OPTIMAL ENDOGENOUS SUSTAINABILITY
WITH AN EXHAUSTIBLE RESOURCE

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Abstract

We study the possibility of a sustainable consumption level under an exhaus-
tible resource constraint for an economy where the resource productivity may be
increased through a dedicated research policy. We give a general condition for
such a research policy to be efficient. Next we characterize the optimal growth
paths of an economy with a CES production function, including as limit cases
the Cobb-Douglas and the Leontiev technologies. In the CES and the Leontiev
cases, an optimal balanced growth path with a constant positive consumption
level exists provided that the research productivity be sufficiently high with re-
spect to impatience. In the Cobb-Douglas case, an optimal path is necessarily a
balanced path, but avoiding an asymptotic decline to zero of the consumption
level requires a more stringent condition involving the output elasticity with
respect to the resource factor. Last we give a complete characterization of the
optimal transitions to the steady state for the CES case.

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1 INTRODUCTION

That the technical progress could be dedicated to improve the efficiency of some specific factors of production is not a quite new idea. Its first formalization goes back at least to Kennedy (1964) (see also Dandrikis and Phelps (1965), Nelson and Phelps (1966), Phelps (1966), Samuelson (1965), von Weizäker (1966), and the recent contributions of Acemoglu (2003-b) and Sato (2006)), and as far as optimal endogenous growth is concerned to the Uzawa (1965) fundamental paper.

Next the idea has been exploited in resource economics. Curiously, in the famous Symposium Issue of the Review of Economic Studies (1974), although dedicated technical progress or factor augmenting technical progress is briefly mentioned by Dasgupta and Heal (see also Dasgupta and Heal (1979)), their seminal paper and the paper of Sitglitz are essentially oriented towards unbiased exogenous technical progress, and the Solow paper, towards Harrod’s neutral exogenous progress.

More recently the idea got some revival for explaining the long run evolution of distribution and the changes in demand for skilled and unskilled labor (see Acemoglu (2002), (2003-a) and (2003-b) for well documented surveys) and again in resource economics (see for example, André and Smulders (2003), Eriksson (2004), Grimaud and Rougé (2003), Smulders (1996)).

A lot of papers in resource economics adopted the framework of the so-called new growth theory, with a proliferation of either intermediate goods or final consumption goods, in which the technological knowledge is embodied. Although this formulation could appear as an inescapable detour for a positive theory, it tends to blur more fundamental relationships which was enlightened in the Uzawa (1965) paper. We choose in the present paper to go back to the way pioneered by Uzawa in order to isolate the specificity of dedicated investment process in relation to the non renewable resource problem, bypassing the intermediate good sector and assuming that some Hicksian aggregate consumption good can be produced directly from labor and some non renewable resource.


\footnote{Stiglitz (1974) assumes Cobb-Douglas production functions so that the technical progress cannot be dedicated under the additional assumption that any factor $x$, measured in efficiency units, takes the form $ax^a$, where $a$ is some positive efficiency index.}

\footnote{However this proliferation would be questioned by most historians versed in technological history. To anybody which would not be convinced we suggest to visit any technological museum, on pre-industrial art and craft. The most striking evidence of such visits is that, given a level of development of general knowledge, each niche is fully exploited, giving rise to a lot of either intermediate or final goods. For example in the "Musée de l’outil", in the small town of Troyes (Champagne county, France) we denumbered sixty types of planes used by wood workers, and it is a mere sample of such capital goods because there exists a continuum of such goods, each one adapted to some specific task and to the characteristics of each wood worker (skill, morphology, and so on...)}

\footnote{Concerning the strong differences between non renewable and renewable resource economies, see Amigues, Long and Moreaux (2004), Amigues and Moreaux (2004), and Moreaux}
In the present model there exist two assets, the stock of non-renewable natural resource and the stock of dedicated technical knowledge. The stock of knowledge is a capital stock which can be accumulated like physical capital and for which a higher rate of accumulation implies that some consumption has to be given up, because more labor has to be employed in the research sector, which is no more available for the consumption good production sector. But this is also a capital the production function of which presents special features.

There are two extreme conceptions of the world of (fruitful) ideas. According to the first one, the set of ideas is finite or at least bounded, and its exploration is more and more costly. According to the second one, the set of ideas is unbounded and more accumulated ideas facilitates the discovery of new ones, so that there is some learning by doing in the research sector but without the deceleration of the cumulative effects of the traditional learning by doing theory (see Arrow (1962)), so that indefinite improvements of the efficiency of the primary production factors can be sustained at non-increasing marginal costs. This is the assumption retained in the present paper, like in most papers on endogenous growth theory.

We first develop a general model of dedicated technical change and derive the main conditions for efficiency and optimality of an R&D policy. Next we specialize this general framework to specific economies e.g. the constant elasticity class, in order to give a full characterization of the optimal paths, both the steady states and the transitions towards the balanced growth path. We develop the study for three kinds of productive functions exhibiting different levels of imperfect input substitution: the Cobb-Douglas, the Leontiev and the general CES functions. The Cobb-Douglas form has been intensively studied in the endogenous growth and natural resource literature since the seminal work of Dasgupta et Heal (1974), the results for the Leontiev and CES cases are new.

We show that even if knowledge and resource are poor substitute for labor (as in the Leontiev limit case), some strictly positive per capita consumption level can be sustained in the long run along an optimal path, provided that the productivity in the research sector be sufficiently high relative to the impatience rate.

Along the optimal transition path, a positive growth of consumption can be achieved through a permanent increase in the labor input devoted to production, compensated by a decrease in the level of R&D effort. This general pattern is not qualitatively modified by the degree of substituability between the inputs. The dynamics of the resource extraction path may be however more complex. It may be the case that the resource use expands during a first time period before declining in the long run.

5 Thus the society is technically sustainable, and economically sustained provided that its impatience rate be not too high and its management be optimal. On the sustainability definitions and concepts see the seminal work of Pezzey (1992-a,1992-b), and the more recent surveys of Pezzey and Toman (2002) and Neumayer (2003).
In the Cobb-Douglas case, where technical progress is no more dedicated, we show that an optimal path is necessarily a balanced path, a feature already identified by Schou (2002). This is a consequence of perfect substituability in situations where there does not exist possibilities to accumulate a productive capital stock over time (see by contrast the analysis of Scholz and Ziemes (1999)). A positive growth in the long run may be achieved in the Cobb-Douglas case provided that the research labor productivity be sufficiently high and the resource elasticity be sufficiently low. However the high input substituability does not help sustainability. Actually, we show that the basic condition for sustaining a strictly positive consumption level in the long run is more stringent in the Cobb-Douglas case than in the poorer substituability or the Leontief or CES functions.

2 THE MODEL

We consider an economy in which the population is constant over time and the labor supply is inelastic. Without loss of generality we may assume that the labor endowment is equal to unity.

The economy produces an aggregate consumption good from labor and some non-renewable resource. Let \( q \) be the instantaneous production level of this consumption good, and \( l \) and \( s \) be respectively the amount of labor and resource inputs used in the consumption good production sector.

The efficiency of these inputs is depending upon specific technological knowledges which can be accumulated through specific R&D efforts. Let \( A \) be the stock of knowledge determining the efficiency of the labor input and \( B \) the stock of knowledge determining the efficiency of the resource input so that, denoting respectively by \( x \) and \( y \) the amounts of labor and resource inputs measured in efficiency units:

\[
x = x^f(A, l) \quad \text{and} \quad y = y^f(B, s).
\]

Assumption E.1 The efficiency functions \( x^f \) and \( y^f \), each one \( R^+_x \rightarrow R^+_x \), are \( C^2 \) functions strictly increasing in each argument and such that:
\[
\lim_{A \downarrow 0} x^f(A, l) = 0, \forall l > 0 \quad \text{and} \quad \lim_{l \downarrow 0} x^f(A, l) = 0, \forall A > 0
\]

and
\[
\lim_{B \downarrow 0} y^f(B, s) = 0, \forall s > 0 \quad \text{and} \quad \lim_{s \downarrow 0} y^f(B, s) = 0, \forall B > 0.
\]

Let \( F \) be the production function of the consumption good. We assume that :

**Assumption F.1** The production function \( F : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is a \( C^2 \) function strictly increasing in each argument and quasi-concave, and such that :
\[
\lim_{x \downarrow 0} F(x, y) = 0, \forall y > 0 \quad \text{and} \quad \lim_{y \downarrow 0} F(x, y) = 0, \forall x > 0.
\]

An additional assumption is :

**Assumption F.2** The production function satisfies F.1 and is homogeneous.

For the sake of simplicity, we assume here that only \( B \) may be increased. Let \( n \) be the employment in the R\&D sector aiming at improving the resource efficiency. The instantaneous rate at which \( B \) can be increased is positively related to the research effort and the previously accumulated stock of knowledge \( B \). Furthermore there is neither exogeneous technical progress nor learning effect generated by the mere use of the resource. Let \( b \) be the knowledge accumulation function.

**Assumption B.1** The knowledge accumulation function \( b : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is a \( C^2 \) function strictly increasing within \( \mathbb{R}^2_+ \) such that :
\[
\lim_{n \downarrow 0} b(n, B) = 0, \forall B > 0 \quad \text{and} \quad \lim_{B \downarrow 0} b(n, B) = 0, \forall n > 0.
\]

Labour is homogenous and can be instantaneously and freely transferred from any sector, consumption good production or technological research, to the other one. Hence the employment constraints are :
\[
1 - l - n \geq 0, \quad l \geq 0 \quad \text{and} \quad n \geq 0
\]

Resource extraction is assumed to be costless. Alternatively \( F \) may be understood as describing an integrated production process, the primary inputs of which are labor, resource and technological knowledge. The dynamics of the resource stock \( S \) is given by :
\[
\dot{S} = -s
\]

The instantaneous utility is a strictly concave function of the instantaneous consumption level \( c \) and the social welfare is the sum of the instantaneous utilities discounted at some constant rate \( \rho \).
Assumption U.1 The utility function $u : R_+ \rightarrow R$ is a $C^2$ function strictly increasing and strictly concave, satisfying the Inada condition: $\lim_{c \downarrow 0} u'(c) = +\infty$.

The benevolent social planner maximizes the welfare function subject to the above constraints. Before examining the optimum problem, let us examine first the efficiency problem since any optimal plan must be efficient.

3 EFFICIENCY

Efficiency may be given different equivalent definitions. For example Dasgupta and Heal (1974) define an efficient path of the economy as a path along which it is not possible to increase the consumption over any time interval, however short or long, given an initial stock of capital and an initial amount of resource. In the present context where there is no physical capital but a knowledge capital $B$, it would translate into an impossibility to increase the consumption level given an initial stock of knowledge and an initial stock of resource. An equivalent definition is the following. For any time interval $[t_1, t_2]$ let $\{c^*, t \in [t_1, t_2]\}$ be a consumption path having to be achieved over the interval and let $\{(l^*_t, n^*_t, s^*_t), t \in [t_1, t_2]\}$ be a feasible policy sustaining the consumption path starting from initial values $B_{t_1}$ and $S_{t_1}$ of $B$ and $S$ respectively, given that $B$ must be at least equal to $B^* (\geq B_{t_1})$ at the end of the interval. Then the policy is efficient if the cumulated resource extraction over the interval is minimized, that is $\{(l^*_t, n^*_t, s^*_t), t \in [t_1, t_2]\}$ is a solution of the following problem $(E)$:

\[
(E) \quad \max_{(l,n,s)} - \int_{t_1}^{t_2} s \, dt \\
\text{s.t.} \quad F(x^f(A, l), y^f(B, s)) - c^* \geq 0 \\
\dot{B} = b(n, B), \quad B_{t_1} > 0 \text{ given} \\
1 - l - n \geq 0; \\
l \geq 0, n \geq 0 \text{ and } s \geq 0 \\
B_{t_2} - B^* \geq 0, \quad B^* \geq B_{t_1} \text{ given}
\]

For the sake of simplicity let us assume that $c^*_t > 0, t \in [t_1, t_2]$. Then, under E.1 and F.1, we must have both $l > 0$ and $s > 0$ and we may delete the corresponding non negativity constraints. Thus we write the Langrangian $L^E$ as follows:
\[
\mathcal{L}^E = -s + \pi^E [F(x^f(A, l), y^f(B, s)) - c^*] + \nu^E b(n, B) + \omega^E [1 - l - n] + \gamma^E n
\]

First order conditions:

\[\frac{\partial \mathcal{L}^E}{\partial l} = 0 \iff \pi^E F_1 = \omega^E\] (3.1)
\[\frac{\partial \mathcal{L}^E}{\partial n} = 0 \iff \nu^E b_n = \omega^E - \gamma^E\] (3.2)
\[\frac{\partial \mathcal{L}^E}{\partial s} = 0 \iff \pi^E F_s = 1\] (3.3)

Complementary slackness conditions:

\[\pi^E \geq 0 \quad \text{and} \quad \pi^E [F(x^f(A, l), y^f(B, s)) - c^*] = 0\] (3.4)
\[\omega^E \geq 0 \quad \text{and} \quad \omega^E [1 - l - n] = 0\] (3.5)
\[\gamma^E \geq 0 \quad \text{and} \quad \gamma^E n = 0\] (3.6)

Dynamics of the costate variable:

\[\dot{\nu}^E = -\frac{\partial \mathcal{L}^E}{\partial B} \iff \dot{\nu}^E = -\pi^E F_B - \nu^E b_B\] (3.7)

Transversality condition:

\[\nu^E_{t^2} \geq 0 \quad \text{and} \quad \nu^E_{t^2} [B_{t^2} - B^*] = 0\] (3.8)

Note that would \(t^2\) be equal to \(\infty\) the terminal constraint \(B_{t^2} - B^* \geq 0\) would have to be deleted, that is \(B_{\infty}\) would have to be free, and the transversality condition at infinity would be:

\[\lim_{t \to \infty} \nu^E B = 0.\] (3.9)

This alternative characterization for the case \(t^2 = \infty\) will be useful later (see the proof of the Proposition 3, section 5).

Let us consider a time sub-interval within which \(n > 0\) so that \(\gamma^E = 0\). Then by (3.1) \(\cup\) (3.2), noting that \(\pi^E = F_{s}^{-1}\) by (3.3), we get:

\[n > 0 \Rightarrow F_1 F_s^{-1} = \nu^E b_n.\] (3.10)

Time differentiating this equation while making use of (3.1) \(\cup\) (3.2) and (3.7), we obtain:

\[\frac{\dot{F}_1}{F_1} - \frac{\dot{F}_s}{F_s} = \frac{\dot{\nu}^E}{\nu^E} + \frac{\dot{b}_n}{b_n} = -b_n \frac{F_B}{F_t} - b_B + \frac{\dot{b}_n}{b_n}\] (3.11)
Proposition 1: Under E.1 and F.1, over any time interval within which $c > 0$: 

$$n > 0 \Rightarrow \frac{\dot{F}_I}{F_I} - \frac{\dot{F}_S}{F_S} = -b_n \frac{F_B}{F_I} - b_B + \frac{\dot{b}_n}{b_n}.$$  

(3.12)

An economic interpretation of the above condition (3.12) is the following. Consider an efficient trajectory, $(c^*, l^*, n^*, B^*, S^*)$ and a time interval $\Theta = [t, t+h+dt]$, $h > dt > 0$ during which $n^* > 0$ and the following perturbation of the research policy $n^*$ over the three subintervals $\Theta_1 = [t, t+dt]$, $\Theta_2 = [t + dt, t + h)$ and $\Theta_3 = [t + h, t + h + dt]$, sustaining the same consumption path $c^*$.

Along the first subinterval $\Theta_1$, the society increases its research effort by $dn > 0$ at each point of time and decreases by the same amount the employment in the consumption good production sector while keeping the consumption at its reference level $c^*$ by increasing the use of the resource input. At $t + dt$, the end of the subinterval, the resource productivity has been increased say by $d_1B > 0$ and the resource stock is lower than along the reference path, say by $d_1S < 0$.

During the second subinterval $\Theta_2$ the difference $B_\tau - B^*_\tau$ is kept constant and equal to $d_1B$. This allows for a decrease of the resource effort by an amount $dn_\tau < 0, \tau \in \Theta_2$. The released labor is allocated to the consumption good production sector allowing for a decrease of the resource extraction rate with respect to $s^*$ while keeping the consumption level equal to $c^*$. Let $d_3S > 0$ be the amount of resource saved during the second subinterval.

During the third subinterval $\Theta_3$ the society drives back $B_\tau$ to its reference level, $B_{t+h+dt} = B^*_{t+h+dt}$, by cutting the research effort from its reference level, $dn_\tau < 0, \tau \in \Theta_3$. The saved labor is once again allocated to the production of the consumption good for another period of decrease of the extraction rate with respect to $s^*$, $ds < 0, \tau \in \Theta_3$. Let $d_3S$ be the amount of resource saved over this third subinterval.

Since the perturbation is assumed to be feasible, we must have $dS \equiv d_1S + d_2S + d_3S \leq 0$. But, would the inequality be strict, it would mean that we have built a policy sustaining $c^*$ and using less resource than $\int_0^\infty s^* dt$, a contradiction since the initial policy is an efficient policy. Thus we must have $dS = 0$. We show in Appendix A.2 that (3.12) is nothing but this condition $dS = 0$.

4 OPTIMALITY

4.1 The social planner problem

Let $(SP)$ be the problem of the social planner, that is:
(SP) $\max_{c,l,n,s} \int_0^\infty u(c)e^{-\rho t} \, dt$

s.t $F(x^f(A,l),y^f(B,s)) - c \geq 0$

$\dot{S} = -s$, $S_0 > 0$ given

$1 - l - n \geq 0$

$c \geq 0, l \geq 0, n \geq 0$ and $s \geq 0$

$S \geq 0$

Since under U.1, clearly $c$ must be positive at each point of time, then under E.1 and F.1, $l$ and $s$ have to be positive too, hence $S > 0$. Thus we may delete the corresponding non negativity constraints when writing the Lagrangian $L$ of the problem (SP):

$$L = u(c)e^{-\rho t} + \pi [F(x^f(A,l),y^f(B,s)) - c] + \nu b(n,B) - \lambda s$$

$$+ \omega [1 - l - n] + \gamma n$$

First order conditions:

$$\frac{\partial L}{\partial c} = 0 \quad \iff \quad u' e^{-\rho t} = \pi \quad (4.1)$$

$$\frac{\partial L}{\partial l} = 0 \quad \iff \quad \pi F_{x} x^f = \pi F_{l} = \omega \quad (4.2)$$

$$\frac{\partial L}{\partial s} = 0 \quad \iff \quad \pi F_{s} = \pi F_{y} y^f = \pi F_{s} = \lambda \quad (4.3)$$

$$\frac{\partial L}{\partial n} = 0 \quad \iff \quad \nu b_n = \omega - \gamma \quad (4.4)$$

Complementary slackness conditions:

$$\pi \geq 0 \quad \text{and} \quad \pi [F(x^f(A,l),y^f(B,s)) - c] = 0 \quad (4.5)$$

$$\omega \geq 0 \quad \text{and} \quad \omega [1 - l - n] = 0 \quad (4.6)$$

$$\gamma \geq 0 \quad \text{and} \quad \gamma n = 0 \quad (4.7)$$

Dynamics of the costate variables:

$$\dot{\nu} = -\frac{\partial L}{\partial B} \quad \iff \quad \dot{\nu} = -\pi F_{B} - \nu b_{B} \quad (4.8)$$

$$\dot{\lambda} = -\frac{\partial L}{\partial S} \quad \iff \quad \dot{\lambda} = 0 \iff \lambda \text{ constant} \quad (4.9)$$

Transversality conditions:

$$\lambda \lim_{t \to \infty} S = 0 \quad (4.10)$$

$$\lim_{t \to \infty} \nu B = 0 \quad (4.11)$$
The implications of the transversality conditions are different for $S$ and for $B$, due to the different natures of the two stocks. Since clearly $\lambda > 0$, then we must have $\lim_{t \to \infty} S = 0$. $S$ is decreasing and must be exhausted. The stock of knowledge $B$ is positive and is non decreasing implying that $\lim_{t \to \infty} \nu = 0$. $B$ is increasing and its imputed shadow price must decrease at a sufficiently high rate in the long run, so that the imputed value of the dedicated knowledge capital must go down to zero.

4.2 The Hotelling Rule

From (4.1) and (4.3) we get:

$$u' e^{-\rho t} F_s = \lambda, \quad t \geq 0$$  \hspace{1cm} (4.12)

Along an optimal path the marginal social benefit, in terms of discounted utility, from extracting one more unit of resource must be the same at each point of time. This is the standard arbitrage condition similar to the Hotelling rule in a partial equilibrium analysis.

Time differentiating this arbitrage condition, we obtain:

$$\rho - \frac{u''}{u'} \dot{c} = \frac{\dot{F}_s}{F_s} \hspace{1cm} (4.13)$$

4.3 Accounting identities

Let $\zeta$ be the marginal effect, in terms of discounted utility, of an increase of $B$, that is:

$$\zeta \equiv \pi F_y y_B = \pi F_B$$

Substituting into (4.8) and integrating over $[t, \infty]$, we get:

$$\lim_{\theta \to \infty} \nu_0 - \nu_t = - \int_t^\infty \zeta d\tau - \int_t^\infty \nu_B d\tau$$
and since \( \nu \to 0 \), we obtain:

\[
\nu_t = \int_t^\infty \zeta d\tau + \int_t^\infty \nu_B d\tau
\]  

(4.14)

The marginal value of a small increase of \( B_t \) is the sum of two terms. The first term, a direct effect, is the discounted value of the additional consumption stream, from \( t \) onwards up to infinity, generated by the small increase of the knowledge at time \( t \). The second term, an indirect effect, is the discounted value of a marginally more efficient knowledge accumulation process at each point of time from \( t \) onwards.

>From (4.2) and (4.3), we conclude that:

\[
\frac{\zeta}{y_B} = \frac{\lambda}{y_s}.
\]  

(4.15)

The equation (4.15) is the imputation rule linking the value of the natural resource to the value of the stock of knowledge. If the production function \( F \) is homogeneous, then \( c = F_x x + F_y y \). Multiplying both sides by \( \pi \) and making use of (4.2) and (4.3) , results in the following accounting identity:

\[
\pi c = \omega \frac{x}{x_f} + \lambda \frac{y}{y_s}
\]

(4.16)

We conclude as follows:

**Proposition 2**: Under E.1, F.1, and U.1, optimally requires that at any point of time, the general Hotelling rule holds:

\[
\frac{\dot{F}_s}{F_s} = \rho - \frac{u^c}{u^t} \dot{c}
\]

and

\[
\nu_t = \int_t^\infty \zeta d\tau + \int_t^\infty \nu_B d\tau.
\]

Under the stronger assumption F.2, the following accounting condition is satisfied:

\[
\pi_c = \omega \frac{x}{x_f} + \lambda \frac{y}{y_s}.
\]
5 THE CE ECONOMY

By a CE economy we mean an economy in which all the functions are constant elasticity functions.

**Assumption E.2**: The efficiency functions \( x^f \) and \( y^f \) are the product functions:

\[
x^f(A, l) = Al \quad \text{and} \quad y^f(B, s) = Bs.
\]

**Assumption F.3**: The production function is the CES function referred to as the general case in what follows:

\[
q = \left[ \alpha_1 x^{-\eta} + \alpha_2 y^{-\eta} \right]^{-1/\eta}, \quad 0 < \alpha_1 < 1 \quad \text{and} \quad \alpha_2 = 1 - \alpha_1.
\]

**Assumption F.4**: The production function is the Leontiev function, the limit case of the above CES function for \( \eta \to \infty \):

\[
q = \min\{x, y\}.
\]

**Assumption F.5**: The production function is the Cobb-Douglas function, the limit case of the above CES function for \( \eta \downarrow 0 \):

\[
q = x^{\alpha_1} y^{\alpha_2}, \quad 0 < \alpha_1 < 1 \quad \text{and} \quad \alpha_1 = 1 - \alpha_2.
\]

**Assumption B.2**: The instantaneous knowledge capital accumulation function is proportional to the research effort and to the accumulated knowledge:

\[
\dot{B} = bnB, \quad b > 0.
\]

**Assumption U.2**: The instantaneous utility function is the isoelastic function:

\[
u(c) = (1 - \epsilon)^{-1} c^{1-\epsilon}, \quad \epsilon > 0 \quad \text{and} \quad \epsilon \neq 1.
\]

A first strong implication of both E.2 and B.2 is that, along an efficient path, over any time interval within which \( n > 0 \), the marginal rate of substitution between \( l \) and \( s \) may be defined as a function of \( S \) and \( S \) only. The point to be emphasized is that we don’t need that \( F \) be homogeneous.

\[\text{[5]}\] A general formulation of the CES should include the case \( \eta \in (-1, 0) \), the CES function tending to a linear form as \( \eta \to -1 \). Such formulations may lead to local instability of the differential system determining the variable dynamics. Indeed it can be shown that it is the case in the present model.
Proposition 3: Under E.2, F.1 and B.2, along any efficient path, the marginal rate of substitution between $l$ and $s$ is proportional to the stock of resource over any time interval within which $n > 0$:

$$n > 0 \Rightarrow F_l F_s^{-1} = bS. \quad (5.1)$$

Consider a path solution of the efficiency problem $(E)$. Under B.2, the condition is now $\dot{\nu}^E = -\pi^E F_B - \nu^E b m$. Multiplying both sides of the equation by $B$ and rearranging the terms, we get:

$$-(\dot{\nu}^E) = \pi^E F_B$$

Under E.2, $F_B = F_s s$ and $F_s = F_y B$. Thus $F_B B = F_y s B = F_s B$. By (3.3) $\pi^E F_s = 1$, hence $\pi^E F_B B = s$, so that:

$$-(\dot{\nu}^E) = s$$

Integrating over $[t, \infty)$ and making use of the transversality condition (3.9) results in:

$$-\int_t^\infty -(\dot{\nu}^E) d\tau = - \lim_{\theta \to \infty} \nu^E F_B B = \int_t^\infty s \ d\tau = S$$

Under B.2, $\nu^E b_n = \nu^E b B$, hence $\nu^E b_n = bS$ and from (3.1) and (3.3) we get $F_l F_s^{-1} = bS$ provided that $n > 0$.

A second strong implication of both E.2 and B.2 is that the stock of resource $S$ and the stock of dedicated knowledge $B$ must have the same imputed values along an optimal path. This must hold under F.1, thus for the general case and the Cobb-Douglas case, and under F.4. There again we don’t need that F be homogeneous.

Proposition 4: Under E.2, F.1, B.2 and U.1, and also for the case of a Leontiev technology, the shadow value of the stock of resource and the stock of technological knowledge must have the same imputed values along any optimal path, that is:

$$\lambda S = \nu B. \quad (5.2)$$

Consider a path solution of the social planner problem $(SP)$. Then multiply both sides of (4.8) by $B$ and use E.2 and B.2 to get:

$$-\dot{\nu} B = \pi^E F_s B + \nu b m B,$$

\footnote{For any product of time functions $x, y$, we denote by $(x|y)$ the time derivative of the product: $(x|y) = \dot{x} y + x \dot{y}$}
hence
\[-(\nu B) = \pi F_y s B.\]

>From (4.3) and $F_s = F_y s B$ under E.2, this is equivalent to:
\[-(\nu B) = \lambda s.\]

Integrating over $[t, \infty)$, we get:
\[-\int_t^\infty (\nu B) d\tau = -[\lim_{\theta \uparrow \infty} \nu_\theta B_\theta - \nu t B_t] = \lambda \int_t^\infty s \ d\tau = \lambda S\]

Last by the transversality condition (4.11), $\lim_{\theta \uparrow \infty} \nu_\theta B_\theta = 0$ and we obtain (5.2). The proof is roughly the same for the Leontiev technology, taking care of the non differentiability problems, and is given in Appendix A.3.
6 OPTIMAL REGULAR PATHS IN THE CE ECONOMY

We define a regular path as a path along which the state variables $B$ and $S$ are growing at a constant rate over time. Hence the employment in the research sector is constant, $g^R = bn$, so that the employment in the consumption good sector is constant too.

6.1 The general case

6.1.1 Characteristics of the optimal path

Let $z$ be the input ratio $xy^{-1}$, the inputs being measured in efficiency units; $R ≡ BS$ be what we call the resource potential, that is the resource stock measured in efficiency units, given the technological level $B$; last $Z(R)$ be the following pure notational device:

$$Z(R) \equiv \left[\frac{\alpha_1 A}{\alpha_2 b}\right]^{1/(1+\eta)} R^{-1/(1+\eta)}, \quad (6.1)$$

so that:

$$dZ(R)/Z = -(1 + \eta)^{-1} dR/R, \lim_{R \downarrow 0} Z(R) = +\infty \quad \text{and} \quad \lim_{R \uparrow \infty} Z(R) = 0. \quad (6.2)$$

The following Proposition 5 summarizes the main characteristics of the optimal path when both sectors, consumption good production and research, are active, whereas Proposition 6 summarizes the same characteristics for any time period during which the sole active sector is the consumption good production sector.

Proposition 5 Under F.3, B.2 and U.2, along an optimal path, if $n > 0$ over some time interval, then within this interval:

$$z = Z(R) \quad \text{and} \quad g^z = -b(1 + \eta)^{-1}[1 - l(1 + \alpha_1^{-1}\alpha_2 Z(R)^\eta)] \quad (6.3)$$

$$g^R = b[1 - l(1 + \alpha_1^{-1}\alpha_2 Z(R)^\eta)] \quad (6.4)$$

$$g^c = c^{-1}[ b(1 + \alpha_1\alpha_2^{-1} Z(R)^{-\eta})^{-1} - \rho ] \quad (6.5)$$

$$g^l = \frac{b}{1 + \eta} \left[ \frac{1 + \eta - \epsilon}{\epsilon(1 + \alpha_1\alpha_2^{-1} Z(R)^{-\eta})} - \frac{\rho(1 + \eta)}{eb} + \alpha_1^{-1}\alpha_2 Z(R)^\eta \right] \quad (6.6)$$

$$g^s = \frac{\eta b}{1 + \eta} \left[ \frac{1 + \eta - \epsilon}{\epsilon(1 + \alpha_1\alpha_2^{-1} Z(R)^{-\eta})} - \frac{\rho(1 + \eta) + \epsilon \eta \rho}{eb} \right] \quad (6.7)$$

Proof See Appendix A.4.

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9We denote by $g^x$ the instantaneous growth rate of any time function $x : g^x = \dot{x}x^{-1}$.

10Along an optimal path the consumption cannot be equal to zero, hence necessarily $l > 0$. 

14
Proposition 6  Under F.3, B.2 and U.2, along an optimal path, if \( n = 0 \) over some time interval, then within this interval:

i. \( s_t \equiv \tilde{s}(t, \lambda) \) with:

\[
\frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda} = -\frac{\tilde{s}(t, \lambda)}{\lambda} \frac{1 + \alpha_1^{-1} \alpha_2 z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta} < 0 \quad (6.8)
\]

\[
\frac{\partial \tilde{s}(t, \lambda)}{\partial t} = \rho \lambda \frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda} < 0 \quad (6.9)
\]

ii.

\[
g^s = -g^z = -b \frac{1 + \alpha_1^{-1} \alpha_2 z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta} < 0 \quad (6.10)
\]

\[
g^c = -\frac{\alpha_2}{\alpha_1} \frac{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta} < 0 \quad (6.11)
\]

Proof  See Appendix A.5.

Note that if \( n = 0 \), then \( z < Z(R) \) as shown in Appendix A.7.2 (see (A.7.6)).

6.1.2 Efficient regular paths

Let us examine now the regular paths. By definition along such paths \( g^B \) is constant, and \( g^S \) is also constant implying that \( g^s = g^S \). Furthermore, if the path is efficient, then (5.1) : \( F_l F_s^{-1} = bS \), has to be satisfied, and under F.1 this equation may be rewritten as (cf (A.1.2.b) and (A.1.3.b)):

\[
\frac{\alpha_1 A}{\alpha_2 B} (A l)^{(1+\eta)}/(B s)^{(1+\eta)} = bS \implies \frac{s}{S} = \frac{b \cdot \alpha_2 (B s)^{-\eta} / \alpha_1 A^{-(1+\eta)}}
\]

Since \( sS^{-1} \) and \( l \) are constants along a regular path, then \( Bs \) has to be constant too. But \( Al \) is also constant so that \( c \) must also be constant. Last since \( Bs \) is a constant and \( g^s = g^S \), then \( BS \), that is \( R \), is also constant.

Now consider an efficient regular path beginning at time \( \tau \), then:

\[
B = B_\tau e^{b n (t-\tau) \tau} \geq \tau.
\]

and since \( Bs \) is constant, then:

\[
s = s_\tau e^{-b n (t-\tau)} \geq \tau.
\]

Noting that \( l = 1 - g^B b^{-1} \), then, provided that \( b > g^B \), for any given \( B_\tau \) and \( s_\tau \), the following constant consumption level \( c \):

\[
c = [\alpha_1 (A (1 - g^B b^{-1}))^{-\eta} + \alpha_2 (B s_\tau)^{-\eta}]^{-1/\eta}
\]

can be sustained from \( \tau \) onwards provided that:

\[
S_\tau \geq \int_{\tau}^{\infty} s_\tau e^{-g^B (t-\tau)} dt = s_\tau / g^B.
\]
Efficiency requires that this weak inequality be satisfied as an equality: $s_\tau = g^B S_\tau = bn S_\tau$. Thus, measured in efficiency units, the constant resource input in the consumption good sector must be equal to $bn S_\tau B_\tau = bn R_\tau$.

Summing up we conclude as follows:

**Proposition 7** Under F.3 and B.2, at any point of time $\tau$, the economy can switch to any efficient regular path along which the consumption level is constant and equal to:

$$c = \alpha_1 (A(1-n))^{-\eta} + \alpha_2 (bn R_\tau)^{-\eta} \quad n \in (0, 1) \quad (6.12)$$

Along such a regular path the resource potential and the input ratio (measured in efficiency units) are both constant:

$$R_t = R_\tau \quad \text{and} \quad z = A(1-n)(bn R_\tau)^{-1}, \quad t \geq \tau \quad (6.13)$$

The regular level of consumption is an increasing function of $A$, $b$ and $R_\tau$. Since:

$$dc/dn <, =, > 0 \iff n >, =, < \hat{n}(R_\tau) \quad t \geq \tau$$

where:

$$\hat{n}(R) \equiv (1 + (\alpha_1 \alpha_2^{-1})^{1/(1+\eta)}(A^{-1} b R)^{\eta/(1+\eta)})^{-1},$$

then as a function of $n$, $c$ is first increasing and next decreasing with a maximum at $\hat{n}(R_\tau)$, maximum we denote by $\hat{c}(R_\tau)$:

$$\hat{c}(R_\tau) \equiv \alpha_2^{-1/\eta} b R_\tau [1 + (\alpha_1 \alpha_2^{-1})^{1/(1+\eta)}(b R_\tau A^{-1})^{\eta/(1+\eta)}]^{-1-\eta/\eta}.$$ 

Note that $\hat{n}(R_\tau)$ is a decreasing function of $R_\tau$, $\alpha_1$ and $b$.

### 6.1.3 Optimal regular paths

Amongst the continuum of regular paths, the optimal ones must satisfy the condition (A.4.4) and $g^c = -s^\eta = -bn$, hence:

$$bl z^\eta = \alpha_1 \alpha_2^{-1} b[1 - l]$$

which gives:

$$z^\eta = \alpha_1 \alpha_2^{-1}[l^{-1} - 1] \quad (6.14)$$

that is:

$$l = [1 + \alpha_1^{-1} \alpha_2 z^\eta]^{-1}. \quad (6.15)$$

Furthermore the optimality condition (6.5) for $g^c = 0$ implies that:

$$\alpha_1 \alpha_2^{-1} z^{-\eta} = b \rho^{-1} - 1,$$

so that $l = 1 - \rho b^{-1}$. From (6.5) for $g^c = 0$, again, and $c = \alpha_1^{-1/\eta} A[l + \alpha_1^{-1} \alpha_2 z^\eta]^{-1/\eta}$ by (A.1.7.a), and from $1 + \alpha_1^{-1} \alpha_2 z^\eta = l^{-1}$ as just shown in (6.15),
we get \( c = \alpha_1^{-1/\eta} A^{(1+\eta)/\eta} = \alpha_1^{-1/\eta} A [(b - \rho)b^{-1}]^{(1+\eta)/\eta} \). Next from the expression of \( z^\eta \) given by (6.14) we get:

\[
z = (\alpha_1 \alpha_2^{-1})^{1/\eta} [(b - \rho) \rho^{-1}]^{1/\eta},
\]

which leads to:

\[
\bar{R} = \alpha_1 (\alpha_2 b)^{-1} A (\alpha_1 \alpha_2^{-1})^{-(1+\eta)/\eta} [(b - \rho) \rho^{-1}]^{(1+\eta)/\eta}
\]

\[
= Ab^{-1}(\alpha_1 \alpha_2^{-1})^{-1/\eta} [(b - \rho) \rho^{-1}]^{(1+\eta)/\eta}.
\]

Since we must have \( l < 1 \), there exists an optimal regular path if \( b > \rho \), in which case it is unique. Thus we may conclude as follows:

**Proposition 8** Under F.3, B.2 and U.1 there exists an optimal regular path if \( b > \rho \), in which case it is unique. Along the optimal regular path the main characteristics of the economy take the following values:

\[
\bar{l} = 1 - \rho b^{-1}, \text{ and } \bar{n} = \rho b^{-1} \quad (6.16)
\]

\[
\bar{c} = \alpha_1^{-1/\eta} A [(b - \rho)b^{-1}]^{(1+\eta)/\eta} \quad (6.17)
\]

\[
\bar{R} = Ab^{-1}(\alpha_1 \alpha_2^{-1})^{-1/\eta} [(b - \rho) \rho^{-1}]^{(1+\eta)/\eta} \quad (6.18)
\]

\[
\bar{z} = (\alpha_1 \alpha_2^{-1})^{1/\eta} [(b - \rho) \rho^{-1}]^{-1/\eta} \quad (6.19)
\]

Let us remark that \( \bar{n} = \hat{n}(\bar{R}) \) and \( \bar{c} = \hat{c}(\bar{R}) \), that is the optimal regular value \( \bar{n} \) of \( n \) maximizes the efficient constant level of \( c \) if \( R = \bar{R} \).

Clearly the optimal trajectory requires that, at the time at which it is taken, the resource potential be precisely equal to \( \bar{R} \). Hence we shall have to examine what is happening in the most probable case in which \( R^0 \neq \bar{R} \). This is examined in Section 7 below.

Note also that, as usual in this type of model, the regular values \( \bar{l}, \bar{c}, \bar{R} \) and \( \bar{z} \) do not depend upon the elasticity \( \epsilon \). But as it will be shown in Section 7, the speed of convergence towards the regular path is commanded by \( \epsilon \).

### 6.2 The Leontiev case

The Leontiev case could be thought of a special case of the CES case, taking the limit of the main characteristics of the optimal regime when \( \eta \uparrow \infty \). But the fact that the derivatives of \( F \) are not always defined implies a more careful treatment.

Since any optimal policy must be efficient, then \( Al = Bs \) and \( l = 1 - n \). Substituting for \( l \) its value as a function of \( n \) and for \( s \) the expression \( AB^{-1}(1-n) \),

---

11 This is an immediate consequence of the stationarity of the regular value of \( c_t \).
the Lagrangian of the problem \((SP)\) may be now written as:

\[
L = (1 - \epsilon)^{-1}[A(1 - n)]^{1-\epsilon}e^{-\rho t} - \lambda AB^{-1}(1 - n) + \nu b n B + \gamma n \tag{6.20}
\]

The optimality condition with respect to \(n\) is given by:

\[
A c^{-\epsilon} e^{-\rho t} = \lambda AB^{-1} + \nu b B,
\]

whence \(n > 0\) along the optimal trajectory. According to Proposition 4, this is equivalent to:

\[
c^{-\epsilon} e^{-\rho t} = \lambda B^{-1}[1 + b RA^{-1}] \tag{6.21}
\]

Differentiating this relation with respect to time, we get:

\[
\epsilon g c + \rho = b n - b A^{-1} \dot{R}[1 + b RA^{-1}]^{-1} \tag{6.22}
\]

Since \(R = BS\) and \(s = A[1 - n]B^{-1}\), the growth rate of \(R\) is given by:

\[
g R = g R + g S = b n - s S^{-1} = b n - A[1 - n]R^{-1} \tag{6.23}
\]

In (6.22), substituting for \(\dot{R}\) its expression \(g R\) where \(g R\) is given by (6.23), and rearranging the terms, we obtain:

\[
g c = \epsilon^{-1}[b(1 + b R A^{-1})^{-1} - \rho]. \tag{6.24}
\]

Since \(c = A l\), we get also \(g l = g c\) and since \(c = B s\), then:

\[
g s = g c - g B = \epsilon^{-1}[b(1 + b R A^{-1})^{-1} - \rho] - b n. \tag{6.25}
\]

Furthermore, since \(g s = -s S^{-1}\) and using the above expression of \(s\), we get:

\[
g S = -A[1 - n]R^{-1}. \tag{6.26}
\]

Remember that efficiency requires that the input ratio \(z\) be equal to one and hence be independent of \(R\), unlike the general case. Similarly, the comparison between the expressions of \(g R\), \(g c\), \(g l\) and \(g S\) in the general case and in the Leontiev case shows significant differences. These expressions cannot be obtained for the Leontiev case merely by making \(\eta \uparrow \infty\) in the corresponding expressions for the general case.

Let us turn to the issue of regular paths. Since \(l\) must be a constant, then \(c\) must also be a constant along a regular path. This implies, from (6.24), that \(R\) must also be a constant \(\bar{R}\):

\[
\bar{R} = A[b - \rho](b \rho)^{-1} \tag{6.27}
\]

Since \(R_i\) is constant, then \(g R = -g S\), that is, using (6.26) : \(bn = A[1 - n]\bar{R}^{-1}\). Substituting for \(\dot{R}\) its expression (6.27) we obtain:

\[
\bar{l} = [b - \rho]b^{-1}, \quad \bar{n} = \rho b^{-1} \quad \text{and} \quad \bar{c} = A[b - \rho]b^{-1}. \tag{6.28}
\]

Hence the sectoral employment is the same in the Leontiev case and in the general case. Also \(\bar{c}, \bar{R}\) and \(\bar{z}\) may be obtained from the general case by making \(\eta \uparrow \infty\).
Note that a lower level of substitutability between the inputs does not imply a lower level of sustainable consumption level. Comparing (6.28) and (6.17), it is easily verified that the long term consumption level in the CES case is higher than the Leontiev case consumption level provided that \( \alpha_2 b > \rho \). Since \( \alpha_2 < 1 \), this implies that the R&D productivity must be sufficiently high to get a better consumption level in the general CES case compared to the Leontiev case.

### 6.3 The Cobb-Douglas case

Last let us consider the case of a Cobb-Douglas technology. In this case the technical progress is no more dedicated. The natural resource is still necessary, that is F.1 holds. But, at least in some models like those developed by Dasgupta and Heal (1979) in which there is no technical progress, the resource may be, or not, essential according to the value of \( \alpha_2 \) the output elasticity with respect to the resource factor. Note that, as pointed out by Dasgupta and Heal (1974), the average product of the resource is unbounded in the Cobb-Douglas case while it is bounded in the case of a CES technology with \( \eta \in (0, \infty) \). As we shall see the main consequence of the unboundedness of the average product of the resource factor is to allow for a permanent positive growth rate of the consumption over time, at least if the productivity in the research sector is sufficiently high, which is never the case if \( \eta < \infty \). Furthermore we also show that any optimal path is an optimal regular path. Thus irrespective of the initial conditions about the stock of knowledge and the stock of resource, any optimal path is a balanced path. This is a sharp difference with the analysis of Garg and Sweeney (1978) which is also based upon a Cobb-Douglas technology but with physical capital accumulation and exogenous technical progress. In their model, balanced growth requires specific values of the initial state variables of the economy.

The main characteristics of the Cobb-Douglas economy are summarized in the following Proposition 9.

**Proposition 9** Under F.5, B.2 and U.2, there exists a unique optimal path iff \((1 - \epsilon)\alpha_2 b < \rho < \alpha_1^{-1} \alpha_2 b\). Along the optimal path:

\[
\begin{align*}
z &= (\alpha_1 A) (\alpha_2 b R)^{-1} \equiv Z(R) \quad (6.29) \\
g^z &= b (\alpha_1^{-1} l - 1) \quad (6.30) \\
g^c &= \epsilon^{-1} (\alpha_2 b - \rho) \quad (6.31) \\
l &= \bar{l} - \alpha_1 (\epsilon \alpha_2 b)^{-1} [\rho - (1 - \epsilon) \alpha_2 b] \quad (6.32) \\
n &= \bar{n} = (\epsilon \alpha_2 b)^{-1} [\alpha_2 b (\alpha_1 + \epsilon \alpha_2) - \alpha_1 \rho] \quad (6.33) \\
g^R &= (\epsilon \alpha_2)^{-1} [\alpha_2 b - \rho] = \alpha_2^{-1} g^c = -g^z \quad (6.34) \\
g^B &= (\epsilon \alpha_2)^{-1} [\alpha_2 b (\alpha_1 + \epsilon \alpha_2) - \alpha_1 \rho] \quad (6.35) \\
g^S &= g^* = -\epsilon^{-1} [\rho - (1 - \epsilon) \alpha_2 b] \quad (6.36)
\end{align*}
\]

**Proof** See Appendix A.6.
Proposition 9 shows that an optimal path is necessarily a regular path. Thus irrespective of the initial conditions $B^0$ and $S^0$, the optimal policy is to set $l$ and $n$ to their optimal values $\bar{l}$ and $\bar{n}$ from $t = 0$ onwards. This implies to set the initial consumption level $c_0$ and the initial extraction rate $s_0$ to the following values (see (A.6.18) and (A.6.3), Appendix A.6)) :

$$c_0 = \left[\epsilon^{-1}(\rho - (1 - \epsilon)\alpha_2 b)\right]^{\alpha_2} (A\bar{l})^{\alpha_1} (B^0 S^0)^{\alpha_2} \quad \text{and} \quad s_0 = \alpha_1^{-1} \alpha_2 b\bar{l} S^0 \quad (6.37)$$

In view of (6.31) we can distinguish three types of optimal paths according the consumption is increasing, constant or decreasing. The following proposition summarizes the main features of each one of these types of optimal paths.

**Proposition 10** Under F.5, B.2 and U.2, provided that $(1 - \epsilon)\alpha_2 b < \rho < \alpha_1^{-1} \alpha_2 b$ then :

i. If $\rho = \alpha_2 b$, then $c = \bar{c}$, $t \geq 0$, a constant given by :

$$\bar{c} = A^{((b - \rho)/b) R_0^b} (b - \rho) b^{-1} [b \rho (b - \rho)^{-1}]^{1/b}, \quad (6.38)$$

an increasing and concave function of $R^t = B^0 S^0$ since $\alpha_2 < 1$ implies that $\rho < b$. Furthermore the resource potential is constant, $R = R_0$, $t \geq 0$, hence $g^B = -g^S = \rho$, and $\bar{l} = (b - \rho)b^{-1}$ and $\bar{n} = \rho b^{-1}$ in this case.

ii. If $\rho < \alpha_2 b$, then $g^c > 0$ and $g^R > 0$. The consumption increases along the optimal path because the resource potential increases as a result of the relatively high productivity of the research effort. Moreover $\bar{l} < \alpha_1 < (b - \rho)b^{-1}$ and $\bar{n} > \alpha_2 > \rho b^{-1}$. Note also that $g^B > \rho$ and $g^S = g^c < -\rho$ in this case.

iii. If $\rho > \alpha_2 b$ then $g^c < 0$ and $g^R < 0$. The consumption level decreases asymptotically to zero together with the resource potential. We have $\bar{l} > \alpha_1 > 1 - \rho b^{-1}$ and $\bar{n} < \alpha_2 < \rho b^{-1}$. Moreover $g^B < \rho$ and $g^S = g^c > -\rho$.

**Proof.** This is an immediate implication of Proposition 9 and the above comments.

For the pivotal values of $\rho$, $\alpha_2$ and $b : \rho = \alpha_2 b$, the allocation of labor between the consumption good production sector and the research sector, is the same than in the general case $\eta > 0$. A direct implication is that $B$ is growing at the same rate in the both cases. But $S$ is decreasing at the same rate too. Thus for these critical values the essential difference between the general economy and the Cobb-Douglas economy is that, in the first one, the steady state requires a specified value of the resource potential $R$, while in the later one the steady state trajectory may be taken from any value of the resource potential. Then given that $\bar{l}$ and $\bar{n}$ have determined values which do not depend upon $R_0$, the adjustment is born by $\bar{c}$ through the choice of the extraction rate of the resource.

The last point worth to be noticed is that, as expected, for $\rho \neq \alpha_2 b$, along the optimal trajectories, the allocation of labor between the production sector and the research sector, the growth rate of $R$, $B$ and $S$ and the input ratio $z$
are all depending upon the elasticity of the marginal utility $\epsilon$. This is a direct implication of the mere fact that, if $\rho \neq \alpha_2 b$, then the steady state is not a stationary state.

7 TRANSITION PATHS

Let us assume that $\rho < b$ so that the optimal regular path exists and let us consider first the transitional dynamics in the general case. As it will appear in the sequel of this section, the qualitative behavior of the variables remains roughly the same in the limit case of a Leontiev technology. We split the study of the transitional dynamics into two parts. First we use the phase diagram technique to describe the dynamics of $c$ and $R$ in the $(R, c)$ plane. We prove that the regular path $(\bar{R}, \bar{c})$ is a saddle-point provided that $\rho < b$ and $\eta > 0$. Hence if an optimal path exists it is necessarily unique. Then we use again the phase diagram technique in the $(R, l)$ plane to study the dynamics of $l$, $n$ and $s$. We end this section by a proof of the existence of an optimal path.

7.1 The dynamics of $c$ and $R$

The dynamics of the consumption path and the resource potential path are solutions of the differential system (6.4) and (6.5) as far as $n > 0$, that is:

$$\dot{R} = bR[1 - cA^{-1/\alpha_1}(1 + \alpha_1^{-1} \alpha_2 Z(R)^\eta)(^{(1+\eta)/\eta}]$$

(7.1)

$$\dot{c} = \epsilon c \big[ b(1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})^{-1} - \rho \big]$$

(7.2)

Let $c^R(R)$ be the curve $\dot{R} = 0$, that is, given (7.1):

$$c^R(R) = A\alpha_1^{-1/\eta}(1 + \alpha_1^{-1} \alpha_2 Z(R)^\eta)^{-(1+\eta)/\eta}$$

(7.3)

Log-differentiating (7.3) results in:

$$\frac{dc^R(R)}{c^R(R)} = -(1 + \eta) \frac{dZ(R)}{Z(R)} \frac{\alpha_1^{-1} \alpha_2 Z(R)^\eta}{1 + \alpha_1^{-1} \alpha_2 Z(R)^\eta}$$

Substituting for $dZ(R)Z(R)^{-1}$ its value given by (6.2), we get:

$$\frac{dc^R(R)}{dR} = c^R(R)R^{-1}(1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})^{-1} > 0,$$

with $\lim_{R \to 0} c^R(R) = 0$ and $\lim_{R \to \infty} c^R(R) = \alpha_1^{-1/\eta} A$. The curve $\dot{c} = 0$ is the vertical through $\bar{R}$. We conclude that provided that $\ell < 1$ or $n > 0$:

$$\dot{R} \begin{cases} > 0 & \text{if } c < c^R(R) \\ < 0 & \text{if } c > c^R(R) \end{cases} \quad \text{and} \quad \dot{c} \begin{cases} > 0 & \text{if } R < \bar{R} \\ < 0 & \text{if } R > \bar{R} \end{cases}$$

(7.4)
It can be checked (see Appendix A.7 for a formal proof) that the differential system (7.1)-(7.2) exhibits the local saddle point property provided that $\rho < b$, the point $(\bar{R}, \bar{c})$ of the $(R, c)$ plane being the unique steady-state of the system.

We denote by $\hat{c}^1(R)$ the function implicitly defined by the stable arms converging to the steady state, that is the optimal trajectory in the $(R, c)$ plane for $n > 0$.

Let $\hat{c}(R)$ be the function determining the upper bound, in the $(R, c)$ plane, of validity of the system (7.1)-(7.2). From (A.1.7.a) (Appendix A.1), we get for $\ell = 1$:

$$\hat{c}(R) = \alpha_1^{-1/\eta}A[1 + \alpha_1^{-1}\alpha_2Z(R)^\eta]^{-1/\eta}.$$  (7.5)

Log-differentiating (7.5) and taking (6.2) into account results in:

$$\frac{d\hat{c}(R)}{dR} = (1 + \eta)^{-1}\hat{c}(R)R^{-1}(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1} > 0,$$

with $\lim_{R \downarrow 0} \hat{c}(R) = 0$ and $\lim_{R \uparrow \infty} \hat{c}(R) = \alpha_1^{-1/\eta}A$, and it is easily checked that $\hat{c}^0(R) < \hat{c}(R)$.

For $(R, c)$ above the $\hat{c}(R)$ frontier, that is for $n = 0$, the dynamics of the system is driven by (6.10)-(6.11) instead of (6.4)-(6.5). We show in Appendix A.7 that the curve $\hat{c}^1(R)$ crosses the curve $\hat{c}(R)$ from below at $R = \tilde{R} > \bar{R}$. Thus for $R_0 > \tilde{R}$ there exists some initial time period $[0, T)$ during which $n = 0$ and we denote by $\hat{c}^2(R)$ this part of the optimal trajectory located above the curve $\hat{c}(R)$.

The phase diagram is pictured in Figure 1.

The following Proposition 11 summarizes the main properties of optimal transition paths in the $(R, c)$ plane.

**Proposition 11** Under F.3, B.2 and U.2 and assuming that $\rho < b$, along the optimal transition path:

1. If $R_0 < \tilde{R}$:
   - $R$ increases over time, and converges asymptotically to its steady state level $\tilde{R}$;
   - $c$ increases over time, and converges asymptotically to the long run consumption level $\tilde{c}$;
   - $z = Z(R)$ decreases over time and converges asymptotically to $\tilde{z}$.

2. If $\tilde{R} < R_0 \leq \tilde{R}$:
   - $R$ decreases over time and converges towards $\tilde{R}$;
- $c$ decreases over time and converges towards $\bar{c}$;
- $z$ increases over time and converges towards $\bar{z}$.

iii. If $\bar{R} < R_0$:
- $R$ decreases over time and converges towards $\bar{R}$;
- $c$ decreases over time and converges towards $\bar{c}$;
- There exists an unique date $T$ defined by $R_T = \bar{R}$ and such that $c = \bar{c}^2(R) > \bar{c}(R)$ for $t < T$ and $c = \bar{c}^1(R) < \bar{c}(R)$ for $t > T$. Furthermore $n = 0$ if $t \in [0, T)$ and $n > 0$ for $t > T$.
- $z$ increases over time and converges towards $\bar{z}$.

In the limit case of a Leontiev technology, the dynamics of $c$ and $R$ along an optimal path is the solution of the differential system (6.23)-(6.24). The locus $\dot{R} = 0$, that is the $c^R(R)$ curve, is now defined as:

$$c^R(R) = bAR(A + bR)^{-1}$$

an increasing function of $R$ varying between 0 and $A$ for $R$ varying between 0 and infinity. The feasibility condition $l \leq 1$ defines $\bar{c}(R) = A$ as the upper bound of the feasibility domain in the $(R, c)$ plane.

It can be shown that $(\bar{R}, \bar{c})$ defined by (6.27) and (6.28) exhibits the saddle point property (see Appendix A.7) provided that $\rho < b$. There also exists a critical value of $R$, $\tilde{R}$ such that for $R_0 > \tilde{R}$ the optimal policy is to set $n = 0$ until the level $\tilde{R}$ is reached and then to follow the $\bar{c}(R)$ trajectory corresponding to the stable arm pointing to the steady state $(\bar{R}, \bar{c})$. The qualitative properties of the dynamics of $c$ and $R$ are the same as in the general case, that is the results of Proposition 11 remain valid under F.4 with clearly the exception of the dynamics of $z$ which is constant and equal to one in this case.

### 7.2 The dynamics of $l$, $n$ and $s$

The input factors dynamics are easier to describe in the $(R, l)$ plane. From (6.4) and (6.6) the dynamics of $l$ and $R$, for $n > 0$, are the solutions of the following differential system:

$$\begin{align*}
\dot{R} &= Rb[1 - l(1 + \alpha_1^{-1}\alpha_2Z(R)^\eta)] \\
\dot{l} &= l \frac{b}{1 + \eta} \left[ 1 + \eta \left( 1 + \alpha_1^{-1}\alpha_2^{-1}Z(R)^{-\eta} \right) - \frac{\rho(1 + \eta)}{eb} + \alpha_1^{-1}\alpha_2Z(R)^\eta \right] 
\end{align*}$$

(7.6)

(7.7)

Let $l^R(R)$ be the curve $\dot{R} = 0$. We get from (7.6):

$$l^R(R) = (1 + \alpha_1^{-1}\alpha_2Z(R)^\eta)^{-1}$$

(7.8)

Log-differentiating (7.8), we obtain:

$$\frac{dl^R(R)}{dR} = \eta(1 + \eta)^{-1}l^R(R)R^{-1}(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1} > 0$$
with \( \lim_{R \to 0} l^R(R) = 0 \) and \( \lim_{R \to \infty} l^R(R) = 1 \). \( l^R(R) \) is an increasing function of \( R \) with an horizontal asymptote at \( l = 1 \).

Let \( l^i(R) \) be the curve \( l = 0 \). From (7.7):

\[
l^i(R) = \frac{\rho \alpha_1 (1 + \eta)}{eb \alpha_2 Z(R)^\eta} - \frac{1 + \eta - \epsilon}{\epsilon (1 + \alpha_1^{-1} \alpha_2 Z(R)^\eta)}
\]  

(7.9)

Differentiating (7.9) results in:

\[
\frac{dl^i(R)}{dR} = \frac{\rho \alpha_1 \eta}{eb \alpha_2 RZ(R)^\eta} - \frac{\eta (1 + \eta - \epsilon) \alpha_1^{-1} \alpha_2 Z(R)^\eta}{(1 + \eta - \epsilon R (1 + \alpha_1^{-1} \alpha_2 Z(R)^\eta))^2}
\]

From (7.9) its is easily checked that \( l^i(R) \geq 0 \) for any \( R \geq 0 \) iff \( (b - \rho)b^{-1} < \epsilon (1 + \eta)^{-1} \) and that \( dl^i(R)/dR \geq 0 \) in this case. If \( (b - \rho)b^{-1} > \epsilon (1 + \eta)^{-1} \), \( l^R(R) \) is only defined in the positive orthant of the \((R, l)\) space for \( R \geq R_0 = \alpha_1^{-1/\eta} \alpha_2^{-1/\eta} (b - \rho)^{-1} (1 + \eta)^{-1/\eta} (1 + \eta)^{-1/\eta} \). Note that \( dl^i(R)/dR > 0 \) for \( R > R_0 \) and that \( \lim_{R \to \infty} l^i(R) = +\infty \). Clearly:

\[
\begin{align*}
    \dot{R} &> 0 \quad \text{if} \quad l < l^R(R) \quad \text{and} \quad \dot{l} \begin{cases} 
        > 0 & \text{if} \quad l > l^i(R) \\
        < 0 & \text{if} \quad l < l^i(R)
    \end{cases}
\end{align*}
\]

(7.10)

The phase diagram is pictured in Figure 2.

**Figure 2 about here**

We are now in position to describe the dynamics of \( l \) and \( s \). The following Proposition 12 summarizes the results:

**Proposition 12** Under F.3, B.2 and U.2, provided that \( \rho < b \), along the optimal path:

i. \( l \) increases over time for \( R < \hat{R} \) and decreases if \( \hat{R} < R < \hat{R} \). For \( R > \hat{R} \), \( l \) is constant and equal to one.

ii. The R&D effort \( n_t \) decreases over time if \( R < \hat{R} \) and increases over time if \( \hat{R} < R < \hat{R} \). If \( R > \hat{R} \), \( n = 0 \) and \( R \) decreases up to that time at which \( R = \hat{R} \).

iii. The extraction rate \( s \) decreases over time if, first, \( R \geq \hat{R} \), or if \( R \leq \hat{R} \) and \( 1 + \eta < \epsilon \), or last if \( R \leq \hat{R} \), \( \epsilon < 1 + \eta \) and \( b (1 - \epsilon) < \rho \).

iv. If \( \epsilon < 1 + \eta \) and \( \rho < b (1 - \epsilon) \) (note that this implies \( \epsilon < 1 \)), there exists a critical value of \( R_0 \), we denote by \( R^* \), such that \( s \) increases over time while \( R < R^* \) (this implies \( R_0 < R^* \)) and decreases over time if \( R > R^* \).

**Proof** The claims i and ii are immediate consequences of the geometry of the phase diagram in the \((R, l)\) space and the full employment condition fulfilled along the optimal path which entails that \( l = 1 - n \). A formal proof of the claims iii and iv of the proposition is given in Appendix A.8.
Along the optimal transition path, both the research effort path and the employment of labor in the consumption good sector follow monotonic trajectories. In the economically most interesting case, that is when the resource potential is initially low ($R_0 < \bar{R}$), the research effort $n$ decreases along the optimal transition path. This is because when $\rho < b$, it is in the society interest to invest in research right from the start in order to benefit from the increasing returns over the accumulation of knowledge $B$. Note also that the resource extraction pattern is not necessarily monotonic along the optimal transition path. More precisely, if first the society has a low incentive to flatten its consumption profile over time, that is if $\epsilon$ is low, second if the social rate of impatience $\rho$ is also low with respect to $b$, and third if the initial level of the resource potential $R_0$ is low, the optimal path of resource extraction increases over time during a first time period before decreasing asymptotically to zero as the economy reaches the steady state (see André and Smulders (2003) for a similar result).

7.3 Existence of the optimal path

The local stability property, together with the concavity of both the utility function and the production function (under F.3) imply that if an optimal path exists, it is unique. To prove the existence of the optimal path, it remains to find trajectories for the costate variables $\{(\lambda, \nu), t \geq 0\}$ and the adjoint variables $\{((\omega, \gamma), t \geq 0\}$ satisfying the set of necessary conditions.

Let us denote by $\{(c^*, R^*), t \geq 0\}$ the solution of the system (7.1)-(7.2) starting from $R_0 \leq \bar{R}$ and corresponding to the stable arms converging to $(\bar{R}, \bar{c})$ in the $(R, c)$ plane, that is $c^* = \bar{c}(R^*)$. In this case $\gamma^*$, the optimal value of $\gamma$, is equal to 0, $\forall t \geq 0$. Making use of (A.4.8) and integrating over $[0, \infty)$ we get the optimal value of $\lambda$, $\lambda^*$:

$$
\lambda^* = S_0^{-1} \int_0^\infty c^{(1-\epsilon)}(1 + \alpha_1 \alpha_2^{-1} Z(R^*)^{-\eta})^{-1} e^{-\rho t} dt \quad (7.11)
$$

Moreover, making use of (A.4.7) and (7.11) we obtain the optimal value of $\omega$, $\omega^*$:

$$
\omega^* = e^{-\rho t} \int_0^\infty c^{(1-\epsilon)}(1 + \alpha_1 \alpha_2^{-1} Z(R^*)^{-\eta})^{-1} e^{-\rho(\tau-t)} d\tau \quad (7.12)
$$

>From (A.1.7.a), we conclude that the optimal value of $l$, $l^*$, is defined by:

$$
l^* = \alpha_1^{1/\eta} A^{-1}[1 + \alpha_1^{-1} \alpha_2 Z(R^*)^{\eta}]^{1/\eta} c^*
$$

in an unique way and hence that the optimal value of $n$, $n^* = 1 - l^*$, is determined by $\{(c^*, R^*), t \geq 0\}$. Hence the optimal value of $B$, $B^*$ is given by:

$$
B^* = B_0 \exp\left\{\int_0^t bn^* dt\right\},
$$

and the optimal value of $S$, $S^*$ is defined by $S^* = R^* B^*$. From (5.2) we get the optimal value of $\nu$, $\nu^* = \lambda^* S^* B^*$. 

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Let us turn now to the case \( R_0 > \bar{R} \). Here the optimal path is made of two consecutive phases. During the first phase \([0,T)\), the extraction rate is given by \( s = \tilde{s}(t,\lambda) \), a function of time and \( \lambda \) and the consumption level is given by (A.5.2), where \( s = \tilde{s}(t,\lambda) \) and \( z = A(B_0\tilde{s}(t,\lambda))^{-1} \), hence \( c \) is also a function of time and \( \lambda \).

At \( t = T \), \( R_T = \bar{R} \) defines the available resource stock \( \tilde{S} \), which is given by \( \tilde{S} \equiv \bar{R}B_0^{-1} \). Remember that \( \lambda \) is a constant over the optimal path. Hence to \( \tilde{S} \) is associated a unique value of \( \lambda \), we denote by \( \hat{\lambda} \), solution of (7.11) for \( S = \tilde{S} \). This defines the extraction rate over \([0,T)\), \( s = \tilde{s}(t,\hat{\lambda}) \), the consumption good production level \( c \) and the input ratio \( z \) as functions of time and \( \lambda \). Since \( R = B_0S \) in the present case and \( S = S_0 - \int_0^T \tilde{s}(\tau,\hat{\lambda})d\tau \), then \( c \) is also defined as a function of \( R \), that is \( c = \hat{c}^2(R) \).

Let us define \( \hat{V}(T) \) as :

\[
\hat{V}(T) = \int_T^{\infty} e^{t(1-\epsilon)}(1 + \alpha_1\alpha_2^{-1}Z(R^*)^{-\eta})^{-1} e^{-\rho(t-T)} dt
\]

From (7.11) we get \( \hat{\lambda} = e^{-\rho T} \tilde{S}^{-1} \hat{V}(T) \). It remains to compute the optimal length of the first phase \( T^* \). From the stock condition over the entire horizon \([0,\infty)\), we obtain :

\[
S_0 - \tilde{S} = \int_0^T \tilde{s}(t,\hat{\lambda}) dt = \int_0^T \tilde{s}(t, e^{-\rho T} \tilde{S}^{-1} \hat{V}(T)) dt \equiv I(T; \tilde{S}) \quad \text{(7.13)}
\]

Differentiating (7.13) with respect to \( T \) we get :

\[
\frac{dI(T; \tilde{S})}{dT} = \tilde{s}(T, \hat{\lambda}) + e^{-\rho T} \tilde{S}^{-1} \frac{d\hat{V}(T)}{dT} \bigg|_{\hat{\lambda}=\hat{\lambda}} dt.
\]

Differentiating \( \hat{V}(T) \) with respect to \( T \) results in :

\[
\frac{d\hat{V}(T)}{dT} = -\hat{c}^2(\bar{R})^{1-\epsilon}(1 + \alpha_1\alpha_2^{-1}Z(\bar{R})^{-\eta})^{-1} + \rho \hat{V}(T)
\]

Hence the derivative \( dI/dT \) simplifies to :

\[
\frac{dI(T; \tilde{S})}{dT} = \hat{s}(T, \hat{\lambda}) - e^{-\rho T} \tilde{S}^{-1} \hat{c}^2(\bar{R})^{1-\epsilon}(1 + \alpha_1\alpha_2^{-1}Z(\bar{R})^{-\eta})^{-1} \int_0^T \frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda}|_{\lambda=\hat{\lambda}} dt
\]

which is strictly positive since \( \partial \tilde{s}(t, \lambda)/\partial \lambda < 0 \) (see Proposition 6). Furthermore \( \lim_{T \to 0} I(T; \tilde{S}) = 0 \) and \( \lim_{T \to \infty} I(T; \tilde{S}) = +\infty \). Hence we conclude that (7.13) has a unique solution \( T^* \), a function of the initial resource stock \( S_0 \) and the initial knowledge level \( B_0 \) through \( \tilde{S} \).

Since \( d\tilde{S} = -\bar{R}B_0^{-2} dB_0 \), the differentiation of (7.13) with respect to \( S_0 \) and \( B_0 \) leads to :

\[
\frac{dT}{dS_0} = \left[ \frac{dI(T)}{dT} \right]^{-1} > 0 \quad \text{(7.14)}
\]

\[
\frac{dT}{dB_0} = \left[ \frac{dI(T)}{dT} \right]^{-1} \bar{R}B_0^{-2} \left[ 1 - e^{-\rho T} \hat{V}(T) \tilde{S}^{-2} \int_0^T \frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda} dt \right] > 0 \quad \text{(7.15)}
\]
The length of the first phase during which the society is embodied with a sufficiently large initial potential $R_0$ to prefer consuming without doing any R&D effort, increases with $B_0$ and $S_0$, that is with $R_0$.

Last we compute the optimal values $\omega^*$, $\gamma^*$ and $\nu^*$ of $\omega$, $\gamma$ and $\nu$ during the first phase $[0, T)$. Making use of (4.2), (4.3) and (A.4.1) we obtain:

$$(\alpha_1 A)(\alpha_2 B_0)^{-1}z^{-(1+\eta)} = \omega \tilde{\lambda}^{-1}$$

Using the definition of $z = A(B_0 \tilde{s}(t, \tilde{\lambda}))^{-1}$ we derive the following expression for $\omega^*$:

$$\omega^* = (\alpha_1 B_0^\eta)(\alpha_2 A^\eta)^{-1}\tilde{s}(t, \tilde{\lambda})^{1+\eta} \quad t \in [0, T)$$

(7.16)

Furthermore from (4.4) and making use of (5.2) we get: $\gamma = b\lambda S - \omega$, from which, substituting for $\omega$ its expression (7.16), we obtain the optimal value of $\gamma$ over the first phase $[0, T)$$^{12}$:

$$\gamma^* = \tilde{\lambda}[bS - (\alpha_1 B_0^\eta)(\alpha_2 A^\eta)^{-1}\tilde{s}(t, \tilde{\lambda})^{1+\eta}] \quad t \in [0, T)$$

(7.17)

Last, making use of (5.2) again, we get $\nu^* = \tilde{\lambda}SB_0^{-1}$.

8 CONCLUSION

The paper has examined the possibility of optimal sustained consumption levels under an exhaustible resource constraint when a dedicated technical progress may be applied to the improvement of the natural resource efficiency. We first give a general condition for a dedicated research policy to be efficient and next to be optimal.

Next we turn to the study of what we call a constant elasticity economy. We have shown that in this case, the imputed values of the resource stock and the knowledge stock have to be equal at each point of time along an optimal path and that efficiency requires that the marginal rate of substitution between labor and resource in the consumption good production sector has to be proportional to the resource stock level.

We have characterized the set of efficient and optimal balanced growth paths. In the case of a CES or a Leontiev technology, there exists only one optimal regular path provided that the research productivity be higher than the social rate of impatience. Along this regular path, the allocation of labor to the consumption good production sector and the research sector is constant and the consumption level and the resource potential, a measure of the resource stock in efficiency units, are all constant over time. The Cobb-Douglas is a very special case, in which the optimal plan is necessarily a balanced growth path. This is an immediate consequence of the fact that in this case technical the progress necessarily

\footnote{Note that $S$ is perfectly determined over the first phase by the stock condition $S = S_0 - \int_0^T \tilde{s}(\tau, \lambda) d\tau$.}
boosts at the same rate the productivity of all the inputs. Hence provided that the productivity be sufficiently high in the research sector, what is sustainable in the long run is an ever increasing consumption rate which is forbidden in the CES case for $\eta < \infty$. Note that since the output elasticity is strictly lower than one in the Cobb-Douglas case, the necessary condition to get a non declining consumption level for such a technology is not less stringent than in the CES or the Leontiev case. Hence one should not be tempted to think that a sustainable consumption level in the long run would be more easily obtained in an economy where the natural resource is a better substitute for labor.

Last we study the transition to the optimal regular path in the CES and the Leontiev cases. Along the transition path, the consumption level and the resource potential move in the same direction, increasing if the resource potential is initially lower than its optimal long term level and decreasing in the opposite case. We show the existence of a critical level of the resource potential over which no research effort has to be undertaken, and under which the society must invest in research. The research effort must evolve in the opposite direction with respect to the resource potential. That is, an economy initially embodied with a low resource potential should initially make a high research effort, and then decrease monotonically its R&D investment while reaching the steady state. The released labor is transferred to the consumption good production sector, and the productive labor and consumption both increase during the transition. Note however that there does not exist any simple substitution rule between the resource factor and the labor factor. Under some parameters configurations, the extraction rate of the natural resource may increase during an initial period of time alongside with the labor devoted to production, before having to decrease.
APPENDIX

A.1 Properties of the CES production function

The following properties of the CES function are repeatedly used and listed here for the sake of selfcontainedness.

Let us consider the case \( c = F(Al, Bs) = F(x, y) \) where :

\[
F = [\alpha_1(Al)^{-\eta} + \alpha_2(Bs)^{-\eta}]^{-1/\eta} = [\alpha_1 x^{-\eta} + \alpha_2 y^{-\eta}]^{-1/\eta}
\]  
(A.1.1)

with :

\( \eta > 0 \), \( 0 < \alpha_1 < 1 \), \( \alpha_2 = 1 - \alpha_1 \) and \( A > 0 \).

Then :

\[
F_x = \alpha_1 c^{1+\eta}x^{-1+\eta}
\]  
(A.1.2.a)

\[
F_l = AF_x = \alpha_1 Ac^{1+\eta}x^{-1+\eta} = \alpha_1 c^{1+\eta}l^{-1-\eta}x^{-\eta}
\]  
(A.1.2.b)

\[
F_A = lF_x = \alpha_1 c^{1+\eta}l^{-1+\eta} = \alpha_1 c^{1+\eta}A^{-1}x^{-\eta}
\]  
(A.1.2.c)

\[
F_y = \alpha_2 c^{1+\eta}y^{-1+\eta} = (1 - \alpha_1)c^{1+\eta}y^{-1+\eta}
\]  
(A.1.3.a)

\[
F_s = BF_y = \alpha_2 c^{1+\eta}By^{-1+\eta} = \alpha_2 c^{1+\eta}s^{-1}-y^{-\eta}
\]  
(A.1.3.b)

\[
F_B = sF_y = \alpha_2 c^{1+\eta}sy^{-1+\eta} = \alpha_2 c^{1+\eta}B^{-1}y^{-\eta}
\]  
(A.1.3.c)

\[
F_x F_y^{-1} = \alpha_1 \alpha_2 c^{1+\eta}z^{-1+\eta} = \alpha_1 (1 - \alpha_1) c^{1+\eta}z^{-1+\eta},
\]  
(A.1.4)

where \( z = xy^{-1} \).

\[
xF_x(yF_y)^{-1} = lF_l(sF_s)^{-1} = AF_A(BF_B)^{-1} = \alpha_1 \alpha_2 c^{1+\eta}z^{-1+\eta} = \alpha_1 (1 - \alpha_1) c^{1+\eta}z^{-1+\eta}
\]  
(A.1.5)

\[
c = xF_x + yF_y = lF_l + sF_s = AF_A + BF_B
\]  
(A.1.6)

\[
c = \alpha_1^{-1/\eta}x[1 + \alpha_1^{-1} \alpha_2 z^{-\eta}]^{-1/\eta}
\]  
(A.1.7.a)

\[
c = \alpha_2^{-1/\eta}y[1 + \alpha_1 \alpha_2^{-1}z^{-\eta}]^{-1/\eta}
\]  
(A.1.7.b)
A.2 Interpretation of the efficiency condition (3.12)

The increment \(d_1B(>0)\) of the resource productivity factor (w.r.t the reference path) over the first subinterval \(\Theta_1\), that is at \(t+dt\), and the decrement \(d_1S(<0)\) of the stock of resource induced by the perturbation of the policy are given by:

\[
d_1B \simeq b^*_n,t dndt \quad \text{and} \quad d_1S \simeq -(F^r_{l,t}/F^r_{s,t})dndt
\]

where a star means that the function is evaluated along the reference path.

Since over the second sub-interval \(\Theta_2\), the difference \(B_{t} - B^*_t\) is kept constant and equal to \(d_1B < 0, \tau \in \Theta_2\). Assuming that this labor is now allocated to the consumption good production sector for the sake of the efficiency of the perturbation, and taking into account that the efficiency of the resource factor is now higher, the society can save the resource. Let \(d_2S\) be the amount of resource saved over the sub-interval. We get:

\[
d_2S = d_1B \int_{t}^{t+h} \left[ \frac{F^r_{l,t} \ b^*_B}{b^*_n,t} - \frac{F^r_{B,t}}{F^r_{s,t}} \right] d\tau
\]

For \(h\) sufficiently small, hence for \(dt\) small too, we obtain:

\[
d_2S \simeq d_1B \left[ \frac{F^r_{l,t+dt} \ b^*_B \ t+dt}{F^r_{s,t+dt} \ b^*_n,t+dt} - \frac{F^r_{B,t+dt}}{F^r_{s,t+dt}} \right] (h - dt)
\]

Over the third sub-interval \(\Theta_3\), the society reduces the research effort, \(dn_r = -(b^*_n,t/h^*_n,t)\) \(dn_r \in \Theta_3\) for driving back \(B_t\) to its reference level at the end of the sub-interval, \(B_{t+h+dt} = B^*_t\). \(-dn_t\) is allocated to the physical good production sector while keeping its production level to its reference level. Thus the resource saved amounts to:

\[
d_3S \simeq \frac{b^*_n,t}{b^*_n,t+h} \ F^r_{l,t+h} dndt
\]

Let \(dS = d_1S + d_2S + d_3S\), be the amount of resource saved w.r.t. the reference path. Adding and substracting \((F^r_{l,t+h}/F^r_{s,t+h})\) to the expression of \(dS\) results in:
\[dS \approx \left[ \frac{F_{l,t+h}^*}{F_{s,t+h}^*} - \frac{F_{l,t}^*}{F_{s,t}^*} - \frac{b_{n,t+h}^*}{b_{n,t}^*} \frac{F_{l,t+h}^*}{F_{s,t+h}^*} \right] + b_{n,t}^* \left( \frac{F_{l,t+h}^*}{F_{s,t+h}^*} \frac{b_{B,t+dt}^*}{b_{n,t+dt}^*} + \frac{F_{B,t+dt}^*}{F_{s,t+dt}^*} \right) (h - dt) \] 

For \( h \) sufficiently small and \( dt \) infinitely smaller than \( h \) we get the following approximations:

\[ h - dt \approx h \]

\[ \frac{b_{B,t+dt}^*}{b_{n,t+dt}^*} \approx \frac{b_{B,t}^*}{b_{n,t}^*} \text{ and } \frac{F_{B,t+dt}^*}{F_{s,t+dt}^*} \approx \frac{F_{B,t}^*}{F_{s,t}^*} \]

\[ \frac{F_{l,t+h}^*}{F_{s,t+h}^*} \approx \frac{F_{l,t}^*}{F_{s,t}^*} + \left( \frac{F_{l,t}^*}{F_{s,t}^*} \right) h \] \(^{13}\) and \( b_{n,t+h}^* \approx b_{n,t}^* + \frac{\dot{b}_{n,t}^*}{b_{n,t}^*} h \)

\[ \frac{F_{l,t+h}^*}{F_{s,t+h}^*} \approx \frac{F_{l,t}^*}{F_{s,t}^*} \frac{b_{n,t}^*}{b_{n,t+h}^*} \text{ and } \frac{b_{n,t}^*}{b_{n,t+h}^*} \approx 1 - \frac{\dot{b}_{n,t}^*}{b_{n,t}^*} h \]

hence:

\[ dS \approx \frac{F_{l,t}^*}{F_{s,t}^*} \left[ \frac{\dot{F}_{l,t}^*}{F_{s,t}^*} - \frac{\dot{F}_{s,t}^*}{F_{s,t}^*} - \frac{\dot{b}_{n,t}^*}{b_{n,t}^*} \frac{F_{l,t}^*}{F_{s,t}^*} + b_{B,t}^* + b_{n,t}^* \frac{F_{B,t}^*}{F_{s,t}^*} \right] h \ dn \ dt 

- \left\{ \frac{\dot{b}_{n,t}^*}{b_{n,t}^*} \left[ E_1 - E_2 h \right] \right\} h^2 \ dn \ dt \]

where:

\[ E_1 = \left( \frac{\dot{F}_{l,t}^*}{F_{s,t}^*} \right) / \frac{F_{l,t}^*}{F_{s,t}^*} - \frac{\dot{b}_{n,t}^*}{b_{n,t+h}^*} \text{ and } E_2 = \frac{\dot{b}_{n,t}^*}{b_{n,t+h}^*} \left( \frac{\dot{F}_{l,t}^*}{F_{s,t}^*} \right) / \frac{F_{l,t}^*}{F_{s,t}^*} \]

Letting \( h \downarrow 0 \), the second term of the above sum converges to 0 before the first term.

Now note that since the policy is the policy of the reference path over the interval \([0, t) \cup [t + dt + h, \infty)\) then \( \int_0^t s_\tau d\tau + \int_{t+dt+h}^\infty s_\tau d\tau \) is not affected by the policy perturbation. Thus if the perturbation is feasible, we must have:

\[ dS = \int_t^{t+h+dt} s_\tau d\tau - \int_t^{t+dt} s_\tau d\tau \leq 0. \]

\(^{13}\)For any ratio \( \left( \frac{\dot{x}}{x} \right) \), we denote by \( \left( \frac{\dot{x}}{x} \right) \) the time derivative \( \frac{d}{dt} \left( \frac{\dot{x}}{x} \right) \)
But clearly $dS < 0$ would imply that the reference path is not efficient, hence we must have $dS = 0$ that is:

$$\frac{\dot{F}_{l,t}}{F_{l,t}} = \frac{\dot{F}_{s,t}}{F_{s,t}} + b_{n,t} + b_{B,t} = 0,$$

that is (3.12).

A.3 Proof of Proposition 4 for the Leontiev technology

As shown in subsection 6.2, in the Leontiev case the Lagrangian of Problem ($SP$) is given by (6.20).

The dynamics of the costate variable $\nu_t$ is given by:

$$\dot{\nu} = -\nu bn - \lambda AB[1 - n] \quad (A.3.1)$$

while $\lambda_t = \lambda$. The transversality conditions are the conditions (4.10) and (4.11).

Let us define $h$ as $h = \lambda S - \nu B$, so that:

$$\dot{h} = \lambda \dot{S} - \dot{\nu} B - \nu \dot{B}.$$

In this expression let us substitute for $\dot{S}$ its value $-AB^{-1}[1 - n]$, for $\dot{\nu}$ its value given by (A.3.1) and for $\dot{B}$, its value $bn B$. We get:

$$\dot{h} = 0 \implies h = h_0, \ t \geq 0.$$

By the transversality conditions, we must have $\lim_{t \to \infty} h = 0$, hence $h_0 = 0$, so that:

$$\lambda S = \nu B, \ t \geq 0$$

that is (5.2).

A.4 Proof of Proposition 5

A.4.1 Proof of (6.3)

$(A.1.2.b) \cup (A.1.3.b)$ results in:

$$F_l F_s^{-1} = \frac{\alpha_1 t^{-1} x^{-\eta}}{\alpha_2 s^{-1} y^{-\eta}} = \frac{\alpha_1 A(Al)^{-1} x^{-\eta}}{\alpha_2 B(Bs)^{-1} y^{-\eta}} = \alpha_1 A(\alpha_2 B)^{-1} x^{-\eta} (1 + \eta) \quad (A.4.1)$$

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(5.1) \(\cup\) (A.4.1) implies that :
\[
\alpha_1\alpha_2^{-1}Az^{-(1+\eta)} = bBS = bR ,
\]  
(A.4.2)
hence :
\[
z = (\alpha_1\alpha_2^{-1})^{1/(1+\eta)}A^{1/(1+\eta)}b^{-1/(1+\eta)}R^{-1/(1+\eta)} \equiv Z(R) .
\]  
(A.4.3)
Note that \(z^{-1}(1+\eta)AB^{-1} = z^{-\eta}s\)\(l^{-1}\) which, together with (A.4.2), implies that :
\[
z^\eta = \alpha_1\alpha_2^{-1}s(bS)^{-1}.
\]  
(A.4.4)
From (A.4.3), we get :
\[
g^z = \dot{z}z^{-1} = -(1+\eta)^{-1}g^R = -(1+\eta)^{-1}[g^B + g^S]
\]  
and since (under B.2) \(g^B = bn\), and \(g^S = -sS^{-1}\), then the above expression, together with (A.4.4), results in
\[
g^z = -(1+\eta)^{-1}[bn - sS^{-1}] = -(1+\eta)^{-1}[b(1-l) - bl\alpha_1^{-1}\alpha_2z^\eta]
\]  
that is :
\[
g^z = -b(1+\eta)^{-1}[1 - l(1 + \alpha_1^{-1}\alpha_2z^\eta)].
\]  
(A.4.5)

A.4.2 Proof of (6.4)

>From (A.4.3) and (A.4.5) we obtain :
\[
g^z = -(1+\eta)^{-1}g^R = -(1+\eta)^{-1}[1 - l(1 + \alpha_1^{-1}\alpha_2Z(R)^\eta)],
\]
hence the first equation (6.4) :
\[
g^R = b[1 - l(1 + \alpha_1^{-1}\alpha_2Z(R)^\eta)].
\]
>From (A.1.7.a) and \(z = Z(R)\) we get \(l = cA^{-1}\alpha_1^{-1/\eta}[1 + \alpha_1^{-1}\alpha_2z^\eta]^{1/\eta}\). Thus substituting for \(l\) into the above expression of \(g^R\) we get the second inequality (6.4) :
\[
g^R = b[1 - cA^{-1}\alpha_1^{1/\eta}(1 + \alpha_1^{-1}\alpha_2Z(R)^\eta)^{(1+\eta)/\eta}].
\]

A.4.3 Proof of (6.5)

(4.2) \(\cup\) (4.3) \(\Rightarrow\) \(\pi F_{t,t} + \pi sF_s = \omega l + \lambda s\), that is, under F.2, by the Euler theorem :
\[
\pi q = \pi e = \omega l + \lambda s ,
\]
which may be rewritten, using (4.1), as :
\[
cue^{-\rho t} = \lambda s[1 + \omega l(\lambda s)^{-1}] .
\]  
(A.4.6)
By (5.2) and (4.3) with $\gamma = 0$:

$$\omega = b\lambda S.$$  \hspace{1cm} \text{(A.4.7)}

Substituting for $\omega$ in (A.4.6) and using (A.4.4), we get:

$$cu' e^{-\rho t} = \lambda s[1 + bS^{-1}] = \lambda s[1 + \alpha_1\alpha_2^{-1}z^{-\eta}].$$  \hspace{1cm} \text{(A.4.8)}

Now substitute for $c$ in the above expression its value given by (A.1.7.b), remembering that $y = Bs$, then:

$$u' e^{-\rho t} = \alpha_1^{1/\eta}\lambda B^{-1}[1 + \alpha_1\alpha_2^{-1}z^{-\eta}]^{(1+\eta)/\eta},$$  \hspace{1cm} \text{(A.4.9)}

from which we get under U.2:

$$\epsilon g^c + \rho = g^B + (1 + \eta)g^z[1 + \alpha_1^{-1}\alpha_2^z\eta]^{-1}.$$  \hspace{1cm} \text{(A.4.10)}

A.4.4 Proof of (6.6)

>From (A.1.7.a), remembering that $g^x = g'$, we get:

$$g^c = g' - g^z\frac{\alpha_1^{-1}\alpha_2^z\eta}{1 + \alpha_1^{-1}\alpha_2^z\eta},$$

and substituting for $g^z$ its expression (6.3), we obtain:

$$g^c = g' + b(1 + \eta)^{-1}(1 + \alpha_1\alpha_2^{-1}z^{-\eta})^{-1} - b(1 + \eta)^{-1}\alpha_1^{-1}\alpha_2^zl^\eta.$$  \hspace{1cm} \text{(A.4.10)}

In the above equation let us substitute for $b(1 + \alpha_1\alpha_2^{-1}z^{-\eta})^{-1}$ its expression (A.4.10) we get:

$$g' = g'[1 - (1 + \eta)^{-1}\epsilon] - \rho(1 + \eta)^{-1} + b(1 + \eta)^{-1}\alpha_1^{-1}\alpha_2^z\eta.$$  \hspace{1cm} \text{(A.4.10)}

Last substitute for $g'$ its expression (A.4.11), we obtain:

$$g' = \frac{b}{1 + \eta}\left[1 - \epsilon\eta b - \frac{\rho(1 + \eta)}{\epsilon b} + \alpha_1^{-1}\alpha_2^z\eta\right],$$

that is (6.6).
A.4.5 Proof of (6.7)

>From (A.1.7.b) we get:
\[
g^c = g^B + g^s + g^z \frac{\alpha_1 \alpha_2^{-1} z^{-\eta}}{1 + \alpha_1 \alpha_2^{-1} z^{-\eta}}.
\]

Substituting for \(g^z\) its expression (6.3) gives:
\[
g^c = g^B + g^s - b(1 + \eta)^{-1}(1 - l(1 + \alpha_1^{-1} \alpha_2 z^\eta))(1 + \alpha_1^{-1} \alpha_2 z^\eta)^{-1}.
\]

Remembering that \(g^B = b[l - l]\) and rearranging the terms, we obtain:
\[
g^c = b - bl + g^s - b(1 + \eta)^{-1}(1 + \alpha_1^{-1} \alpha_2 z^\eta)^{-1} + b(1 + \eta)^{-1} l,
\]
that is:
\[
g^c = b(1 + \eta)^{-1} + b\eta(1 + \eta)^{-1} - b(1 + \eta)^{-1}[1 + \eta]^l - l
\]
\[
- b(1 + \eta)^{-1}(1 + \alpha_1^{-1} \alpha_2 z^\eta)^{-1} + g^s
\]
\[
= b(1 + \eta)^{-1}[1 - (1 + \alpha_1^{-1} \alpha_2 z^\eta)^{-1}] + b\eta(1 + \eta)^{-1}[1 - l] + g^s
\]
\[
= b(1 + \eta)^{-1}(1 + \alpha_1 \alpha_2^{-1} z^{-\eta})^{-1} + b\eta(1 + \eta)^{-1}[1 - l] + g^s.
\]

Let us substitute for \(b(1 + \alpha_1 \alpha_2^{-1} z^{-\eta})^{-1}\) its expression (A.4.10) to get:
\[
g^s = -b\eta(1 + \eta)^{-1}[1 - l] - (1 + \eta)^{-1}[\epsilon g^c + \rho] + g^e
\]
that is:
\[
g^s = -\frac{b\eta}{1 + \eta}[1 - l] + \frac{1 + \eta - \epsilon}{1 + \eta} - \frac{\rho}{1 + \eta} g^c - \frac{\rho}{1 + \eta}.
\]

Last, substituting for \(g^c\) its expression (A.4.11) we get:
\[
g^s = \frac{b(1 + \eta - \epsilon)}{\epsilon(1 + \eta)(1 + \alpha_1 \alpha_2^{-1} z^{-\eta})} - \frac{\rho(1 + \eta - \epsilon)}{\epsilon(1 + \eta)} - \frac{\rho}{1 + \eta} - \frac{b\eta}{1 + \eta}[1 - l]
\]
that is, rearranging the terms:
\[
g^s = \frac{b\eta}{1 + \eta} \left[ \frac{1 + \eta - \epsilon}{\epsilon(1 + \alpha_1 \alpha_2^{-1} z^{-\eta})} - \frac{\rho(1 + \eta + \epsilon \eta b)}{\epsilon \eta b} + l \right]
\]
(A.4.12)

which is (6.7).

A.5 Proof of Proposition 6

Let \(T = [\theta, \theta']\), \(\theta < \theta'\), be a time interval during which \(n = 0\) and \(l = 1\), so that:
\[
B = B_\theta \; , \; y = B_\theta s \; and \; z = AB_\theta^{-1}s^{-1}, over \; T.
\]
(A.5.1)
Now let us remark that:

$(4.3) \cup (4.9) \implies \pi F_s = \lambda$

$(4.1)$ under $U.2 \implies \pi = e^{-\epsilon}e^{-\rho t}$

$(A.1.3.b) \implies F_s = \alpha_2 c^{1+\eta} s^{-(1+\eta)}B_\theta^{-\eta}$

$(A.1.7.b) \implies c = \alpha_2^{-1/\eta} B_\theta s[1 + \alpha_1 \alpha_2^{-1} \lambda^{-\eta}]^{-1/\eta}$ \hspace{1cm} (A.5.2)

from which we get:

$$\alpha_2 c^{1+\eta-\epsilon} s^{-(1+\eta)}B_\theta^{-\eta} = \lambda e^{\rho t}$$

$$\iff \alpha_2^{1-(1+\epsilon)/(1+\eta)} B_\theta^{-\epsilon} s^{-\epsilon}(1 + \alpha_1 \alpha_2^{-1} \lambda^{-\eta})^{-1/(1+\epsilon)} = \lambda e^{\rho t} \hspace{1cm} (A.5.3)$$

We denote by $M(s)$ the L.H.S of the above equation $(A.5.3)$.

**A.5.1 Existence and properties of $\tilde{s}(t, \lambda)$**

Log-differentiating $M$ we get:

$$\frac{dM}{ds} = -\epsilon \frac{ds}{s} + (1 + \eta - \epsilon) \frac{dz}{s} \frac{\alpha_1 \alpha_2^{-1} \lambda^{-\eta}}{1 + \alpha_1 \alpha_2^{-1} \lambda^{-\eta}}$$

and since from $(A.5.1): dz^{-1} = -dss^{-1}$, then:

$$\frac{dM(s)}{ds} = -\frac{M(s)}{s} \left[ \epsilon + \frac{1 + \eta - \epsilon}{1 + \alpha_1^{-1} \alpha_2 z^\eta} \right]$$

$$= -\frac{M(s)}{s} \frac{1 + \eta + \epsilon}{1 + \alpha_1^{-1} \alpha_2 z^\eta} < 0 \hspace{1cm} (A.5.4)$$

Note also that:

$$\lim_{s \to 0} M(s) = +\infty \quad \text{and} \quad \lim_{s \to \infty} M(s) = 0. \hspace{1cm} (A.5.5)$$

>From $(A.5.4)$ and $(A.5.5)$ we conclude that to any pair $(\lambda, t)$ is associated a unique value of $s$ by $(A.5.3)$ so that the function $\tilde{s}(t, \lambda)$ is well defined.

Now $(A.5.3)$ implies that:

$$\frac{dM(s)}{ds} ds = d\lambda e^{\rho t} + \rho \lambda e^{\rho t} dt ,$$

from which we get, taking $(A.5.4)$ into account:

$$\frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda} = -\frac{\tilde{s}(t, \lambda)}{\lambda} \frac{1 + \alpha_1^{-1} \alpha_2 z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta} < 0$$

and:

$$\frac{\partial \tilde{s}(t, \lambda)}{\partial t} = \rho \lambda \frac{\partial \tilde{s}(t, \lambda)}{\partial \lambda} < 0 ,$$

that is $(6.8)$ and $(6.9)$. 36
A.5.2 Expressions of $g^s$, $g^z$ and $g^c$

>From (A.5.3) we get:

$$\frac{dM}{Mdt} = \frac{dM}{ds} \frac{1}{M} \frac{ds}{dt} = \rho,$$

and substituting for $dM/ds$ given by (A.5.4), we obtain:

$$g^s = -\rho \frac{1 + \alpha_1^{-1} \alpha_2 z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta}, \quad (A.5.6)$$

and recalling that from (A.5.1) : $g^s = -g^z$, we get (6.10).

Log-differentiating (A.5.2) results in:

$$g^c = g^s + g^z \frac{\alpha_1 \alpha_2^{-1} z^{-\eta}}{1 + \alpha_1 \alpha_2^{-1} z^{-\eta}}.$$

Substituting for $g^z$ its value $-g^s$ and for $g^s$ its above value (A.5.6), we get (6.11):

$$g^c = g^s(1 + \alpha_1 \alpha_2^{-1} z^{-\eta})^{-1} = -\rho \frac{\alpha_2}{\alpha_1} \frac{z^\eta}{1 + \eta + \epsilon \alpha_1^{-1} \alpha_2 z^\eta}.$$

A.6 Proof of Proposition 9

Here:

$$c = q = (Al)^{\alpha_1} (Bs)^{\alpha_2} \quad (A.6.1)$$

hence:

$$F_l F_s^{-1} = \alpha_1 s (\alpha_2 l)^{-1}. \quad (A.6.2)$$

For $n_t > 0$, we get from (5.1) : $\alpha_1 s (\alpha_2 l)^{-1} = bS$, hence:

$$sS^{-1} = b \alpha_1^{-1} \alpha_2 l. \quad (A.6.3)$$

Multiplying the both sides of (A.6.3) by $AB^{-1}$ gives:

$$As(BS)^{-1} = \alpha_1^{-1} \alpha_2 b Al_B^{-1} \iff Al(B_s)^{-1} = z = \alpha_1 A(\alpha_2 b)^{-1} (BS)^{-1} \iff z = \alpha_1 (\alpha_2 b)^{-1} AR^{-1} \quad (A.6.4)$$

that is (6.29), which defines $z \equiv Z(R)$ in the present case.

Using (4.2), (4.3) and (A.6.2), we get also:

$$\omega l = \alpha_1 \alpha_2^{-1} \lambda s. \quad (A.6.5)$$
Inserting this expression of $\omega l$ into the accounting condition $\pi c = \omega l' + \lambda s$ leads to:

$$\pi c = (\alpha_1 + \alpha_2)\alpha_2^{-1}\lambda s = \alpha_2^{-1}\lambda s,$$

that is:

$$\pi = \alpha_2^{-1}\lambda s (Al)^{-\alpha_1} (Bs_l)^{-\alpha_2}$$

$$\iff \pi = \lambda (\alpha_2 B)^{-1} z^{-\alpha_2}. \tag{A.6.7}$$

Differentiating (A.6.4) w.r.t. time, we obtain:

$$g^z = -g^R = -[g^B + g^S] = -[bn - sS^{-1}].$$

Making use of (A.6.3) this is equivalent to:

$$g^z = -[bn - \alpha_1^{-1}\alpha_2 bl] = -b[1 - l - \alpha_1^{-1}\alpha_2 l] = -b[1 - \alpha_1^{-1}l],$$

that is:

$$g^z = b[\alpha_1^{-1}l - 1]. \tag{A.6.8}$$

Since $\ell_t = \bar{\ell}$, where $\bar{\ell}$ is given by (6.32) as shown below, we obtain (6.29).

Differentiating (A.6.7) w.r.t. time gives:

$$g^\pi = -bn - \alpha_1 g^z \iff \epsilon g^c + \rho = bn + \alpha_1 g^z.$$

Making use of (A.6.8), we get:

$$g^c = \epsilon^{-1}[\alpha_2 b - \rho]. \tag{A.6.9}$$

Let us turn now to the dynamics of $l$. Differentiating (A.6.3) through time leads to:

$$g^s - g^S = g^l \iff g^s + sS^{-1} = g^l$$

and making use of (A.6.3) again, we get:

$$g^s = g^l - \alpha_1^{-1}\alpha_2 bl. \tag{A.6.10}$$

Now differentiating (A.6.1) w.r.t. time gives:

$$g^c = \alpha_1 g^l + \alpha_2 g^B + \alpha_2 g^s.$$

Substituting for $g^s$ its expression (A.6.10) results in:

$$g^c = \alpha_2 b - \alpha_1^{-1}\alpha_2 bl + g^l$$

and substituting for $g^c$ its expression (A.6.9), we get:

$$g^l = \epsilon^{-1}[(1 - \epsilon)\alpha_2 b - \rho] + \alpha_1^{-1}\alpha_2 bl.$$

Let us define $\bar{l}$ as:

$$\bar{l} = \alpha_1(\epsilon\alpha_2 b)^{-1} [\rho - (1 - \epsilon)\alpha_2 b]. \tag{A.6.11}$$

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We may rewrite the growth rate of $l$ as:

$$g = \alpha^{-1}_1 \alpha_2 b [l - \bar{l}]$$

(A.6.12)

so that the feasibility condition $0 < \bar{l} < 1$ is:

$$0 < \bar{l} < 1 \iff (1 - \epsilon) \alpha_2 b < \rho < \alpha^{-1}_1 \alpha_2 b.$$  

(A.6.13)

>From now on we assume that (A.6.13) holds.

Next let us turn to the determination of $\lambda$. From (A.6.6) we get:

$$\pi c = e^{1-\epsilon} e^{-\rho t} = \alpha^{-1}_2 \lambda S_{t_0}.$$  

Integrating this relation over $[t_0, \infty)$ gives:

$$\int_{t_0}^{\infty} e^{1-\epsilon} e^{-\rho (t-t_0)} dt = \alpha^{-1}_2 \lambda e^{\rho t_0} S_{t_0}.$$  

that is:

$$c_t \int_{t_0}^{\infty} e^{[(1-\epsilon)g - \rho](t-t_0)} dt = \alpha^{-1}_2 \lambda e^{\rho t_0} S_{t_0}.$$  

(A.6.14)

Let $\delta \equiv \rho - (1-\epsilon)g^c$:

$$\delta = \rho - (1-\epsilon) e^{-1}(\alpha_2 b - \rho) = e^{-1} [\rho - (1-\epsilon) \alpha_2 b] = \alpha^{-1}_1 \alpha_2 b,$$

and, under (A.6.13), $\delta > 0$. Hence (A.6.14) may be rewritten as:

$$c_{t_0} = \alpha^{-1}_2 \lambda e^{\rho t_0} \delta S_{t_0}.$$  

(A.6.15)

(A.6.7) evaluated at $t = t_0$ gives:

$$\pi_{t_0} = e^{-\epsilon} e^{-\rho t_0} = \alpha^{-1}_2 \lambda B_{t_0} e^{-\alpha_1 t_0},$$

resulting in the following expression of $c_{t_0}$:

$$c_{t_0} = (\alpha^{-1}_2 \lambda B_{t_0}^{-1})^{-1/\epsilon} e^{-\alpha_1 t_0} e^{-\rho t_0/\epsilon}.$$  

(A.6.16)

Inserting this expression of $c_{t_0}$ into (A.6.15) results in:

$$\lambda = \alpha_2 B_{t_0}^{-\epsilon} \frac{\alpha_1 (1-\epsilon) \delta^{-\epsilon} S_{t_0} e^{-\rho t_0}}{z_{t_0}}.$$  

Now let us substitute for $z_{t_0}$ its expression (A.6.4) evaluated at $t = t_0$. We get:

$$\lambda = \alpha_2 \delta^{-\epsilon} [\alpha_1 A (\alpha_2 b)^{-1}]^{\alpha_1/(1-\epsilon)} B_{t_0}^{1-\epsilon} e^{\alpha_2 z_{t_0}} e^{-(\alpha_1 + \epsilon \alpha_2)} S_{t_0} e^{-\rho t_0}.$$  

(A.6.17)

Now dividing (A.6.15) by (A.6.16), we get:

$$c_{t_0} = \delta (B_{t_0} S_{t_0}) e^{\alpha_1}.$$  

that is using $z_{t_0}$ given by (A.6.4):

$$c_{t_0} = \delta (\alpha_1 A)^{\alpha_1} (\alpha_2 b)^{-\alpha_1} (B_{t_0} S_{t_0})^{\alpha_2}.$$  

(A.6.18)
Now consider the expression of \( c_{t_0} \) given by (A.6.1) and substitute for \( s_{t_0} \) its expression coming from (A.6.3): 
\[
s_{t_0} = b\alpha - \frac{1}{1 + \alpha Z(R_{t_0})}. \tag{A.6.19}
\]
Comparing (A.6.18) to (A.6.19) gives:
\[
\delta[\alpha_1(\alpha_2b)^{-1}]^{-1} = l_{t_0},
\]
that is:
\[
l_{t_0} = \bar{l}.
\]
Since \( t_0 \) is arbitrary, we conclude that:
\[
l_t = \bar{l} \quad \text{and} \quad n_t = \bar{n}, \quad t \geq 0,
\]
that is (6.32) and (6.33). (6.35) is a trivial implication of \( g^B = bn \), that is:
\[
g^B = (\epsilon\alpha_2)^{-1}[\alpha_2b(\alpha_1 + \epsilon\alpha_2) - \alpha_1\rho]. \tag{A.6.20}
\]
From (A.6.3), we get (6.36):
\[
g^S = -sS^{-1} = -\alpha_1^{-1}\alpha_2b\bar{l} = -\epsilon^{-1}[\rho - (1 - \epsilon)\alpha_2b]. \tag{A.6.21}
\]
Last from (A.6.20) and (A.6.21) we get (6.34):
\[
g^R = g^B + g^S = (\alpha_2\epsilon)^{-1}[\alpha_2b - \rho]. \tag{A.6.22}
\]

### A.7 Local stability analysis and proof of the existence of \( \bar{R} \)

#### A.7.1 Local stability analyses

Let us show now that the point \( (\bar{R}, \bar{c}) \) exhibits the saddle point property. Let us denote by \( \Phi^R(R, c) = \bar{R} \) and \( \Phi^c(R, c) = \bar{c} \) the functions of \((R, c)\) defined by (7.1) and (7.2) respectively. After differentiation we get:

\[
\frac{\partial \Phi^R(R, c)}{\partial R} = b[1 - cA^{-1}\alpha_1^{-1/\eta}(1 + \alpha_1^{-1}\alpha_2Z(R)^{\eta})^{(1+\eta)/\eta}] + bcA^{-1}\alpha_1^{-1+1/\eta}\alpha_2Z(R)^{\eta}(1 + \alpha_1^{-1}\alpha_2Z(R)^{\eta})^{1/\eta}\tag{A.7.1}
\]

\[
\frac{\partial \Phi^R(R, c)}{\partial c} = -bRA^{-1}\alpha_1^{-1/\eta}(1 + \alpha_1^{-1}\alpha_2Z(R)^{\eta})^{(1+\eta)/\eta}\tag{A.7.2}
\]

\[
\frac{\partial \Phi^c(R, c)}{\partial R} = -\eta bc(1 + \eta)R^{-1}\alpha_1^{-1}\alpha_2^{-1}Z(R)^{-\eta}(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-2}\tag{A.7.3}
\]

\[
\frac{\partial \Phi^c(R, c)}{\partial c} = \epsilon^{-1}[b(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1} - \rho]\tag{A.7.4}
\]
which evaluated at \((\bar{R}, \bar{c})\) gives the following jacobian matrix \(J\):

\[
J \equiv \begin{bmatrix}
\frac{\partial \Phi^R(R, c)}{\partial R}|_{(\bar{R}, \bar{c})} & \frac{\partial \Phi^R(R, c)}{\partial c}|_{(\bar{R}, \bar{c})} \\
\frac{\partial \Phi^c(R, c)}{\partial R}|_{(\bar{R}, \bar{c})} & \frac{\partial \Phi^c(R, c)}{\partial c}|_{(\bar{R}, \bar{c})}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\rho & -\alpha_2^{-1/\eta}(b\rho^{-1})^{(1+\eta)/\eta} \\
-\eta(\epsilon(1+\eta))^{-1}\alpha_2^{-1/\eta}\rho^{2+1/\eta}(b - \rho)b^{-(1+\eta)/\eta} & 0
\end{bmatrix}
\]

The determinant of \(J\) is thus given by:

\[
|J| = -\eta(\epsilon(1+\eta))^{-1}\rho(b - \rho) < 0 \text{ if } \rho < b \text{ and } \eta > 0.
\]

Hence, provided that \(\rho < b\) and \(\eta > 0\), the corresponding eigenvalues alternate in sign and the system (7.1)-(7.2) exhibits the saddle point property.

In the Leontiev case, analogous computations for the system (6.23)-(6.24) around the steady state \((\bar{R}, \bar{c})\), where \(\bar{R}\) and \(\bar{c}\) are in this case defined respectively by (6.27) and (6.28), lead to the following jacobian matrix \(K\):

\[
K = \begin{bmatrix}
\rho & -b\rho^{-1} \\
-(\epsilon b)^{-1}\rho^2(b - \rho) & 0
\end{bmatrix}
\]

The determinant of \(K\) is thus given by \(|K| = -\epsilon^{-1}\rho(b - \rho)\) which is negative if the condition \(\rho < b\) is fulfilled so that the dynamical system (6.23)-(6.24) in the Leontiev case exhibits also the saddle-point property.

A.7.2 Existence of \(\hat{R}\)

Now, we show that, in the general case, there exists some critical value of \(R\), we denote by \(\hat{R}\), such that the curve \(\hat{c}(R)\) crosses the curve \(\bar{c}(R)\) from below.

For any pair \((R, c)\) let \(\Gamma(R, c)\) be the slope at \((R, c)\) of the curve of the trajectory satisfying the system (7.1)-(7.2) and going through \((R, c)\), that is :

\[
\Gamma(R, c) \equiv \frac{\hat{c}}{\hat{R}} \bigg|_{R} = \frac{c \Phi^c}{\hat{R} \Phi^R} \bigg|_{R} =
\]

\[
= \frac{\rho - b(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1}}{\epsilon[\alpha_2^{-1/\eta}(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{(1+\eta)/\eta} - \epsilon^{-1}\alpha_1\alpha_2^{-1}Z(R)^{-(1+\eta)}]}\tag{A.7.5}
\]

where we use (6.1) to get an expression of \(R\) as a function of \(Z(R)\).
Let us restrict the attention to the \((R, c)\) pairs such that \(R \geq \bar{R}\) and \(c \geq c(R)\). From the previous study of the dynamics of \(R\) and \(c\), we know that the trajectories are such that \(\dot{c} < 0\) and \(\bar{R} < 0\) in this case. Hence the numerator and the denominator of \(\Gamma(R, c)\) are both strictly positive for such \((R, c)\) pairs.

First note that:

\[
\frac{\partial \Gamma(R, c)}{\partial c} = -\frac{\Gamma(R, c)A\alpha_1\alpha_2^{-1}Z(R)^{-(1+\eta)}}{e^{2[\alpha_2^{1/\eta}(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})(1+\eta)/\eta} - e^{-1}A\alpha_1\alpha_2^{-1}Z(R)^{-(1+\eta)}}\]

the denominator being strictly positive, as already noted. Hence \(\Gamma(R, c)\) is a decreasing function of \(c\). Let us denote by \(\bar{\Gamma}(R) = \lim_{c \uparrow \infty} \Gamma(R, c)\) the lower bound of \(\Gamma(R, c)\) for any \(R \geq \bar{R}\):

\[
\bar{\Gamma}(R) = e^{-1}\alpha_2^{-1/\eta}\left[\rho - b(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1}\right](1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1/\eta}.
\]

It is easily verified that \(\lim_{R \uparrow \bar{R}} \bar{\Gamma}(R) = \lim_{R \downarrow \infty} \bar{\Gamma}(R) = 0\) and that the function \(\bar{\Gamma}(R)\) is first increasing and then decreasing when \(R\) increases from \(\bar{R}\) to infinity. Hence \(\Gamma(R, c) > \bar{\Gamma}(R) > 0\) for any finite value of \(R > \bar{R}\).

Now consider the function \(\tilde{\Gamma}(R) \equiv \Gamma(\tilde{c}(R), R)\) which gives the slopes corresponding to the different trajectories along the frontier curve \(\tilde{c}(\bar{R})\):

\[
\tilde{\Gamma}(R) = e^{-1}\alpha_2^{-1/\eta}\left[\rho - b(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1}\right](1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta})^{-1/\eta}.
\]

Some elementary manipulations show that:

\[
\frac{d\tilde{c}(R)}{dR} < \tilde{\Gamma}(R) \iff A\epsilon[(1 + \eta)RZ(R)]^{-1} < \rho(1 + \alpha_1\alpha_2^{-1}Z(R)^{-\eta}) - b.
\]

Making use of (6.1) this is equivalent to:

\[
\rho(\alpha_1\alpha_2^{-1})^{1/(1+\eta)}(Ab^{-1})^{-\eta/(1+\eta)}R^{\eta/(1+\eta)} - (b - \rho)
\]

\[-A\epsilon(1 + \eta)^{-1}(\alpha_1\alpha_2^{-1})^{-1/(1+\eta)}(Ab^{-1})^{-1/(1+\eta)} > 0.
\]

Multiplying both sides by \(R^{\eta/(1+\eta)}(\alpha_1\alpha_2^{-1})^{-1/(1+\eta)}\rho^{-1}(Ab^{-1})^{-\eta/(1+\eta)}\) and inserting the expression of \(\tilde{R}\) given by (6.18) results in:

\[
\tilde{R}^\eta/(1+\eta) - \tilde{R}^{\eta/(1+\eta)}R^{\eta/(1+\eta)} - A\epsilon[(1 + \eta)\rho^{-1}(Ab^{-1})^{\eta-1/(1+\eta)}(\alpha_1\alpha_2^{-1})^{-2/(1+\eta)} > 0
\]

which defines a second order polynomial in \(\tilde{R}^\eta/(1+\eta)\), we denote by \(P(R^\eta/(1+\eta))\). \(P\) has only one strictly positive root \(\tilde{R}^\eta/(1+\eta)\) given by:

\[
\tilde{R}^\eta/(1+\eta) = (1/2)[\tilde{R}^\eta/(1+\eta) + \sqrt{[\tilde{R}^\eta/(1+\eta) + 4A\epsilon\rho(1 + \eta)^{-1}(Ab^{-1})^{\eta-1/(1+\eta)}]^{1/2}}]
\]

We conclude that:

\[
\tilde{\Gamma}(R) \begin{cases} 
< \frac{d\tilde{c}(R)}{dR} & \text{if } R \in (\bar{R}, \bar{R}) \\
\geq \frac{d\tilde{c}(R)}{dR} & \text{if } R \geq \bar{R}
\end{cases}
\]
Hence the slopes of the curves depicting the different trajectories are strictly higher than the slope of the \( \bar{c}(R) \) curve for sufficiently high levels of \( R \). Since first \( \bar{c}(R)\alpha_1^{-1/\eta}A \), second, \( \lim_{R \to \infty} d\bar{c}(R)/dR = 0 \) and third, \( \Gamma(R, c) > 0 \) for \( c \leq \alpha_1^{-1/\eta}A \), we conclude that the curves corresponding to the different trajectories cut from below the \( \bar{c}(R) \) frontier at some finite value of \( R \). Hence, as any one of these curves, the curve \( \bar{c}(R) \) crosses the frontier curve \( \bar{c}(R) \) at some finite value of \( R \), we denote by \( \bar{R} \).

In the Leontiev case, making use of (6.23) and (6.24) results in the following expression of \( \Gamma(R, c) \):

\[
\Gamma(R, c) = Ac^{-1}[\rho - Ab(A + bR)^{-1}][A + bR - Ac^{-1}bR]^{-1},
\]
a decreasing function of \( c \) which is strictly positive for \( R > \bar{R} \). Setting \( c = A \) we obtain the expression of the slopes of the curves along the feasibility frontier:

\[
\bar{\Gamma}(R) = c^{-1}[\rho - Ab(A + bR)^{-1}],
\]
slopes which are strictly positive if \( R > \bar{R} \). Hence the \( \bar{c}(R) \) curve crosses the horizontal \( c = A \) at some finite value \( \bar{R} \) of \( R \).

We end this section by showing that for \( R_t > \bar{R} \), the consumption path must be located above the \( \bar{c}(R) \) frontier in the general case. If \( R_t > \bar{R} \), \( \gamma_t \geq 0 \) entails that:

\[
z_t^{-(1+\eta)\alpha_1}(\alpha_2B_0)^{-1} = F_{t,t}F_{s,t}^{-1} \geq bs_t
\]

\[
\iff z_t \leq (\alpha_1A)^{1/(1+\eta)}(\alpha_2bR_t)^{-1/(1+\eta)} = Z(R_t).
\]

(A.7.6)

This implies that for \( R_t > \bar{R} \):

\[
\alpha_t = A\alpha_1^{-1/\eta}(1 + \alpha_1^{-1}\alpha_2z_t^{1/\eta})^{-1/\eta} \geq A\alpha_1^{-1/\eta}(1 + \alpha_1^{-1}\alpha_2Z(R_t)^{1/\eta})^{-1/\eta} = \bar{c}(R_t).
\]

If \( R_0 \uparrow \infty \), we get \( \bar{s}(0, \lambda) \uparrow \infty \) and \( z_0 \downarrow 0 \), hence \( \lim_{R \to \infty} c_0 = \alpha_1^{-1/\eta}A \). As illustrated on Figure 1, the consumption path \( \bar{c}(R) \) for \( R > \bar{R} \) is located below the horizontal line \( c = \alpha_1^{-1/\eta}A \) and above the frontier curve \( \bar{c}(R) \).

### A.8 Proof of claims iii and iv of Proposition 12

Let us consider first the general case. For \( \bar{R} < R < \bar{R} \), we have \( g^z > 0 \) and \( g^l < 0 \) so that (from the definition of \( z \)) \( B:t,s \) must decrease. But since \( g^R > 0 \), it follows that \( g^z < 0 \). Moreover we know from (6.10) that \( g^z < 0 \) if \( R > \bar{R} \), proving the first part of claim iii of the proposition.

Next, consider the case \( 1 + \eta < \epsilon \) and \( R \leq \bar{R} \). Since \( l \leq 1 \), this implies, from (6.7), that \( g^z < 0 \). In the case \( \epsilon < 1 + \eta \) and \( R \leq \bar{R} \), we get \( g^c \geq 0 \), that is
(1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta}) \leq b \rho^{-1}. Moreover for \( R_0 > 0 \), \( 1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta} \to 1. \\
Hence, a sufficient condition for \( g^* < 0 \) is :

\begin{equation}
l \leq l^R(R) \leq 1 - \frac{1 + \eta - \epsilon}{\epsilon \eta (1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})} + \frac{\rho(1 + \eta)}{\epsilon \eta b} \nonumber\end{equation}

\begin{equation}
\iff (1 + \alpha_1^{-1} \alpha_2 Z(R)^{\eta})^{-1} \leq \frac{\epsilon - 1 - \eta}{\epsilon \eta (1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})} + \frac{\rho(1 + \eta)}{\epsilon \eta b} \nonumber\end{equation}

\begin{equation}
\iff Z(R)^{-\eta} \geq \alpha_2 [b(1 - \epsilon) - \rho(\alpha_1 \rho)^{-1}] \nonumber\end{equation}

which is always satisfied provided that \( b(1 - \epsilon) < \rho \). Claim \( iii \) of the proposition is proved.

Last, in the case \( \epsilon < 1 + \eta \) and \( \rho < b(1 - \epsilon) \) (note that this implies that \( \epsilon < 1 \)), \( g^* = 0 \) defines implicitly a function \( l^*(R) \) given by :

\begin{equation}
l^*(R) = \frac{\rho(1 + \eta) + \epsilon \eta b}{\epsilon \eta} - \frac{1 + \eta - \epsilon}{\epsilon \eta (1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})}. \quad \text{(A.8.1)}
\end{equation}

Differentiating (A.8.1) and making use of (6.2) we obtain :

\begin{equation}
\frac{dl^*(R)}{dR} = \frac{(1 + \eta - \epsilon) \alpha_1 \alpha_2^{-1} Z(R)^{-\eta}}{(1 + \eta)(1 + \alpha_1 \alpha_2^{-1} Z(R)^{-\eta})^2 R} > 0 \quad \text{since} \quad \epsilon < 1 + \eta. \nonumber\end{equation}

Let us denote by \( R^*_0 \) the solution of \( l^*(R) = 0 \) and by \( R^*_1 \) the solution of \( l^*(R) = l^R(R) \). It is easily checked that :

\begin{equation}
R^*_0 \quad = \quad Ab^{-1}(\alpha_1 \alpha_2^{-1})^{-1/\eta} [(1 + \eta)[b(1 - \epsilon) - \rho]]^{(1 + \eta)/\eta} \quad \text{(A.8.2)}
\end{equation}

\begin{equation}
R^*_1 \quad = \quad Ab^{-1}(\alpha_1 \alpha_2^{-1})^{-1/\eta} [(b(1 - \epsilon) - \rho)^{-1}]^{(1 + \eta)/\eta} < \hat{R}. \quad \text{(A.8.3)}
\end{equation}

Hence the locus \( g^* = 0 \), that is the function \( l^*(R) \), crosses the stable arm converging towards \( (\hat{R}, \epsilon) \) at some critical value of \( R \), we denote by \( R^* \), such that \( R^*_0 < R^* < R^*_1 < \hat{R} \). Since \( g^* > 0 \) if \( l < l^*(R) \) and \( g^* < 0 \) if \( l > l^*(R) \), we get the claim \( iv \) of the proposition.

REFERENCES


DANDRAKIS, E and E. S. PHELPS. (1965), A Model of Induced Innovation, Growth and Distribution, Economic Journal, 76, 823-840.


SCHOU, P. (1996), A Growth Model with Technological Progress and Non-renewable Resources, mimeo, University of Copenhagen.


Fig. 1 – The phase diagram in the \((R, c)\) plane for the general case
Fig. 2 – The phase diagram in the $(R, \ell)$ plane for the general case when $\epsilon < 1 + \eta$ and $\rho < b(1 - \epsilon)$