Global dynamics in a growth model with exhaustible resource*

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Abstract

We revisit the seminal model of endogenous growth with exhaustible resources, the so called Dasgupta Heal Stiglitz Solow model (DHSS). For this optimal control problem with two state variables, we explicitly characterize the dynamics of all the variables in the model and from all possible initial values of the stocks.

We determine the condition under which the consumption is initially increasing with time and the condition under which initial investment is positive implying overshooting of the man-made capital. We provide an example where investment is initially positive increasing with time, reaches a maximum, declines and eventually becomes negative.

Key words: endogenous growth, exhaustible resources, exponential integral

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1 Introduction

We provide a closed form solution to the Dasgupta-Heal-Solow-Stiglitz (from here on DHSS) model. The DHSS model is based on seminal articles by Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974). It describes an economy with two assets, man-made capital and a nonrenewable resource stock. Together with man-made capital the raw material from the resource is used as an input in the production of a commodity that can be used for consumption and for net investments in man-made capital. In this framework some essential questions have been addressed. For instance, in the case where the objective is to maximize the minimum rate of consumption throughout the time horizon a central question is whether despite the resource limitation there exists a sustainable constant positive rate of consumption (Solow, 1974). Another stream of the literature adopts a utilitarian objective and studies the optimal consumption and investment paths that maximize a discounted sum of utility from consumption (Stiglitz, 1974). In the present paper we give the optimal paths for the DHSS economy under a utilitarian objective. In the literature attention has mainly been given to the case where the production function is Cobb-Douglas and instantaneous utility is logarithmic. Even for these specifications no closed form solutions of the optimum have been found so far. There has been some progress regarding the characterization of the optimal solution to the DHSS problem, in particular it has been shown without explicitly finding the solution that consumption can be single peaked\(^1\) (see Pezzey and Withagen (1998) and Hartwick et al. (2003)). However, in the absence of a closed-form solution it is not possible to address other relevant issues such as understanding the relationship between the instant of time where the peak takes place and the initial stocks of capital and the resource and how this phenomenon depends on the model parameters. Moreover, to actually calculate the optimal path more information is needed on the co-state variables associated with the stocks, which amounts to having a complete solution of the model. In this paper we provide a closed form solution to the DHSS problem using the exponential integral function. The exponential integral is a special case of a family of 'special functions' which are extensively used in mathematical physics and probability theory. They are particularly helpful to determine solutions to differential equations encountered in physics (see e.g., Temme (1996) Ch 5 and Ch 7). The use of special functions in economic theory is relatively recent. Boucekkine et al. 2007 and 2008 show that the solution to a two-sector Lucas-Uzawa model of endogenous growth can be expressed in terms of a specific type of 'special functions': the hypergeometric functions. In our problem of finding the solution to the canonical growth model with resource constraints, the DHSS model, it is another type of special function, the

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\(^1\)For high rates of pure time preference consumption always decreases over time and for low rates of time preference consumption monotonically increases during an initial interval of time reaches a maximum and eventually decreases.
exponential integral\(^2\), that turned out to be instrumental in expressing the solution in a closed form. Thus, along with Boucekkine et al. 2007 and 2008, this paper is a proof that special functions can be play a key role in analyzing dynamic economic problems and characterizing the transition dynamics of all the variables in a dynamic problem and from all possible initial values of the state variables\(^3\).

We exploit the explicit form of the solution to study the behavior of the optimal consumption and investment paths as functions of the parameters of the model. We then compare the solution to the DHSS model, under a utilitarian objective, to the solution of the problem where the objective is to maximize the minimum rate of consumption over the whole time horizon. The solution to the latter problem is called the maximin rate of consumption. Sustainability in the DHSS context requires that the Hartwick rule, i.e. zero genuine savings, holds: the investment rate must equal the extraction rate times the marginal product of the resource. Asheim (1994) shows that this condition should hold at all instants of time. Hence, if at some instant of time Hartwick’s rule holds it does not mean that the economy is on a sustainable path. The argument used by Asheim rests on the assumption that in a utilitarian optimum the initial rate of consumption is below the maximin rate of consumption if the pure rate of time preference is low, and above the maximin rate of consumption if the pure rate of time preference is high. By continuity there is a rate of time preference for which both initial consumption rates coincide, and for which the utilitarian rate of consumption is increasing for an initial period of time. Asheim (1994) refers to a graph in Dasgupta and Heal (1979) to support this assumption with respect to the rates of time preference. However, Dasgupta and Heal do not provide a formal proof of their claim. Also, no proof is given of continuity. In our case we are able to provide a proof to both claims. Moreover, and more importantly, we show that in our case not only consumption can be single peaked, but also investments in man-made capital. Investments can initially increase and then decrease, and become eventually negative. To the best of our knowledge this feature of the optimal path of the DHSS model has not been shown in the literature before. We provide the condition under which optimal initial investment is positive which implies overshooting in the man-made capital.

The outline of the paper is as follows. The model is introduced in section 2. Section 3 contains the characterization of the optimum in a series of lemmata and propositions. The proofs of these is relegated to the mathematical appendix. Section 4 goes into some sensitivity analyses. Finally, section 5 concludes.

\(^2\)The exponential integral belongs to a class of special functions called the ‘confluent hypergeometric functions’ which solve the Kummer differential equation, a confluent of the Gauss hypergeometric differential equation. For more details we refer the reader to Temme 1996, Ch 5 and Ch 7.

\(^3\)Dynamic systems in economics, in particular those involving more than one state variable, have been so far treated rigorously but mostly using qualitative techniques such as phase diagrams accompanied with an analytical study of the behaviour in the neighborhood of a steady state, or using numerical techniques.
2 The problem and conditions for optimality

Consider the following optimal control problem of the DHSS economy, which we refer to as the DHSS optimal growth problem:

\[
\text{Max} \int_0^\infty e^{-\rho t} \ln C(t) dt
\]  
subject

\[
K(t)^\alpha R(t)^{1-\alpha} = C(t) + I(t)
\]  
\[
\dot{S}(t) = -R(t)
\]  
\[
\dot{K}(t) = I(t)
\]

with

\[
K(0) = K_0 > 0 \text{ and } S(0) = S_0 > 0.
\]

Here \(K(t)\) and \(S(t)\) denote the stock of man-made capital and the nonrenewable resource at instant of time \(t\) respectively, \(C(t), I(t)\) and \(R(t)\) are rates of consumption, net investment in man made capital and resource extraction at instant of time \(t\), and \(\rho\) and \(\alpha\) are the rate of pure time preference (assumed positive) and the production elasticity of man made capital \((0 < \alpha < 1)\) respectively. For any variable \(x(t)\) we adopt the convention \(\dot{x}(t) = dx(t)/dt\). When there is no danger of confusion we omit the time argument.

A solution of the optimal growth problem above is described by a quintuplet of paths \(\{C, I, K, R, S\}\).

Let \(\lambda(t)\) and \(\mu(t)\) denote the co-state variables associated with respectively the stock of capital and the natural resource stock. The current value Hamiltonian is given by

\[
H(K, R, I, \lambda, \mu) = \ln \left( K^\alpha R^{1-\alpha} - I \right) + \lambda - \mu R
\]

The maximum principle gives

\[
-\frac{1}{K^\alpha R^{1-\alpha} - I} + \lambda = 0
\]  
\[
\frac{(1 - \alpha) K^\alpha R^{\alpha-\alpha}}{K^\alpha R^{1-\alpha} - I} - \mu = 0
\]  
\[
\dot{\lambda} = \rho \lambda - H_K = \rho \lambda - \frac{\alpha K^{\alpha-1} R^{1-\alpha}}{K^\alpha R^{1-\alpha} - I}
\]  
\[
\dot{\mu} = \rho \mu - H_S = \rho \mu
\]

Any solution that satisfies the above system along with the following transversality conditions

\[
\lim_{t \to \infty} e^{-\rho t} \lambda(t) K(t) = 0
\]  
\[
\lim_{t \to \infty} e^{-\rho t} \mu(t) S(t) = 0
\]

is a solution to the optimal control problem (1).
3 The solution

We give an explicit solution to the DHSS optimal growth problem. It turns out that the solution can be expressed in terms of a special function, i.e. the exponential integral defined as

\[ \text{Ei}(a, \theta) \equiv \int_1^\infty e^{-\theta u}u^{-a}du \]

with \( \text{Re}(\theta) > 0 \) (see e.g., Abramowitz and Stegun, 1972). A special case of the exponential integral is

\[ \text{Ei}(1, z) \equiv \int_0^\infty e^{-v}v^{-1}dv. \]

One of the properties of the exponential integral that will prove useful for our purposes is (see Abramowitz and Stegun, 1972 p.229, inequality 5.1.20)

\[ \frac{1}{2}e^{-z} \ln \left(1 + \frac{2}{z}\right) < \text{Ei}(1, z) < e^{-z} \ln \left(1 + \frac{1}{z}\right). \] (12)

This property will also be useful to determine the behavior of the optimal paths that solve the problem above.

**Lemma 1:** Let

\[ h(x_0) \equiv x_0^{-\alpha} \left( \frac{1}{x_0 e^{x_0} \text{Ei}(1, x_0)} - 1 \right) \] (13)

Then

(i) \( h \) is strictly decreasing over \((0, \infty)\) and \( h((0, \infty)) = (0, \infty) \)

(ii) for any \( \bar{h} > 0 \) there exists a unique \( x_0 > 0 \) such that

\[ h(x_0) = \bar{h} \] (14)

**Proof:** Appendix A.

Before stating the main theorem of the paper, we introduce the following variables that we use throughout the paper; they allow to improve the presentation of several lengthy expressions:

\[ \kappa \equiv \frac{\mu(0)}{1 - \alpha}, \varphi \equiv (1 - \alpha)\kappa^{\frac{\alpha - 1}{\alpha}}, \sigma(t) \equiv e^{-\rho t} \lambda(t), \sigma_0 \equiv \sigma(0) \]

\[ \pi(t) \equiv \sigma_0^{\frac{\alpha - 1}{\alpha}} + \varphi t, \pi_0 \equiv \pi(0), \chi(t) \equiv \frac{\rho \pi(t)}{\varphi}, \chi_0 \equiv \chi(0). \] (15)

**Main Theorem:**

*For any \( K_0 \geq 0, S_0 \geq 0, \rho > 0 \) and \( \alpha \in (0, 1) \), \( x_0 \) is the unique solution to*

\[ h(x_0) = A \] (16)
where
\[
A = \frac{S_0}{K_0} \left( \frac{(1 - \alpha) \tilde{\alpha}}{\rho} \right)^{\frac{\alpha}{1 - \alpha}} > 0.
\]
and we have
\[
\mu(0) \equiv \mu_0 = \frac{e^{x_0} (e^{-x_0} - x_0 E(1,x_0))}{\rho S_0} \quad \text{and} \quad \lambda_0 \equiv \lambda(0) = \frac{x_0 e^{x_0}}{\rho K_0} E(1,x_0).
\]
Moreover
i) The optimal consumption path is given by
\[
C(t) = e^{-\rho t} \pi(t)^{-\frac{\alpha}{1 - \alpha}}
\] (17)
ii) The optimal investment path is given by
\[
I(t) = \pi(t)^{\frac{\alpha}{1 - \alpha}} \left( \frac{1}{\kappa} \right)^{1 - \alpha} \left( K_0 \left( \frac{1}{\pi_0} \right)^{1 - \alpha} - \frac{1}{\varphi} e^{x_0} (E(1,x_0) - E(1,x(t))) \right) - e^{-\rho t}
\] (18)
iii) The optimal path of the stock of capital is given by
\[
K(t) = \pi(t)^{\frac{\alpha}{1 - \alpha}} \left( K_0 \left( \frac{1}{\pi_0} \right)^{1 - \alpha} - \frac{1}{\varphi} e^{x_0} (E(1,x_0) - E(1,x(t))) \right)
\] (19)
iv) The optimal extraction rate is given by
\[
R(t) = \left( \frac{1}{\kappa} \right)^{\frac{\alpha}{1 - \alpha}} \left( K_0 \left( \frac{1}{\pi_0} \right)^{1 - \alpha} - \frac{1}{\varphi} e^{x_0} (E(1,x_0) - E(1,x(t))) \right)
\] (20)
v) The optimal path of the stock of resource is given by
\[
S(t) = S(0) \left( \frac{1}{\kappa} \right)^{\frac{\alpha}{1 - \alpha}} \left( K_0 \left( \frac{1}{\pi_0} \right)^{1 - \alpha} - \frac{1}{\varphi} e^{x_0} E(1,x_0) \right) t
\]
\[
- \left( \frac{1}{\kappa} \right)^{\frac{\alpha}{1 - \alpha}} \frac{1}{\varphi \rho} (1 - e^{-\rho t})
\]
\[
- \left( \frac{1}{\kappa} \right)^{\frac{\alpha}{1 - \alpha}} \frac{1}{\varphi \rho} e^{x_0} ((x_0 + \rho t) E(1,x_0 + \rho t) - x_0 E(1,x_0)).
\]
Proof: See the next section.

There are two approaches to prove this Theorem. A first approach is a direct check that the above paths, along with corresponding paths of the costate variables \( \lambda(.) \), \( \mu(.) \), do satisfy the necessary conditions of the maximum principle and the transversality conditions\(^4\). Another approach is to present the steps that were followed to obtain the optimal paths above. This is the approach we follow in the next section.

\(^4\)The conditions from the maximum principle along with the transversality conditions constitute a set of sufficient conditions of optimality.
4 Solving the optimal growth problem

In this section, we provide the steps to determine the solution to the set of conditions given by the maximum principle (6)-(9). We first express the optimal paths as functions of two ‘parameters’ \( \lambda_0 \equiv \lambda(0) \) and \( \mu_0 \equiv \mu(0) \). The determination of \( \lambda_0 \) and \( \mu_0 \) is presented at the end of this section. It will be useful to rewrite the conditions (6)-(9) in the following form

\[
\frac{1}{K^\alpha R^{1-\alpha} - I} = \lambda \quad (22)
\]

\[
\mu = (1 - \alpha) K^\alpha R^{-\alpha} \lambda \quad (23)
\]

\[
\dot{\lambda} = \lambda \left( \rho - \alpha K_1^{\alpha-1} R^{1-\alpha} \right) \quad (24)
\]

\[
\dot{\mu} = \rho \mu \quad (25)
\]

4.1 The rate of consumption

Lemma 2

\[
\sigma(t) = \left( \sigma_0^{1-\frac{1}{\bar{\sigma}}} + \varphi t \right)^{\frac{1}{1-\bar{\sigma}}} \quad (26)
\]

Proof: See Appendix B.

It is now straightforward to determine the consumption path from (15), since \( C(t) = \frac{1}{\lambda(t)} \).

Proposition 1

The optimal consumption path is given by

\[
C(t) = e^{-\rho t} \pi(t)^{-\frac{\bar{\sigma}}{1-\bar{\sigma}}}.
\]

4.2 The stock of man-made capital

We now turn to the optimal path of man-made capital.

Lemma 3

The stock of capital is the solution to the following first order differential equation

\[
\dot{K}(t) + f(t) K(t) = g(t) \quad \text{with} \quad K(0) = K_0
\]

where

\[
f(t) \equiv - \left( \frac{\kappa}{\sigma(t)} \right)^{-\frac{1}{1-\alpha}} \quad \text{and} \quad g(t) \equiv - \frac{1}{\lambda(t)}
\]

Proof: See Appendix C.
Lemma 4
The optimal stock of capital is given by
\[
K(t) = K_0 e^{-\int_0^t f(z) dz} + \int_0^t g(z) e^{-\int_0^t f(s) ds} dz
\]  
(27)
Proof: See Appendix D.

The following lemma gives an expression for \( \int_z^t f(z) ds \).

Lemma 5
\[
\int_z^t f(z) dz = -\ln \left( \frac{\pi(t)}{\pi(z)} \right)^{\frac{1}{\tau x_0}}
\]  
(28)
Proof: See Appendix E.

We also have to determine \( \int_0^t g(z) e^{-\int_0^t f(s) ds} dz \).

Lemma 6
\[
\int_0^t g(z) e^{-\int_0^t f(s) ds} dz = -\frac{1}{\varphi} e^{x_0} \pi(t)^{\frac{1}{\tau x_0}} \left( \text{Ei}(1, x_0) - \text{Ei}(1, -x(t)) \right)
\]  
(29)
Proof: See Appendix F.

We can now describe the path of the capital stock.

Proposition 2
The optimal path of the stock of capital is given by
\[
K(t) = \pi(t)^{\frac{1}{\tau x_0}} \left( K_0 \left( \frac{1}{\pi_0} \right)^{\frac{1}{\tau x_0}} - \frac{1}{\varphi} e^{x_0} \pi(t)^{\frac{1}{\tau x_0}} \left( \text{Ei}(1, x_0) - \text{Ei}(1, x(t)) \right) \right)
\]  
(30)
Proof:
From lemma 4 we have (27). Substituting (28) and (29) from lemmata 5 and 6 gives
\[
K(t) = K_0 \left( \frac{\pi(t)}{\pi_0} \right)^{\frac{1}{\tau x_0}} - \frac{1}{\varphi} e^{x_0} \pi(t)^{\frac{1}{\tau x_0}} \left( \text{Ei}(1, x_0) - \text{Ei}(1, x(t)) \right).
\]
Factoring \( \pi(t)^{\frac{1}{\tau x_0}} \) yields (30).
4.3 The extraction and investment paths

We determine the resource path from (23). After substitution of $\mu$ and $\lambda$ using (25), (26) and (15) we arrive at

$$R(t) = \left(\frac{1}{\kappa}\right)^{\frac{1}{\pi}} \pi(t) \frac{1}{\pi-1} K(t)$$

So a direct substitution from Proposition 2 we have

**Proposition 3:**

The optimal extraction rate is given by

$$R(t) = \left(\frac{1}{\kappa}\right)^{\frac{1}{\pi}} \left( K_0 \left( \frac{1}{\pi_0} \right)^{\frac{1}{\pi-1}} - \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x_0) - \text{Ei}(1, x(t))) \right).$$

The optimal investment path can be obtained by direct substitution of $C(t)$, $K(t)$ and $R(t)$ from respectively (17), (19), (20) into (2) to obtain (18).

4.4 The resource stock

The optimal path of the stock of resource is the unique solution to

$$\dot{S} = -R \text{ with } S(0) = S_0.$$

**Proposition 4.**

The optimal path of the stock of resource is given by

$$S(t) = S(0) - \left(\frac{1}{\kappa}\right)^{\frac{1}{\pi}} \left( K_0 \left( \frac{1}{\pi(0)} \right)^{\frac{1}{\pi-1}} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) t$$

$$- \left(\frac{1}{\kappa}\right)^{\frac{1}{\pi}} \frac{1}{\varphi \rho} (1 - e^{-\rho t})$$

$$- \left(\frac{1}{\kappa}\right)^{\frac{1}{\pi}} \frac{1}{\varphi \rho} e^{x_0} \left( (x_0 + \rho t) \text{Ei}(1, x_0 + \rho t) - x_0 \text{Ei}(1, x_0) \right)$$

Proof: See Appendix G

4.5 Solving for $\mu_0$ and $\lambda_0$

All the paths above are written as functions of $\mu_0$ and $\lambda_0$ since $\pi_0 = \sigma_0^{\frac{\alpha-1}{\alpha}} = \lambda_0^{\frac{\alpha-1}{\alpha}}$. To fully characterize the optimal paths of consumption, the rate of extraction, and the stocks of capital and the resource we still need to determine $\mu_0$ and $\lambda_0$. We use the transversality conditions (11) and (10) to do so.
Lemma 7: The vector \((\mu_0, \lambda_0)\) is the solution to

\[
\frac{1}{\mu_0} e^{x_0} \frac{e^{-x_0} - x_0 \text{Ei}(1, x_0)}{\rho} = S_0
\]

(31)

\[
K_0 \lambda_0^{1/\alpha} - \frac{\mu_0^{1-\alpha}}{(1-\alpha)^{1/\alpha}} e^{x_0} \text{Ei}(1, x_0) = 0
\]

(32)

Proof: See Appendix H.

Lemma 8: \(x_0\) is the unique solution to (16).

Proof: Appendix I.

Once \(x_0\) is determined we have from Appendix I

\[
\mu_0 = \frac{e^{x_0} (e^{-x_0} - x_0 \text{Ei}(1, x_0))}{\rho S_0}
\]

\[
\sigma_0 = \frac{x_0}{\rho K_0} e^{x_0} \text{Ei}(1, x_0).
\]

Propositions 1-4 and Lemmata 7 and 8 constitute the Main Theorem.

5 Sensitivity analysis

We exploit the analytical tractability of the solution to the DHSS model to establish some key features of the optimal paths: we focus on the consumption and the investment paths.

5.1 The optimal consumption path

We first study the conditions under which consumption is increasing for some initial period of time. We highlight the role of the parameters of the model, like the pure rate of time preference and the initial stocks of capital and the natural resource on the possibility that consumption may rise for an initial period of time. From (17) the time at which maximum consumption is reached is

\[
t^* = -\frac{1}{\rho \varphi} \left( \frac{\alpha}{\alpha - 1} + \rho \pi_0 \right) = \frac{1}{\rho} \left( \frac{\alpha}{1 - \alpha} - x_0 \right)
\]

(33)

Clearly \(t^* > 0\) iff \(x_0 < \frac{\alpha}{1 - \alpha}\) which holds iff \(A > \bar{A} \equiv h \left( \frac{\alpha}{1 - \alpha} \right) > 0\). Indeed, given \(A\) and \(x_0\) such that \(h (x_0) = A\) we have

\[
x_0 < \frac{\alpha}{1 - \alpha} \text{ iff } A > \bar{A}
\]

since \(h\) is a strictly decreasing function. Therefore consumption is initially (i.e. for \(t < t^*\)) increasing over time, if and only if \(A > \bar{A}\). For a given \(\alpha \in (0, 1)\) we have \(A \equiv \frac{x_0}{K_0} \left( \frac{(1-\alpha)}{\rho} \right)^{1/\alpha} > \bar{A}\) when \(\frac{x_0}{K_0}\) is large.
enough or $\rho$ is small enough. More precisely, from (16), (33) and using the implicit function theorem we have

$$\frac{dx_0}{d\rho} = \frac{dA}{h'(x_0)} > 0 \text{ since } h'(x_0) < 0 \text{ and } \frac{dA}{d\rho} < 0 \tag{34}$$

and thus

$$\frac{dt^*}{d\rho} = -\frac{1}{\rho^2} \left( \frac{\alpha}{1-\alpha} - x_0 \right) - \frac{1}{\rho} \frac{dx_0}{d\rho} < 0.$$ 

We can also determine

$$\frac{dt^*}{dA} = \frac{1}{\rho} \left( -\frac{dx_0}{dA} \right) = -\frac{1}{\rho h'(x_0)} > 0$$

and therefore an increase of $\frac{S_0}{K_0}$ implies a larger $t^*$.

Note that the existence of a phase where consumption is increasing with time depends on the ration $\frac{S_0}{K_0}$ and not the absolute values of $S_0$ or $K_0$.

The peak of consumption can be expressed, after manipulations, as

$$C^* = \rho^{\frac{1}{1-\alpha}} \alpha^{-\frac{\alpha}{1-\alpha}} e^{\frac{\alpha}{1-\alpha}} \frac{S_0}{(e^{-x_0} - x_0 E(1,x_0))}$$

or in terms of the initial stock of capital as

$$C^* = \rho \left( \frac{\alpha}{1-\alpha} \right)^{-\frac{\alpha}{1-\alpha}} e^{\frac{\alpha}{1-\alpha}} K_0 \frac{1}{E(1,x_0)} \left( \frac{1}{x_0} \right)^{\frac{1}{1-\alpha}}.$$ 

Consider now the rate of consumption that solves the following growth problem

$$\max_{C} \{ \min U(C) \} \tag{35}$$

subject to (2)-(5). It can be shown (Solow, 1974) that the solution to this problem, referred to as the maximin rate of consumption, is

$$C_{\text{max min}} = \left( \frac{2\alpha - 1}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \frac{S_0}{\alpha} \left( \frac{1-\alpha}{K_0} \right)^{\frac{2\alpha-1}{\alpha}} \text{ where } \frac{1}{2} < \alpha < 1.$$ 

The utilitarian criterion’s main criticism is that it discounts future consumption heavily and gives a greater weight to present consumption. It is intuitive to think that the solution under such a criterion, which favors present consumption relative to future consumption, would result in larger initial consumption than a Consumption rate that would be sustained at all time. We show below that this may not be true. There exists a rate of pure time preference $\hat{\rho} > 0$ such that if $\rho < \hat{\rho}$ the initial rate of consumption is below the maximin rate of consumption.

The initial rate of consumption in the utilitarian framework is

$$C_0 \equiv C(0) = \frac{1}{\lambda_0} = \frac{\rho K_0}{x_0 e^{x_0} E(1,x_0)} \tag{36}$$
which can also be written as

\[ C_0 = \rho \left( S_0 (1 - \alpha) \frac{1}{\rho} \left( \frac{x_0}{\rho} \right)^{\frac{1}{\alpha}} + K_0 \right). \]  

(37)

Taking the derivative of \( C_0 \) with respect to \( \rho \) and using (34) we have

\[ \frac{dC_0}{d\rho} = \frac{C_0}{\rho} + \rho S_0 (1 - \alpha) \frac{x_0}{\rho} \left( \frac{x_0}{\rho} \right)^{-1} \frac{d}{d\rho} \left( \frac{x_0}{\rho} \right) > 0. \]  

(38)

The initial rate of consumption under a utilitarian objective is thus a continuous monotonically increasing function of \( \rho \). We now determine the \( \lim_{\rho \to 0} C_0 \) and \( \lim_{\rho \to \infty} C_0 \). These two limits are not straightforward
to determine from (36) since \( \lim_{\rho \to 0} x_0 = 0 \) (as was shown in the appendix I) and \( \lim_{\rho \to \infty} x_0 = \infty \). To
determine these two limits we use the property (12) of the exponential integral which after using a
property of the logarithmic function that

\[ \frac{x}{1 + x} < \ln (1 + x) < x \text{ for } x > -1, x \neq 0 \]
gives

\[ \frac{1}{x_0 + 2} < \frac{1}{2} x_0 \ln \left( 1 + \frac{2}{x_0} \right) < x_0 e^{x_0} \text{Ei}(1, x_0) < x_0 \ln \left( 1 + \frac{1}{x_0} \right) < 1 \]  

(39)

and thus

\[ \rho K_0 < C_0 = \frac{\rho K_0}{x_0 e^{x_0} \text{Ei}(1, x_0)} < (x_0 + 2) \rho K_0. \]  

(40)

Therefore \( \lim_{\rho \to 0} C_0 = 0 \) and \( \lim_{\rho \to \infty} C_0 = \infty \). This along with (38) allows to conclude that there exists
a unique rate of pure time preference \( \hat{\rho} > 0 \) such that \( C_0 < C_{\text{max min}} \) iff \( \rho < \hat{\rho} \). Note that similarly it can
be shown that \( \lim_{\rho \to 0} C_0 = 0 < C_{\text{max min}} \) and \( \lim_{\rho \to \infty} C_0 = \infty > C_{\text{max min}} \).

Sustainability in de DHSS context requires that the Hartwick rule holds: zero genuine savings, entail-
ing that the investment rate is equal to the extraction rate times the marginal product of the resource. Asheim (1994) shows that this condition should hold at all instants of time. Hence, if at some instant of
time Hartwick’s rule holds it does not mean that the economy is on a sustainable path. The argument
used by Asheim rests on the assumption that in a utilitarian optimum the initial rate of consumption is
below the maximin rate of consumption if the pure rate of time preference is low, and above the maximin
rate of consumption if the pure rate of time preference is high. By continuity there is a rate of time preference
for which both initial consumption rates coincide, and for which the utilitarian rate of consumption is
increasing for an initial period of time. Asheim (1994) refers to a graph in Dasgupta and Heal (1979)
to support this assumption with respect to the rates of time preference. Asheim’s example then shows
that at some instant of time along the interval where consumption is rising, the Hartwick rule is satisfied,
but obviously the rate of consumption at that instant of time is not sustainable. However, Dasgupta
and Heal do not provide a formal proof of their claim. Also, no proof is given of continuity. Pezzey and
Withagen (1998) provide an example where both claims are satisfied. In their example the production function is Cobb-Douglas with constant returns to scale and the elasticity of marginal utility is equal to the production elasticity of capital. This result is further elaborated upon by Hartwick et al. (2003), who make the same assumption with respect to these elasticities. They also derive a closed form solution for consumption in this particular case, which is not explicitly given by Pezzey and Withagen. We consider the case of an elasticity of marginal utility equal to unity and no restriction on the production elasticity of capital (except having it in the unit interval). In this case we are able to provide a proof to both claims as well. One of the contributions of the present paper is therefore that it provides another example justifying Asheim’s approach, for the case of constant returns to scale and a logarithmic instantaneous utility function. One could argue that the choice of a logarithmic utility function is rather restrictive. However, our choice of the parameter set allows for a sensitivity analysis with regard to the production elasticity, independent of the elasticity of marginal utility. Moreover, we have optimistic expectations regarding the possibility of generalizing our approach with respect to latter elasticity. Finally, we can give closed form solutions of all other relevant variables, which will be exploited in the next subsection.

5.2 The optimal investment path

We now turn to the optimal investment path. We show that the investment path can also be initially increasing over time and decline after reaching a peak and eventually become negative. To the best of our knowledge, this has not been shown in the literature. The optimal path of capital reaches a peak and then gradually declines over time: there is overshooting in human made capital. The stock of capital asymptotically converges to zero. We illustrate these possibilities by means of a numerical example: $S_0 = 3$, $K_0 = 1$, $\alpha = \frac{1}{2}$ and $\rho = 0.03$. We plot the optimal consumption and investment paths below.

It can be shown that, when $\alpha = \frac{1}{2}$, investment at time zero is equal to  
\[
I(0) = -\frac{1}{2} \frac{2e^{x_0}Ei(1, x_0) - x_0}{e^{x_0}Ei(1, x_0) - \frac{1}{2} S_0}
\]  
From (39) we know the denominator is negative. Therefore, we have positive investment at time zero if  
\[
e^{x_0}Ei(1, x_0) > \frac{1}{2}
\]  
We have  
\[
\frac{d(e^{x_0}Ei(1, x_0))}{dx_0} = -\frac{1}{x_0} + e^{x_0}Ei(1, x_0) < 0
\]  
from (39). Therefore, there exists $\bar{x}_0 \approx 1.289$ such that $I(0) > 0$ iff $x_0 < \bar{x}_0$, i.e. iff $\rho$ or $K_0$ small enough or $S_0$ large enough.
6 Conclusion

We gave a closed form solution to the seminal model of endogenous growth with exhaustible resources, based on Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974). For this two state variables optimal control problem, we give a closed form representation of the dynamics of all the variables in the model and from all possible initial values of the state variables. We establish several features that the solution may exhibit. We determine the condition under which the consumption is initially increasing with time and the condition under which initial investment is positive. We provide an example where investment is initially positive increasing with time, reaches a maximum, declines and eventually becomes negative.
Appendices

Appendix A: Proof of Lemma 1

We compute the derivative of the function \( h \) and find

\[
h'(x_0) = \frac{(e^{-x_0})^2 (1 - \alpha) - (e^{-x_0} - x_0 (1 - \alpha) + e^{-x_0}) \text{Ei}(1, x_0) + (\text{Ei}(1, x_0))^2 x_0 \alpha}{(\text{Ei}(1, x_0))^2 x_0^{x_0 + 1} (1 - \alpha)}
\]

The sign of the denominator is positive. Therefore the sign of \( h'(x_0) \) is the same as the sign of the numerator, denoted by \( N(x_0) \). Using the property of the exponential integral that

\[
\frac{1}{2} e^{-x_0} \ln \left( 1 + \frac{2}{x_0} \right) < \text{Ei}(1, x_0) < e^{-x_0} \ln \left( 1 + \frac{1}{x_0} \right)
\]

it holds for all \( x_0 \in (0, \infty) \) that

\[
N(x_0) \leq e^{-2x_0} \left( 1 - \alpha - (x_0(1-\alpha) + 1) \frac{1}{2} \ln \left( 1 + \frac{2}{x_0} \right) + \left( \ln \left( 1 + \frac{1}{x_0} \right) \right)^2 x_0 \alpha \right)
\]

For the derivative of \( Z(x_0) \) we get

\[
Z'(x_0) = -\frac{1}{2} (1 - \alpha) \ln(1 + \frac{2}{x_0}) + \frac{x_0 (1 - \alpha) + 1}{x_0 (x_0 + 2)} + \alpha \left( \ln(1 + \frac{1}{x_0}) \right)^2 - 2 \alpha - \frac{1}{1 + x_0} \ln(1 + \frac{1}{x_0})
\]

The second derivative is

\[
Z''(x_0) = -2 \frac{2x_0 - 3x_0 \alpha + x_0^2 - 2x_0^2 \alpha + (4x_0 \alpha + 4x_0^2 \alpha + x_0^3 \alpha) \ln \left( \frac{1}{x_0} (x_0 + 1) \right) + 1}{(x_0 + 1)^2 (x_0 + 2)^2 x_0^2}
\]

Using

\[
\frac{1}{1 + \frac{1}{x_0}} < \ln \left( 1 + \frac{1}{x_0} \right) < \frac{1}{x_0}
\]

we have

\[
\left( 2x_0 - 3x_0 \alpha + x_0^2 - 2x_0^2 \alpha + (4x_0 \alpha + 4x_0^2 \alpha + x_0^3 \alpha) \ln \left( \frac{1}{x_0} (x_0 + 1) \right) + 1 \right)
\]

and

\[
\left( 2x_0 - 3x_0 \alpha + x_0^2 - 2x_0^2 \alpha + (4x_0 \alpha + 4x_0^2 \alpha + x_0^3 \alpha) \frac{1}{1 + \frac{1}{x_0}} + 1 \right)
\]
and thus
\[ Z''(x_0) < (-2) \frac{\left( 2x_0 - 3x_0 \alpha + x_0^2 - 2x_0^2 \alpha + (4x_0 \alpha + 4x_0^2 \alpha + x_0^3 \alpha) \frac{1}{1 + \alpha} + 1 \right)}{(x_0 + 1)^2 (x_0 + 2)^2 x_0^2} \]
which after simplifications becomes
\[ Z''(x_0) < (-2) \frac{(3 + \alpha) x_0 + (3 - \alpha) x_0^3 + x_0^2 (1 - \alpha) + 1}{(x_0 + 1)^4 (x_0 + 2)^2 x_0^2} < 0 \]
So \( Z''(x_0) < 0 \) for all \( \alpha \in [0, 1] \) and all \( x_0 \in (0, \infty) \) and therefore
\[ Z'(x_0) \in (Z' (\infty), Z' (0)) \]
with
\[ \lim_{x_0 \to 1} (Z' (x_0)) = 0 \text{ and } \lim_{x_0 \to 0^+} (Z' (x_0)) = \infty \]
that is \( Z'(x_0) > 0 \) for all \( x_0 \in (0, \infty) \) and therefore
\[ Z(x_0) \in (Z(0), Z(\infty)) \]
with
\[ \lim_{x_0 \to \infty} (Z(x_0)) = 0 \text{ and } \lim_{x_0 \to 0^+} (Z(x_0)) = -\infty \]
that is
\[ \lim_{x_0 \to \infty} (Z(x_0)) = 0 \]
\( Z(x_0) < 0 \) all \( x_0 \in (0, \infty) \) and therefore \( N(x_0) < 0 \) and \( h(x_0) < 0 \) all \( x_0 \in (0, \infty) \).

To complete the proof it can be checked that
\[ \lim_{x_0 \to \infty} (h(x_0)) = 0 \text{ and } \lim_{x_0 \to 0^+} (h(x_0)) = \infty \]
Appendix B: Proof of Lemma 2

Rewriting (24) to get
\[ \dot{\lambda} = \lambda \left( \rho - \alpha \left( (KR^{-1})^{\alpha} \right)^{\frac{\alpha - 1}{\alpha}} \right) \]

After substitution of \( KR^\alpha \) from 23 into 24 and using the definitions of \( \kappa \) and \( \sigma \) we have
\[ \dot{\lambda}(t) = \rho \lambda(t) - \lambda(t)\alpha \left( \frac{\mu(t)}{(1 - \alpha)\lambda(t)} \right)^{\frac{\alpha - 1}{\alpha}} = \rho \lambda(t) - \lambda(t)\alpha \left( \frac{\kappa}{\sigma(t)} \right)^{\frac{\alpha - 1}{\alpha}} \]

Hence
\[ \frac{\dot{\sigma}(t)}{\sigma(t)} = -\rho + \frac{\dot{\lambda}(t)}{\lambda(t)} = -\alpha \left( \frac{\kappa}{\sigma(t)} \right)^{\frac{\alpha - 1}{\alpha}} \]

The solution reads
\[ \sigma(t) = \left( \sigma_0^{1 - \frac{1}{\alpha}} + \varphi t \right)^{\frac{1}{\alpha}} \]

as is easily verified taking into consideration that \( \kappa \equiv \frac{\mu_0}{1 - \alpha} \) and \( \varphi \equiv (1 - \alpha)\kappa^{\frac{\alpha - 1}{\alpha}} \)
Appendix C: Proof of Lemma 3

Using 22 we have

\[
\left(\frac{1}{\lambda} + I\right)K^{-\alpha} \frac{1}{1 - \alpha} = R
\]

Along with \(\mu = (1 - \alpha)K^\alpha R^{-\alpha} \lambda\) from 23 we have

\[
\frac{\kappa}{\sigma(t)} = K^\alpha R^{-\alpha} = K^\alpha \left(\left(\frac{1}{\lambda} + I\right)K^{-\alpha} \frac{1}{1 - \alpha}\right)^{-\alpha}
\]

\[
= \left(\frac{1}{\lambda} + \hat{K}\right)^{-\alpha} K^{\frac{1}{1 - \alpha}}
\]

Hence

\[
\left(\frac{\kappa}{\sigma(t)}\right)^{\frac{1 - \alpha}{\alpha}} = \left(\frac{1}{\lambda} + \hat{K}\right)^{-1} K
\]

and

\[
g(t) = -\frac{1}{\lambda} = \hat{K} - K \left(\frac{\kappa}{\sigma(t)}\right)^{\frac{1 - \alpha}{\alpha}} = \hat{K} - K f(t)
\]
Appendix D: Proof of Lemma 4

Indeed \( e^{-\int_0^t f(z)dz} \) is a solution to the differential equation \( \dot{K} + f(t)K = 0 \). And \( \int_0^t g(z) e^{-\int_0^t f(s)ds}dz \) is a particular solution to the equation \( \dot{K} + f(t)K = g(t) \), as can be seen as follows.

\[
\left( \int_0^t g(z) e^{-\int_0^t f(s)ds}dz \right)' = g(t) + \left( \int_0^t g(z) (-f(t)) e^{-\int_0^t f(s)ds}dz \right)
= g(t) - f(t) \left( \int_0^t g(z) e^{-\int_0^t f(s)ds}dz \right)
\]
Appendix E: Proof of Lemma 5

\[ \int_0^t f(z) \, dz = - \int_0^t \left( \frac{\kappa}{\sigma(z)} \right)^{-\frac{1-\alpha}{\alpha}} \, dz \]

\[ = -\kappa^{\frac{1-\alpha}{\alpha}} \int_0^t \frac{1}{\pi(z)} \, dz \]

Integrating gives

\[ \int_0^t f(z) \, dz = -\kappa^{\frac{1-\alpha}{\alpha}} \left( \frac{\ln \pi(t)}{\varphi} - \frac{\ln \pi(0)}{\varphi} \right) \]

and

\[ \int_z^t f(z) \, dz = -\kappa^{\frac{1-\alpha}{\alpha}} \left( \frac{\ln \pi(t)}{\varphi} - \frac{\ln \pi(z)}{\varphi} \right) \]

\[ = -\kappa^{\frac{1-\alpha}{\alpha}} \frac{1}{\varphi} \ln \frac{\pi(t)}{\pi(z)} \]

\[ = -\ln \left( \frac{\pi(t)}{\pi(z)} \right)^{\frac{1}{\alpha}} \]

since

\[ \kappa^{\frac{1-\alpha}{\alpha}} \frac{1}{\varphi} = \frac{1}{1-\alpha} \]
Appendix F: Proof of Lemma 6

We have from the previous lemma

\[ e^{-\int_{t}^{t} f(s)ds} = \left( \frac{\pi(t)}{\pi(z)} \right)^{\frac{1}{\rho}} \]

Hence, since \( g(t) = -\frac{1}{\lambda(t)} \) with \( \lambda(t) = e^{\rho(t)} \), it holds that

\[
\int_{0}^{t} g(z) e^{-\int_{z}^{t} f(s)ds} dz = \int_{0}^{t} -\frac{1}{e^{\rho z} \pi(z)} \left( \frac{\pi(t)}{\pi(z)} \right)^{\frac{1}{\rho}} dz
\]

\[
= -\pi(t)^{\frac{1}{\rho}} \int_{0}^{t} e^{-\rho z} \pi(z)^{-1} dz
\]

We have \( z = \frac{\pi(z) - \pi(0)}{\varphi} \) and \( d\pi(z) = \varphi dz \). Hence

\[
\int_{0}^{t} g(z) e^{-\int_{z}^{t} f(s)ds} dz = -\pi(t)^{\frac{1}{\rho}} \int_{\pi(0)}^{\pi(t)} e^{-\rho \frac{\pi(z) - \pi(0)}{\varphi}} \pi(z)^{-1} \frac{d\pi(z)}{\varphi}
\]

\[
= -\frac{1}{\varphi} e^{\rho \pi(0)} \pi(t)^{\frac{1}{\rho}} \int_{\pi(0)}^{\pi(t)} e^{-\frac{\pi(z) - \pi(0)}{\varphi}} u^{-1} du
\]

\[
= -\frac{1}{\varphi} e^{\rho \pi(0)} \pi(t)^{\frac{1}{\rho}} \int_{\pi(0)}^{\pi(t)} e^{-\frac{\varphi}{\rho} u^{-1}} du
\]

\[
= -\frac{1}{\varphi} e^{\rho \pi(0)} \pi(t)^{\frac{1}{\rho}} \int_{\pi(0)}^{\pi(t)} e^{-y} y^{-1} du
\]

\[
\blacksquare
\]
Appendix G: Proof of Proposition 4

\[ S(t) = S(0) - \int_0^t \left( \frac{1}{\kappa} \right)^{1/\alpha} \left( K_0 \left( \frac{1}{\pi(0)} \right)^{1/\alpha} - \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x_0) - \text{Ei}(1, x(z))) \right) \, dz \]

\[ S(t) - S(0) = -\left( \frac{1}{\kappa} \right)^{1/\alpha} \int_0^t \left( K_0 \left( \frac{1}{\pi(0)} \right)^{1/\alpha} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) \, dz \]

\[ - \left( \frac{1}{\kappa} \right)^{1/\alpha} \frac{1}{\varphi} e^{x(t)} \int_0^t \text{Ei}(1, x(z)) \, dz \]

\[ = - \left( \frac{1}{\kappa} \right)^{1/\alpha} \left( K_0 \left( \frac{1}{\pi(0)} \right)^{1/\alpha} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) t \]

\[ - \left( \frac{1}{\kappa} \right)^{1/\alpha} \frac{1}{\varphi} e^{x_0} \int_0^t \text{Ei}(1, x(z)) \, dz \]

We first evaluate

\[ \int_0^t \text{Ei}(1, x(z)) \, dz \]

The antiderivative of the exponential integral can be written as

\[ \int_0^t \text{Ei}(1, a + bu) \, du = \frac{e^{-a} - e^{-(a+bt)}}{b} + (a + bt) \text{Ei}(1, a + bt) - a \text{Ei}(1, a) \]

and therefore

\[ \int_0^t \text{Ei}(1, x_0 + \rho z) \, dz = \frac{1}{\rho} (e^{-x_0} (1 - e^{-\rho t})) + \frac{1}{\rho} (x_0 + \rho t) \text{Ei}(1, x_0 + \rho t) - \frac{1}{\rho} x_0 \text{Ei}(1, x_0) \]

We can now write the path of the stock of resource along the optimal trajectory

\[ S(t) = S(0) - \left( \frac{1}{\kappa} \right)^{1/\alpha} \left( K_0 \left( \frac{1}{\pi(0)} \right)^{1/\alpha} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) t \]

\[ - \left( \frac{1}{\kappa} \right)^{1/\alpha} \frac{1}{\varphi} (1 - e^{-\rho t}) \]

\[ - \left( \frac{1}{\kappa} \right)^{1/\alpha} \frac{1}{\varphi} e^{x_0} (x_0 + \rho t) \text{Ei}(1, x_0 + \rho t) - x_0 \text{Ei}(1, x_0) \]

\[ \blacksquare \]
Appendix H: Proof of Lemma 7

Since
\[ e^{-\rho t} \lambda(t) \pi(t) \frac{1}{\varphi} = \pi(t) \]
the transversality condition for capital boils down to
\[ \lim_{t \to \infty} \pi(t) \left( K_0(\frac{1}{\pi(0)}) \frac{1}{\varphi} - \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x_0) - \text{Ei}(1, x(t))) \right) = 0 \]
We evaluate
\[ \lim_{t \to \infty} \pi(t) \left( \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x(t))) \right) = \lim_{t \to \infty} \frac{\varphi x(t)}{\rho} \left( \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x(t))) \right). \]

It can be shown that
\[ \lim_{z \to \infty} \left( z \frac{1}{\varphi} e^{x_0} \text{Ei}(1, z) \right) = 0 \]
Therefore the transversality condition gives
\[ \lim_{t \to \infty} \pi(t) \left( K_0(\frac{1}{\pi(0)}) \frac{1}{\varphi} - \frac{1}{\varphi} e^{x_0} (\text{Ei}(1, x_0)) \right) = 0 \]
This is satisfied if
\[ K_0(\frac{1}{\pi(0)}) \frac{1}{\varphi} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) = 0 \]
Written in terms of \((\mu_0, \lambda_0)\) this gives (32).

To show (31) we use the condition (11) which implies exhaustion of the resource stock, that is

\[ -S(0) = - \lim_{t \to \infty} \left( \frac{1}{\kappa} \right)^\frac{1}{2} \left( K_0(\frac{1}{\pi(0)}) \frac{1}{\varphi} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) t \]
\[ - \lim_{t \to \infty} \left( \frac{1}{\kappa} \right)^\frac{1}{2} \frac{1}{\varphi \rho} (1 - e^{-\rho t}) \]
\[ - \lim_{t \to \infty} \left( \frac{1}{\kappa} \right)^\frac{1}{2} \frac{1}{\varphi \rho} e^{x_0} (x_0 + \rho t) \text{Ei}(1, x_0 + \rho t) \]
\[ + \lim_{t \to \infty} \left( \frac{1}{\kappa} \right)^\frac{1}{2} \frac{1}{\varphi \rho} e^{x_0} x_0 \text{Ei}(1, x_0) \]

We have
\[ \lim_{t \to \infty} (x_0 + \rho t) \text{Ei}(1, x_0 + \rho t) = 0 \]
and
\[ \lim_{t \to \infty} e^{x_0} (e^{-x_0} (1 - e^{-\rho t}) - x_0 Ei(1, x_0)) = e^{x_0} (e^{-x_0} - x_0 Ei(1, x_0)) \]
So the transversality condition (11) above becomes

\[
S(0) = - \lim_{t \to \infty} \left( \frac{1}{\kappa} \right)^{1/2} \left( K_0 \left( \frac{1}{\pi(0)} \right) \frac{1}{\varphi t} - \frac{1}{\varphi} e^{x_0} \text{Ei}(1, x_0) \right) t
\]

\[
- \lim \left( \frac{1}{\kappa} \right)^{1/2} \frac{1}{\varphi \rho} e^{x_0} \left( e^{-x_0} - x_0 \text{Ei}(1, x_0) \right)
\]

Using (32) from (10), (11) gives (31) ■
Appendix I: Proof of Lemma 8

We have from the Lemma 7

\[ \frac{1}{\mu_0} e^{x_0} \frac{e^{-x_0} - x_0 \text{Ei}(1, x_0)}{\rho} = S_0 \]

\[ K_0 = \frac{(\pi(0))^{\frac{1}{\alpha}}}{(\mu_0)^{\frac{\alpha-1}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}}} e^{x_0} \text{Ei}(1, x_0) = 0 \]

This gives after substitution of \( \pi(0) \)

\[ \mu_0 = \frac{e^{x_0} (e^{-x_0} - x_0 \text{Ei}(1, x_0))}{\rho S_0} \]

\[ \sigma_0 = \frac{x_0}{\rho K_0} e^{x_0} \text{Ei}(1, x_0) \]

Hence the ratio \( \frac{\sigma_0}{\mu_0} \) is

\[ \frac{\sigma_0}{\mu_0} = \frac{S_0 x_0 \text{Ei}(1, x_0)}{K_0 (e^{-x_0} - x_0 \text{Ei}(1, x_0))} \]

So \( x_0 \) is the solution to this nonlinear equation. Hence

\[ \left( \frac{\sigma_0}{\mu_0} \right)^{1-\frac{1}{\alpha}} = \left( \frac{S_0 x_0 \text{Ei}(1, x_0)}{K_0 (e^{-x_0} - x_0 \text{Ei}(1, x_0))} \right)^{1-\frac{1}{\alpha}} \quad (41) \]

We now use (15) to write \( \frac{\sigma_0}{\mu_0} \) in terms of \( x_0 \). Indeed we have

\[ x_0 \equiv x(0) = \frac{\rho \pi(0)}{\varphi} \]

which after substitution of \( \pi(0) \) and \( \varphi \) gives \( \frac{\sigma_0}{\mu_0} = \frac{x_0 (1-\alpha)^{\frac{1}{\alpha}}}{\rho} \). Thus (41) gives

\[ \frac{x_0 (1-\alpha)^{\frac{1}{\alpha}}}{\rho} = \left( \frac{S_0 x_0 \text{Ei}(1, x_0)}{K_0 (e^{-x_0} - x_0 \text{Ei}(1, x_0))} \right)^{1-\frac{1}{\alpha}} \]

which becomes

\[ h(x_0) = x_0^{\frac{\alpha}{\alpha-1}} \left( \frac{1}{x_0 e^{x_0} \text{Ei}(1, x_0)} - 1 \right) = \frac{S_0}{K_0} \left( \frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho} \right)^{1-\frac{1}{\alpha}} = A \]

25
References


