The Hartwick rule and the characterization of constant consumption paths in the presence of an exhaustible resource

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Abstract

This paper deals with the interpretation of the Hartwick rule in the particular Dasgupta-Heal-Solow model with one capital good and one non-renewable resource. It is argued that the rule is more a descriptive property of constant consumption paths than a sustainability indicator. It is shown that investing the rents from the resource use into man-made capital is a necessary condition for an efficient resource depletion but it is not a sufficient condition for sustainability. A sustainable consumption indicator is described to characterize the sustainability of constant consumption paths.

Key words: the Hartwick rule, non-renewable resources, sustainable consumption

JEL Classifications: Q01, Q30, D99

* I wish to acknowledge the helpful comments and criticisms of Professors Gilles Rotillon and Jean-Marc Bourgeois. Moreover, some results of Martinet and Doyen (2003) are used, but the present opinions are mine, as all remaining errors.

Preprint submitted to Conference on SURED, young economists sessions 2004
1 Introduction

Is an economy with an exhaustible resource involved in production able to provide a constant (and positive) consumption for an infinite time? This kind of sustainability concern has been discussed several times since the Meadows report (1972).

Solow (1974a) showed that, under specific conditions, the Maximin criterion allows to determine the maximum sustainable consumption for the initial capital stock $K_0$ and resource stock $S_0$. Hartwick (1977) argued that the Hartwick investment rule, which consists in investing the rents from the resource use in reproducible capital, leads to a constant efficient consumption path. This rule is often argued to lead to sustainability as it maintains the total stock of capital and the consumption constant along time. Is the Hartwick rule an indicator of sustainability? Is a society investing the rents from the use of an exhaustible resource going to enjoy a constant consumption (or utility)? These questions are important in the debate about sustainable development and a lot of papers are dealing with these issues and the Hartwick rule.

Hartwick (1978) generalized the rule by considering many exhaustible resources. When substitution among resources is possible, investing the returns from these resources into reproducible capital leads to a constant consumption. It corresponds to a zero net investment. Otherwise, Dixit et al. (1980) generalized the Hartwick rule arguing that a constant net investment (positive or negative) is a sufficient condition to have an equitable path. As the result obtained by Solow (1974a) requires the Hartwick rule (and not the generalized one), Mitra (2002) showed that the rule given by Dixit et al. (1980) indicates equity, but efficient equitable paths require the Hartwick rule.

On the one hand, Solow (1986) argued that the Hartwick investment rule indicates sustainability, and he said that “a society that invests aggregate resource rents in reproducible capital is preserving its capacity to sustain a constant level of consumption” (Solow, 1993, p.170). But on the other hand, Asheim et al. (2003) concluded that this rule does not indicate sustainability and is more a description than a prescription for policies. Moreover, Mitra (2002) showed that a competitive path which is efficient and equitable follows the Hartwick rule, but a path that follows the rule can be inefficient.

The purpose of the present paper is to provide an interpretation of the Hartwick rule and to characterize constant consumption paths when an exhaustible resource is a component of the production function. This work focuses on the “efficient” use of an exhaustible resource and is mainly based on the

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1 Criterion interrelating with the Rawls (1971) criterion of justice.
2 “Efficient” means that the extraction path minimizes the quantity of resource.
Dasgupta-Heal-Solow model. Asheim et al. (2003) considered the generalized Hartwick rule and the present study only deals with the classic Hartwick rule. Nevertheless, the objective is the same: It is argued that the Hartwick investment rule is a description of constant consumption paths and should not be considered as an indicator of sustainability. In other words, the paper attempts to demonstrate that a sustainable “equitable” path, with an efficient resource use, follows the Hartwick rule; but a path that follows the Hartwick rule can be unsustainable. For such a purpose, and to shed a new light on this rule, the paper is not based on the optimal control framework, but on the viable control analysis (Aubin, 1991) and its results on the sustainability issue (Martinet and Doyen, 2003).

Thereafter, as it is argued that the Hartwick rule is not a sufficient indicator for sustainability, a new indicator is introduced: the maximal sustainable consumption indicator. The study of its evolution along time allows some new comments about this issue.

The paper is organized as follows. In section 2, the framework is presented and some results of Martinet and Doyen (2003) are summarized. Then, in section 3, these results are used to present a new interpretation of the Hartwick Rule. In section 4, the “sustainable consumption indicator” is described and numerical simulations illustrate the general discussion. To conclude, some general observations are given in section 5.

2 Hartwick rule and viability

The aim of this section is to present the framework that will permit to link the Solow (1974a) and Hartwick (1977) results to the viability results (Martinet and Doyen, 2003). In the first subsection, the economic model is presented. Then, subsection 2.2 recall the Hartwick (1977) results. And finally some viability results are given in subsection 2.3.

2.1 The model

The model used is the so-called Dasgupta-Heal-Solow model, as it is described in Dasgupta and Heal (1974) and Solow (1974a). The capital $K_t$ and the resource stock $S_t$ evolve under the dynamic equations

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3 In this paper, a constant consumption path will be called “equitable” to match with the referred literature. Nevertheless, the equity issue can not be reduced to the seek of constant consumption paths.
\[ \dot{K} = F(K, r) - c_t \]  
\[ \dot{S} = -r_t \]  

(1)  
(2)

where \( F \) is the production function, \( c_t \) the consumption at time \( t \) and \( r_t \) the extraction of the natural resource. For the sake of simplicity, and to be consistent with the Solow (1974a) and Hartwick (1977) analysis\(^4\), \( F \) is defined as a Cobb-Douglas production function, i.e. \( F = K^\alpha r^\beta \). The initial stocks are denoted respectively \( K_0 \) for the capital and \( S_0 \) for the exhaustible resource and are supposed to be given and exogenous.

In this paper, a level of consumption is said to be sustainable if it can go on forever. Thus, a generation has a sustainable consumption if its level of consumption is possible for every following generation. This conception of sustainability is a technical one, and is not based on an interpretation of the sustainable development concept; it is not argued that sustainable development requires a constant consumption but only that a level of consumption is sustainable if it can go on forever.

For such a model, Solow (1974a) proved that a constant positive consumption is possible if the share of the capital in production is greater than that of the resource, i.e. \( \alpha > \beta \). Then, the maximum sustainable consumption is given by the equation

\[ c_0^+ = (1 - \beta) \left( S_0(\alpha - \beta) \right)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha - \beta}{1-\beta}}. \]  

(3)

The sustainable consumption depends on the initial stocks \( K_0 \) and \( S_0 \). Along this path, the Hartwick rule holds true and the extraction is defined by the equation

\[ r_t = \left( \frac{c_0^+}{1-\beta} \right)^{\frac{1}{\beta}} K_t^{-\frac{\alpha}{\beta}}. \]

2.2 The Hartwick results

Hartwick (1977) argued that a sufficient condition to enjoy a constant consumption path is to invest in reproducible capital all the returns from the exhaustible resource use. Within our framework, the marginal productivity of the resource is \( F'_r = \beta K^{\alpha_r-1} \) and the rents from the resource use are \( r \beta K^{\alpha_r-1} \). Thus, the Hartwick investment rule requires \( \dot{K} = \beta K^{\alpha_r \beta} \).

\(^4\) Moreover, it is shown in Dasgupta and Heal (1979) that for CES production functions, the study of the intertemporal use of an exhaustible resource is relevant only when the elasticity of substitution is equal to one. On the one hand, for a higher elasticity, the resource is not essential to production and the sustainability is not really a problem. On the other hand, if the elasticity is lower than one, there is no sustainable consumption path.
The Hartwick result (Hartwick, 1977) requires a) the Hotelling local efficiency rule and b) the Hartwick investment rule. Under these two conditions, Hartwick proved that the consumption is constant (without specifying if it is positive or not).

Can the Hartwick investment rule be used as a sustainability indicator? If one considers sustainability as a constant and positive consumption for each generation, the Hartwick rule is an indicator of sustainability if the following conditions are satisfied.

1. The Hotelling efficiency rule holds true locally.
2. A constant and positive consumption path exists.
3. And finally, if the current consumption is a sustainable (optimal?) one.

The first condition does not rely on a strong assumption. It only requires that the resource price (whatever it is) increases at the interest rate.\(^5\)

The second point represents the “true” sustainability condition. Roughly speaking, a “sustainable path” can be followed only if it exists. In an economic model, the existence of such a path relies on the way the economy is represented, and in particular on the form of the production function. In the present work, we consider a model in which a constant consumption path exists (Solow, 1974a). Thus, the sustainability concern is not addressed here.\(^7\)

Our thought is based on the third point. We wonder how to know if the current consumption is sustainable. We will see that in fine, this point is linked to the Hotelling rule (efficiency of the whole path).\(^8\)

2.3 The viability analysis results

As an important objective of the paper is to characterize constant consumption paths in quite a new way, we need to describe some results from Martinet and Doyen (2003). They used the viable control analysis to deal with the sustainability issue in an economic model with a non-renewable resource. This

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5 The resource price must increase at the interest rate (marginal productivity of the capital), i.e. \(\frac{d \log(F'_r)}{dt} = F'_k\).
6 We can suppose that the resource owners are rational enough to follow this rule.
7 From our point of view, the main point to deal with the sustainability issue should be the debate about the substitutability between natural resources and “man-made” capital. Neumayer (2003) argues the same.
8 It is a stronger assumption to assume that the resource market is perfect, e.g. agents may be unable to foresee far enough in the future and are not perfectly informed to compute the true resource price.
approach focuses on the intertemporal feasible paths of a dynamic system
under some state and control constraints: a set of constraints that represents
the “good health” of the system is defined, and a path is said to be viable if it
respects the whole constraints. In a sustainability concern, the state variables
are the stocks of man-made capital and an exhaustible resource, while the
controls are the consumption and extraction levels.

The purpose of the approach is thus to determine the set of states from which
there exists at least one viable path, i.e. the set of states \((K_t, S_t)\) from which
there exist control decisions \((c(.), r(.))\) keeping the system evolving under (1)
and (2) into the set of constraints. The set of all viable states is called the
viability kernel. If, for any \(t\), the state \((K_t, S_t)\) of the system is outside the
viability kernel, then, there does not exist any path starting from \((K_t, S_t)\) that
respects the whole constraints forever, and this, whatever are the following
decisions \((c(.), r(.))\). Such a crisis situation is irreversible. Then, the viability
kernel is the set of states from which at least a sustainable path exists, but
the actual sustainability of a path depends on the decisions that apply. Thus,
a second step is to determine the controls associated with viable paths. For
a given viable state, the set of all viable decisions upon the admissible ones
is defined. This set describes viable policies (policies that will not lead the
system into crisis).

In our framework, some viable paths exhibit constant consumption along
time.\(^9\) Such paths can be interpreted as intergenerationally equitable. The
aim of this paper is to consider these constant consumption paths, to charac-
terize them and to examine how the Hartwick rule applies.

In the Cobb-Douglas case, with \(\alpha > \beta\) and \(\beta < 1\), Martinet and Doyen (2003)
showed that a constant level of consumption \(c_\sharp\) is sustainable if, for a given
level of capital \(K_t\), the resource stock \(S_t\) is greater than a threshold \(V(c_\sharp, K)\)
which depends on the current stock of capital and the level of consumption.

**PROPOSITION 1.** The minimal stock of resource required to sustain a con-
stant consumption \(c_\sharp\) forever after time \(t\), given a level of capital \(K_t\), is defined
by the equation

\[
V(c_\sharp, K_t) = \frac{1}{\alpha - \beta} \left( \frac{c_\sharp}{1 - \beta} \right)^{\frac{1}{\beta} - 1} K_t^{1 - \frac{\beta}{\alpha - \beta}}.
\]

\(^9\) A wider description of all viable paths is given in Martinet and Doyen (2003). If
sustainability is defined with respect to a guaranteed level of consumption \(c_{min}\) for
every generation, the sustainable consumptions are not reduced to constant ones,
and the extraction/investment decisions are not limited to the one defined by the
Hartwick rule.
Proof. This result is closely linked to the definition of the viability kernel of the system. The proof is a part of the main proposition of Martinet and Doyen (2003, Proposition 3.3), which is too long to be described here. □

Moreover, Martinet and Doyen (2003) showed that all constant consumption paths with \( c_t = c^*_t \leq c^+_0 \) are feasible and that, for such paths, a part of the natural resource can be preserved forever.

Corollary 1. The maximal quantity of resource that can be preserved forever along a constant consumption path with \( c_t \leq c^+_0 \) is given by the following expression

\[
S^*_t = S_0 - \frac{1}{\alpha - \beta} \left( \frac{c^*_t}{1 - \beta} \right)^{\frac{1}{\beta - 1}} K_0^{1 - \frac{\beta}{\alpha}}. \tag{5}
\]

Proof. From the initial state \((K_0, S_0)\), the minimal quantity of resource used to sustain the constant consumption \( c_t \) is \( V(c_t, K_0) \). Thus, the maximal quantity of resource that can be preserved is given by \( S_t = S_0 - V(c_t, K_0) \). □

If we consider optimality as the maximization of the sustainable consumption (maximin criterion), the resource will be exhausted (at infinite time) so to maximize the maintainable consumption. Thus we have \( V(c^+_0, K_0) = S_0 \). Making some algebra, this equality leads to the same result as in Solow (1974a).\(^{10}\) Thus, the viability results encompass the Solow results.

Thanks to these results, we will examine in the next section if the satisfaction of the Hartwick investment rule along a constant consumption path means that the economy is following a sustainable path.

3 What is the Hartwick rule?

The purpose of this section is to shed a new light on the debate about the Hartwick investment rule. First, it is shown that the Hartwick investment rule characterize efficient resource use. Then, we prove that the Hartwick investment rule (investing the rents from the depletion of a non-renewable resource into man-made capital) can not be used as an indicator of sustainability neither as a policy prescription.

\(^{10}\) Equation (3) in the present paper.
3.1 Efficient use of natural resources

It’s worthwhile to specify what “efficient” means in this paper. The efficiency of a path is defined with respect to the resource use. It is not defined by a marginal condition (a price condition for example). The efficient use of the resource is the level of extraction that minimizes the quantity of resource used along a given consumption path. Thus, we do not explicitly refer to the Hotelling rule.

The following proposition determines the efficient extraction rule for constant consumption paths. Such an efficient extraction has to include the choice of the investment level that maintains the capacity of production.

**Proposition 2.** The efficient extraction rule, which minimizes the quantity of resource used along a constant consumption path $c_t = c^*$, is given by the equation

$$r^*(c^*_t, K_t) = \left(\frac{c^*_t}{1-\beta}\right)^{\frac{1}{\beta}} K_t^{-\frac{2}{\beta}}. \quad (6)$$

**Proof.** As established in Proposition 1, the minimal quantity of resource used to sustain a constant consumption $c_t$ is given by $V(c_t, K)$ defined by the equation (4). Such an optimality is obtained if the extraction $r(t)$ satisfies the following Hamilton-Jacobi-Bellman equation

$$V'(K) \dot{K} + r = 0.$$  

As $V'(K) = -\frac{1}{\beta} \left(\frac{c}{1-\beta}\right)^{\frac{1-\alpha}{\beta}} K^{-\frac{2}{\beta}}$ and $\dot{K} = K^\alpha r^\beta - c$, one can verify that the solution is

$$r(t) = \left(\frac{c}{1-\beta}\right)^{\frac{1}{\beta}} K_t^{-\frac{2}{\beta}}. \quad \square$$

This extraction rule is nothing else than the Hartwick investment rule in the Cobb-Douglas case. As the Harwick rule is given by $\dot{K} = r \beta K^\alpha r^\beta - 1 = \beta F$, and using the dynamic equation (1) we get $c = (1-\beta) F = (1-\beta) K^\alpha r^\beta$ which is the same as (6).

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\[11\] As we do not consider a price system in our approach, this description of the Harwick rule does not deal with the question of competitiveness. The possibility of attaining the optimal allocation via a decentralized mechanism and the maximization of individual behaviors is not considered.
Thus, we have stated that along a constant consumption path, an efficient resource use means an extraction rule that leads to the Hartwick investment rule. We have here the so-called converse of the Hartwick rule. An efficient use of the resource along a constant consumption path is characterized by the Hartwick investment rule. While searching the maximum sustainable consumption, Solow (1974a) showed, in the optimal control framework, that minimizing the quantity of resource \( \int_{0}^{\infty} r(t)dt \) used along a constant consumption path requires the satisfaction of the Hotelling local efficiency rule \( \frac{d(log f)}{dt} = f_k \) and leads to the same result than our Proposition 2.

3.2 The “Hartwick investment rule” does not indicate sustainability

The following proposition states that the Hartwick investment rule is not a sufficient condition to indicate the sustainability of a constant consumption path. If the initial extraction level is greater than \( r^*(c_0^+) \) as it is defined by (6) and the Hartwick investment rule followed, sustaining the initial level of consumption is impossible as the resource stock will be exhausted in a finite time.

PROPOSITION 3. A path that follows the Hartwick investment rule with a constant consumption is not necessarily sustainable.

Proof. We give the proof of this proposition by providing a counterexample. The level of extraction at time 0 is noted \( R_0 \) and is supposed to be greater than the extraction \( r^*(c_0^+) \). The production is then \( K_0^a R_0^b \); the investment (given by the Hartwick rule) is \( \beta K_0^a R_0^b \) and the consumption is \( C_0 = (1-\beta)K_0^a R_0^b \). If equity is required, the consumption will be maintained at \( c_t = C_0 > c_0^+ \). Such a consumption can be maintained for a time while the investment follows the Hartwick rule.

The minimal quantity of resource required to sustain this consumption forever is 

\[
V(C_0, K_0) = \frac{1}{\alpha - \beta} \left( \frac{C_0}{1-\beta} \right)^{\frac{1}{\beta} - 1} K_0^{1-\frac{\alpha}{\beta}} > V(c_0^+, K_0) = S_0
\]

Thus, the stock of resource will be exhausted in a finite time. The consumption can not be sustained forever. \( \Box \)

\(^{12}\) Withagen and Asheim (1998) and Withagen et al. (2003) deal with this issue.

\(^{13}\) This could be the case for example if the price of the resource is under-evaluated because of some market inefficiency.
3.3 Some comments about this example

The previous counterexample is similar to the one given by Asheim et al. (2003, p. 138-139). Nonetheless, whenever the resource is “essential” to production,\textsuperscript{14} it is not optimal to exhaust the resource stock in a finite time $T$. The efficiency condition (Hotelling rule) does not hold if the resource is exhausted as the marginal productivity of the resource is infinite after time $T$. In our framework, the question of efficiency is not addressed with respect to a price condition. But if one considers this issue, the path described in the previous counterexample is not efficient. Asheim et al. (2003) provided another counterexample in which the Hotelling rule is satisfied for the whole path. This example depends on an important hypothesis of their model: the capital accumulation can be negative. In this case, after a first phase with $c_t = c(0) > c_0^+$ and an investment that follows the Hartwick rule, a second phase with a negative constant value of investment (generalized Hartwick rule) allows to satisfy the Hotelling rule in spite of the resource exhaustion: if the capital is exhausted in the same time as the resource, there is no profitable arbitrage opportunities in $T$.

Why is the resource exhausted in a finite time?

The problem arises from a wrong evaluation of the resource price. If the Hotelling rule were respected for the whole path, the initial price of the resource would be higher and less resource would be used with respect to the efficiency condition. Thus, a necessary condition for sustainability is to verify that the constant consumption path is possible, i.e. that the extraction path does not exhaust the resource in a finite time. Solow (1974a,b) noticed this point and searched the level of consumption for which the extraction rule (6) exhausts the resource in an infinite time. Doing so, he determined the maximal sustainable consumption (Maximin path). This path is associated with the “true” resource price. Thus, we can say that the Hartwick investment rule is not sufficient to describe sustainability in our framework. Sustainability requires to verify that the levels of extraction, and so, the consumption are not too high; i.e. the whole path satisfies the Hotelling rule.

We can conclude this section by arguing that the Hartwick investment rule can be used as an indicator of resource use efficiency, but not as a sustainability indicator. In the next section, a “maximal sustainable consumption” indicator is introduced. It allows to characterize the sustainability of constant consumption paths.

\textsuperscript{14}We refer here to the economic interpretation given by Dasgupta and Heal (1974). It corresponds to the condition $\lim_{r \to 0} f_r' = +\infty$ and does not necessarily require the production to be nil when no resource is used ($f(K, 0) = 0$). In the Cobb-Douglas case, both are true.
4 Characterization of constant consumption paths

We have shown in the previous section that the Hartwick rule is not a satisfying indicator for sustainability in the Weak Sustainability paradigm. In this section, a new indicator is proposed to characterize the sustainability of constant consumption path.

4.1 A sustainable consumption indicator

If sustainability requires a non-decreasing consumption along time, one can argue that a generation can choose to consume less than the previous ones without compromising the sustainable development. Then, one way to describe the sustainability of a consumption path is to examine how the sustainable consumption evolves along this path. For such a purpose, we define the maximal sustainable consumption $c^+_t$ which represents the level of consumption that can be sustained after time $t$. This level of “possible” consumption is used as a sustainability indicator, and does not describe the actual consumption.

Definition 1. The indicator of sustainable consumption is defined as follows

$$c^+_t = c^+(K(t), S(t)) = (1 - \beta)(\alpha - \beta)^{\frac{1}{1-\beta}} S(t)^{\frac{1}{1-\beta}} K(t)^{\frac{\alpha-\beta}{1-\beta}}.$$  (7)

At $t = 0$, this indicator is equal to the maximal sustainable consumption $c^+_0$ defined by Solow (1974a). The value of $c^+_t$ depends on the stocks level. If these stocks change, the value of $c^+_t$ changes too. The purpose of this section is to study how $c^+_t$ evolves along time.

Why $c^+_t$ can be used as a sustainability indicator?

Suppose that current decisions (consumption, extraction and investment) make the indicator decreasing, then, the sustainable consumption for all future generations will be reduced. Thus we can say that current choices are not sustainable. On the contrary, if current choices keep the indicator constant or increasing, all future generations will be able to consume at least as much as it was possible before the current generation choices. Thus, current choices are sustainable.

To characterize the sustainability of a constant consumption path, we consider the evolution of the sustainable consumption indicator along time. The time
The derivative of equation (7) is

\[
\frac{dc}{dt}^+ = c_t^+ \left[ \frac{\beta}{1 - \beta} \dot{S} S_t^{-1} + \frac{\alpha - \beta}{1 - \beta} K K_t^{-1} \right]
\]  

(8)

For a constant consumption \(c_t = c_0\), and when the Hartwick investment rule is followed, we have

\[
\dot{K} = \frac{\beta c_0}{1 - \beta}
\]

and

\[
-\dot{S} = r_t = \left( \frac{c_0}{1 - \beta} \right)^{\frac{1}{\beta}} K_t^{-\frac{\alpha}{\beta}}.
\]

Then, equation (8) becomes

\[
\frac{dc}{dt}^+ = c_t^+ \frac{\beta c_0}{(1 - \beta)^{\frac{1}{\beta}}} K_t^{-\frac{\alpha}{\beta}} S_t^{-1} \left[ (c_t^+)^{\frac{1-\beta}{\beta}} - c_0^{-\frac{1-\beta}{\beta}} \right]
\]

(9)

or equivalently (for \(c_0 \neq 0\)),

\[
\frac{dc}{dt}^+ = c_t^+ \left[ \left( \frac{c_t^+}{c_0} \right)^{\frac{1-\beta}{\beta}} - 1 \right] \frac{\beta}{1 - \beta} r_t S_t.
\]

(10)

We obviously deduce from this equation that the *sustainable consumption indicator* increases when \(c_0 < c_t^+\) and decreases otherwise. The quantity \(\frac{dc}{dt}^+\) represents the losses/gains of all futures generations. This quantity depends on the level of the sustainable consumption, the difference between the actual consumption and the sustainable one, and the resource extraction ratio \(\frac{r_t}{S}\). This last point means that the smaller the extraction is with respect to the resource stock size, the less the consequences on sustainability are important (see Neumayer, 2003, p. 152-153 for a wider discussion on this topic).

Let us define the value \(x = c_0 - c^+_t(t)\). \(x\) is the over consumption of the generation living in \(t\) (if \(x < 0\), it is equivalent to a lower consumption than the sustainable one). We can determine how evolve the gains/losses with respect to \(x\).

\[
\frac{d dc}{dx} = c^+ - \frac{\beta}{(1 - \beta)^{\frac{1-\beta}{\beta}}} K^{-\frac{\alpha}{\beta}} S \left( (c^+)^{\frac{1-\beta}{\beta}} - 1 \right) \left( c^+ + x \right)^{\frac{1-\beta}{\beta}}.
\]

(11)

For \(\beta < 1\), the sign of this expression depends on the sign of the expression in brackets. It is negative if \(x > \lambda c^+\) where \(\lambda = \beta^{\frac{1-\beta}{\beta}} - 1 < 0\). Figure 1 illustrate this point.
It means that if $x > 0$, the greater is $x$, the greater the losses for all future generations are. On the contrary, the gains from a limitation of the consumption ($x < 0$) are bounded and a large restriction won’t compensate for an over-consumption. Moreover, the “loss-effect” is cumulative as too high a consumption implies an increase along time of the difference $x$ between the actual and sustainable consumptions.\footnote{Furthermore, it is easy to show that the losses are increasing more than proportionally to $x$ as the derivative $\frac{d^2 L}{dx^2}$ is negative.}

The next subsection illustrates these results.

4.2 Simulations an illustrations

For an illustrative purpose of previous results, we consider a continuous time dynamic model with three time intervals. On each interval, the consumption is constant and the extraction is \textit{efficient} (as defined by (6)); consequently, the Hartwick investment rule is followed.

Consider that the consumption is constant at levels $c_a$ on the interval $I_a = [0; t_a]$ and $c_b$ on the interval $I_b = [t_a; t_b]$, with $t_a < t_b$. Consider that $c_a > c_0^+$ and that $c_b < c_{t_a}^+$. After $t_b$, the consumption is constant and fixed at $c_{t_b}^+$. This case corresponds to an unsustainable initial consumption on $I_a$ and to the limitation of consumption for the generations living on interval $I_b$. After $t_b$, the consumption is the maximal sustainable consumption.

During the first period $I_a$, the consumption is greater than the maximal sustainable consumption. Such a situation reduces the level of the sustainable consumption given by the indicator $c_{t_a}^+$. After $t_a$, the consumption is lower than the sustainable one. The indicator of sustainable consumption $c_{t_b}^+$ increases.

The figure 2 illustrates this example. We take $t_a = 20$ and $t_b = 40$. The production function parameters are $\alpha = 2/3$ and $\beta = 1 - \alpha$. The initial stocks are $K_0 = 2$ and $S_0 = 10$. At $t = 0$, the maximal sustainable consumption is then $c_0^+ \simeq 1.72$. The consumption is $c_a = 1.8 > c_{t_b}^+$ during the first period. At $t = 20$, the sustainable consumption is $c_{t_0}^+ \simeq 0.77$. The consumption on the second period is $c_b = 0.5 < c_{t_b}^+$.
consumption, the sustainable consumption indicator does not reach its initial value. It would be useless to repeat such a simulation with different parameters values as this kind of simple models can not be used to get quantitative information. Nevertheless, it informs us that an over-consumption is very harmful for future generations as it requires a longer limitation period to compensate for a “prodigal” one.

5 Some concluding observations

In this framework, the Hartwick investment rule is not sufficient to characterize sustainability. Sustainability also depends on the level of consumption that has to be lower than or equal to the indicator $c_t^+$. The Hartwick rule provides a necessary condition. It characterizes an efficient resource use.

Asheim et al. (2003) argued that

(1) The Hartwick rule does not indicate sustainability.
(2) The Hartwick rule does not require substitutability between man-made and natural capital.

The present work concludes in the same way as their first result. Nevertheless, the second assertion is false in this framework. A constant positive consumption is possible, in an economy with an exhaustive resource, only if man-made capital is a substitute for natural capital. A proof is given in Dasgupta and Heal (1979) and in Martinet and Doyen (2003). Just note that in Asheim et al. (2003) the proof of the second assertion is given, in a more general framework, only for a renewable resource.

The results of this paper are a complement to the conclusions of Mitra (2002) that states that the Hartwick Rule is quite different from an equity concern and would rather be considered as an indicator of the inefficiency for equitable paths. Furthermore, it is shown here that, in the Dasgupta-Heal-Solow model framework, the Hartwick rule does not indicate sustainability but the efficient use of the exhaustible resource along constant consumption paths.

From a more general point of view, sustainability is often defined as ‘Equity + Efficiency’ (Heal, 1998). These two points are considered within the Hartwick result. Nevertheless, a lot of studies and results about the Hartwick rule do not deal with the question of the existence of a maximin path. This existence is taken as an hypothesis. Thus, these studies results are depending on this assumption and are often local results: the Hotelling rule is required

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16 Efficiency is here what we have called Optimality with respect to a criterion previously in the paper.
locally and the results depend on local properties of the model. It is not a new topic of conversation. Solow (1974b, p.12) noticed that

“many patterns of exploitation of the exhaustible-resource pool obey Hotel ling’s fundamental principle myopically, from moment to moment, but are wrong from a very long run point of view. Such mistaken paths may even stay very near the right path for a long time, but eventually they veer off and become bizarre in one way or another.”

The present paper attempts to describe the dynamic behavior of a simple model in the presence of such a mistake. It is stated that another important issue for sustainability is the ‘feasibility’ of paths. Solow (1974a) and Cass and Mitra (1991) consider this important issue and describe the existence conditions of such constant consumption paths (and thus, maximin paths).

A significant contribution of this paper is to describe the dynamic evolution of the sustainable consumption indicator. If sustainability requires the current consumption to be attainable by all future generations, this indicator provides a good information. If it does not decrease, the current decisions are “sustainable”. Such an indicator can take into account changes in the production function (value of the parameters $\alpha$ and $\beta$) and discovery of new resource stock. Nevertheless, the sustainability concern can not be reduced to such a consumption indicator. Further research needs to be done. In particular, we aim for a generalization of this approach to any consumption path (not only constant ones) and for the study of a decentralized economic model.

References


Fig. 1. Value of the time derivative of $c^+(t)$ with respect to the difference between actual consumption and the sustainable one, $x = c_0 - c^+(t)$. 
Fig. 2. Evolution of the “sustainable consumption indicator” when the consumption is different from the sustainable one.