 2 adds an *anonymity requirement* to the previous point: The expected payoff of player i_1 when his proposal θ is on the floor is the same as the expected payoff of player i_2 when his proposal $\pi_{i_1 \leftrightarrow i_2}(\theta)$ is on the floor.

- Point 4 in Definition 2 puts *stationarity and anonymity restrictions* on the voting behavior when a proposal on the floor is pitted against an amendment: First, whenever the same proposal on the floor and the same amendment are pitted against each other, the winner is the same. Second, if the amendment is a simple swap of the proposal on the floor, then the amendment wins. Third, the majority's decision for an amendment or a proposal on the floor is unresponsive to a change in the players' "labels."

Consider a k–candidate with simple swaps which is an SSPE in the simplified ORBG. From that k–candidate with simple swaps, let us construct a generalized k–candidate by preserving the same *anonymous proposal,* the same *amendment rule,* and the same *voting rule* but adding the following *selection rule*: Whenever players choose between a proposal on the floor and an amendment, all players vote in favor of the amendment. It is trivially true that no unilateral deviation from this selection rule can improve a player's payoff.[17] Hence, it is intuitive that the generalized k–candidate so constructed is an SSPE in the ORBG. This is formally stated in Proposition 9.

Proposition 9. *If the k-candidate with simple swaps is an SSPE of the simplified ORBG, then there is a generalized k-candidate with simple swaps that is an SSPE of the ORBG.*

The proof of Proposition 9 can be found in the appendix.

Proposition 9 tells us that the SSPE found by studying the simplified ORBG corresponds to SSPE of the original ORBG. While the insight of Proposition 9 is admittedly trivial, the crucial question in the remainder of this section is whether the converse is also true: Can a generalized k–candidate be an SSPE of the ORBG

[17] Alternatively, we could also use a construction where all players except the author of the proposal on the floor vote in favor of the amendment. Along an equilibrium path of play, such behavior would be optimal even in the presence of a "trembling hand."

without corresponding to a k–candidate with simple swaps that is an SSPE in the simplified ORBG? Loosely speaking, the issue in this section is what we have missed by restricting attention to the simplified ORBG.

This question is far from trivial. To see what the problem is, note that a strategy profile has to specify for each pair of a proposal and an amendment how each player would vote between the two. A player's choice is *not* merely determined by the amount of surplus which he receives under the proposal on the floor and the amendment. Rather, it depends on the *entire* proposal and the *entire* amendment through the expected length of further delays after one or the other option wins. We will make this issue tractable by focusing on properties of the sequences of amendments which occur after a deviation from the supposed equilibrium path of play.

More specifically, this section will proceed in the following steps:

1. We consider the generalized $\widehat{k}$–candidate for some $\widehat{k}$. Suppose that it is optimal for the initial proposer to deviate unilaterally from the generalized $\widehat{k}$–candidate by making the proposal that he would make when playing according to the generalized k–candidate for some $k \neq \widehat{k}$. We show that, if this were indeed optimal, then any amendment to the proposal would be a simple swap of it. As a result, the initial proposer's unilateral one-shot deviation leads to a path of play that resembles the generalized k–candidate until a proposal is endorsed and voted upon. This is shown in Propositions 10 and 11 below.

2. We compute the payoff which the initial proposer could achieve by the aforementioned deviation. Due to the premise that this deviation is optimal for the initial proposer, it follows that the payoff we compute must be no less than the initial proposer's payoff from the generalized $\widehat{k}$–candidate. This gives us a necessary condition for the generalized $\widehat{k}$–candidate to be an SSPE. For the case with δ sufficiently close to one, we show that this necessary condition puts $\widehat{k}$ in a neighborhood around $\sqrt{n}$. This is Proposition 12 below.

3. Based on this insight, we argue that (except in a knife–edge case) there is

only one $\widehat{k} \in \{1, \ldots, n-1\}$ so that the generalized $\widehat{k}$–candidate satisfies the necessary condition for an SSPE. (This is Corollary 4 below.)

4. Again for the case with δ close to one, we have already found a k such that the generalized k–candidate is an SSPE. We did so when we found an SSPE in the simplified ORBG and showed that it corresponded to one in the ORBG. As a result, the "uniqueness" that follows from Corollary 4 implies that our findings from the simplified ORBG carry over to the ORBG.

We introduce the following notation: Let σ be a generalized k-candidate. Take any sequence $(\theta^0, \theta^1, \ldots, \theta^T)$ such that with strictly positive probability, σ induces a path of play along which the proposal θ^0 is made at some history $h^0 \in H^{\emptyset}$ by player i^0. Then, the game reaches history h^1, where player i^1 makes the amendment θ^1, and it reaches history h^2, where player i^2 makes the amendment θ^2, and so on, until eventually player i^{T+1} endorses θ^T at history h^{T+1}.

Proposition 10. *Let $h \in H^{\emptyset}$ and $h' \in H^{i,\theta}$, for some $\theta \in \Delta^n$. Let player i be the proposer at h, and player j be the proposer at h'. Suppose that it is optimal for player i to propose θ at history h, provided that σ is played at all histories following h. Consider the choice of player j at history h'. Provided that σ is played at all histories following h', either it is optimal for player j to endorse the proposal θ at history h', or it is optimal to make the amendment $\pi_{i\leftrightarrow j}(\theta)$.*

Proof. Suppose that, at history h', it is strictly better for player j to make an amendment $\widetilde{\theta} \neq \pi_{i\leftrightarrow j}(\theta)$, instead of the amendment $\pi_{i\leftrightarrow j}(\theta)$. Thus $p_j(\widetilde{\theta}, \sigma) > p_j(\pi_{i\leftrightarrow j}(\theta), \sigma)$. By definition of a generalized k-candidate (see Definition 2), it follows that $p_i(\pi_{i\leftrightarrow j}(\widetilde{\theta}), \sigma) > p_i(\theta, \sigma)$. This implies that it is not optimal for player i to propose θ at history h, and the proof of the proposition is complete. $\square$

Repeating the same line of argument, we can also show the next proposition:

Proposition 11. *Let $h \in H^{(i,\theta)}$ and $h' \in H^{(j_1, \pi_{i\leftrightarrow j_1}(\theta))}$. Let player j_1 be the proposer at h, and player j_2 be the proposer at h'. Suppose that it is optimal for player j_1 to make the amendment $\pi_{i\leftrightarrow j_1}(\theta)$. Consider the choice of player j_2 at history h'.*

Provided that σ is played at all histories following h', either it is optimal for player j_2 to endorse $\pi_{i\leftrightarrow j_1}(\theta)$, or it is optimal to make the amendment $\pi_{j_1\leftrightarrow j_2}(\pi_{i\leftrightarrow j_1}(\theta))$.

The two propositions above lead to the following conclusions: Suppose that players $i^0, i^1, \ldots, i^T$ choose the initial proposal and the amendments optimally. Then, θ^t can be described as a simple swap of θ^{t-1} for any $t = 1, \ldots, T$. Moreover, if player i^0 at history h^0 has an expected payoff of V, then any player i^t who makes an amendment at history h^t with $t = 1, \ldots, T$, has an expected payoff of δV. Finally, player i^{T+1} endorses θ^T because that proposal gives him δV.

We compute the payoffs that would result if it were optimal for the initial proposer to deviate from the generalized $\widehat{k}$–candidate by making the anonymous proposal associated with the generalized k–candidate for some $k \neq \widehat{k}$. For this computation, we need the following system of equations:

$$V_k^{\widehat{k}} = \left(\frac{k}{n-1}\right) X_k^{\widehat{k}} + \left(\frac{n-1-k}{n-1}\right) \delta W_k^{\widehat{k}}, \tag{14}$$

$$W_k^{\widehat{k}} = \frac{Y_k - V_k^{\widehat{k}}}{n-1}, \tag{15}$$

$$X_k^{\widehat{k}} = 1 - k\delta V_k^{\widehat{k}} - \max\left\{0, \frac{n-1}{2} - k\right\} \left(\frac{\delta}{n}\right) Y_{\widehat{k}}, \tag{16}$$

$$Y_{\widehat{k}} = \frac{\widehat{k}}{\delta\widehat{k} + (1-\delta)(n-1)}, \tag{17}$$

$$Y_k = \frac{k}{\delta k + (1-\delta)(n-1)}. \tag{18}$$

Eqns. (14)–(16) are analogous to Eqns. (1)–(3). Given the premise that the initial proposer's deviation is optimal, we have shown in Propositions 10 and 11 that play proceeds according to the generalized k–candidate until a proposal is endorsed. Note that $\widehat{k}$ enters Eqns. (14)–(16) only through $\frac{\delta}{n} Y_{\widehat{k}}$, which is the continuation utility after some endorsed proposal has been voted on and rejected. In that case, play reverts to the generalized $\widehat{k}$–candidate.[18]

[18] Note that the one-shot deviation principle is applied differently than in previous sections of this paper. In the present section, the choice between a proposal on the floor and an amendment is no longer rendered trivial: Therefore, it is no longer true that all histories at which a particular player is the proposer are "equivalent." It is true, however, that all histories in the set $H^{\emptyset}$ at

From the above system of equations, we can compute the payoff $V_k^{\widehat{k}}$ for any pair $(k, \widehat{k})$.

Suppose that, for some pair $(k, \widehat{k})$, we find $V_k^{\widehat{k}} \leq V_{\widehat{k}}^{\widehat{k}}$ for all $k \neq \widehat{k}$. Then the initial proposer's unilateral deviation which we have discussed above cannot be optimal. Clearly, this is a necessary condition for the generalized $\widehat{k}$–candidate to be an SSPE.

This is the gist of Proposition 12 below.

Proposition 12. *If the generalized $\widehat{k}$-candidate is an SSPE of the open rule legislative bargaining game $G(\delta, n)$, then $\widehat{k} \in \arg\max_{k \in \{1,\ldots,n-1\}} V_k^{\widehat{k}}$, where $V_k^{\widehat{k}}$ is the solution to Eqns. (14)-(18).*

Suppose that we wanted to test which of the generalized k-candidates satisfies the necessary condition established in Proposition 12. This would require the computation of the variables $(V_k^{\widehat{k}}, W_k^{\widehat{k}}, X_k^{\widehat{k}}, Y_{\widehat{k}}, Y_k)$ for $(n-1)^2$ possible pairs $(k, \widehat{k})$. Hence, testing for an SSPE is now considerably more complicated than it was in Section 6. Matters simplify a lot, however, when we focus on the case with patient players. Indeed, $Y_1, \ldots, Y_{n-1}$ all converge to one as $\delta \to 1$. As a result, $V^{\widehat{k}}, W^{\widehat{k}}$, and $X_k^{\widehat{k}}$ converge to limits that are independent of $\widehat{k}$. This allows us to simplify the Eqns. (14)-(16). For each $k = 1, \ldots, n-1$, let $\overline{V}_k, \overline{W}_k$, and $\overline{X}_k$ be the solution to the following system of equations:

$$\overline{V}_k = \left(\frac{k}{n-1}\right)\overline{X}_k + \left(\frac{n-1-k}{n-1}\right)\overline{W}_k, \tag{19}$$

$$\overline{W}_k = (1-\overline{V}_k)/(n-1), \tag{20}$$

$$\overline{X}_k = 1 - k\overline{V}_k - \max\left\{0, \frac{n-1}{2} - k\right\}\left(\frac{1}{n}\right). \tag{21}$$

Note that Eqns. (19)-(21) are the same system of linear equations as in Section 5, now applied at $\delta = 1$.

Now we are ready to state the main result of this section.

which the same player proposes are followed by the same continuation game. Hence, we think of the open rule legislative bargaining game as a stochastic game. It moves to a new state whenever a history in $H^{\emptyset}$ is reached. If the proposer at that history is player i, then the game is in state i.

Theorem 2. *Suppose that $n \geq 15$ and δ is sufficiently close to one. If a generalized k-candidate is an SSPE of the ORBG, then it holds that*

$$k \in \{1, \ldots, n-1\} \cap (\sqrt{n} - 1, \sqrt{n} + 1).$$

The proof of Theorem 2 is relegated to the appendix.

In order to assess the implications of Theorem 2, let us first consider the case where n is such that $\sqrt{n}$ is an integer. In that case, $\sqrt{n}$ is the only integer contained in the interval $(\sqrt{n} - 1, \sqrt{n} + 1)$. Hence, the generalized $\sqrt{n}$-candidate is the only generalized k-candidate that can be an SSPE.

Now consider the case where n is such that $\sqrt{n}$ is not an integer. In that case, the interval $(\sqrt{n} - 1, \sqrt{n} + 1)$ contains two integers. Let us denote them by k^* and $k^* + 1$. Now we use Eqns. (19)-(21) to compute $\overline{V}_{k^*}$ and $\overline{V}_{k^*+1}$. There is no reason to expect that these two amounts are generally equal. If $\overline{V}_{k^*} > \overline{V}_{k^*+1}$, then Proposition 12 implies that the generalized $(k^* + 1)$-candidate cannot be an SSPE. Similarly, if $\overline{V}_{k^*} < \overline{V}_{k^*+1}$, then Proposition 12 implies that the generalized k^*-candidate cannot be an SSPE.

From these observations, we obtain the following corollary:

Corollary 4. *Suppose that $n \geq 15$ and δ is sufficiently close to one. Suppose that the k^*-candidate with simple swaps is an SSPE of the ORBG. Moreover, suppose that there is some $k^{**} \neq k^*$ such that the generalized k^{**}-candidate is an SSPE. Then, k^* and k^{**} are successive integers and it holds that $\overline{V}_{k^*} = \overline{V}_{k^{**}}$.*

The interpretation is as follows: Consider the case where $n \geq 15$ and δ is close enough to one. In our analysis of the simplified ORBG, we have found one k such that the k–candidate with simple swaps is an SSPE. In Proposition 7, we have shown that this SSPE of the simplified ORBG easily extends to an SSPE of the ORBG. Hence, we already have found one particular k such that the generalized k–candidate is an SSPE of the ORBG. Theorem 2 and Corollary 4 tell us that the SSPE which we have already found is (except in a knife–edge case) the *only* generalized k–candidate that is an SSPE. In addition, even in that knife–edge case, there can at most be two values of k such that the generalized k–candidate that is an SSPE, these two values of k must be successive integers, and the proposer's payoffs in both potential equilibria are equal. The conclusion is that our main results from the simplified ORBG carry over to the ORBG.

10 Conclusion

While we confirm Baron and Ferejohn's insight that open rules tend to lead to less efficient and more egalitarian outcomes, we find that they may be even less efficient and also less egalitarian than suggested by Baron and Ferejohn.

Moreover, our analysis may have important implications for the design of legislatures and their committees. For instance, the tendency of open rules to produce egalitarian outcomes, even after the proposer has been selected at the cost of significant delays, opens up a more detailed comparison of the egalitarian efficiency trade-offs between closed and open rules. Smaller legislatures yield less delay and a more egalitarian allocation than larger legislatures under open rules.

The size of the legislature may need to be quite large for other reasons than examined in this paper, e.g. to be sufficiently representative of the underlying electorate. Then, surplus division could be first delegated to a smaller committee that itself is representative of the legislature. If the committee uses the open rule and the committee decision is put to a final vote in the legislature under a closed rule, the efficiency and equality advantages of open rules could be preserved.

Another finding is that patient players induce more delays, even to the extreme that the expected delay becomes arbitrarily large. Hence, from an efficiency perspective, it would be useful if players were more impatient when deciding about surplus divisions than they actually are. This could be achieved by limiting the time members of a legislature can spend on committees to take, or at least to prepare, decisions on surplus divisions.

Many closed rule bargaining models allow for players to differ in their recognition probabilities or for a general acceptance rule rather than simple majority. These extensions would further complicate the analysis, although we would expect a general acceptance rule to be more easily feasible than an extension to heterogeneous recognition probabilities.

Finally, one potential avenue for future research would be to investigate experimentally the size of "coalitions" that emerge in open vs. closed rule bargaining.[19]

Appendix

Proof of Proposition 1. The proof consists of three steps. First, we show part (i) and part (ii) for the case $k \geq (n-1)/2$. Second, we show part (ii) for the complementary case $k \leq (n-3)/2$. Third, we show part (iii).

Step 1. Suppose first that $k \geq (n-1)/2$. From a direct computation of the solution of Eqns. (1)-(4), we find

$$\begin{aligned} \beta_k V_k/k &= (n-1)^2 - \delta(n-2)(n-1-k), \\ \beta_k W_k/k &= \delta(n-1-k+k^2), \end{aligned}$$

where β_k is given by

$$\beta_k = [(1-\delta)(n-1) + \delta k][(n-1)^2 + \delta(n-1)(1+k^2) - \delta k].$$

We observe that β_k is strictly positive since $(n-1)^2 > \delta k$. Hence, the difference $V_k - W_k$ has the same sign as $(\beta_k/k)(V_k - W_k)$. The latter expression can be written as

$$\begin{aligned} (V_k - W_k)(\beta_k/k) &= (n-1)^2 - \delta(n-2)(n-1-k) - \delta(n-1-k+k^2) \\ &= (1-\delta)(n-1)^2 - \delta k^2 + \delta k(n-1) \\ &= (1-\delta)(n-1)^2 + \delta k(n-1-k). \end{aligned}$$

We see that $V_k = W_k > 0$ if $\delta = 1$ and $k = n-1$. If at least one of the inequalities $\delta \leq 1$ and $k \leq n-1$ holds strictly, we see that $V_k > W_k > 0$.

[19]The well-known paper by Fréchette et al. (2003) uses parameter values under which the open and closed rule model predict the same coalition size, see Table 1 in their paper.

Step 2. Suppose now that $k \leq (n-3)/2$. In that case, from Eqns. (1)-(4), we can directly compute

$$\begin{aligned}\gamma_k V_k/k &= 2n(n-1)^2 - \delta[2n(n^2-3n+2) - k(n^2-2n-1) - 2k^2(n-1)],\\ \gamma_k W_k/k &= \delta[2n(n-1) - (n+1)k + 2k^2(n-1)],\end{aligned}$$

where γ_k is given by

$$\gamma_k = 2n[(1-\delta)(n-1) + \delta k][(n-1)^2 + \delta(n-1)(1+k^2) - \delta k].$$

Since $\gamma_k > 0$, we can conclude that $V_k - W_k$ has the same sign as the expression

$$\begin{aligned}(\gamma_k/k)(V_k - W_k) &= 2n(n-1)^2 - \delta[2n(n-1)^2 - k(n^2-n)]\\ &= 2n(1-\delta)(n-1)^2 + \delta k(n^2-n) > 0.\end{aligned}$$

Now it remains to show that $W_k > 0$. To this end, notice that

$$\begin{aligned}\gamma_k W_k/(k\delta) &= 2n(n-1) - (n+1)k + 2(n-1)k^2\\ &= k^2(2n-2) - k(n+1) + (2n-2)n.\end{aligned}$$

Since $n+1 < 2n-2$ and $k < n$, this expression is strictly positive, and hence $W_k > 0$, as desired.

Step 3.

We have already shown that, for any $k = 1, \ldots, n-1$, it holds that $W_k > 0$ and $V_k - W_k \geq 0$, implying in particular that $V_k > 0$ and $V_k - \delta W_k \geq 0$. From the explicit Eqn. (4), it follows directly that $Y_k > 0$. Finally, Eqn. (1) can be rewritten as

$$kX_k = (n-1)(V_k - \delta W_k) + k\delta W_k,$$

implying that $X_k > 0$ as well.

$\square$

Proof of Corollary 1.

Part (i) of the corollary follows directly from substituting $k = n-1$ and $\delta = 1$ into Eqn. (1). In order to see why part (ii) of the corollary is true, recall from Proposition 1 that for any choices of k and δ other than $k = n-1$ and $\delta = 1$, we have $V_k > W_k$. Thus, Eqn. (1) implies the inequality $(n-1)V_k < kX_k + (n-1-k)\delta V_k$. Rearranging yields

$$X_k \left(\frac{k}{(1-\delta)(n-1)+\delta k} \right) > V_k.$$

It only remains to note that $\frac{k}{(1-\delta)(n-1)+\delta k} \leq 1$, since $k \leq n-1$ and the denominator is larger than k. Hence, we can conclude that $X_k > V_k \geq \delta V_k$, as desired.

$\square$

Proof of Proposition 3.

Suppose that there is a vector $\widehat{\theta} \in \Delta^n$ such that proposing $\widehat{\theta}$ instead of the proposal prescribed by the k-candidate with simple swaps is a profitable deviation for the proposer, say player i. Suppose that there is a player $j \in N \setminus \{i\}$ such that $0 < \widehat{\theta}_j < \frac{\delta}{n}((n-1)W_k + V_k)$. Player j neither endorses the proposal $\widehat{\theta}$, nor does he vote in its favor. Consequently, it would also be a profitable deviation for player i to offer zero to player j, and offer $\widehat{\theta}_l$ to all players $l \in N \setminus \{j\}$. By the same token, suppose that there is a player $j \in N \setminus \{i\}$ such that $\frac{\delta}{n}((n-1)W_k + V_k) < \widehat{\theta}_j < \delta V_k$. In that case, player j is willing to vote in favor of $\widehat{\theta}$, but not willing to endorse it. This would not change if he was offered $\frac{\delta}{n}((n-1)W_k + V_k)$ instead of $\widehat{\theta}_j$. Thus, it would also be a profitable deviation for player i to offer player j only $\frac{\delta}{n}((n-1)W_k + V_k)$, and offer each player $l \in N \setminus \{j\}$ the amount $\widehat{\theta}_l$. Repeating the same argument, we see that if the proposer has any profitable deviation $\widehat{\theta}$, then he has a profitable deviation to a proposal $\widetilde{\theta}$ which gives each player other than the proposer either zero, or $\frac{\delta}{n}((n-1)W_k + V_k)$, or δV_k.

Let $\mathcal{P}_k$ be the set of vectors $\theta \in \Delta^n$ that, for some $k' \in \{1, \ldots, n-1\}$, contain k' components equal to δV_k and $\max\{0, \frac{n-1}{2} - k'\}$ components equal to $\frac{\delta}{n} Y_k$, where V_k and Y_k are as defined in Eqns. (1)-(4). Moreover, a vector $\theta \in \mathcal{P}_k$ has one component equal to $1 - k'\delta V_k - \max\{0, \frac{n-1}{2} - k'\}\frac{\delta}{n} Y_k$. Any remaining components are equal to zero.[20]

[20]Recall that the set Δ^n consists of vectors that are non-negative in all components and sum up to (at most) one. It follows that the number k' satisfies the inequality $1 - k'\delta V_k - \max\{0, \frac{n-1}{2} -$

Let λ_k^{+m} be the gain which the proposer can make by offering δV_k to $k+m$ players instead of to k players, starting from a proposal $\theta \in \mathcal{P}_k$, where $m \geq 2$. We want to show that $\lambda_k^{+m} > 0$ implies $\lambda_k^{+} > 0$. Suppose that $k \geq (n-1)/2$. Then we have

$$\lambda_k^{+m} = -\left(\frac{k}{n-1}\right) m\delta V_k + \left(\frac{m}{n-1}\right)(X_k - \delta W_k - m\delta V_k).$$

If this is strictly positive, then dividing by m yields

$$-\left(\frac{k}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) - \left(\frac{m}{n-1}\right)\delta V_k > 0,$$

which can be rewritten equivalently as

$$-\left(\frac{k}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k - \delta V_k) > \left(\frac{m-1}{n-1}\right)\delta V_k,$$

hence

$$\lambda_k^{+} > \left(\frac{m-1}{n-1}\right)\delta V_k.$$

We have shown earlier that $V_k > 0$. Thus, it follows that $\lambda_k^{+} > 0$, as desired.

Now consider the case where $k \leq (n-3)/2$. Then, we have

$$\lambda_k^{+m} \leq -\left(\frac{k}{n-1}\right) m\delta V_k + \left(\frac{m}{n-1}\right)(X_k - \delta W_k - m\delta V_k).$$

Suppose that $\lambda_k^{+m} > 0$, then dividing by $m/(n-1)$ and rearranging terms, it follows that

$$-(k+m)\delta V_k + X_k - \delta W_k > 0.$$

Since $m \geq 2$, we can conclude that also

$$-(k+1)\delta V_k + X_k - \delta W_k > 0.$$

Again, a fortiori, we have that

$$-(k+1)\left(\delta V_k - \frac{\delta}{n}Y_k\right) + X_k - \delta W_k > 0,$$

$k'\}\frac{\delta}{n}Y_k \geq 0$. In other words, it is ensured that for any proposal in $\mathcal{P}_k$, the share of surplus for the proposer remains non-negative. We also show in Corollary 1 that the proposer's share is larger than that of any other player.

and hence $\lambda_k^+ > 0$, as desired.

An analogous argument can be used to show that $\lambda_k^{-m} > 0$ implies $\lambda_k^- > 0$. $\square$

Proof of Proposition 5. For $k = n - 1$, the system of Eqns. (1)-(4) yields the following solutions:

$$\begin{aligned} V_{n-1} &= \frac{1}{1+\delta(n-1)}, \\ W_{n-1} &= \frac{\delta}{1+\delta(n-1)}, \\ X_{n-1} &= \frac{1}{1+\delta(n-1)}, \\ Y_{n-1} &= 1. \end{aligned}$$

By definition, $\lambda_{n-1}^+ = 0$, so it remains to show that for δ sufficiently small, $\lambda_{n-1}^- \leq 0$. Substituting the above equations, we obtain

$$\lambda_{n-1}^- = \frac{\delta(n-2) - 1 + \delta^2}{(n-1)(1+\delta(n-1))}.$$

Indeed, it follows that $\lambda_{n-1}^- \leq 0$ if and only if $\delta^2 + \delta(n-2) - 1 \leq 0$. This inequality is satisfied when

$$\delta \leq \sqrt{\left(\frac{n-2}{2}\right)^2 + 1} - \left(\frac{n-2}{2}\right).$$

$\square$

Proof of Proposition 6. Suppose that there is a $k \geq (n+1)/2$ such that some k-candidate is an SSPE. Now consider a deviation by the initial proposer from the supposed SSPE. Under the deviation, the proposer offers δV_k to $k-1$ instead of to k players, and offers zero to the remaining $n-k$ players, where V_k is as before. This deviation gives the proposer an expected payoff of

$$\left(\frac{k-1}{n-1}\right)(1 - k\delta V_k + \delta V_k) + \left(\frac{n-k}{n-1}\right)\delta W_k,$$

while the proposer's expected payoff when playing according to the supposed SSPE is

$$V_k = \left(\frac{k}{n-1}\right)(1 - k\delta V_k) + \left(\frac{n-1-k}{n-1}\right)\delta W_k.$$

Clearly, a necessary condition for the k-candidate with simple swaps to be an SSPE is the inequality

$$\left(\frac{k-1}{n-1}\right)\delta V_k - \left(\frac{1}{n-1}\right)(1-k\delta V_k) + \delta\left(\frac{1}{n-1}\right)W_k \leq 0.$$

Since $\delta\left(\frac{1}{n-1}\right)W_k \geq 0$, it is necessary that

$$\left(\frac{k-1}{n-1}\right)\delta V_k - \left(\frac{1}{n-1}\right)(1-k\delta V_k) \leq 0. \tag{22}$$

The inequality can be rearranged to

$$\delta V_k \leq \frac{1}{2k-1}.$$

It follows that

$$1-k\delta V_k \geq \frac{k-1}{2k-1}.$$

Recall that V_k is the expected payoff of the proposer induced by the supposed SSPE. Under that strategy profile, the proposer offers himself $1-k\delta V_k$, and the proposal is endorsed (and then implemented) with probability $k/(n-1)$. Thus, we obtain

$$V_k \geq \left(\frac{k}{n-1}\right)(1-k\delta V_k) \geq \left(\frac{k}{n-1}\right)\left(\frac{k-1}{2k-1}\right).$$

Combining the above inequalities, we obtain

$$\frac{1/\delta}{2k-1} \geq V_k \geq \left(\frac{k}{n-1}\right)\left(\frac{k-1}{2k-1}\right).$$

This leads to the condition

$$\delta \leq \frac{n-1}{k^2-k} \leq \frac{4(n-1)}{(n-1)(n+1)} = \frac{4}{n+1}.$$

The last inequality follows from the premise that $k \geq (n+1)/2$. Canceling $(n-1)$, we obtain $\delta(n+1) \leq 4$, as desired. □

Proof of Proposition 7. In view of Proposition 6, we only have to consider the case where $k = (n-1)/2$. Indeed, fix some value of $\delta \in (0,1)$ and a number n of players, and suppose that the $\frac{n-1}{2}$-candidate with simple swaps is an SSPE.

From these premises, we are going to derive an implicit upper bound on n. On the path of play of the supposed SSPE, every proposal is endorsed with probability 1/2, and is accepted with certainty once it is endorsed. Thus, we find the following expression for the expected size of the total surplus divided:

$$V_k + (n-1)W_k = \left(\frac{1}{2}\right)\sum_{t=0}^{\infty}\left(\frac{\delta}{2}\right)^t = 1/(2-\delta).$$

Due to sincere voting (Point 3 in Definition 1), a player accepts an endorsed proposal if and only if it gives him at least $\left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)$. Consider a deviation from the supposed SSPE by the current proposer. This deviation consists of changing the offer to one player from δV_k to $\left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)$. Consequently, that player will no longer endorse the proposal but will still vote for it once it has been endorsed by some other player.

This deviation gives the proposer an expected payoff of

$$\left(\frac{n-3}{2(n-1)}\right)\left(X_k + \delta V_k - \frac{\delta}{n}\frac{1}{2-\delta}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right)\delta W_k.$$

Recall that playing according to the supposed SSPE gives a proposer an expected payoff of

$$V_k = \frac{1}{2}X_k + \frac{1}{2}\delta W_k.$$

Consequently, the proposer's gain from the deviation can be written as

$$\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)\right) + \left(\frac{1}{n-1}\right)\delta W_k - \left(\frac{1}{n-1}\right)X_k.$$

Taking into account that $X_k = 1 - \frac{n-1}{2}\delta V_k$, we find the following necessary condition for the $\frac{n-1}{2}$-candidate with simple swaps to be an SSPE:

$$\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)\right) - \left(\frac{1}{n-1}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right) + \left(\frac{1}{n-1}\right)\delta W_k \leq 0.$$

In view of the fact that $W_k \geq 0$, this implies the necessary condition

$$\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \frac{\delta}{n}\frac{1}{2-\delta}\right) - \left(\frac{1}{n-1}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right) \leq 0.$$

This yields

$$\delta V_k \leq \frac{2n(2-\delta) + \delta(n-3)}{2n(2-\delta)(n-2)}.$$

In the supposed SSPE, any proposal is endorsed with probability 1/2. If a player's proposal is endorsed, then it is also accepted, and so the expected payoff of a proposer can be bounded as follows:

$$V_k \geq \left(\frac{1}{2}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right).$$

Using the bound previously derived for δV_k, we can write

$$V_k \geq \frac{1}{2} - \left(\frac{n-1}{4}\right)\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right).$$

Now we have bounded V_k both from above and below, as follows:

$$\left(\frac{1}{\delta}\right)\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right) \geq V_k \geq \frac{1}{2} - \left(\frac{n-1}{4}\right)\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right).$$

This readily implies the inequality

$$\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right)\left(\frac{1}{\delta} + \frac{n-1}{4}\right) \geq 1/2.$$

Arranging this inequality in the quadratic form yields

$$\delta^2\left(3n^2 - 10n + 3\right) - \delta\left(4n^2 - 8n + 12\right) + 16n \geq 0. \tag{23}$$

Recall that we have assumed throughout the paper that $n \geq 5$, which implies that $n^2 - 2n > 0$ and $3n^2 - 10n + 3 > 0$. Therefore, it follows that

$$\delta^2 - \delta\left(\frac{4n^2 - 8n + 12}{3n^2 - 10n + 3}\right) + \left(\frac{16n}{3n^2 - 10n + 3}\right) \geq 0. \tag{24}$$

It is easily verified that for any $n \geq 13$,[21] we have

$$\left(\frac{4n^2 - 8n + 12}{6n^2 - 20n + 6}\right)^2 - \left(\frac{16n}{3n^2 - 10n + 3}\right) > 0, \tag{25}$$

and consequently, the quadratic Inequality (23) has two distinct real roots

$$\overline{\delta}(n) = \frac{4n^2 - 8n + 12}{6n^2 - 20n + 6} + \sqrt{\left(\frac{4n^2 - 8n + 12}{6n^2 - 20n + 6}\right)^2 - \left(\frac{16n}{3n^2 - 10n + 3}\right)}, \tag{26}$$

$$\underline{\delta}(n) = \frac{4n^2 - 8n + 12}{6n^2 - 20n + 6} - \sqrt{\left(\frac{4n^2 - 8n + 12}{6n^2 - 20n + 6}\right)^2 - \left(\frac{16n}{3n^2 - 10n + 3}\right)}. \tag{27}$$

[21] In fact, if we consider the left-hand side of Ineq. (25) as a continuous function of n, then we find that it is equal to zero for $n \approx 3.52$ and $n \approx 11.99$, negative for values of n between these two roots, and positive otherwise. Recall that in our model, we assume that n is an odd integer and that $n \geq 5$.

Existence of the supposed SSPE requires that either $0 < \delta \leq \underline{\delta}(n)$ or $\overline{\delta}(n) \leq \delta \leq 1$. However, in the limit, as $n \to \infty$, we find that $\overline{\delta}(n)$ converges to $4/3 > 1$, while $\underline{\delta}(n)$ converges to zero. This implies that Inequality (23) cannot be satisfied for n sufficiently large. In turn, this implies that a k-candidate with $k \geq (n-1)/2$ ("majority endorsement") is not an SSPE if n is sufficiently large. $\square$

Proof of Proposition 8.

From the previous proposition, it follows that a k-candidate with super-majority endorsement cannot be an SSPE if δ is sufficiently close to one. Hence, we only have to deal with the case where $k = (n-1)/2$.

We substitute $\delta = 1$ and $k = (n-1)/2$ into Eqns. (1)-(4). To ease notation, we write x instead of $X_{(n-1)/2}$, and similarly use v and w. This yields

$$\begin{aligned} 2v &= x + w, \\ (n-1)w &= 1 - v, \\ 2x &= 2 - (n-1)v. \end{aligned}$$

The solution to this system is

$$\begin{aligned} x &= \frac{3n-1}{n^2+2n-1}, \\ v &= \frac{2n}{n^2+2n-1}, \\ w &= \frac{n+1}{n^2+2n-1}. \end{aligned}$$

Now we can use Eqns. (5)-(6) to check whether the candidate under consideration is an SSPE. We find

$$\begin{aligned} \lambda^+_{(n-1)/2} &= \frac{-n^2+3n-2}{(n-1)(n^2+2n-1)}, \\ \lambda^-_{(n-1)/2} &= \frac{(n-1)(n-3)-4n}{2n(n^2+2n-1)}. \end{aligned}$$

We note that $\lambda^+_{(n-1)/2} > 0$ for any $n \geq 3$. Now turn to $\lambda^-_{(n-1)/2}$. Clearly, the denominator is strictly positive, so we only have to sign the numerator. Indeed, $(n-1)(n-3) - 4n < 0$ for $n \in \{3, 5, 7\}$ and $(n-1)(n-3) - 4n > 0$ for any $n \geq 9$.

$\square$

Proof of Corollary 2.

We need to show that the interval $\left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right]$ contains exactly one integer. Since this interval is closed and of unit length, it can at most contain two integers. It is sufficient to show that it does not contain two integers. Indeed, suppose towards a contradiction that the endpoints of the interval are integers. In particular, let $z = -\frac{1}{2}+\sqrt{n+\frac{1}{4}}$ be an integer. Solving for n yields $n+\frac{1}{4} = (z+\frac{1}{2})^2$. This can be rewritten as $n+\frac{1}{4} = z^2+z+\frac{1}{4}$ or as $n = z(z+1)$. Either z is even, or $z+1$ is even. Therefore, the product $z(z+1)$ is even, and so n is even. This contradicts our assumption that n is odd.

□

Proof of Proposition 9

The proof of this proposition consists of two steps:

We construct a generalized k-candidate in the following way: Suppose that the stationary strategy profile σ^* is a k-candidate with simple swaps and that it is an SSPE of the simplified ORBG. The profile σ^* consists of anonymous proposals η^{i*} for every $i \in N$, amendment rules ψ^{i*} for every $i \in N$, and acceptance rules A^{i*} for every $i \in N$. Let σ^{**} be a stationary strategy profile for the ORBG, which consists of these same anonymous proposals η^{i*} for every $i \in N$, amendment rules ψ^{i*} for every $i \in N$, and acceptance rules A^{i*} for every $i \in N$, and, in addition, features the following selection rule χ^{**}, used by every player $i \in N$: *"Whenever a vote takes place between an amendment and a proposal on the floor, every player votes in favor of the amendment."* It is trivially true that a unilateral deviation by any player from the selection rule χ^{**} does not change the outcome of a vote, and therefore cannot be profitable for the deviating player. If there were profitable unilateral deviations at histories other than the ones governed by the selection rule, then the k-candidate with simple swaps would not be an SSPE of the simplified ORBG. Hence, we have shown that the stationary strategy profile σ^{**} is an SSPE of the ORBG.

Finally, we observe that the stationary strategy profile σ^{**} is a generalized k-candidate: The anonymous proposal and the acceptance rule under σ^{**} satisfy

Points 1 and 5, respectively, of Definition 2. Note that they correspond exactly to Points 1 and 3 of Definition 1. Under σ^{**}, every player always votes in favor of any amendment against the proposal on the floor, which trivially satisfies Point 4 of Definition 2. The amendment rule under σ^{**} corresponds to that in a k-candidate with simple swaps as defined in Point 2 of Definition 1. This amendment rule, combined with the trivial selection rule, ensures that Points 2, 3, and 4 of Definition 2 are satisfied.

$\square$

Proof of Theorem 2.

We solve Eqns. (19)-(21) for $\overline{V}_k$:

$$\overline{V}_k = \begin{cases} \frac{2k^2(n-1)+2(n-1)n+k(n^2-2n-1)}{2n(k^2(n-1)+n(n-1)-k)}, & \text{if } k \leq (n-3)/2, \\ \frac{n+k(n-2)-1}{k^2(n-1)+n(n-1)-k}, & \text{if } k \geq (n-1)/2. \end{cases} \tag{28}$$

We have already argued that the generalized $\widehat{k}$-candidate can only be an SSPE if $\overline{V}_{\widehat{k}} \geq \overline{V}_k$ for every $k = 1, \ldots, n-1$.

As an auxiliary, it is useful to define the continuous function

$$\nu(\kappa) := \frac{2\kappa^2(n-1) + 2(n-1)n + \kappa(n^2 - 2n - 1)}{2n(\kappa^2(n-1) + n(n-1) - \kappa)}$$

for the real-valued variable κ. We observe that $\nu(\kappa)$ exists for all $\kappa \in [1, (n-3)/2]$. Its derivative can be written as

$$\nu'(\kappa) = -\frac{(n-1)^3(\kappa^2 - n)}{2n(\kappa^2(n-1) - \kappa + n(n-1))},$$

and we can easily verify that

$$\begin{aligned} \nu'(\kappa) &> 0 \text{ if } \kappa \in [1, \sqrt{n}), \\ \nu'(\kappa) &= 0 \text{ if } \kappa = \sqrt{n}, \\ \nu'(\kappa) &< 0 \text{ if } \kappa \in (\sqrt{n}, (n-3)/2]. \end{aligned}$$

We see that $\nu(\kappa)$ attains its unique maximum at the point $\kappa = \sqrt{n}$. So far, we have shown that the generalized $\widehat{k}$-candidate can only be an SSPE if either $\sqrt{n} - 1 < \widehat{k} < \sqrt{n} + 1$ or $\widehat{k} \geq \frac{n-1}{2}$. In order to complete the proof of the theorem, we want to exclude the latter possibility.

To this end, we first note that $\overline{V}_{\widehat{k}} \leq \nu(\widehat{k})$ for any $\widehat{k} \geq \frac{n-1}{2}$. In order to see this, consider the following sequence of implications:

$$\begin{aligned}
\widehat{k} &\geq \frac{n-1}{2}, \\
0 &\leq 2\widehat{k} - (n-1), \\
0 &\leq \widehat{k}^2\left(\frac{n-1}{n}\right) - \left(\frac{\widehat{k}}{2}\right)\left(\frac{(n-1)^2}{n}\right), \\
0 &\leq \widehat{k}^2\left(\frac{n-1}{n}\right) + \widehat{k}\left(\frac{n^2-2n-1}{2n}\right) - \widehat{k}(n-2), \\
n + \widehat{k}(n-2) - 1 &\leq \widehat{k}^2\left(\frac{n-1}{n}\right) + (n-1) + \frac{\widehat{k}}{2n}\left(n^2-2n-1\right), \\
\overline{V}_{\widehat{k}} &\leq \nu(\widehat{k}).
\end{aligned}$$

Now suppose by way of contradiction that for some $\widehat{k} \geq \frac{n-1}{2}$, the generalized $\widehat{k}$-candidate is an SSPE. Then, for any $k \in \{1, \ldots n-1\} \setminus \{\widehat{k}\}$ it holds that

$$\overline{V}_k \leq \overline{V}_{\widehat{k}} \leq \nu(\widehat{k}).$$

Due to the premise that $n \geq 15$, we have that $\sqrt{n}+1 < \frac{n-1}{2}$. Thus there is an integer $\widetilde{k} \in (\sqrt{n}-1,\ \sqrt{n}+1)$ such that

$$\nu(\widetilde{k}) = \overline{V}_{\widetilde{k}} > \nu(\widehat{k}) \geq \overline{V}_{\widehat{k}}.$$

Since $\overline{V}_{\widetilde{k}} > \overline{V}_{\widehat{k}}$, we have obtained the desired contradiction and the proof of the theorem is complete.

□

References

Anesi, V. (2010) Noncooperative foundations of stable sets in voting games, *Games and Economic Behavior, 70,* 488–493.

Ansolabehere, S., J.M. Snyder, and M.M. Ting (2005), Legislative bargaining under weighted voting, *American Economic Review, 95,* 981–1004.

Ansolabehere, S. J.M. Snyder, A.B. Strauss, and M.M. Ting (2005), Voting weights and formateur advantages in the formation of coalition governments, *American Journal of Political Science, 49,* 550–563.

BANKS, J. AND J. DUGGAN (2000), A bargaining model of collective choice, *American Political Science Review, 94,* 73–88.

BANKS J. AND J. DUGGAN (2006), A general bargaining model of legislative policy–making, *Quarterly Journal of Political Science, 1,* 49–85.

BARON, D. AND J. FEREJOHN (1989), Bargaining in legislatures, *American Political Science Review, 83,* 1181–1206.

BATTAGLINI, M. AND S. COATE (2007) Inefficiency in legislative policy-making, *American Economic Review, 97,* 118–149.

BATTAGLINI, M., S. NUNNARI, AND T. PALFREY (2014) Dynamic free-riding with irreversible investments, *American Economic Review, 104,* 2858–71.

BOWEN, T.R. AND Z. ZAHRAN (2012), On dynamic compromise, *Games and Economic Behavior, 76,* 391–419.

BRITZ, V., P.J.J. HERINGS, AND A. PREDTETCHINSKI (2014), On the convergence to the Nash bargaining solution for action–dependent protocols, *Games and Economic Behavior, 86,* 178–183.

DIERMEIER, D. AND P. FONG (2011), Legislative Bargaining with reconsideration, *Quarterly Journal of Economics, 126,* 947–985.

DIERMEIER, D. AND P. FONG (2012), Characterization of the von Neumann–Morgenstern stable set in a non–cooperative model of dynamic policy–making with a persistent agenda–setter, *Games and Economic Behavior, 76,* 349–353.

FAHRENBERGER, T. AND H. GERSBACH (2007), Legislative process with open rules, CER–ETH Working Paper 07/64, ETH Zurich.

DIERMEIER, D. AND T.J. FEDDERSEN (1998), Cohesion in legislatures and the vote of confidence procedure, *American Political Science Review, 92,* 611–621.

ERASLAN, H. (2002), Uniqueness of stationary equilibrium payoffs in the Baron–Ferejohn model, *Journal of Economic Theory, 148,* 2195–2222.

ERASLAN, H. AND K. EVDOKIMOV (2019), Legislative and Multilateral Bargaining, *Annual Review of Economics, 11,* 443-472.

Falconieri, S. (2004), Lobbying and the optimality of delegation: Learning from political institutions, Working Paper.

Fréchette, G., J.H. Kagel, and S.F. Lehrer (2003), Bargaining in legislatures: An experimental investigation of open versus closed amendment rules, *American Political Science Review, 97,* 221–232.

Laruelle, A. and F. Valenciano (2008), Noncooperative foundations of bargaining power in committees and the Shapley–Shubik index, *Games and Economic Behavior, 63,* 341–353.

McCarty, N. (2000), Proposal rights, veto rights, and political bargaining, *American Journal of Political Science, 44,* 506–522.

Oleszek, W. (2011), Congressional Procedures and the Policy Process, Fourth Edition (CQ Press, 1995).

Primo, D. (2003), Open vs. closed rules in budget legislation: A result and an application, Working Paper.

Primo, D. (2007), A comment on Baron and Ferejohn (1989): The open rule equilibrium and coalition formation, *Public Choice, 130,* 129–135.

Rubinstein, A. (1982), Perfect equilibrium in a bargaining model, *Econometrica, 50,* 97–110.

Working Papers of the Center of Economic Research at ETH Zurich

(PDF-files of the Working Papers can be downloaded at www.cer.ethz.ch/research/working-papers.html).

20/346 V. Britz, H. Gersbach
Open Rule Legislative Bargaining

20/345 A. Brausmann, E. Grieg
Resource Discoveries and the Political Survival of Dictators

20/344 A. Jo
The Elasticity of Substitution between Clean and Dirty Energy with Technological Bias

20/343 I. van den Bijgaart, D. Cerruti
The effect of information on market activity; evidence from vehicle recalls

20/342 H. Gersbach, R. Wattenhofer
A Minting Mold for the eFranc: A Policy Paper

20/341 L. Bretschger
Getting the Costs of Environmental Protection Right

20/340 J. A. Bingler, C. Colesanti Senni
Taming the Green Swan: How to improve climate-related fi

nancial risk assessments

20/339 M. Arvaniti, T. Sjögren
Temptation in Consumption and Optimal Redistributive Taxation

20/338 M. Filippini, S. Srinivasan
Voluntary adoption of environmental standards and limited attention: Evidence from the food and beverage industry in Vietnam

20/337 F. Böser, C. Colesanti Senni
Emission-based Interest Rates and the Transition to a Low-carbon Economy

20/336 L. Bretschger, E. Grieg, P. J.J. Welfens, T. Xiong
Corona Fatality Development and the Environment: Empirical Evidence for OECD Countries

20/335 M. Arvaniti, W. Habla
The Political Economy of Negotiating International Carbon Markets

20/334 N. Boogen, C. Daminato, M. Filippini, A. Obrist
Can Information about Energy Costs Affect Consumers Choices? Evidence from a Field Experiment

20/333 M. Filippini, N. Kumar, S. Srinivasan
Nudging the Adoption of Fuel-Efficient Vehicles: Evidence from a Stated Choice Experiment in Nepal

20/332 L. Bretschger, E. Grieg
Exiting the fossil world: The effects of fuel taxation in the UK

20/331 H. Gersbach, E. Komarov
Research Bubbles

20/330 E. V. Dioikitopoulos, C. Karydas
Sustainability traps: patience and innovation

19/329 M. Arvaniti, C. K. Krishnamurthy, A. Crepin
Time-consistent resource management with regime shifts

19/328 L. Bretschger, K. Pittel
Twenty Key Questions in Environmental and Resource Economics

19/327 C. Karydas, A. Xepapadeas
Climate change financial risks: pricing and portfolio allocation

19/326 M. Filippini, S. Srinivasan
Investments in Worker Health and Labor Productivity: Evidence from Vietnam

19/325 H. Gersbach
Democratizing Tech Giants! A Roadmap

19/324 A. Brausmann, M. Flubacher and F. Lechthaler
Valuing meteorological services in resource-constrained settings: Application to smallholder farmers in the Peruvian Altiplano

19/323 C. Devaux and J. Nicolai
Designing an EU Ship Recycling Licence: A Roadmap

19/322 H. Gersbach
Flexible Majority Rules for Cryptocurrency Issuance

19/321 K. Gillingham, S. Houde and A. van Benthem
Consumer Myopia in Vehicle Purchases: Evidence from a Natural Experiment

19/320 L. Bretschger
Malthus in the Light of Climate Change

19/319 J. Ing and J. Nicolai
Dirty versus Clean Firms' Relocation under International Trade and Imperfect Competition

19/318 J. Ing and J. Nicolai
North-South diffusion of climate-mitigation technologies: The crowding-out effect on relocation

19/317 J. Abrell, M. Kosch and S. Rausch
How Effective Was the UK Carbon Tax? - A Machine Learning Approach to Policy Evaluation

19/316 D. Cerruti, C. Daminato and M. Filippini
The Impact of Policy Awareness: Evidence from Vehicle Choices Response to Fiscal Incentives

19/315 M. Filippini, N. Kumar and S. Srinivasan
Energy-related financial literacy and bounded rationality in appliance replacement attitudes: Evidence from Nepal

19/314 S. Houde and E. Myers
Heterogeneous (Mis-) Perceptions of Energy Costs: Implications for Measurement and Policy Design

19/313 S. Houde and E. Myers
Are Consumers Attentive to Local Energy Costs? Evidence from the Appliance Market

19/312 N. Kumar
A model-based clustering approach for analyzing energy-related financial literacy and its determinants

19/311 C. Karydas and A. Xepapadeas
Pricing climate change risks: CAPM with rare disasters and stochastic probabilities

19/310 J.Abrell,S. Rausch and C. Streitberger
Buffering Volatility: Storage Investments and Technology-Specific Renewable Energy Support

19/309 V. Britz
Negotiating with frictions

19/308 H. Gersbach and S. Papageorgiou
On Banking Regulation and Lobbying

18/307 V. Britz, A. Ebrahimi and H. Gersbach
Incentive Pay for Policy-makers?

18/306 C. Colesanti Senni and N. Reidt
Transport policies in a two-sided market