Measuring the Wealth of Nations: Income, Welfare and Sustainability in Representative-Agent Economies

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Abstract

In a finite-horizon general equilibrium model national income is the value of output at supporting prices and a perturbation increases welfare if and only if it raises national income. We show how to extend these results to an infinite horizon representative agent model, and in the process relate them to a debate about how to measure welfare in a dynamic model, how to measure “green national income,” and how to measure “sustainability.” The obvious extension has all the right properties - it measures national income, provides an if and only if welfare increase criterion, and acts as a good indicator of sustainability. Our measure is observable and has been measured for a number of countries. Our index is a Fisherian wealth measure and our results represent the completion of a research agenda set out by Samuelson in 1961.

JEL Classification:
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1 Context

In finite horizon general equilibrium models it is quite clear how to measure national income and how to judge whether a small change is a potential welfare improvement. Indeed both problems have the same solution. Under the standard assumptions about completeness of markets and absence of externalities, if \( p^* \) is a competitive equilibrium price vector and \( c_i^* \) are the equilibrium consumption levels of the \( n \) consumers \( i = 1, ..., n \) then national income is \( p^* \sum_i c_i^* \), the total value of consumption at equilibrium prices, and
a small change in the allocation of resources is a potential Pareto improvement if and only if it increases the value of this inner product. This measure of national income is a linear approximation to a true welfare measure, which is the source of the need to consider only small changes in resource allocation.\(^1\) With infinite horizon models, in contrast, there is no general agreement on how to measure changes in welfare, or on what is the right measure of national income, or on how the two relate.

Here we show how to extend the finite results to an infinite horizon representative agent model, and in the process relate them to the debate about how to measure welfare in a dynamic model, how to measure “green national income,” and how to measure “sustainability.”\(^2\) The obvious extension has all the right properties - it measures national income, provides an if and only if welfare increase criterion, and acts as a good indicator of sustainability. Our preferred measure is a measure of wealth rather than income, and in this respect is in keeping with Samuelson’s remark that in extending national income to a dynamic context, “we find that the only valid approximation to a measure of welfare comes from computing wealth-like magnitudes .." (Samuelson [8] page 57 line 4 on). Perhaps surprisingly, it also relates closely to an empirical measure of sustainability that is increasingly widely used, the World Bank’s measure of “genuine savings.” Because of this, our measure is observable and has been measured for a number of countries (see Arrow et al [1]). The extension of the standard welfare measure to the infinite horizon case is not straightforward: the finite dimensional version is proven by using separating hyperplane theorems, which are not easily used in the infinite dimensional case as hyperplanes can have counterintuitive properties and separation arguments are treacherous because of emptiness of the interiors of many preferred-or-indifferent sets. Our arguments circumvent these problems by drawing instead on results from dynamic programming and optimal control, and can be thought of as completing a research program set out in Samuelson’s insightful early contribution [8], which has been neglected in subsequent literature.

2 Representative agent model

Let the vector \(c(t)\) be an \(m\)-vector of flows of goods consumed and giving utility at time \(t\), and \(s(t)\) be an \(n\)-vector of stocks at time \(t\), also possibly

\(^1\)For an early contribution see Samuelson [7].
\(^2\)For contributions to that debate see Asheim and Weitzman [2], Dasgupta and Maler [3] and Heal [5]. For a survey of the field see Heal and Kristrom [6].
sources of utility. Each stock \( s_j(t) \) changes over time in a way which may depend on the values of all stocks and of all flows:

\[
s_i(t) = d_i (c(t), s(t)), \quad i = 1, \ldots, n
\]

The economy’s objective is to maximize the discounted (at discount rate \( \delta \)) integral of utilities (2):

\[
\max \int_0^\infty u (c(t), s(t)) e^{-\delta t} dt
\]

subject to (1). Utility is a function of \( s \) as well as \( c \) because some stocks may affect welfare directly as well as through their impact on consumption, as for example is the case with human capital or with forests. The utility function \( u \) is assumed strictly concave and the reproduction functions \( d_i (c(t), s(t)) \) concave. This general formulation makes it possible to include human capital and other assets. Note that if \( n = m = 1 \) and \( u = u (c) \) and \( s = f (s) - c \) we have the Ramsey-Solow model. When \( u \) does not depend on \( s \) and (1) takes the form \( s_i(t) = -c_i(t) \) we have the Hotelling model. The general formulation we have chosen captures the possible contributions of environmental stocks to consumer utility and to productive efficiency. This is a general representative agent or optimal growth model, a distinct framework from the general equilibrium model we started with, but it nevertheless allows us to establish an analogous set of results.

To solve this problem we construct a Hamiltonian which takes the form

\[
H(t) = u (c(t), s(t)) e^{-\delta t} + \sum_{i=1}^{n} \lambda_i(t)e^{-\delta t}d_i (c(t), s(t))
\]

where the \( \lambda_i(t) \) are the shadow prices of the stocks.

The first order conditions for optimality can be summarized as

\[
\frac{\partial u (c(t), s(t))}{\partial c_j} = -\sum_{i=1}^{n} \lambda_{i,t} \frac{\partial d_i (c(t), s(t))}{\partial c_j}
\]

\[
\dot{\lambda}_i(t) - \delta \lambda_i(t) = -\frac{\partial u (c(t), s(t))}{\partial s_i} - \sum_{k=1}^{n} \lambda_k(t) \frac{\partial d_k (c(t), s(t))}{\partial s_i}
\]
2.1 Measuring future welfare

We make use of the state valuation function \( V(s) \), which we define in the usual manner:

\[
V(s_0) = \max_{\{c_t\}} \int_0^\infty u(c, s) e^{-\delta t} dt, \quad \dot{s}_{i,t} = d_i(c_t, s_t), \quad i = 1, \ldots, n, \quad s_0 \text{ given}
\]

\( V \) is the true non-linear welfare measure and we will show that an extension of the finite horizon national income to the infinite case is a linear approximation to \( V \), just as \( P^* \sum_i c_i^* \) is a linear approximation to the true welfare measure in the finite case. By standard results,

\[
\frac{\partial V}{\partial s_i} = \lambda_i
\]  

so that the shadow price of the \( i \)-th stock is the marginal social productivity of that stock. It immediately follows that

\[
\frac{dV}{dt} = \sum_i \lambda_i \dot{s}_i
\]  

We can obtain a second expression for the rate of change of the state valuation function by differentiating under the integral sign in the definition of \( V \). Equating the two expressions for \( \frac{dV}{dt} \) gives \( \delta V = H \).

A hyperplane which supports the optimal path is one that separates the set of paths preferred to an optimum from those which are feasible.\(^3\) This is a time path of prices for stocks and flows \( p_{c,j}(t) \) and \( p_{s,i}(t) \) which satisfies two conditions: any path at least as good as the optimum has a value at these prices at least as great as the optimal path, and any feasible path costs no more than the optimum.

**Definition 1** A set of prices \( p_{c,j}(t) \) and \( p_{s,i}(t) \) supporting the optimal path will be called optimal prices and will be used to define National Wealth \( NW \) as follows: National Wealth along the optimal path is

\[
NW = \int_0^\infty \{\langle p_c(t), c^*(t) \rangle + \langle p_s(t), s^*(t) \rangle\} e^{-\delta t} dt.
\]

\(^3\)We shall assume that the functions \( d_i(c(t), s(t)), i = 1, \ldots, n \) are such that the set of feasible paths for \( c_j(t) \) and \( s_i(t) \) is bounded: reasonable conditions sufficient for this are presented for the models used here in Heal [5].
Here \( \langle p_c(t), c^*(t) \rangle \) represents the inner product of the price vector \( p_c(t) \) at time \( t \) with the consumption vector \( c^*(t) \) at time \( t \). NW is just the infinite horizon analog of the conventional finite horizon national income measure, and Fisher was the first to use the term wealth for such a magnitude.\(^4\) Samuelson [8] sets out the need for working with preferences over consumption paths that are functions of time if we are to measure national income correctly in a dynamic context but, not surprisingly given when he was writing, shies away from doing this.

We want to establish that any small change which increases this measure is a welfare improvement. Next we characterize a set of optimal prices which are quite intuitive: they are the marginal utilities of the stocks and flows along an optimal path.

**Proposition 2** The sequence of prices defined by the derivatives of the utility function along an optimal path, i.e.,

\[
\{p_{c,j}(t), p_{s,i}(t)\} = \left\{ \frac{\partial u(c^*(t), s^*(t))}{\partial c_j(t)}, \frac{\partial u(c^*(t), s^*(t))}{\partial s_i(t)} \right\} \forall j, i, t
\]

form a set of optimal prices in the sense of definition 1.

The derivatives of the utility function can be used to define a price system at which national wealth can be computed. It is immediate that any small change in a path which has a positive present value at these optimal prices will increase welfare, and vice versa. An increase in NW is therefore a necessary and sufficient condition for a welfare increase:

**Theorem 3** A variation \( \{\Delta c(t), \Delta s(t)\}_{t=0}^{\infty} \) on optimal path \( \{c^*(t), s^*(t)\}_{t=0}^{\infty} \) has positive present value at the optimal prices \( \{p_{c,j}(t), p_{s,i}(t)\}_{t=0}^{\infty} \), and so is an increase in NW, if and only if the implementation of this variation leads to an increase in welfare.

This theorem, though immediate given what precedes it, is important: it tells us that any other linear index indicates a small increase in welfare if and only if it agrees locally with NW, which is therefore the benchmark for welfare indices. The alternative measure of wealth, used inter alia by Dasgupta and Mäler (2000), is the value of stocks at shadow prices, \( W = \sum_i \lambda_i s_i \).

\(^4\)This definition was used in Heal [5]. Note that Irving Fisher defines wealth at the present discounted value of all future consumption - see Samuelson [8] page 51. So this is in fact a Fisherian definition of wealth. He did not define wealth as the value of assets at market prices - an alternative definition that has tempted some.
Note from equation (7) that the rate of change of the state valuation function is the rate of change of $W$ at constant prices. Building on this and the earlier definitions we can establish an important relationship between the rates of change of national wealth $NW$ and the state valuation function $V$: they are equal up to a first order approximation.

**Theorem 4** The rate of change of $NW$ equals that of $V$ up to a first order approximation.

**Proof.** We have as the current value Hamiltonian of the basic system $H = u(c, s) + \sum \lambda \cdot s_i$ so using the results on the properties of $V$

$$H = u(c, s) + \frac{dV}{dt}$$

and so $\delta V = u(c, s) + \frac{dV}{dt}$  

(8)

so

$$-p_c c - p_s s = \frac{dV}{dt} - \delta V + O(2)$$

(9)

where $O(2)$ is a second order term. Differentiating the expression for $NW$ and using (9) we have

$$\frac{dNW}{dt} - \delta NW = \frac{dV}{dt} - \delta V + O(2)$$

(10)

which can be rewritten as

$$\frac{d}{dt} \left( NW e^{-\delta t} \right) = \frac{d}{dt} \left( V e^{-\delta t} \right) + O(2) e^{\delta t}$$

(11)

Note that (11) implies that

$$\frac{dNW}{dt} = \frac{dV}{dt} + O(2)$$

(12)

The proof is that as

$$\frac{d(NW e^{-\delta t})}{dt} = -\delta NW e^{-\delta t} + e^{-\delta t} \frac{dNW}{dt}$$

(13)

equation (11) implies that

$$-\delta NW e^{-\delta t} + e^{-\delta t} \frac{dNW}{dt} = -\delta V e^{-\delta t} + e^{-\delta t} \frac{dV}{dt} + O(2)$$

(14)

and if $NW = V$ then

$$\frac{dNW}{dt} = \frac{dV}{dt} + O(2)$$

(15)
So the equality of the rates of change of the discounted values implies the equality of the rates of change of the undiscounted values provided that at the initial time these undiscounted values are equal. As the origins of the functions $NW$ and $V$ are arbitrary, we can add or subtract a constant from for example $NW$ to ensure that at $t = 0$ we have $NW = V$. ■

3 Implications

Change in $NW$ is a linear approximation to change in $V$, the true welfare measure, and $NW$ is a linear index. $NW$ therefore plays a role exactly analogous to the standard general equilibrium measure in the finite horizon case. From (7) we have also that

$$\frac{dNW}{dt} = \sum \lambda_i s_i + O(2) \quad (16)$$

What is interesting here is that $\sum \lambda_i s_i$ is observable in principle, depending only on current values of variables, and indeed is what Hamilton and Clements have termed "genuine savings." (See Hamilton and Clements [4] and also Arrow et al. [1]) So the change in $NW$ is observable and equals, to a first order approximation, a measure that has been suggested as appropriate for measuring how an economy’s long-term productive potential is changing, how sustainable is its development. This is an insight that Samuelson [8] missed - he states of the $NW$ measure that “I know of no way of even approximating from market valuations of factors what the values of consumption quantities $Q$ at prices $P$ would be.” Perhaps this is because he focussed on measuring wealth rather than the change in wealth.

There is a direct connection to the literature on sustainability, as $V$ measures the economy’s long-term welfare potential and that this be non-decreasing has been suggested as an interpretation of sustainability (Heal [5], Arrow et al [1]). In this context sustainability is equivalent to non-decreasing national wealth (Dasgupta and Maler [3] amongst others).

References


