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Poverty and sustainable development around the world during transition periods[☆]

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ABSTRACT

This paper addresses the question of whether countries can escape from poverty in a sustainable manner. For this purpose, we introduce an endogenous growth model that incorporates human, physical, and natural capital, as well as subsistence consumption. We provide a closed-form solution of the model to exploit the entire transitional phase of countries with unequal initial endowments of human capital and natural resources. We calibrate this model for 108 countries using data from the World Bank on countries' physical capital and natural resource endowments. Using a battery of established consumption-based sustainability tests, we assess sustainability during the transition towards the economies' steady-state. We find that most countries are characterized by sustainable development. For those countries not qualifying for sustainable development, we are able to quantify by how much initial capital endowments fall short of minimum requirements implied by sustainability tests.

1. Introduction

Given they are endowed with natural resources, is it possible that countries can escape from poverty in a sustainable manner? This paper addresses this question — both, theoretically as well as empirically, by focusing on sustainability during transition periods in the presence of subsistence needs for a sample of 108 countries.

An answer to this question is particularly relevant for resource-rich but otherwise poor countries.[1] The relevant literature suggests many possible sustainability tests to judge whether or not an economy behaves sustainably. Based on an optimal growth model, we employ these tests but we exploit the entire development path of countries with unequal initial endowments of human capital and natural resources. Controlling for possibly false signals of the tests, along the adjustment path, we find that most countries in our sample behave in a sustainable manner.

As stressed by the development literature, savings in developing countries is determined not only by the willingness to save but also by the ability to save (e.g. see Steger, 2000). Assuming a constant-intertemporal-elasticity-of-substitution (CIES) formulation of preferences abstracts from the requirement of a minimum consumption level that restricts the possibilities to substitute consumption between time periods, and, hence, to save. To shed light on our above raised question, it is necessary to account for such minimum consumption requirements. This idea is supported by (Pezzy and Anderies, 2003), who stated that ignoring subsistence consumption in a resource-economic modeling environment might lead to a biased analysis of institutional evolutions and biased policy recommendations.

To systematically reconcile sustainable development with escape from poverty, we introduce an endogenous growth model with the following attributes: First, we introduce a human capital sector following the famous Uzawa–Lucas endogenous growth model (Uzawa, 1965; Lucas, 1988), where physical and human capital are subject to depreciation whereas population of the economy grows at a constant exponential rate. Second, we allow for minimum consumption requirements by introducing Stone–Geary preferences (Stone, 1954; Geary, 1950) to acknowledge the possibility that some (resource-rich) countries are restricted by the ability to save.[2] Third, we link the economy

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[1] Venables (2016) reports on numbers issued by the IMF that 29 out of 51 resource-rich economies belong to the group of low or lower-middle income countries.

[2] Alternatively, we might be thinking of habit formation (Carroll et al., 2000).

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to the resource sphere by borrowing elements from the continuous time Dasgupta–Heal–Solow–Stiglitz model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974, DHSS from here on).

We show that our model allows for a closed form solution that enables us to calibrate the economy's transition towards its steady-state using country specific data.[3] This is necessary to investigate the possible sustainable evolution at the beginning of an economy's development phase. We are, thus, able to exploit the entire dynamics of the resource-economy environment starting from its initial endowment. In turn, using a model linearized around the economy's steady-state would not allow for this. The analytical solutions allow us to apply several sustainability tests along the adjustment path.

The sustainability tests that we conduct are consumption based. First, it is the ability of realizing a minimum subsistence level of consumption. Second, we regard a non-decreasing consumption profile as a sustainability criterion. A further contribution of our paper is that we expand Weitzman's (1976, 1997) sustainability test. Weitzman's (1976, 1997) approach has been originally developed for an environment with a constant interest rate, and is therefore limited to steady-state analyses. We adopt Weitzman's approach by transferring the implied sustainability test to the case of a non-constant interest rate during adjustment periods. The motivation for tracing out the required quantities in our model theoretically is to conduct the tests on the calibrated model and to draw conclusions regarding sustainability for particular countries.

We calibrate the model for 108 countries using mainly data from the World Bank's Wealth of Nations database.[4] The data provides us with information on countries' natural resource, physical and human capital endowments.

Our key findings are as follows: 98 out of 108 countries are equipped with sufficient initial endowments of resources, physical and/or human capital to permanently allow subsistence consumption at the least. We choose the World Bank's poverty lines as a measure for this level of subsistence consumption[5]

The countries with insufficient initial endowments are mostly low-income countries. We find further, that 91 out of these 98 economies qualify for positive long-run growth while 7 converge to a zero growth scenario where households can just afford minimum subsistence consumption. Moreover, we find that the model's predictions for most countries in our sample are characterized by sustainable development with respect to consumption growth and Weitzman's (1976, 1997) sustainability test. In the former case, the whole trajectory for welfare maximizing consumption towards the steady-state exhibits non-negative growth rates. In the latter case, consumption is permanently below a consumption annuity level that is equal in its present value.

The structure of the paper is as follows. The next section reviews the literature relevant to our contribution. Section 3 lays out the economic problem that we aim to solve and provides the theoretical results. Section 4 presents the calibration and the results of the sustainability test. Section 5 discusses our findings and, finally, Section 6 concludes.

2. Review of literature

Subsistence consumption that enables individuals to meet their minimum basic needs of life, has been discussed frequently in an economic growth context. Two papers closely related to this one are Steger (2000) and Strulik (2010). Both solve a utility maximization problem with Stone–Geary preferences but with a standard AK-type production technology. Their models are nested in ours if one is setting the output elasticity of the resource equal to zero and taking no account of human capital accumulation. However, these settings ignore that many developing low-income countries are resource rich (Barbier, 2005), facing substantial development needs (see Araujo et al., 2016) and grow less rapidly (Gaitan and Roe, 2012).

Another strand of literature has extensively analyzed the DHSS framework under some specific assumptions. The Cobb Douglas constant returns to scale production structure with reproducible man-made capital and resource input has been employed by Benchekroun and Withagen (2011), Asheim and Buchholz (2004) and others. Mitra et al. (2013) employ a general constant returns to scale technology with reproducible man-made capital and resources. input.[6] Antony and Klarl (2019a,b) introduce minimum subsistence consumption in a utilitarian approach into the DHSS model.

Our approach is also related to the nexus between resources, growth, and development. First, as inter alia argued by Collier et al. (2010) and van der Ploeg and Venables (2011), because of human as well as physical capital scarcity, many resource rich developing countries should use resource rent windfalls to speed up development by accumulating capital. They argue that capital scarcity implies higher return on domestic capital. Hence, it might be beneficial to invest in human and physical capital than investing abroad. However, empirical findings in Venables (2016) seem to suggest that this is not happening. We add to this literature by asking the underlying question whether initial endowments of physical and human capital together with initial resource stocks are sufficient to allow the permanent realization of subsistence level consumption at the least. If yes, we proceed by investigating how far can development take countries above this subsistence level.

Accumulation of knowledge in terms of human capital together with non-renewable resource depletion has been also analyzed in, among others, Barbier (1999), Scholz and Ziemes (1999), Groth and Schou (2002), Schou (2000), Grimaud and Rouge (2003) and Bretschger and Smulders (2012), while the relationship between natural resource abundance and human capital development is discussed by Leamer et al. (1999), Gylfason (2001) or Stijns (2006).

A recent strand of the literature analyzes the implications of non-renewable resources in an environment with directed technical change. The most relevant contributions are Di Maria and Valente (2008), Acemoglu et al. (2012) and André and Smulders (2014). Although the economic set-up in this series of contributions differs from ours, these paper stress the importance of incentives to produce new ideas that are able to augment a scarce non-renewable resource.[7] We are dealing with the problem of efficient human capital accumulation that is increasing the marginal productivity of a non-renewable resource. Effort can be allocated to final goods production or to human capital accumulation and, hence, we are dealing with the endogenous direction of such activities.

Acemoglu et al. (2012) and André and Smulders (2014) are calibrating their theoretical models to trace out their dynamic implications reflecting particular scenarios. While Acemoglu et al. (2012) focus on optimal policies regarding taxes and subsidies and their implications for R&D, André and Smulders (2014) use their calibration to understand general stylized facts on the treatment of non-renewable resources. Our focus differs as we want to calibrate our model to particular country scenarios. This allows us to draw more specific conclusions

[3] The R code and the calibration data are available from the authors upon request.

[4] The database is available at https://datacatalog.worldbank.org/dataset/wealth-accounting.

[5] Hence, we interpret subsistence as a consumption mode that is directly related the concept of the poverty line that in turn is used to identify the fraction of the population that is absolutely poor (see Steger, 2000).

[6] See Antony and Klarl (2019a,b) for a more detailed review of this literature.

[7] The set-up in these contribution analyzes an economy with a clean and a dirty production sector where only the latter uses non-renewable resources such as fossil fuels while both sectors can substitute for each other. In our set-up, production always uses non-renewable resources and consequently, the economy is more dependent on them.

at the country level, in particular on quantities often used to judge sustainability.

Focusing on periods after the industrial revolution (where subsistence consumption might be still a relevant issue), we are totally aware of the literature that explain the historical phases of development. This development starts from the Malthusian Stagnation to the Industrial Revolution and ends with sustained growth of per capita income (Galor and Weil, 2000; Galor, 2005, 2011). Similar to our setting, in the benchmark unified growth theory model, population growth affects natural resource scarcity (and labor productivity). Focusing on population dynamics and resource scarcity, Peretto and Valente (2015) discuss a Schumpetarian growth model with endogenous fertility. Inter alia, they show that if labor and resources are substitutes (complements), the economy can be described to be in a steady-state in the long-run (ends up with a demographic explosion or collapse).[8] The economy might converge to three possible steady-states: First, the economy grows at positive rates in the long-run. Consumption and production per capita grow without bound and tend to infinity. Second, the economy realizes the subsistence level and there is no long-run growth. Third, the economy follows a long-run zero growth scenario where per capita production and consumption is above the subsistence level. This reflects the implication of a weak scale effect of population size on human capital in the Uzawa–Lucas model (Jones, 2005) that helps overcoming Malthusian stagnation.

We add to this literature as our focus is on the question whether economies that escape from poverty are able to do so in a sustainable manner. Typically, the aforementioned contributions focus on the steady-state behavior and linearization techniques to discuss transitional dynamics (close to the steady-state). However, this strategy comes at the cost that some interesting dynamics during the adjustment phase are neglected, particularly during the early stages of development. To overcome this caveat, we derive a closed-form solution of our model along the entire optimal trajectory.[9]

We add further to the discussion on the appropriateness of competing measures of sustainability. Regarding the concept of sustainability, we apply several approaches. First, we follow Holden et al. (2014) who give an interpretation of the Brundtland Report (WCED, 1987) that regards a development as sustainable if basic human needs are guaranteed in an intergenerational way. Second, we are investigating consumption growth. Asheim and Buchholz (2004) regard any consumption path characterized by non-decreasing consumption as sustainable.

Finally, exploiting the closed-form solution of our model, we extend Weitzman's (1997) sustainability test on the economy's consumption pattern for the case of a non-constant interest rate.[10] Recently Cairns and Martinet (2014, 2021) pioneered the concept of the sacrifice which we apply to the two last mentioned tests. The sacrifice can be seen as a foregone quantity that would be potentially available without endangering the goal of sustainability traded in for future returns. We calibrate our model and investigate the mentioned sustainability concepts using data for 108 countries largely drawn from the World Bank's data bases. In contrast to e.g. Rodgríguez and Sachs (1999) who calibrate their model only for Venezuela, we are able to draw conclusions about possible sustainable development for the majority of countries in the world.

[8] Further relevant contributions on economic development include Smulders (2005), Xepapadeas (2005), Schäfer (2014) and Bretschger (2013).

[9] We do so by making use of the Gaussian hypergeometric function. This approach has been used in economics by e.g. Boucekkine and Ruiz-Tamarit (2008), Boucekkine et al. (2008), Ruiz-Tamarit (2008), Hiraguchi (2009, 2014), Guerrini (2010) and Perez-Barahona (2011).

[10] Weitzman (1997) mentions that besides a constant interest rate, other discount rates should be considered, but "he would hate to be the one who has to make such recalculations in practice" Weitzman (1997, fn. 6, p. 6).

3. The model

In this section, we lay out the intertemporal utilitarian problem that we aim to solve. Preliminary calculations that are helpful in finding a solution to the problem together with necessary conditions for its existence are presented. The aim of this section is to derive a closed-form solution of the model.

3.1. The optimization problem

The economy is populated by a mass 1 of infinitively living representative households with the following Stone–Geary intertemporal utility function

$$U_t = \int_0^\infty \frac{(c_t - \underline{c})^{1-\eta} - 1}{1-\eta} L_t e^{-\rho t} dt, \tag{1}$$

where c_t is consumption per capita at time t, $\underline{c}$ is the minimum subsistence level of consumption, $\eta > 0$ and $\rho > 0$ is the rate of time preference. $L_t = L_0 e^{nt}$ is the household size at time t which is growing at rate n. We will refer to $c_t - \underline{c}$ as the excess consumption in the sense that is taking place in excess of a subsistence level $\underline{c}$.

We consider a social planer to maximize households' lifetime utility given the relevant budget constraints. These constraints are given by the accumulation of reproducible physical capital, the accumulation of human capital, and by the use of a non-renewable resource that is necessary for production.

We assume that production is given by the aggregate Cobb–Douglas production technology

$$Y_t = A K_t^\alpha (H_t u_t L_t)^\beta R_t^\gamma, \tag{2}$$

where K_t denotes the stock of physical capital and H_t is the level of human capital. Each household member supplies inelastically one unit of raw labor of which the fraction u_t is employed in the final goods production. Total effective labor input into final goods production is therefore $H_t u_t L_t$. R_t is the use and extraction of the resource. We assume constant returns to scale, i.e. $\alpha + \beta + \gamma = 1$, and $0 < \alpha, \beta, \gamma < 1$. A denotes a constant level of total factor productivity. (2) shows the potential of long-run growth in case human and physical capital accumulation occurs fast enough to compensate for the scarcity problem reflected by the presence of the non-renewable resource.[11]

Physical capital is produced from foregone final output with unit productivity and depreciates at a rate $\delta_1 > 0$. The net increase in the stock of reproducible capital is therefore

$$\frac{\partial K_t}{\partial t} = \dot{K}_t = Y_t - C_t - \delta_1 K_t. \tag{3}$$

Human capital is accumulated by foregone labor supply in production of final output

$$\dot{H}_t = B(1 - u_t)H_t - \delta_2 H_t, \tag{4}$$

where $B > 0$ is a constant productivity parameter, $\delta_2 \geq 0$ is the constant rate of depreciation of human capital and $(1 - u_t)$ is the fraction of labor supply not used in final goods production but spent on learning and accumulation of human capital. (4) is linear in H_t which corresponds to the preferred specification in Lucas (1988).

The model uses human capital accumulation as the engine that allows for permanent positive growth. As an alternative, one could build on the (semi-)endogenous growth literature and where growth

[11] Note that the production function (2) takes no account of any aggregate human capital externality in the sense of Lucas (1988). For such an externality, (2) would have to be multiplied by an additional factor $H_{a,t}^\gamma$ with $\gamma > 0$ and $H_{a,t}$ denoting an aggregate stock of human capital which would be identical to H_t in an analysis focusing on a centralized solution. However, even without this type of aggregate externality, the economy is still fully capable of producing long-run positive growth.

originates from embodied technical change. Typically, this would stress the importance of research and development (R&D) in intermediate input factors required for final good production. Human capital accumulation and embodied technological change are undoubtedly different types of development. From the mathematical perspective, however, the present representation also captures dynamics that can be produced by (semi-)endogenous growth models.

The following representation of our set-up motivates this. Let final output be given by[12]

$$Y_t = A L_{Y,t}^{\beta} \int_0^{H_t} x_{i,t}^{1-\beta} di,$$
$$\text{with} \quad x_{i,t} = K_{i,t}^{\mu} R_{i,t}^{1-\mu}, 0<\mu<1, K_t = \int_0^{H_t} K_{i,t} di, R_t = \int_0^{H_t} R_{i,t},$$

and the development of H_t by

$$\dot{H}_t = H_t \frac{L_{H,t}}{L_t} - \delta_2 H_t. \tag{5}$$

Now, labor is distributed across final output production, $L_{Y,t} = u_t L_t$, and R&D activities, $L_{H,t} = (1-u_t)L_t$, to promote H_t as in many (semi-)endogenous growth models (see e.g. Jones, 1999). Consequently, H_t would now be interpreted as the level of technology embodied in horizontally differentiated intermediates and progress would take place by increasing this variety of inputs. In our setting, intermediate input factors would have to be produced using physical capital and the non-renewable resource with a constant returns to scale Cobb–Douglas technology. This representation is obviously one that is familiar from the literature on (semi-)endogenous growth.

The R&D Eq. (5) is a modification of the basic formulation of the production function for ideas in Romer (1987, 1990). The modification follows the approach in Barro and Sala-i-Martin (2004) that has also been discussed already by Jones (1995a) and eliminates scale effects originating from population size in the steady-state growth rate of the economy.[13] Summarizing the argument, the chosen Uzawa–Lucas set-up is also able to pick up dynamics that are known from the modern semi-endogenous growth literature. This is important for our purpose as we analyze among other things the growth behavior of economies. Scale effects in growth rates in the long-run which are at odds with empirical findings would render our theoretical results useless for practical calibration purposes.[14]

Production requires the use of R_t units of a non-renewable resource at time t. The stock S_t of the resource develops according to

$$\dot{S}_t = -R_t. \tag{6}$$

[12] We elaborate here on the issue focusing on technological change based on an increasing variety of intermediate input factors. One could alternatively consider growth models using quality ladders, i.e. models based on Aghion and Howitt's (1992) Schumpeterian approach. In reduced form, such models deliver dynamics very similar to the case of increasing variety in input factors.

[13] Following the original idea in Romer (1987, 1990) would assume a production function for ideas as $\dot{H}_t = H_t L_{H,t}$ and, hence, $\frac{\dot{H}_t}{H_t}$ would depend on the extend of $L_{H,t}$. A larger population would yield, ceteris paribus, a higher growth rate for H_t and purely endogenous growth. This implication gave rise to the well-known Jones (1995a,b) critique and the development of the semi-endogenous approach to growth. For an overview, we would like to refer to Jones (1999, 2005). By letting $\frac{\dot{H}}{H_t}$ depend on $\frac{L_{H,t}}{L_t} = 1-u_t$ instead of $L_{H,t}$ is equivalent to follow Barro and Sala-i-Martin (2004, Sec. 6.1.7) leading to a semi-endogenous growth model where population size no longer influences steady-state growth. Steady-state growth, though not growth during transition, is influenced by quantities that are typically assumed to be exogenous such as population growth.

[14] Predicting a country's development based on a theoretical model with population scale effects in steady-state growth rates would paint an overwhelmingly rosy picture if such growth rates are used to judge factors such as sustainability based on non-declining consumption growth as we will do further down below.

The problem under consideration is to maximize the present value Hamiltonian for the representative household that reads as

$$\begin{aligned}\mathscr{H}_t = \frac{(c_t-\underline{c})^{1-\eta}-1}{1-\eta} e^{-\rho t} L_t \; &+ \lambda_{1,t}[Y_t - c_t L_t - \delta_1 K_t] \\ &+ \lambda_{2,t}[B(1-u_t)H_t - \delta_2 H_t] \\ &+ \lambda_{3,t}[-R_t],\end{aligned} \tag{7}$$

where $\lambda_{i,t}$, $i = 1,2,3$ are the co-state variables associated with the constraints of the problem. At the same time, they represent the present value shadow values of physical and human capital as well as the non-renewable resource. The first order conditions for a solution to the problem read as

$$\frac{\partial \mathscr{H}_t}{\partial c_t} = (c_t-\underline{c})^{-\eta} e^{-\rho t} L_t - \lambda_{1,t} L_t = 0, \tag{8}$$
$$-\frac{\partial \mathscr{H}_t}{\partial K_t} = \dot{\lambda}_{1,t} = -\lambda_{1,t}\frac{\partial Y_t}{\partial K_t} + \lambda_{1,t}\delta_1, \tag{9}$$
$$\frac{\partial \mathscr{H}_t}{\partial u_t} = \lambda_{1,t}\frac{\partial Y_t}{\partial u_t} - \lambda_{2,t} B H_t = 0, \tag{10}$$
$$-\frac{\partial \mathscr{H}_t}{\partial H_t} = \dot{\lambda}_{2,t} = -\lambda_{1,t}\frac{\partial Y_t}{\partial H_t} - \lambda_{2,t} B(1-u_t) + \lambda_{2,t}\delta_2, \tag{11}$$
$$\frac{\partial \mathscr{H}_t}{\partial R_t} = \lambda_{1,t}\frac{\partial Y_t}{\partial R_t} - \lambda_{3,t} = 0, \tag{12}$$
$$-\frac{\partial \mathscr{H}_t}{\partial S_t} = \dot{\lambda}_{3,t} = 0. \tag{13}$$

The corresponding transversality conditions are

$$\lim_{t\to\infty} \lambda_{1,t} K_t = 0, \quad \lim_{t\to\infty} \lambda_{2,t} H_t = 0, \quad \lim_{t\to\infty} \lambda_{3,t} S_t = 0. \tag{14}$$

Conditions (8) and (9) give rise to the well-known Keynes–Ramsey rule. (10) decides on the optimal allocation of human capital towards final good production and human capital accumulation by equating its corresponding marginal productivity in both activities. It is identical to the corresponding condition in Lucas (1988) for the present specification for human capital creation and guarantees an efficient allocation of effort in final goods production and human capital allocation. The development of the shadow value of human capital is given by (11). It accounts for human capital's marginal contribution to final good production and its own accumulation. (12) equates the marginal product of resource use in production with its shadow value $\lambda_{3,0}$ which characterizes an efficient use of the resource. Together, (12), (9) and (13) give rise to the well-known Hotelling's rule.

For the purpose of formulating the solution to our problem, it is helpful to use the following notation

$$\begin{aligned}\psi &= \frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}, \\ \varphi_1 &= A^{-\frac{1}{\alpha}} \lambda_{1,0}^{\frac{\alpha-1}{\alpha}} \left(\frac{\lambda_{2,0}}{\beta}\right)^{\frac{\beta}{\alpha}} \left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}} \left(\frac{L_0}{B}\right)^{-\frac{\beta}{\alpha}}, \\ \varphi_2 &= \frac{1-\alpha}{\psi}, \quad \zeta = \frac{\varphi_2-\varphi_1}{\varphi_2} = 1-\frac{\varphi_1}{\varphi_2}, \quad x_t = e^{-\psi t}.\end{aligned} \tag{15}$$

From a technical point of view, it might be relevant to note that as time t runs from zero to infinity in our economy, x_t decays from 1 to zero. This enhances the use of special functions in their integral representations and is therefore a technical variable that serves as a substitute for time. ψ is the decay parameter that essentially governs how fast or slow the economy adjusts to the steady-state.

Furthermore, we introduce prices for resources and human capital in terms of final output, i.e. we define $p_{R,t} = \frac{\lambda_{2,t}}{\lambda_{1,t}}$ and $p_{H,t} = \frac{\lambda_{3,t}}{\lambda_{1,t}}$.

3.2. Results

This section presents the results obtained from solving the model. We follow the typical approach by solving the model in real terms. All quantities are implicitly measured in real units of final output except those representing the use and the stock of the resource as well as human capital. The price for final goods serves as the numeraire and is normalized to unity. We will have to take account of this for the calibration the model to match real world data which are denominated in US $ at prices of 2014. The prices $p_{R,t}$ and $p_{H,t}$ are necessary to translate resource and human capital quantities into final output equivalents.

From the first order conditions (8) and (9) it is obvious that consumption is governed by a typical Keynes–Ramsey rule and from (9) and (12) we see that resource prices follow Hotelling's rule

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\eta}\frac{c_t}{c_t - \underline{c}}\left(\alpha\frac{Y_t}{K_t} - \delta_1 - \rho\right), \tag{16}$$

$$\frac{\dot{p}_{R,t}}{p_{R,t}} = \frac{\partial Y_t}{\partial K_t} - \delta_1, \tag{17}$$

where we observe in (16) a non-constant intertemporal elasticity of substitution as long as minimum subsistence consumption $\underline{c}$ is larger zero. From (16) it is clear that capital productivity $\frac{Y_t}{K_t}$ is the key to solve for the consumption path taken by the economy in its optimum.The following definition will be useful for the formal representation of the solution to the problem.

Definition 1. Provided a solution to the problem exists, the initial conditions for the co-states $\lambda_{1,t}$, $\lambda_{2,t}$ and $\lambda_{3,t}$ characterizing this solution are denoted by $\lambda^*_{1,0}$, $\lambda^*_{2,0}$ and $\lambda^*_{3,0}$. The corresponding values for ζ and φ_1 characterizing the solution to the problem will be denoted by ζ^* and φ^*_1 which follow the notation in (15).

Definition 1 provides us a notation for the optimal trajectory of the economy if a solution exists. The conditions for the existence of such a solution will be worked out further down below. We proceed this way as we already regard the existence of a solution, i.e. the existence of an optimal trajectory for the economy that guarantees at least minimum subsistence consumption as one possible test for sustainability as discussed in the literature review. Additional corresponding sustainability indicators will be worked out in the next section.[15]

It is obvious from (15) that ζ^*, φ^*_1 and φ_2 are reflecting initial conditions and several of the model's parameters. They will explicitly or implicitly appear in all trajectories characterizing the solution. Hence, it is important to understand these quantities. Lemma 1 below gives the trajectory for the capital productivity $\frac{Y_t}{K_t} = \frac{y_t}{k_t}$ where $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$ denote corresponding per capita variables. As we will see, this Lemma allows for an interpretation of ζ^*, φ^*_1 and φ_2.

Lemma 1. *If a solution of the problem exists, the trajectory for capital productivity is given by*

$$\frac{y_t}{k_t} = \frac{1}{\varphi_2}(1 - \zeta^* x_t)^{-1}.$$

Hence,

$$\frac{1}{\varphi^*_1} = \frac{y_0}{k_0} \quad \text{and} \quad \frac{1}{\varphi_2} = \lim_{t\to\infty}\frac{y_t}{k_t}.$$

Furthermore, the net interest rate $i_t - \delta_1 = \alpha\frac{1}{\varphi_2}(1 - \zeta^* x_t)^{-1} - \delta_1$ *and*

$$i_0 - \delta_1 = \alpha\frac{y_0}{k_0} - \delta_1 \quad \text{and} \quad \lim_{t\to\infty} i_t - \delta_1 = \frac{\alpha}{\varphi_2} - \delta_1 = \frac{\beta}{1-\alpha}(B - \delta_2 + n).$$

Proof. The underlying problem is to maximize (7) and Appendix B derives capital productivity's trajectory. The results for φ^*_1 and φ_2 follow from taking limits as $t \to 0$ and $t \to \infty$ and using Definition 1. The existence of a solution depends on further considerations that will be worked out in the main text below. Formally, the conditions for existence and uniqueness of the solution are derived in Appendix E.

Note. The expression of the net interest rate in this Lemma clearly show the absence of any scale effects as $t \to \infty$ arising from the size of the economy. This distinguishes our approach from those employing purely endogenous growth models such as Di Maria and Valente (2008), Acemoglu et al. (2012) and André and Smulders (2014).

Following Lemma 1, $\frac{1}{\varphi_1}$ ($\frac{1}{\varphi_2}$) measure the initial (asymptotic) capital productivity. Consequently, $\zeta^* = \frac{\varphi_2-\varphi_1}{\varphi_2} = \frac{1/\varphi_1 - 1/\varphi_2}{1/\varphi_1}$ reflects the relative distance of the economy's initial capital productivity from its asymptotic counterpart. If $\zeta^* > (<) 0$, the initial capital productivity is above (below) its asymptotic value, i.e., if the initial endowment with physical capital is relatively low (high). $\zeta^* = 0$ implies a constant capital productivity from $t = 0$ onward. It is also interesting to note that capital productivity given by Lemma 1 is following a logistic pattern over time.

With the help of Lemma 1 it is now also possible to understand the behavior of $p_{R,t}$ and p_{H_t}, i.e. the prices of resources and human capital in terms of final output. p_{R_t} grows at a rate equal to the net interest rate implied by Hotelling's rule (17). Appendix A shows that $\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} = B - \delta_2$. Using the first order condition (9), we find that $p_{H,t}$ grows at the rate $i_t - \delta_1 - (B - \delta_2)$ which is a no-arbitrage condition. $B - \delta_2$ is the net rate of return on an investment in human capital. This return is composed of the value of new human capital that can be created by this investment and the marginal product in terms of final output that the investment in human capital yields. The return has two sources as human capital use is divided between production and human capital creation. Whenever the net interest rate on physical capital exceeds $B - \delta_2$, this has to be compensated by a growing $p_{H,t}$ to make human capital investments equally attractive.

As usual, the initial values for the co-states and, hence, ζ^* and φ^*_1 are determined by the respective initial values of the state variables if they allow for a solution. As we will see later, the existence of a solution depends on initial endowments with physical capital and resources to guarantee at least minimum subsistence consumption. In Section 4 we will calibrate the model to a particular scenario reflecting countries' initial endowments and behavior. The initial endowment will contain the per capital physical capital stock k_0 while the economies' initial behavior will be reflected by the initial per capita production y_0. Hence, such a scenario will pin down ζ^* right from the beginning.[16]

With help of Lemma 1 we are now capable of solving for the consumption trajectory presented in Proposition 1 below. This is possible as the capital productivity allows us to compute the interest rate which can then be used in the Keynes–Ramsey rule (16).

Proposition 1. *If a solution to the problem exists, the trajectory for optimal consumption as a function of t is given by*

$$c_t = \underline{c} + (c_0 - \underline{c})\left(1 - \zeta^*\right)^{-\frac{\alpha}{(1-\alpha)\eta}}\left(1 - \zeta^* x_t\right)^{\frac{\alpha}{(1-\alpha)\eta}} x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}},$$

with

$$c_0 - \underline{c} = (\lambda^*_{1,0})^{-\frac{1}{\eta}}, \quad x_t = e^{-\psi t}$$

where c_0 *denotes initial consumption at* $t = 0$.

[15] There is also a technical reason for proceeding this way. One has, first, to solve for the trajectories of all state and co-state variables of the model and, second, to impose the transversality conditions associated with the Hamiltonian. Following this way, it is possible to work out the conditions for the existence of a solution.

[16] It has, of course, to be checked whether the implied ζ^* is indeed a solution in the sense that it guarantees at least minimum subsistence consumption. We will return to this issue further down below in Section 3.3. Of course, ζ^* can be equivalently determined by the initial values of the state variables as will be demonstrated in Appendix E.

Proof. Appendix A derives the behavior of $\lambda^*_{1,t}$ over time. Using this in the first order condition (8) together with Lemma 1 and the definition of x_t delivers the result.

In general, the behavior of c_t is non-monotonic as it is highly nonlinear in x_t and, hence, t. The part in consumption exceeding $\underline{c}$ is governed by two components. First, the term involving $1-\zeta^*$ reflects the deviation from the path that characterizes the dynamic behavior as $t \to \infty$. As $t \to \infty$, $x_t \to 0$ and $1-\zeta^* x_t \to 1$. Second, the term involving x_t only reflects the dynamic behavior as $t \to \infty$. This can be best seen by looking at the limiting behavior of c_t. Depending on the model's parameters, in the limit ($t \to \infty$, $x_t \to 0$), we might obtain three different general consumption profiles that read as

$$\lim_{t\to\infty} c_t = \begin{cases} \underline{c} & \text{if } \frac{\beta}{1-\alpha}(B-\delta_2+n)-\rho<0, \\ \underline{c} + (\lambda^*_{1,0})^{-\frac{1}{\eta}} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} & \text{if } \frac{\beta}{1-\alpha}(B-\delta_2+n)-\rho=0, \\ \underline{c} + (\lambda^*_{1,0})^{-\frac{1}{\eta}} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} \times & \\ \lim_{t\to\infty} x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}} \to \infty & \text{if } (B-\delta_2+n)-\rho>0. \end{cases} \tag{18}$$

From (18) we observe one case with unbounded growth and two cases of asymptotic stationary consumption. Which one is realized depends on whether $\frac{\beta}{1-\alpha}(B-\delta_2+n)-\rho \gtreqqless 0$. This condition has a clear economic interpretation as it reflects the usual trade-off spanned by the return on investments and time preference. In the limit, the net returns to human capital and physical capital investments are identical and equal to $\frac{\beta}{1-\alpha}(B-\delta_2+n)$.[17] If the net return outweighs the rate of time preference, consumption is postponed. This allows for additional investments which will produce positive consumption growth in the limit. If the net return on investments is smaller than or equal to the rate of time preference, consumption will be asymptotically constant either at $\underline{c}$ or above. The low incentive for investments prevents their accumulation at a sufficient speed that would allow for permanent growth.

As the limiting net rate of return appears in further results derived below, we need to discuss this quantity in more detail. It is the productivity in human capital accumulation and a positive externality that gives rise to this expression. With a constant rate of return and human capital utilization u_t, the contribution of a marginal unit of human capital to output growth is $\beta(B-\delta_2+n)$. The last term n enters as human capital provides a positive externality to all household members as they all benefit from its development at the same time. Human capital's contribution to growth is amplified through pre-multiplication with $\frac{1}{1-\alpha} = \frac{1}{\beta+\gamma}$ as human capital complements resource inputs and allows for permanent growth of resource prices by the net rate of return on investments.

From the first derivative of the consumption trajectory given in Proposition 1 with respect to x_t we see that consumption can behave non-monotonically. This applies to the cases of limiting unbounded positive growth as well as limiting subsistence consumption. With unbounded limiting growth, we observe exactly one consumption minimum if $\zeta^* < 0$. The initially high relative endowment with physical capital leads to initially high consumption. Consumption subsequently falls before rising again approaching an asymptotic positive growth rate. In the limiting subsistence consumption case, we observe exactly one maximum if $\zeta^* > 0$. This is because high initial capital productivity encourages investments in the beginning before consumption starts to decline to its subsistence level. All remaining cases are characterized by either monotonically increasing, decreasing or constant consumption profiles. The latter occurs if and only if it happens that $\zeta^* = 0$.

Next, we turn to the dynamic behavior of per capita input factors devoted to final goods production. Lemma 2 traces the behavior of $k_t = \frac{K_t}{L_t}$ over time by making use of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$.[18]

Lemma 2. *(a) The development of the stock of physical capital per capita over time is given by*

$$\begin{aligned} k_t = {} & \frac{(\lambda^*_{1,0})^{-\frac{1}{\eta}}}{\psi} \left(1-\zeta^*\right)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}} \left(1-\zeta^* x_t\right)^{\frac{1}{1-\alpha}} \\ & \times x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}} \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^* x_t)}{\tilde{b}_1} \\ & + \left(1-\zeta^* x_t\right)^{\frac{1}{1-\alpha}} \frac{\underline{c}}{\psi} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^* x_t)}{\tilde{b}_2}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \tilde{a}_1 &= \frac{\eta-\alpha}{\eta(1-\alpha)}, \\ \tilde{b}_1 &= \frac{1}{\psi}\left(\frac{\rho}{\eta} + \frac{\eta-\alpha}{(1-\alpha)\eta}\psi + \frac{1-\eta}{\eta}\delta_1 - n\right) \\ &= 1 + \frac{(1-\alpha)(\rho-n)+(\eta-1)\left[\beta(B-\delta_2+n)-(1-\alpha)n\right]}{(1-\alpha)\eta\psi}, \\ \tilde{a}_2 &= \frac{1}{1-\alpha} > 1, \\ \tilde{b}_2 &= \frac{1}{1-\alpha} - \frac{\delta_1}{\psi} - \frac{n}{\psi} = 1 + \frac{\beta(B-\delta_2+n)-(1-\alpha)n}{(1-\alpha)\psi}. \end{aligned}$$

The solution demands $\tilde{b}_1 > 0$ and $\tilde{b}_2 > 0$.

(b) Given ζ^*, the initial value $\lambda^*_{1,0}$ and, hence, the implied initial value of excess consumption is given by

$$\begin{aligned} \lambda^*_{1,0} &= (c_0 - \underline{c})^{-\eta} \\ &= \left[k_0 - \left(1-\zeta^*\right)^{\frac{1}{1-\alpha}} \frac{\underline{c}}{\psi} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2}\right]^{-\eta} \\ &\quad \times \left\{\frac{(1-\zeta^*)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}}{\psi} \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{\tilde{b}_1}\right\}^{\eta}. \end{aligned}$$

Proof. For part (a) see Appendix C. The proof of part (b) follows from applying part (a) at $t=0$ and solving for $\lambda^*_{1,0}$.

Note. From Lemma 2 we learn that a violation of either $\tilde{b}_1 > 0$ or $\tilde{b}_2 > 0$ would imply that the economy would not be able to accumulate sufficient physical capital to solve the problem. If $\tilde{b}_2 \le 0$, the economy would deplete its physical capital stock in finite time to afford minimum subsistence consumption. If $\tilde{b}_1 \le 0$, the economy cannot afford consumption in excess of its subsistence level over an infinite time horizon. In these cases the parameters of the model, reflected by $\tilde{b}_1$ and $\tilde{b}_2$, would imply an economy that is not productive enough to accumulate physical capital in a sufficiently high manner to allow for the chosen consumption path. For $\lambda^*_{1,0}$ to be strictly positive, the economy has to be endowed with a sufficiently large initial stock of physical capital k_0 in combination with a minimum subsistence level of consumption $\underline{c}$ that is not too large. This has to be reflected by a strictly positive term in squared brackets in the expression for $\lambda^*_{1,0}$ in part (b) of the Lemma.

The appearance of the hypergeometric function is due to compounding over a transition path with a non-constant interest rate. The representation of the results directly in the steady-state with a constant interest rate are much less complex. In this case, $\zeta^* = 0$ and the hypergeometric functions would disappear as ${}_2F_1(a,b;b+1;0)=1$.

[17] In the limit, also the return on holding resources is identical to this expression as the resource price follows Hotelling's rule given by (17).

[18] For a general overview regarding the use of the Gaussian hypergeometric function in economics see e.g. Abadir (1999). In Appendix C, we give a short introduction of using its integral representation for solving the differential equations appearing in the solution of the present problem.

The behavior of the physical capital stock is in general non-monotonic and the limiting behavior of the capital stock is qualitatively identical to the one of consumption.[19] The development is driven by two additive components in (19). The first represents the physical capital stock that is necessary to allow for consumption in excess of $\underline{c}$. The second represents physical capital requirements for realizing $\underline{c}$ via a sufficiently high production. Both components are influenced by a long-run asymptotic and a transitional element. The asymptotic element is represented by the term involving x_t that grows at the same asymptotic rate as does $c_t - \underline{c}$. The terms involving $1-\zeta^* x_t$ reflect the transitional deviations from the asymptotic path. The capital requirement for subsistence consumption is of course governed by $\underline{c}$. As $\underline{c}$ is constant, capital requirements for subsistence consumption are only changing over time due to transitional effects.

The stock of physical capital accumulates past consumption decisions via their compounded value at time t. This is reflected by the two Gaussian hypergeometric functions ${}_2F_1(\cdot)$ in (19) which essentially compute the compounding factor for past consumption. For $\zeta^* \neq 0$ compounding handles the deviation of the compounding rate from its steady-state value. $\tilde{b}_1 > 0$ implies that in the limit the growth rate of $(c_t - \underline{c})L_t$ is smaller than the net capital productivity $\frac{y_t}{k_t} - \delta_1$. This is the typical transversality condition one finds for AK models. Indeed, the present model belongs to this class as all factors of production can be accumulated although resources only at negative rates.[20] Further, $\tilde{b}_2 > 0$ demands that in the limit the growth of $\underline{c}L_t$ is below the net capital productivity. If one of these conditions is violated, total consumption growth would require more than the economy can realize in terms of productivity. $\tilde{b}_1$ and $\tilde{b}_2$ are responsible for the part in compounding due to steady-state growth, while the deviation during transition is captured by $\tilde{a}_1$ and $\tilde{a}_2$, respectively.

With the result in Lemma 2 we are now ready to trace final output per capita y_t over time in Lemma 3.

Lemma 3. *If a solution to the problem exists, final output per capita y_t is given by*

$$y_t = \frac{1}{\varphi_2}(1-\zeta^* x_t)^{-1} k_t,$$

where k_t is provided by Lemma 2.

Proof. The result is obtained by multiplying capital productivity in Lemma 1 by k_t.

Next, we turn to human capital H_t and its development over time. The stock effectively employed in final goods production is $H_t u_t$. Both are measured in human capital units. For our calibration exercise below, however, we need both measured in the same units as final output. The necessary translation of units is done via a simple multiplication by human capital's real price $p_{H,t}$. The corresponding human capital quantity will be denoted by $\tilde{H}_t = p_{H,t} H_t$. The following lemma provides us with their developments over time.[21]

Lemma 4. *The stocks of human capital employed in final goods production and total human capital measured in equivalent units of final output behave according to*

$$\begin{aligned}\tilde{H}_t u_t &= e^{(B-\delta_2+n)t}\frac{\beta\psi}{1-\alpha}\frac{L_0}{B}(1-\zeta x_t)^{-1}k_t \\ &= e^{-\delta_1 t}x_t^{-\frac{1}{1-\alpha}}(1-\zeta^*)^{\frac{\alpha}{1-\alpha}}\frac{\beta}{1-\alpha}\frac{L_0}{B}\times \\ &\times\left[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1}x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right. \\ &\left.+\underline{c}\frac{1}{\tilde{b}_2}x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right],\end{aligned}$$

$$\begin{aligned}\tilde{H}_t &= e^{-\delta_1 t}\frac{\beta}{1-\alpha}\left(\frac{x_t}{1-\zeta^* x_t}\right)^{-\frac{\alpha}{1-\alpha}}L_0\times \\ &\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right. \\ &\left.+\underline{c}\frac{1}{\psi}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\}.\end{aligned}$$

The solution demands $\tilde{b}_1 > 1$ and $\tilde{b}_2 > 1$.

Proof. See Appendix D.

Note. A violation of either $\tilde{b}_1 > 1$ or $\tilde{b}_2 > 1$ would imply that the economy would not be able to accumulate sufficient capital to solve the problem.[22] In this case, the argument applies to human capital. The implied parameter restrictions are stronger compared to physical capital accumulation. $\tilde{b}_1 > 1$ ($\tilde{b}_2 > 1$) demand that in the limit the growth rate of $(c_t - \underline{c})L_t$ ($\underline{c}L_t$) is below the economy's net interest rate. We note further that the positive externality provided by human capital introduces a scale effect into the economy. This is visible as H_t depends positively on the population size L_0.[23]

As effective human capital in production depends linearly on k_t, it shares the non-monotonic behavior of physical capital per capita in general.

Appendix D at the end of the paper shows that $u_t H_t = \frac{u_t \tilde{H}_t}{p_{H,t}}$ and $H_t = \frac{\tilde{H}_t}{p_{H,t}}$ tend to infinity as $t \to \infty$.[24] This distinguishes human from physical capital where we have seen that the latter might become stationary as $t \to \infty$. The cause for this behavior is simultaneously a solution for the economy's problem. The argument is as follows: The marginal contribution of human capital in its own creation is not vanishing as H_t increases (see Eq. (4)). In case of physical capital and due to the Cobb–Douglas production technology, its own marginal contribution in accumulation is diminishing. In case of a limiting bounded k_t, human capital takes the leading role and compensates for the increasing scarcity of the non-renewable resource that is depleted over time. In this case, human capital accumulation allows for a constant level of consumption in the long-run.

Finally, we turn to the use of non-renewable resources in production. For our calibration below, we have to trace the final output equivalent of per capita resource use $\tilde{r}_t = p_{R,t}\frac{R_t}{L_t}$ and its stock $\tilde{s}_t = p_{R,t}\frac{S_t}{L_t}$ presented by the following lemma.[25]

Lemma 5. *Per capita resource use and the stock of resources in final output equivalents are given by*

$$\tilde{r}_t = p_{R,t}\frac{R_t}{L_t} = \gamma y_t = \frac{\gamma}{\varphi_2}(1-\zeta^* x_t)^{-1}k_t,$$

$$\tilde{s}_t = p_{R,t}\frac{S_t}{L_t} = \frac{1}{\varphi_2}\frac{\gamma}{\psi^2}e^{-(\delta_1+n)t}(1-\zeta^*)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta^* x_t}\right)^{-\frac{\alpha}{1-\alpha}}\times$$

[19] The formal details are given in Appendix C.

[20] This becomes obvious as all factors earn the same rate of return implied by the developments of $p_{R,t}$ and $p_{H,t}$ as explained above.

[21] Appendix D also derives the expressions for $H_t u_t$ and H_t.

[22] At first sight, these parameter restrictions seem to be absent from the result obtained for $\tilde{H}_t u_t$. From Lemma 4 we know that it exists if the path for k_t exists and therefore also for $\tilde{b}_1 > 0$ and $\tilde{b}_2 > 0$ but $\tilde{b}_1 \leq 1$ and $\tilde{b}_2 \leq 1$. This case, however, would not be a solution to the problem, as the transversality condition (14) associated with H_t would be violated as shown in Appendix D. Nevertheless, $H_t u_t$ could be computed using the lemma. Doing so would however, hide that both, H_t and u_t would be negative.

[23] See Jones (1995a, 1999) for scale effects in endogenous growth models.

[24] This holds as long as $B - \delta_2 + n > 0$ which is reasonable to assume. If this condition is violated, marginal productivity of human capital in its own creation will be below its rate of depreciation, preventing accumulation of human capital.

[25] Appendix D at the end of the paper also derives the trajectories for R_t and S_t.

$$\left[(\lambda_{1,0}^{*})^{-\frac{1}{\eta}}\left(1-\zeta^{*}\right)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{x_t^{\tilde{b}_1-1}}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^{*}x_t)\right.$$
$$\left.+\tilde{c}\frac{x_t^{\tilde{b}_2-1}}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^{*}x_t)\right],$$

where k_t is provided by Lemma 2 and $\tilde{s}_t$ only exists for $\tilde{b}_1, \tilde{b}_2 > 1$.

Proof. The result for $\tilde{r}_t$ follows as the resource rents' share in final output is constant at γ owed by the Cobb Douglas production technology and using Lemma 3. For $\tilde{s}_t$ see Appendix D.

Note. A violation of either $\tilde{b}_1 > 1$ or $\tilde{b}_2 > 1$ would imply that the economy would need an indefinitely large amount of resources to solve the problem. The implied parameter restrictions are identical to the ones applying to human capital accumulation.

Summarizing this section, we find that the economy tends towards three possible states in the long run. In case $\beta(B-\delta_2+n)-(1-\alpha)\rho > 0$, the economy grows at positive rates in the long-run. Consumption and production per capita grow without bound and tend to infinity. If $\beta(B-\delta_2+n)-(1-\alpha)\rho < 0$, the economy is characterized by subsistence consumption and production per capita in the long-run and there is no long-run growth. If $\beta(B-\delta_2+n)-(1-\alpha)\rho = 0$, the economy is tending towards a long-run zero growth scenario in per capita consumption and production above the subsistence level. Thus, the relevance of the case depends on the parameters of the model. Below, we will calibrate the model to the situations of particular countries and find some countries with zero and some with positive long-run growth.

3.3. Sustainability

We employ three sustainability tests: (1) The existence of a solution to the problem guaranteeing minimum subsistence consumption. (2) We analyze the development of consumption and regard only a non-decreasing behavior as sustainable as e.g. Asheim and Buchholz (2004) among others. (3) We adopt Weitzman's (1976, 1997) test for sustainability and sustainable development.[26]

As we employ different sustainability tests, we have to define properly what we interpret as sustainable or as sustainable development. We follow Weitzman (1976, 1997) and distinguish between sustainability at a point t in time and sustainable development according to the following definition.

Definition 2. An economy is characterized by sustainability at time $t \in [0,\infty)$ according to a particular sustainability test if it passes this test at this particular point in time regardless whether the test is passed at other instances in time. If the economy passes the test for $t \to \infty$, it is governed by asymptotic sustainability. An economy is characterized by sustainable development according to a particular sustainability test if it passes this test for all $t \in [0,\infty)$ and for $t \to \infty$.

In our calibration, we find that some countries are characterized by sustainability over limited periods of time. As the optimal trajectories presented in the previous section are all continuous and differentiable, sustainability can only occur at t in closed subsets of $[0,\infty)$. Therefore, we define a degree of sustainability for these cases.

Definition 3. An economy is characterized by a degree of sustainable development if it is not qualifying for sustainable development according to Definition 2. The degree of sustainability is defined as the length of all intervals on the domain of x_t over which we observe sustainability.

Whenever we observe a degree of sustainable development smaller than one but greater than zero, a test on sustainability at time t might give a false signal as the same test applied at a different instance in time might produce a different result. The following definition distinguishes between the two possible types of false signals. We formulate these false signals taking the forward looking perspective as such a signal is critical if the signal is extrapolated into the future.

Definition 4. A type I false signal occurs at time t if the sustainability test is passed at t but not passed at least at one instance in time $s > t$. A type II false signal occurs at time t if the sustainability test is not passed at t but passed from some instance in time $s > t$ onward.[27]

Next, we need to work out the details regarding the sustainability tests considered. We focus on per capita quantities as we interpret sustainability as a concept applying to the individual although this is only a representative household member in our case.[28]

Sustainability (1) - existence of a solution. The first sustainability test checks whether a solution to the problem of maximizing (1) subject to (3), (4) and (6) exists. Existence of a solution to the problem implies two types of conditions. First, the parameters of the model need to fulfill certain inequalities. Only then finite initial endowments with resources and capital in general allow the economy to cover at least minimum consumption for an infinite time horizon. Second, the initial endowment needs to fulfill particular requirements. Both sets of conditions are necessary, but only if fulfilled together, they are sufficient for the existence of a solution. This is summarized in Proposition 2.

Proposition 2. *For finite and positive initial endowments with physical capital and resources, a unique solution to the problem exists if the parameters of the model satisfy*

$$\tilde{b}_1 > 1 \quad \text{and} \quad \tilde{b}_2 > 1$$

and ζ^* fulfills:

$$\underline{\zeta} < \zeta^* < \bar{\zeta},$$

where

$$\underline{\zeta} = \underset{\zeta \le 1}{\operatorname{argmin}} \left| k_0 - \frac{c}{\psi}(1-\zeta^*)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2} \right|,$$

$$\bar{\zeta} = \underset{\zeta \le 1}{\operatorname{argmin}} \left| \tilde{s}_0 - \frac{\gamma}{1-\alpha}\frac{c}{\psi}(1-\zeta^*)^{\frac{\alpha}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)} \right|.$$

Proof. The parameter restrictions regarding $\tilde{b}_1$ and $\tilde{b}_2$ were already discussed in the preceding section and are derived in Appendices C and D. Appendix E proves that if a solution exists, it is necessarily unique and requires $\underline{\zeta} < \zeta^* < \bar{\zeta}$.

Note. If a solution to the problem exists at an any point in time t, a solution also exists at all subsequent points in time. If an economy is characterized by sustainability according to this test at time t it is also characterized by sustainable development from t onward. This sustainability test is not able to produce false signals.

[26] We abstract from using genuine savings as an indicator for sustainability (see e.g. Hamilton and Naikal, 2014). Our approach is built on an Uzawa–Lucas model with positive human capital externalities. Well established theoretical results on the behavior of genuine savings are not valid in the presence of externalities as pointed out in Hanley et al. (2015). We therefore restrict our analysis to consumption based tests applicable to our theoretical set-up.

[27] Note that just passing at some but not all points in time from s onward is not sufficient to produce a type II false signal as no sustainable development would follow from s onward.

[28] If one looks at the economy-wide level, an additional aspect would have to be considered. Population growth then adds a source of autonomous growth which has to be taken into account. In this case, time has an economic value that adds to sustainability measures (see Weitzman, 1976 or Pezzy, 2004).

We define the indicator $I_{\underline{c}}$ taking the value 1 if the conditions of Proposition 2 are fulfilled and 0 otherwise.

$\underline{\zeta}$ and $\bar{\zeta}$ define a lower and an upper bound for ζ^* which. They also have an economically intuitive meaning. They implicitly define lower bounds for k_0 and $\tilde{s}_0$ required to realize permanent minimum subsistence consumption at the least. These lower bounds are the present value equivalents of subsistence consumption in terms of physical capital and resources necessary for a corresponding level of production. ζ^* and the hypergeometric functions appear as the economy is subject to a non-constant interest rate in transition if $\zeta^* \neq 0$.[29] If the economy falls short of one or both of these requirements, minimum subsistence consumption cannot be realized over the infinite planning horizon. This could be seen as unsustainable according to e.g. Holden et al. (2014). Whether or not an economy is able to realize a solution to the problem is naturally a question that needs the inspection of case specific data.

Sustainability (2) - Consumption growth. A more standard approach to sustainability is to follow the development of consumption. Non-decreasing consumption at time t can be associated with sustainability at t. Consumption growth based on Proposition 1 provides us with the necessary information. Again, knowledge about the values of the model's parameters are necessary to judge whether the development is sustainable in a particular case or how high the degree of sustainability is.

To do so, we need to work out some additional theoretical details. From the Keynes–Ramsey rule (16) and Lemma 1 it follows that $\frac{\dot{c}_t}{c_t} = 0$ for $c_t - \underline{c} \neq 0$ at $x_t(\dot{c}_t = 0) = \frac{1}{\zeta^*}\frac{(1-\alpha)\rho-\beta(B-\delta_2+n)}{(1-\alpha)(\rho+\delta_2)}$. Consequently, the point in time when consumption growth is zero is $t(\dot{c}_t = 0) = -\frac{1}{\psi}\ln\frac{1}{\zeta^*}\frac{(1-\alpha)\rho-\beta(B-\delta_2+n)}{(1-\alpha)(\rho+\delta_2)}$. It is straightforward to show that the optimal path for c_t is strictly concave (convex) at $x_t(\dot{c}_t = 0)$ for $\zeta^* > 0$ ($\zeta^* < 0$), i.e. if the initial physical capital stock is below (above) its steady-state value at $t = 0$.

As soon as we find $0 < x_t(\dot{c}_t = 0) < 1$, we can conclude that the country is not characterized by sustainable development but is only behaving in a sustainable manner over a particular finite period of time. If $\zeta^* > 0$ ($\zeta^* < 0$), the country displays non-negative consumption growth for $t \le t(\dot{c}_t = 0)$ $\left(t \ge t(\dot{c}_t = 0)\right)$.[30] In case $x_t(\dot{c}_t = 0) \le 0$ and $\zeta^* > 0$ ($\zeta^* < 0$), the economy is characterized by non-negative consumption growth only as $t \to \infty$ and increasing (decreasing) consumption before. In case $x_t(\dot{c}_t = 0) \ge 1$ and $\zeta^* > 0$ ($\zeta^* < 0$) we have an economy that is characterized by a monotonic decreasing (increasing) consumption path.

We can therefore define an indicator $I_{\dot{c}\ge 0}$ that measures the degree of sustainability along the adjustment path towards the steady-state based on non-negative consumption growth. More precisely, we define

$$I_{\dot{c}\ge 0} := \begin{cases} x_t(\dot{c}_t = 0) & \text{if} \quad \zeta^* < 0 \quad \text{and} \quad 0 < x_t(\dot{c}_t = 0) < 1, \\ 0 & \text{if} \quad \zeta^* < 0 \quad \text{and} \quad x_t(\dot{c}_t = 0) \le 0, \\ 1 & \text{if} \quad \zeta^* < 0 \quad \text{and} \quad x_t(\dot{c}_t = 0) \ge 1, \\ 1 - x_t(\dot{c}_t = 0) & \text{if} \quad \zeta^* > 0 \quad \text{and} \quad 0 < x_t(\dot{c}_t = 0) < 1, \\ 1 & \text{if} \quad \zeta^* > 0 \quad \text{and} \quad x_t(\dot{c}_t = 0) \le 0, \\ 0 & \text{if} \quad \zeta^* > 0 \quad \text{and} \quad x_t(\dot{c}_t = 0) \ge 1, \\ 1 & \text{if} \quad \zeta^* = 0. \end{cases} \tag{20}$$

Thus, the larger the indicator, the higher is the degree of sustainable development during adjustment. As consumption can behave non-monotonically, this test is able to produce type I and II false signals. However, these can only occur for $0 < x_t(\dot{c}_t = 0) < 1$ as consumption is only then subject to a local maximum. We observe a type I (II) false signal then at all $t < t(\dot{c} = 0)$ in case $\zeta^* > 0$ ($\zeta^* < 0$).

Sustainability (3) - Weitzman (1976, 1997). Finally, we apply Weitzman's (1976, 1997) sustainability test on the economy's consumption pattern. In the original publications of Weitzman (1976, 1997), an economy with a constant interest rate is analyzed. This simplifies the necessary computations considerably but limits, of course, applicability. As we focus on the transitional dynamics of the economy, we adopt Weitzman's idea along the complete adjustment path with a non-constant interest rate.

The idea behind this sustainability test is to compute a hypothetical value for period's t consumption as a sustainability benchmark against which we compare the welfare maximizing consumption. To arrive at this benchmark, one computes a constant value for consumption that is in its present value equal to the present value of the welfare maximizing consumption level. This has been termed as the present value annuity equivalent by Weitzman (1976). As the present value of welfare maximizing consumption path and the annuity equal each other, both could be interchanged on a perfect capital market if they differ. It is the rate of time preference in combination with the decreasing marginal utility that prevents the household from doing so. We take account of minimum subsistence consumption and apply Weitzman's idea on the consumption in excess of its subsistence level.

We are looking at our economy at time t and denote the present value (PV) of a constant excess consumption as $PV(\bar{c}_t - \underline{c})_t$, where we compute the present value over all points in time $s \in [t, \infty)$. $\bar{c}_t$ is carrying a time subscript as this annuity depends on time t from which we start our computations. Consequently, the present value at time t of welfare maximizing excess consumption is denoted by $PV(c_s - \underline{c})_t$ for $s \in [t, \infty)$. The critical benchmark value for the sustainability test is obtained by equating $PV(\bar{c}_t - \underline{c}_t)_t$ with $PV(c_s - \underline{c})_t$ and solving for the annuity $\bar{c}_t$.

According to Weitzman (1976, 1997), sustainability at time t is given if $c_t - \underline{c} \le \bar{c}_t - \underline{c}$ whereas we can speak of sustainable development if the whole consumption trajectory satisfies $c_s - \underline{c} \le \bar{c}_s - \underline{c}$ for all $s \in [t, \infty)$. Proposition 3 summarizes this sustainability test.

Proposition 3. *The economy is characterized by sustainability at time t if* $c_t - \underline{c} \le \bar{c}_t - \underline{c}$ *or equivalently if* $\frac{\bar{c}_t - \underline{c}}{c_t - \underline{c}} \ge 1$ *where*

$$\frac{\bar{c}_t - \underline{c}}{c_t - \underline{c}} = \frac{\tilde{b}_2 - 1}{\tilde{b}_1 - 1}(1 - \zeta^* x_t)^{-\frac{\alpha}{(1-\alpha)\eta}} \frac{{}_2F_1(\tilde{a}_1 - 1, \tilde{b}_1 - 1; \tilde{b}_1; \zeta^* x_t)}{{}_2F_1(\tilde{a}_2 - 1, \tilde{b}_2 - 1; \tilde{b}_2; \zeta^* x_t)}.$$

The economy is characterized by sustainable development if $\frac{\bar{c}_s - \underline{c}}{c_s - \underline{c}} \ge 1$ *for all* $s \in [t, \infty)$.

Proof. The concept is adopted from Weitzman (1976, 1997); the derivation of $\bar{c}_t - \underline{c}$ is given in Appendix F.

The idea in Weitzman (1976, 1997) is also very much related to the sacrifice discussed in Cairns and Martinet (2014, 2021). $\bar{c}_t - c_t$ can be seen as the sacrifice implied by Weitzman's sustainability test. If passed, a part of present consumption is sacrificed to allow for higher future consumption based on the returns foregone earlier.[31] During our calibration exercise we will compare actual consumption in excess of $\underline{c}$ with the benchmark value given in Proposition 3 to see whether sustainability in the sense of Weitzman (1976, 1997) is given.

Based on this sustainability test, we construct an indicator I_W that returns the degree of sustainable development during the transition towards the steady-state. Denote the number of discrete grid points by $N + 1$ at which point we investigate genuine savings. The time grid is

[29] We would have $\zeta^* = 0$ in case the economy is in the steady-state right from the beginning. In this case, ${}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; 0) = 1$ would simplify the expressions significantly. Implicitly, the upper bound $\bar{\zeta}$ also defines a minimum requirement for $\tilde{H}_0$, i.e. the initial resource endowment. See Appendix D for the details.

[30] From the practical perspective, the case $\zeta^* = 0$ is hardly relevant. It would refer to an economy that starts right from the beginning in the steady-state. Moreover, consumption growth would be non-negative for all $t \ge 0$.

[31] Such behavior also increases the present value annuity of consumption at future dates as higher consumption then is discounted less and less.

Table 1
Summary of sustainability tests.

Sustainability test	(1)	(2)	(3)
Criterion	Existence of solution, i.e. $c_t \geq \underline{c} \quad \forall t$	Non-decreasing consumption, i.e. $\dot{c}_t \geq 0$	Consumption not exceeding annuity equivalent (Weitzman, 1976, 1997), i.e. $c_t \leq \bar{c}$
Degree of sustainability (definition 3)	$I_{\underline{c}} \in \{0,1\}$	$I_{\dot{c}\geq 0} \in [0,1]$	$I_W \in [0,1]$
Sustainable development if (definition 2)	$I_{\underline{c}} = 1$	$I_{\dot{c}\geq 0} = 1$	$I_W = 1$
False signals possible (definition 4)	no	yes	yes

Note: See the main text for the definitions of $I_{\underline{c}}$, $I_{\dot{c}\geq 0}$ and I_W.

defined over $t_j = [-\frac{1}{\psi} \ln x_j]$ where $x_j \in [\frac{j}{N}]_{j=0}^{N}$. The indicator is defined as

$$I_W := \frac{\sum_{j=0}^{N} I\left(\frac{\bar{c}_{t_j}-\underline{c}}{c_{t_j}-\underline{c}} \geq 1\right)}{N+1}, \tag{21}$$

where

$$I\left(\frac{\bar{c}_{t_j}-\underline{c}}{c_{t_j}-\underline{c}} \geq 1\right) = \begin{cases} 1 & \text{if } \frac{\bar{c}_{t_j}-\underline{c}}{c_{t_j}-\underline{c}} \geq 1, \\ 0 & \text{if } \frac{\bar{c}_{t_j}-\underline{c}}{c_{t_j}-\underline{c}} < 1. \end{cases}$$

The interpretation of this indicator is qualitatively the same as in case of $I_{\dot{c}\geq 0}$.

From the theoretical perspective, we can conclude that clear cut answers regarding sustainability along the transition path are possible for sustainability tests (1) and (2). For the first test, we were able to work out conditions for initial endowments and parameter values that guarantee sustainability in Proposition 2. For sustainability test (2) based on non-negative consumption growth, we derived explicit expressions for the timing of sustainability. For test (3), we have to rely on numerical analyses during the adjustment period. However, the explicit solutions given in Proposition 3 provide us with the necessary tools. Evaluating the Weitzman (1976, 1997) sustainability test as $t \to \infty$ returns analytically clear cut results. In all possible cases for asymptotic growth, we find that the economy behaves in an asymptotically sustainable manner which is shown in Appendix E at the end of the paper. Table 1 summarizes the sustainability tests and their main properties.

4. Calibration

This section uses the above findings to analyze the full adjustment path of the model economy calibrated to the situation of different country groups and individual countries. Given that we can theoretically pin down the initial conditions for the solution to the problem, this allows us to calibrate the model using recent World Bank data on endowments with different types of capital and an initial level of production.

4.1. Preliminaries

Before starting our calibration of the above model, some words on the units of measurement are in order. As usual in theoretical models, all the quantities in our model are denominated in real units. The data we are using in the below standing sections will be denominated in US $ of 2014. This requires us to use all model's quantities in common units that can be calibrated to match this currency. Due to this, relative prices in the model become relevant and we have to make use of resources' input and stock as well as human capital measured in the common unit of final output.

Our calibration will match a country's actually realized output and its stock of reproducible capital per capita in the base year $t = 0$, i.e. we match y_0 and k_0, with the model's predicted output and its initial capital productivity which implies $1 - \zeta^* = \frac{1}{\varphi_2}\left(\frac{y_0}{k_0}\right)^{-1}$ by Lemma 1 at $t = 0$. We choose the year 2014 as the base year and will trace the models quantities thereafter. 2014 is chosen as the most recent data are available for 2014. The other quantities of interest that we trace over time will be consumption c_t, resource use $\tilde{r}_t$ and its corresponding stock $\tilde{s}_t$ as well as the stock of human capital $\tilde{H}_t$.

Given the initial scenario reflected by ζ^*, we have to solve for $\lambda^*_{1,0}$ by using Lemma 2(b) next and proceed by computing the trajectories for the calibrated quantities of the model.

As we use the initial capital productivity $\frac{y_0}{k_0}$ to calibrate ζ^* as explained above we have to check whether this calibrated value indeed implies a solution to the problem, i.e. whether sustainability test (1) is passed. Based on Proposition 2, we use the calibrated value for ζ^* to compute the amount of initial endowments with physical and human capital as well as resources that are available for covering consumption in excess of its minimum subsistence level. A solution exists if these remaining quantities are positive. Mathematically, we use Proposition 2, and Lemmata 4 and 5 to arrive at the conditions

$$\underline{\tilde{k}}_0 = \frac{\underline{\tilde{K}}_0}{L_0} = \frac{\tilde{K}_0}{L_0} - \frac{\tilde{c}}{\psi}(1-\zeta^*)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta^*)}{\tilde{b}_2} > 0, \tag{22}$$

$$\underline{\tilde{s}}_0 = \frac{\underline{\tilde{S}}_0}{L_0} = \frac{\tilde{S}_0}{L_0} - \frac{\gamma}{1-\alpha}\frac{\tilde{c}}{\psi}(1-\zeta^*)^{\frac{\alpha}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)} > 0, \tag{23}$$

$$\underline{\tilde{h}}_0 = \frac{\underline{\tilde{H}}_0}{L_0} = \frac{\beta}{\gamma}\underline{\tilde{s}}_0 > 0. \tag{24}$$

$\underline{\tilde{k}}_0$ ($\underline{\tilde{K}}_0$), $\underline{\tilde{s}}_0$ ($\underline{\tilde{S}}_0$) and $\underline{\tilde{h}}_0$ ($\underline{\tilde{H}}_0$) denote the parts of initial endowments that are still available to the economy after its needs to cover subsistence consumption are fulfilled. $\underline{\tilde{h}}_0$ is a per capita value that is not to be interpreted as per capita human capital as human capital is benefiting all household members as a positive externality. Rather, it should be interpreted as accumulated investment per capita in human capital (net of depreciation) done in the past up to $t = 0$.

4.2. Data on initial endowments

The data on initial endowments used for the calibration were obtained from the World Bank (2018). The World Bank (2018) provides estimates for stocks of produced, natural and human capital up to 2014 in US $ at current prices. This is part of a comprehensive cross country database on what is termed as "The Wealth of Nations". For our calibration, we chose 2014 as the starting year to make use of the most recent data available.

Although it is clear that such a database only provides estimates, the data are the best available and can be of use for the present purpose. Table 2 provides a summary of the data for 2014 in per capita terms for income based groups of countries.[32]

Table 2 also contains data on the country groups' net foreign assets. For calibration of initial stocks of physical capital, we will add these

[32] Income groups according to the World Bank's thresholds on countries GNI. Details are available from the World Bank's permanent URL http://go.worldbank.org/L547EEP5C0.

Table 2
Capital stocks and GNI per capita in 2014 US$.

World Bank data	No. countries	Prod. capital	Nat. capital (incl. land)	Nat. capital (excl. land)	Net for. assets	GNI
Low-income	24	1,967	6,421	1,236	−322	789
Lower-middle income	37	6,531	6,949	2,187	−650	2,035
Upper-middle income	36	28,527	18,960	8,339	−432	8,563
High-income (non-OECD)	15	59,069	80,104	74,243	14,005	
High-income (OECD)	29	195,929	19,525	12,877	−5,464	
High-income	44	166,438	32,579	26,100	−1269	43,351
World	141	44,760	5,841	8,810	−676	10,987

Note: World Bank (2018, Appendix B) estimates for stocks of different types of capital and net foreign assets per capita in 2014 US $. High-income values are averaged values (weighted by population) for OECD and non-OECD high-income countries reported in World Bank (2018, p. 233). Produced capital: machinery, equipment, structures, urban land; natural capital (incl. land): energy resources (oil, natural gas, hard coal, lignite), mineral resources (bauxite, copper, gold, iron, lead, nickel, phosphate, silver, tin, zinc), timber resources, non-timber forest resources, crop land, pasture land, protected areas. natural capital (excl. land): natural capital (incl. land) less of crop land, pasture land, protected areas. Human capital estimated from expected presented value of labor income. Population in millions. GNI for 2014 in US $ taken from the World Bank data base https://data.worldbank.org/indicator/NY.GNP.PCAP.CD

to the stock of produced capital to arrive at the capital stock which is actually owned by the economies. Consequently, we are investigating the domestic economy as our model analyzes the closed economy case.

Regarding natural capital, we will draw on the data on natural capital excluding land in our calibration. The reason for doing so will become clear further down below where we elaborate on the model's parameters. In case of resources, natural capital excluding land fits quite well with other data used for calibration of the resources output elasticity.

The numbers in Table 2 reflect country groups' averages. However, our calibration can be executed for any single country where we have no missing values in the data base. World Bank (2018) also provides us with estimates of the stock of human capital. However, we are not using them in our calibration. The reasons for this are twofold. First, the World Bank data on human capital do not match with the model's stock of human capital. The World Bank estimates the stocks by computing an expected present value of labor income. Labor income in 2014 is thereby largely determined by the labor share in GDP taken from the Penn World Tables (PWT Feenstra et al., 2015). The expected present value is computed assuming the economy is in a steady-state where growth is constantly exceeding the discount rate by 1.5% p.a. The expectation is reflecting countries demographic characteristics regarding life expectancy. This concept is not reflecting our intention of calibrating the model for economies potentially starting in the base year off the steady-state and adjusting to a balanced growth path over time. Second, we do not need to pick a value for initial human capital as the nominal value of human capital is implicitly calibrated as explained in the preceding section.[33]

In addition to the data taken from World Bank (2018), we are using World Bank data on the countries GNI for calibrating initial output. We decide to choose GNI instead of GDP following the argument in Asheim and Buchholz (2004) who favor national income over domestic production in relation to the DHSS model. Thus, we capture output produced using the production factors owned by the economy. This squares well with correcting produced capital using the net foreign asset position of the economies.

Besides the above World Bank data, we are drawing on the PWT 9.1[34] as we need additional information on countries' labor share in GDP and the depreciation rate of physical capital. Furthermore, we are using additional World Bank data on mortality to calibrate human capital depreciation. We postpone the discussion of these data to the section where we elaborate on the model's parameters.

[33] Note that we are unable to identify real human capital this way. For this we would require data on human capital measured in terms of real output of the economy. To the best of our knowledge, such data are unavailable. Therefore, we can only trace human capital valued at its optimal price $p_{H,t}$.

[34] These data are available at https://www.rug.nl/ggdc/productivity/pwt/.

4.3. Calibration values country groups

Regarding households' preferences, ρ, η, L_0, n and $\underline{c}$ need to be specified. The rate of time preference is a parameter that is frequently calibrated. We feel that an extensive discussion on this parameter's value is not necessary. We will chose $\rho = 0.03$ which seems to be a common choice also used in e.g. Benchekroun and Withagen (2011).

There exist some contributions to the literature that calibrate the type of Stone–Geary utility function used in the present context. Achury et al. (2012) calibrate an intertemporal utility function identical to the present one in (1) for the US and use $\eta = \frac{1}{0.23}$ which is roughly equal to 4.3. They refer to their choice of η as a standard choice in the portfolio literature. Ogaki et al. (1996) provide estimates for $\frac{1}{\eta}$ ranging from 0.569 up to 0.646. In turn, this corresponds to η decreasing from about 1.68 down to 1.55. Álvarez-Peláez and Díaz (2005) calibrate η in a range from 1.5 up to 2.5 in their application of Stone–Geary preferences. Ravn et al. (2006, 2008) analyze the influence of subsistence points such as subsistence consumption on the dynamics of macroeconomic development in general. Despite this, their specification for intertemporal utility is in accordance with the present situation. During calibration of their models they use a value of 2 for η. A value of 2 has also been applied for calibration purposes by Nordhaus (2007) and Acemoglu et al. (2012). We follow this choice and use $\eta = 2$ during our calibration. This is an intermediate value that is in between what has been used in Álvarez-Peláez and Díaz (2005) and Achury et al. (2012).

We calibrate our model on a per capita basis and, hence, normalize L_0 to 1. The population growth rate n is taken from the World Bank. Its value across different groups of countries during 2014 together with the crude mortality rate across all age groups is given in Table 3 below. The mortality rate will be used later on for calibrating human capital depreciation δ_2 (see Eq. (4)).

For the level of subsistence consumption $\underline{c}$, we consider the poverty lines used by the World bank.[35]

As of today, the threshold for extreme absolute poverty is set at 1.90 US $ at 2011 prices and at PPP a day available to an individual for covering basic needs (Ferreira et al., 2016). By now, this is considered to apply to low-income countries. The World Bank recently has

[35] Values for subsistence consumption have also been proposed in Koulovatianos et al. (2007) and Atkeson and Ogaki (1996) which have been used also in Achury et al. (2012) and Ogaki et al. (1996). These numbers, however, reflect very specific countries which is not in accordance with our analysis. Additionally, investigating poverty lines in this context is interesting as they influence economic policy initiatives especially in low-income countries (see e.g. the United Nation's Sustainable Development Goal on poverty, https://www.un.org/sustainabledevelopment/).

Table 3
Demographics, GDP shares and Capital Depreciation 2014 in %.

	No. countries	Pop. growth	Crude mortality	Resource rents' share in GDP	No. countries	Labor income's share in GDP	No. countries	Capital depreciation
Low-income	34	2.6	0.9	12.57	15	51.30	24	4.99
Lower-middle income	47	1.5	0.8	5.57	26	52.87	34	4.58
Upper-middle income	56	0.8	0.7	5.83	37	47.94	35	5.00
High-income	79	0.6	0.8	2.00	55	52.79	44	4.40
World	216	1.2	0.8	3.38	133	51.29	137	4.70

Note: Population growth and mortality in % p.a. from the World Bank's data base: https://data.worldbank.org/indicator/SP.POP.GROW and https://data.worldbank.org/indicator/sp.dyn.cdrt.in. Averages of resource rents in % of GDP calculated using data from https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. Labor income share and depreciation rates on physical capital averages computed using the Penn World Tables 9.1 (variable labsh and delta, https://www.rug.nl/ggdc/productivity/pwt/); country classification in accordance with the World Bank's classification scheme available at https://datahelpdesk.worldbank.org/knowledgebase/articles/906519.

introduced two additional poverty lines applying to lower- and upper-middle-income countries at 3.20 US $ and 5.50 US $ per day at 2011 prices and PPP. For the calculation behind these numbers see Jolliffe and Prydz (2016) who furthermore provide an absolute poverty level for high-income countries at 21.70 US $ per day at 2011 prices and PPP. We convert these numbers into yearly values at prices of 2014 in US $ using the PPP exchange rate. This gives a poverty line of 1,833 (3,631; 3,793; 8,675) US $ using the PPP exchange rate for low (lower-middle, upper-middle, high) income countries.[36]

We turn now to the parameters governing production. The output elasticity γ of resource use R_t is, given the Cobb–Douglas production technology (2), which is set equal to the share of natural resource rents in GDP. Data on this share is available from the World Bank.[37] Table 3 provides a summary of the data for different groups of countries classified according to the country's level of income. It is clearly visible that the resource dependence increases as income decreases. Resources seem to be most important for the low-income countries.

We use the labor income share in GDP for calibrating the output elasticity of effective human capital in production β. Numbers for the labor income share in GDP in 2014 were taken from the Penn World Tables 9.1 and are provided in Table 3. For the labor share we cannot observe a clear pattern and observe values on average of around 0.5 with only moderate variation.[38] Given the assumption of constant returns to scale in production, the capital's share $\alpha = 1 - \beta - \gamma$ follows as a remainder.

Produced capital published in World Bank (2018) and discussed above originates largely from the Penn World Tables. It is estimated employing the perpetual inventory method using country and capital good specific rates of depreciation. The country specific rates vary between 3 and 8% per annum. Table 3 gives the average depreciation rates for the country groups under consideration.

Further, we need to find appropriate values for the parameters governing the creation of human capital. Our specification (4) is similar to the specification originally proposed and calibrated by Lucas (1988). In his specification, depreciation of human capital was excluded, i.e. δ_2 was set equal to zero. Lucas (1988) calibrated B at a value of 0.05 which also has been used e.g. in Funke and Strulik (2000). Chen and Funke (2013) used a higher value of 0.095 for a calibration concerning the Chinese economy. We decided to use 0.05 as a conservative value that is not too optimistic about human capital formation. Regarding δ_2, we choose for the crude mortality rate across all age groups as the unconditional probability for individual human capital ceasing to exist.

Table 4 summarizes our calibration scenario for the different country groups.

4.4. Calibration results country groups

Proceeding as explained above and using the calibration values of the last section, we find that for all country groups the parameter restrictions $\tilde{b}_1, \tilde{b}_2 > 1$ are fulfilled. This means that the problem is properly defined and a solution can potentially exist according to Proposition 2. The second question is then whether such a solution actually exists, i.e. whether initial endowments with physical, natural and human capital are sufficiently large enough (conditions (22), (23) and (24)). If not, we would like to find out by how much initial endowments fall short of their minimum requirements. Table 5 provides the results.

We see that a solution exists for all country groups. Initial endowments with physical and natural capital are sufficiently large to guarantee subsistence consumption at all times. Per capita endowments, $\underline{\tilde{k}}_0$, $\underline{\tilde{s}}_0$ and $\underline{\tilde{h}}_0$, available for consumption in excess of its subsistence level are all positive, i.e. endowments are sufficient to allow for excess consumption.

It is interesting to investigate what is the maximum subsistence consumption that could be afforded by the country groups. To find out, $\underline{\tilde{c}}^{\max}$, we are searching the values for $\underline{\tilde{c}}$ that solve for at least one of the conditions (22), (23) and (24) with equality while the others are not violated. The model's predictions are optimistic with $\underline{\tilde{c}}^{\max}$ clearly above the poverty lines defined by the World Bank.

Looking at the long-run behavior implied by the calibration values, we find that all country groups qualify for positive steady-state growth in per capita quantities. Using the results from Section 3.2, we find these growth rates to vary moderately around 1% p.a. and a rate of interest net of depreciation between 4.3 and 5.4% p.a. Table 6 reports the corresponding findings.

4.5. Calibration results individual countries

For calibration of individual country cases, we proceeded exactly the same way as in case of country groups before by using the same data sources. The calibration values can be found in Table H.1 Appendix H at the end of the paper which lists all 108 countries for which all the required data are available.[39]

[36] Price changes are taken account by using the implicit GDP deflator obtained by dividing the time series for GDP at PPP valued at constant and current prices for low-income countries available at https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.KD and https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD. This results in a growth in prices of 5.34% between 2011 and 2014. PPP exchange rates are implemented by using the implicit exchange rate between GNI per capita in 2014 in int. $ (https://data.worldbank.org/indicator/NY.GNP.PCAP.PP.CD) and current US $ (https://data.worldbank.org/indicator/NY.GNP.PCAP.CD). This results in an adjustment factor of 2.51 (2.95; 1.79; 1.04) for low (lower-middle, upper-middle, high) income countries.

[37] Data are available from the World Bank Data Base at https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. For details on how the numbers are derived see World Bank (2011). Natural resources rents are the sum of oil, natural gas, coal (hard and soft), mineral, and forest rents.

[38] The labor shares reported in Table 3 are low compared with e.g. the traditional $\frac{2}{3}$ that is frequently used. See e.g. the discussion in Karabarbounis and Neiman (2014) on the recently decreasing development of the labor income share.

[39] In total 33 countries were excluded from the World Bank (2018) database. Mostly, this was due to missing data on GNI in the World Bank data and

Table 4
Calibration values.

	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{\tilde{c}}$	η	ρ	n
Low-income	1,980	3,953.1	2,970.2	304	2	0.03	0.026
Lower-middle income	6,001	17,482.5	6,501.3	414	2	0.03	0.015
Upper-middle income	15,358	50,474.3	14,981.5	1,177	2	0.03	0.008
High-income	45,327	173,039.0	27,344.0	7,964	2	0.03	0.006
	L_0	δ_1	B	δ_2	α	β	γ
Low-income	21.892	0.0499	0.05	0.009	0.3613	0.5130	0.1257
Lower-middle income	73.659	0.0458	0.05	0.008	0.4156	0.5287	0.0557
Upper-middle income	61.022	0.0500	0.05	0.007	0.4376	0.4794	0.0583
High-income	30.320	0.0440	0.05	0.008	0.4521	0.5279	0.0200

Note: Calibration values as explained in the main text. All values corresponding to nominal variables are measured in US $ at prices of 2014 per capita. Population L_0 as of 2014 in million people.

Table 5
Calibration results country groups.

	$\tilde{b}_1 > 1$	$\tilde{b}_2 > 1$	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$
Low-income	yes	yes	3,003	2,047	8,354	625
Lower-middle income	yes	yes	15,704	6,092	57,827	3,848
Upper-middle income	yes	yes	44,748	13,755	113,106	8,082
High income	yes	yes	129,982	24,851	655,930	32,006

Note: Results using calibration values from Table 4. All quantities in 2014 US $ per capita.

Table 6
Steady-state growth and interest rates in % p.a.

	$\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} = \lim_{t\to\infty} \frac{\dot{y}_t}{y_t} = \lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$	$\lim_{t\to\infty} i_t - \delta_1$
Low-income	1.19	5.38
Lower-middle income	1.08	5.16
Upper-middle income	0.69	4.35
High-income	0.81	4.62

Note: Growth rates computed according to (16), Lemma1 and $\lim_{t\to\infty} i_t = \frac{\alpha}{1-\alpha}\psi$.

We note that the problem we analyze is not properly defined for 4 of the countries: Iraq, Kuwait, Oman and Qatar. They are all subject to a high resource rents' share in GDP γ which leads to a violation of the conditions $\tilde{b}_1, \tilde{b}_2 > 1$. Production in these economies is simply too dependent on resources and no finite initial endowment could ever fulfill their total resource consumption over time given their initial output.

To apply the first sustainability test, we further have to check whether initial endowments are sufficient to cover at least the subsistence level of consumption. As in case of the analyzed country groups, we compute $\underline{\tilde{k}}_0$, $\underline{\tilde{s}}_0$ and $\underline{\tilde{h}}_0$ defined by (22), (23) and (24). Appendix H reports on the results for all countries where we were able to assemble the full dataset. We find several countries with insufficient endowments in physical and natural capital. Table 7 reports them together with the per capita gap in the initial endowment that prevents countries from realizing at least subsistence consumption. From the total of 108 countries for which we have complete set of data, 98 are equipped with sufficient initial endowments, 6 have insufficient endowments and 4 have a parameter constellation preventing a solution to the problem.

We note in particular that low-income countries in our data sample suffer from insufficient endowments with initial capital stocks. Somewhat surprisingly, Saudi Arabia as a high income country is in such an initial position as well. It is important to remember that we calibrated an initial situation matching actual GNI during 2014. Whenever we find a country with insufficient endowments, it is a combination of reasons behind this finding. Initial production and subsistence consumption is too high combined with the dependence on capital stocks implied by output elasticities. Again, we compute the maximum subsistence consumption $\underline{\tilde{c}}^{\max}$ affordable at the initial and actual GNI during 2014. In most of the cases in Table 7 it is well below the poverty lines defined by the World Bank.

We take the Central African Republic as a particular country case. It is the low-income country with the highest per capita endowments with natural capital. Its endowment with produced capital stood at 2,374 US$ and falls short of the minimum requirements by 2,068 US$ per capita. Hence, this resource rich country cannot afford permanently a subsistence consumption of 437 US$. The latter is the US$ low-income poverty line equivalent at prices of 2014. An initial capital transfer of 2,374 US$ per capita would suffice, ceteris paribus, to lift the country up to an endowment that would allow to escape poverty. We consider as a case study an initial transfer by 2014 that increases per capita produced capital up to 5,000 US$.

This initial transfer allows the country to realize consumption in excess of its subsistence level. Due to the country's parameter constellation, consumption would decrease to its subsistence level. One reason for this is the comparably high share of resource rents in GDP. If the country would be less dependent on resources, long-run positive growth would be possible. The calibration results for the country's consumption path is displayed in Fig. 1. With the country's labor and resource shares $\beta = 0.164$ and $\gamma = 0.124$ as given by the PWT, consumption would peak at $t_1 = 69$ and decline afterwards (scenario 1). A lower resource share and a correspondingly higher labor share imply the peak in consumption to be postponed. With a sufficient low resource share, permanent consumption growth is possible. This, however, would probably demand deeper structural changes in the country's industry.

Next, we turn to countries' sustainability performance with respect to sustainability test (2)-(4). The results for each of the 98 countries with an existing solution can be found in the table in Appendix H. Here we note that 40 of these countries are characterized by full sustainable development according to all indicators.

88 out of the 98 countries with a solution to the problem are characterized by permanent non-negative consumption growth. Hence, sustainability test (2) indicates sustainability at any t and therefore also a sustainable development for these countries. The underlying mechanism leading to permanent non-negative growth is twofold. First, the prerequisite is asymptotically non-negative growth. This is solely determined by calibrating values of the model's parameters (see (18) for the condition). Second, during the transition period, initial endowments are also important. A country always qualifies for non-negative growth if the initial endowment gives rise to a high capital productivity above the steady-state value. If the initial capital productivity is instead lower initially, a high asymptotic interest rate is needed (see (20) for the conditions applying).

Next, we turn to the indicators I_{cg} and I_W for which we find just 10 countries with indicator values smaller than one. We report on them in Table 8 together with the maximum deviation from Weitzman's

the labor share in the PWT 9.1. One country (Togo) was excluded due to inconsistent output shares, i.e. the resource and labor share in GDP added up to more than 1. Malta was excluded as being the only country with a resource share in GDP of exactly zero which is not covered by the model's formulation above.

Table 7
Countries with insufficient initial endowments.

Country	Income group	$\tilde{k}_0$	$\tilde{s}_0$	$\tilde{h}_0$	$\underline{c}$	$\underline{c}^{max}$
Burundi	low	−119	−1,259	−4,482	247	28
Central African Rep.	low	−2,068	8,742	11,621	437	233
Mozambique	low	−34	1,543	4,719	399	294
Niger	low	−317	727	2,283	329	256
Saudi Arabia	high	82,029	−440,228	−298,461	3,884	975
Sierra Leone	low	470	−5,314	−7,945	289	95

Note: Countries with insufficient initial endowments in 2014. Negative numbers in columns 2 to 4 indicate the additional needs of physical, natural or human capital per capita to guarantee subsistence consumption given in column 5. Column 6 gives the maximum subsistence consumption feasible with the given initial endowments. All quantities in 2014 US$.

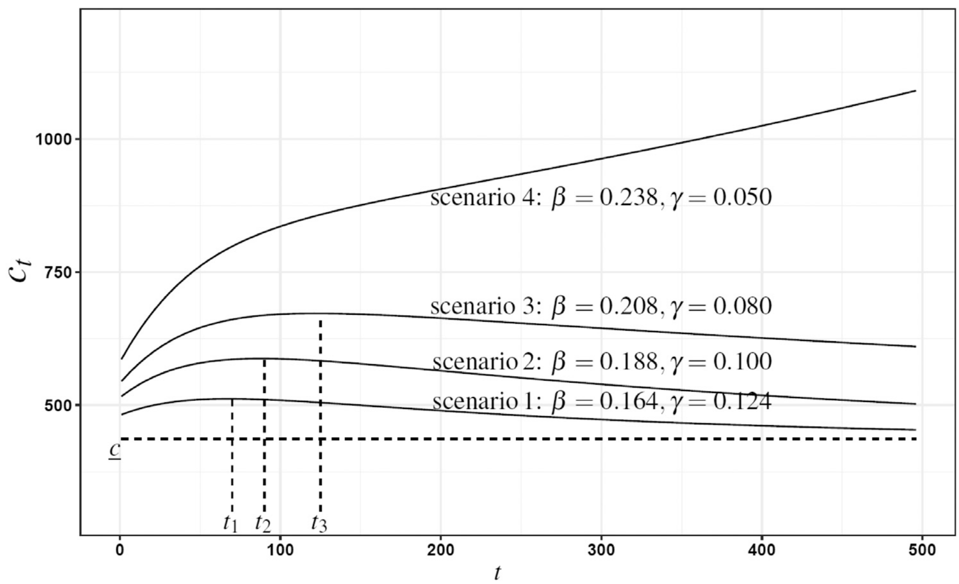

Fig. 1. Potential consumption development Central African Republic.
Note: Consumption paths for the Central African Republic with an initial stock of produced capital of 5,000 US$ with different resource and human capital shares in GDP as indicated in the graph. Scenario 1 reflects GDP shares as given by the PWT. Scenarios 2 - 3 use hypothetical lower resource (γ) and correspondingly higher labor shares (β), $\alpha = 0.712$ throughout the scenarios. Minimum subsistence consumption $\underline{c} = 437$ US$. Consumption peaks at $t_1 = 69$, $t_2 = 89$ and $t_3 = 125$. All quantities are expressed in 2014 US$.

Table 8
Countries with unsustainable development in $I_{\dot{c}\geq 0}$ and/or I_W.

Country	Income group	$I_{\dot{c}\geq 0}$	I_W	$\min(\tilde{c}_s - c_s)$
Azerbaijan	upper-middle	0.984	0.91	−48
Bosnia & Herzegovina	upper-middle	0.937	0.69	−74
Bulgaria	upper-middle	0.967	0.88	−101
Ecuador	upper-middle	0.842	1.00	1,821
Latvia	high	0.441	0.00	−365
Suriname	upper-middle	0.069	0.98	−10
Swaziland	lower-middle	0.570	1.00	952
Switzerland	high	0.612	1.00	19,107
Tajikistan	lower-middle	0.239	1.00	137
Ukraine	lower-middle	0.00	0.00	−156

Note: Countries with negative consumption growth and/or failing Weitzman's (1976, 1997) sustainability test. $\min(\tilde{c}_s - c_s)$ gives maximum deviation from Weitzman's benchmark annuity value for consumption in 2014 US$; negative values imply non-sustainability at least at one point in time.

(1976) benchmark value for consumption. Weitzman's test identifies all countries not capable of sustainable development according to non-negative growth. Non-negative growth, however, identifies countries not conspicuous under the Weitzman test.

We note that mostly middle income countries experience deviations from sustainable development with respect to these tests. The exceptions are Latvia and Switzerland which suffer from negative consumption growth. Latvia is additionally characterized by consumption exceeding Weitzman's benchmark annuity value by, however, a rather moderate amount. The conclusions from Table 8 carry over to the steady-state with respect to Weitzman's test with all countries behaving asymptotically not sustainable.

5. Discussion

The sustainability tests that we applied are consumption based but look at different aspects of consumption. Hence, at least some differences in results are to be expected. Table 9 provides an overview of the results. We can observe that most of the countries failing at one test do also not pass at least one other test.

Naturally, if no solution to the problem exists (sustainability test 1), the country will fail all other tests on sustainable development too. Based on consumption growth, 20 countries fail in case of sustainability test (2) and only 3 do not fail at any test. All countries that fall short of sustainable development according to Weitzman's test also do not pass at least one of the other tests.

False signals arising from sustainability at a particular instance in time are a critical issue. Whether a type I or II false signal is more serious, might depend on the particular question addressed. A type I false signal might influence policy to be inactive even if sustainable development is on its agenda. A false signal of type II might draw an overly pessimistic picture as from some future point in time sustainable development will be realized. This might even be the case if policy aims at sustainable development and sacrifices initial sustainability to allow for this (Cairns and Martinet, 2014, 2021).

We investigate false signals from the perspective of time $t = 0$, i.e. we are identifying false signals that occur during 2014 the year at

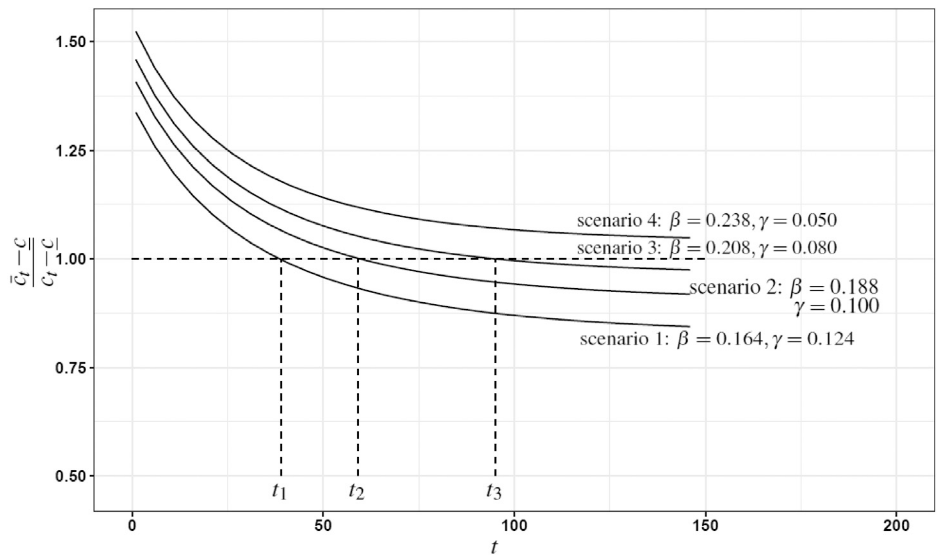

Fig. 2. Weitzman's sustainability test potential scenarios Central African Republic.
Note: Paths of $\frac{\bar{c}_t - \underline{c}}{c_t - \underline{c}}$ for the Central African Republic with an initial stock of produced capital of 5,000 US$ with different resource and human capital shares in GDP as indicated. Scenario 1 reflects GDP shares as given by the PWT. Scenarios 2 - 3 use hypothetical lower resource (γ) and correspondingly higher labor shares (β). Values of 1 or above indicate sustainability. Scenario 1 (2,3) yield a value of 1 at $t_1 = 39$, $t_2 = 59$ and $t_3 = 95$.

Table 9
Number of countries failing at sustainable development.

Also failing — Test:	(1) Existence of solution	(2) Consumption growth	(3) Weitzman (1976, 1997)
None other test	0	3	0
Existence solution	**10**	10	10
Consumption growth	10	**20**	16
Weitzman (1976, 1997)	10	16	**16**

Note: Number of countries on main diagonal in the lower part gives the total number of countries failing a particular test. Off main diagonal table elements give the number of countries failing at least the particular pair of test.

Table 10
Number of false signals.

Test:	(1) Existence of solution	(2) Consumption growth	(3) Weitzman (1976, 1997)
Type I false signal	0	3	3
Type II false signal	0	5	1

Note: Number of countries where false signals of type I and II according to Definition 4 are observed.

which our calibration starts. Sustainability test (1) is unable to produce false signals as explained in Section 3.3. Sustainability tests (2) and (3) are both characterized by moderate numbers on false signals. Table 10 provides the number of type I and II false signals.

For particular cases of type I false signals, we refer back to the case of the Central African Republic. Consumption paths are given in Fig. 1 and we observe consumption peaks for scenarios 1–3 at times t_1, t_2 and t_3. The sustainability test based on non-negative consumption growth would give type I false signals on sustainable development at times $t \in [0, t_1]$, $t \in [0, t_2]$ and $t \in [0, t_3]$, respectively. A similar picture emerges in case of the sustainability test based on Weitzman' test. Fig. 2 shows the development of $\frac{\bar{c}_t - \underline{c}}{c_t - \underline{c}}$. A value of 1 or above at time t indicates sustainability at t. Also in this case we would observe type I false signals on sustainable development at times $t \in [0, t_1]$, $t \in [0, t_2]$ and $t \in [0, t_3]$.

So far, the number of cases not characterized by sustainable development is limited. However, one might criticize the underlying assumptions in our analysis as too optimistic. The Cobb–Douglas production technology (2) naturally implies an elasticity of substitution equal to one which might be seen as overly high. Assuming e.g. a CES production function with a lower elasticity of substitution between all three input factors would necessarily boil down to a steady-state where at best minimum subsistence consumption could be realized. As resource input is then a gross complement to physical and human capital, capital accumulation would lead to a higher demand for resources. This would remove the possibility of positive consumption growth at least in the limit. An interesting intermediate case emerges if we treat physical capital and resource input as gross complements but allow human capital to substitute for resource use. This could be implemented by the following production function

$$Y_t = \left[\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \left((H_t u_t L_t)^{\frac{\beta}{1-\alpha}} R_t^{\frac{\gamma}{1-\alpha}} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{25}$$

which is identical to (2) in case the elasticity of substitution σ equals 1.[40] Appendix G analyzes the economy's steady-state for $\sigma < 1$. In particular, the steady-growth behavior and the conditions for an existence of a solution are derived. We find the following: First, steady-state growth rates are unaffected by σ. Second, for $\sigma < 1$ we observe three possibilities for limiting consumption as given by (18). Third and unsurprisingly, a lower σ influences the existence conditions which now become more stringent. As such, the results from the first sustainability test would be affected in case $\sigma < 1$. Unfortunately, the production technology (25) does not allow for explicit solutions tracing the transitional dynamics.

6. Conclusion

We summarize by highlighting the two major contributions made in this paper. First, the study make a technical contribution by solving complex dynamic problems using special functions. Second, from an economic point of view, the solution method allows us to calibrate the model to individual countries' endowment situation and to trace out the entire dynamics during adjustment towards their steady-state.

This allows us to analyze the behavior of several sustainability tests during the adjustment period as well as in the steady-state. This has not

[40] This specification resembles the steady-state result in Di Maria and Valente (2008) where directed technical change is purely resource augmenting. In our case, human capital accumulation takes the role of technical change.

been done in the literature before as an analytical solution for the adjustment paths is required for this. We find that the model's predictions for most countries in our sample of 108 economies are characterized by sustainable development. We furthermore investigated the issue of false signals that might be sent by sustainability tests. The number of cases where such false signals could arise is, however, moderate.

Our contribution deals with the centralized solution to the welfare maximization problem. Of course, the question arises how this solution can be implemented in a decentralized equilibrium. It would be straightforward to start searching for suitable tools such as taxes or subsidies and analyze issues related to the timing of such policies as in Acemoglu et al. (2012). We leave this task for future research that can provide valuable insights at the individual country level.

CRediT authorship contribution statement

Jürgen Antony: Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Torben Klarl:** Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing.

Appendix A. Co-states

We start by solving for the co-states $\lambda_{i,t}$, $i = 1,2,3$. Note that (13) directly implies $\lambda_{3,t} = \lambda_{3,0}$ which is the initial value for the resource' shadow value.

Proceeding with $\lambda_{2,t}$, we find conditions (10) and (12) to imply

$$\lambda_{1,t}\beta A K_t^{\alpha}(H_t u_t L_t)^{\beta-1} R_t^{\gamma} H_t L_t = \lambda_{2,t} B H_t, \tag{26}$$

$$\lambda_{1,t}\gamma A K_t^{\alpha}(H_t u_t L_t)^{\beta} R_t^{\gamma-1} = \lambda_{3,t}. \tag{27}$$

From (9) and (11) we know that

$$\dot{\lambda}_{1,t} = -\lambda_{1,t}\alpha A K_t^{\alpha-1}(H_t u_t L_t)^{\beta} R_t^{\gamma} + \lambda_{1,t}\delta_1, \tag{28}$$

$$\dot{\lambda}_{2,t} = -\lambda_{1,t}\beta A K_t^{\alpha}(H_t u_t L_t)^{\beta-1} R_t^{\gamma} u_t L_t - \lambda_{2,t} B(1-u_t) + \lambda_{2,t}\delta_2. \tag{29}$$

Using (26) in (29) gives

$$\dot{\lambda}_{2,t} = -\lambda_{2,t} B u_t - \lambda_{2,t} B(1-u_t) + \lambda_{2,t}\delta_2 = -\lambda_{2,t}(B-\delta_2),$$

which directly implies

$$\lambda_{2,t} = \lambda_{2,0} e^{-(B-\delta_2)t}, \tag{30}$$

where $\lambda_{2,0}$ is the initial value of the co-state variable $\lambda_{2,t}$ at time $t = 0$. It is this simple time path for the evolution of human capital's shadow price that allows for a closed form solution of the above problem. (30) takes such a simple form because human capital creation is linear in its own stock given u_t and is not directly depending on K_t, S_t or R_t.

Solving for the path of $\lambda_{1,t}$ is a bit more complex. Dividing both sides of (26) by (27) by each other gives

$$\frac{R_t}{BH_t u_t} = \frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}. \tag{31}$$

Rearranging (26) and using (31) yields

$$\begin{aligned}
\lambda_{1,t}\beta A K_t^{\alpha}(H_t u_t L_t)^{\beta} R_t^{\gamma} &= \lambda_{2,t} B H_t u_t, \\
\beta A \left(\frac{K_t}{BH_t u_t}\right)^{\alpha}\left(\frac{R_t}{BH_t u_t}\right)^{\gamma}\left(\frac{L_t}{B}\right)^{\beta} &= \frac{\lambda_{2,t}}{\lambda_{1,t}} \\
\beta A \left(\frac{K_t}{BH_t u_t}\right)^{\alpha}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}\right)^{\gamma} &= \frac{\lambda_{2,t}}{\lambda_{1,t}}\left(\frac{L_t}{B}\right)^{-\beta} \\
\frac{K_t}{BH_t u_t} &= \left(\frac{\lambda_{2,t}}{\lambda_{1,t}}\frac{1}{\beta A}\right)^{\frac{1}{\alpha}}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{-\frac{\beta}{\alpha}} \\
&= A^{-\frac{1}{\alpha}}\lambda_{1,t}^{-\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{\frac{1-\gamma}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{-\frac{\beta}{\alpha}}
\end{aligned} \tag{32}$$

At this point it is helpful to introduce additional variables that simplify the notation and are useful to solve the model. Define

$$\varphi_1 = A^{-\frac{1}{\alpha}}\lambda_{1,0}^{\frac{\alpha-1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{\alpha}}, \quad \varphi_2 = \frac{1-\alpha}{\psi},$$

$$\zeta = \frac{\varphi_2-\varphi_1}{\varphi_2}, \quad x_t = e^{-\psi t}, \quad \psi = \frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}. \tag{33}$$

As t runs from 0 to ∞, x_t develops from 1 to 0. x_t makes it convenient to solve the model using the Gaussian hypergeometric function. With results (28), (31) and (32), $\lambda_{1,t}$ develops as

$$\begin{aligned}
\dot{\lambda}_{1,t} &= -\lambda_{1,t}\alpha A\left(\frac{K_t}{L_t H_t u_t}\right)^{\alpha-1}\left(\frac{R_t}{L_t H_t u_t}\right)^{\gamma} + \lambda_{1,t}\delta_1, \\
&= -\lambda_{1,t}\alpha A\left(\frac{L_t}{B}\right)^{\beta}\left(\frac{K_t}{BH_t u_t}\right)^{\alpha-1}\left(\frac{R_t}{BH_t u_t}\right)^{\gamma} + \lambda_{1,t}\delta_1, \\
&= -\alpha A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,t}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}} + \lambda_{1,t}\delta_1, \\
&= -\alpha A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}} e^{\frac{\beta(B-\delta_2+n)}{\alpha}t} + \lambda_{1,t}\delta_1.
\end{aligned} \tag{34}$$

(34) takes the form of a Bernoulli equation for $\lambda_{1,t}$. Defining $m_t = \lambda_{1,t}^{1-\frac{1}{\alpha}}$ implies

$$\begin{aligned}
\dot{m}_t &= \frac{\alpha-1}{\alpha}\lambda_{1,t}^{-\frac{1}{\alpha}}\dot{\lambda}_{1,t} \\
&= -\frac{1-\alpha}{\alpha}\delta_1 m_t + (1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}} e^{\frac{\beta(B-\delta_2+n)}{\alpha}t},
\end{aligned}$$

which has the solution

$$\begin{aligned}
m_t &= e^{-\frac{1-\alpha}{\alpha}\delta_1 t}\Bigg[m_0 + (1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}} \\
&\quad \times \left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\int_0^t e^{\frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}z}dz\Bigg], \\
&= e^{-\frac{1-\alpha}{\alpha}\delta_1 t}\left[m_0 + (1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\frac{e^{\psi t}-1}{\psi}\right].
\end{aligned}$$

Using (33) and replacing for m_t gives

$$\begin{aligned}
\lambda_{1,t} &= e^{\delta_1 t}\left(\frac{\varphi_1}{\lambda_{1,0}^{\frac{\alpha-1}{\alpha}}}\right)^{\frac{\alpha}{1-\alpha}}\varphi_2^{\frac{\alpha}{\alpha-1}}e^{-\frac{\alpha}{1-\alpha}\psi t}(1-\zeta x_t)^{\frac{\alpha}{\alpha-1}} \\
&= \lambda_{1,0}e^{\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{\frac{\alpha}{1-\alpha}}.
\end{aligned} \tag{35}$$

Appendix B. Capital productivity

Capital productivity $\frac{Y_t}{K_t} = \frac{y_t}{k_t}$ with $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$ is obtained by rewriting (10) as $\beta Y_t = \frac{\lambda_{2,t}}{\lambda_{1,t}} B H_t u_t$, which implies $\frac{Y_t}{K_t} = \frac{1}{\beta}\frac{\lambda_{2,t}}{\lambda_{1,t}}\frac{BH_t u_t}{K_t}$. Making use of (32) and the results for $\lambda_{1,t}$ and $\lambda_{2,t}$ in (30) and (35) gives

$$\begin{aligned}
\frac{Y_t}{K_t} &= A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1-\alpha}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}} \\
&= A^{\frac{1}{\alpha}}\lambda_{1,0}^{\frac{1-\alpha}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}} \\
&\quad \times \frac{\varphi_1}{\varphi_2}\frac{x_t}{1-\zeta x_t}e^{\frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}t}.
\end{aligned}$$

Using the definitions of ψ, x_t, ζ and φ_1 from (33) gives the result in Lemma 1 for $\zeta = \zeta^*$

$$\frac{y_t}{k_t} = \frac{Y_t}{K_t} = \frac{1}{\varphi_2}(1-\zeta x_t)^{-1}.$$

Appendix C. Physical capital

Solution to the differential eq. (3). The physical capital stock K_t behaves according to

$$\dot{K}_t = Y_t - c_t L_t - \delta_1 K_t = K_t \frac{Y_t}{K_t} - c_t L_t - \delta_1 K_t.$$

Making use of Lemma 1 derived in Appendix B and $L_t = L_0 e^{nt}$ gives

$$\dot{K}_t = \left[\frac{1}{\varphi_2}(1-\zeta x_t)^{-1} - \delta_1\right] K_t - c_t L_0 e^{nt}.$$

This differential equation can be solved in a straightforward manner. To do so, we reformulate this differential equation by making use of Proposition 1's representation of c_t into a standard text book form used for finding the solution

$$\dot{K}_t + f_1(t) K_t = g_1(t), \tag{36}$$

with

$$f_1(t) = -\left[\frac{1}{\varphi_2}(1-\zeta x_t)^{-1} - \delta_1\right],$$

$$g_1(t) = -(c_t - \underline{c})L_t - \underline{c}L_t = -\lambda_{1,t}^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}-n\right)t} L_0 - \underline{c} e^{nt} L_0,$$

where $-f(t)$ is the net return on physical capital at time t. We denote the initial stock of capital at $t=0$ by K_0. The solution to the differential Eq. (36) is given by

$$K_t = K_0 e^{-\int_0^t f_1(z)dz} + \int_0^t g_1(z) e^{-\int_z^t f_1(s)ds} dz. \tag{37}$$

Building the integral $\int_z^t f_1(s)ds$ and using $x_z = e^{-\psi z}$ gives

$$\begin{aligned} -\int_z^t f_1(s)ds &= -\delta_1(t-z) + \int_z^t \left[\frac{1}{\varphi_2}(1-\zeta e^{-\psi s})^{-1}\right] ds \\ &= -\delta_1(t-z) + \frac{1}{1-\alpha} \ln\left[\frac{(1-\zeta x_t)x_t^{-1}}{(1-\zeta x_z)x_z^{-1}}\right]. \end{aligned} \tag{38}$$

Using (38) in (36) gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t}\left(\frac{\varphi_1 + \varphi_2(x_t^{-1}-1)}{\varphi_1}\right)^{\frac{1}{1-\alpha}} \\ &- \int_0^t (c_z - \underline{c}) L_z e^{-\delta_1(t-z)} \left(\frac{(1-\zeta x_t)x_t^{-1}}{(1-\zeta x_z)x_z^{-1}}\right)^{\frac{1}{1-\alpha}} dz \\ &- \int_0^t \underline{c} L_z e^{-\delta_1(t-z)} \left(\frac{(1-\zeta x_t)x_t^{-1}}{(1-\zeta x_z)x_z^{-1}}\right)^{\frac{1}{1-\alpha}} dz. \end{aligned}$$

Inserting (8) and rearranging delivers

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t}(1-\zeta)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_0^t \lambda_{1,z}^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}-n\right)z} L_0 e^{-\delta_1(t-z)} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz \\ &- \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_0^t \underline{c} e^{nz} L_0 e^{-\delta_1(t-z)} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz. \end{aligned}$$

Using (35) and rearranging gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t}(1-\zeta)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- e^{-\delta_1 t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &\times L_0 \int_0^t e^{-\left(\frac{\rho}{\eta}-n-\delta_1+\frac{\delta_1}{\eta}\right)z} x_z^{\frac{\eta-\alpha}{(1-\alpha)\eta}} (1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}} dz \\ &- e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \underline{c} L_0 \int_0^t e^{(n+\delta_1)z} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz. \end{aligned}$$

Using $x_z = e^{-\psi z}$ gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t}(1-\zeta)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- e^{-\delta_1 t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &\times L_0 \int_0^t x_z^{\frac{1}{\psi}\left(\frac{\rho}{\eta}+\psi\frac{\eta-\alpha}{(1-\alpha)\eta}-\delta_1+\frac{\delta_1}{\eta}-n\right)} (1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}} dz \\ &- e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \underline{c} L_0 \int_0^t x_z^{\frac{1}{\psi}\left(\frac{\psi}{1-\alpha}-n-\delta_1\right)} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz. \end{aligned}$$

Changing the domain of integration from z to dx_z with $dz = -\frac{1}{\psi}x_z^{-1}dx_z$ and integrating from x_t to 1 instead of 0 to t gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t}(1-\zeta)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- e^{-\delta_1 t}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &\times \frac{L_0}{\psi} \int_{x_t}^1 x_z^{\frac{1}{\psi}\left(\frac{\rho}{\eta}+\psi\frac{\eta-\alpha}{(1-\alpha)\eta}-\delta_1+\frac{\delta_1}{\eta}-n\right)-1} (1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}} dxz \\ &- e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \underline{c}\frac{L_0}{\psi} \int_{x_t}^1 x_z^{\frac{1}{\psi}\left(\frac{\psi}{1-\alpha}-n-\delta_1\right)-1} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dx_z. \end{aligned} \tag{39}$$

The integrals in (39) – as long as they converge – can be evaluated using the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ which has in general the integral representation

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \tag{40}$$

This integral representation is valid for $\mathbb{R}(c) > \mathbb{R}(b) > 0$ where $\mathbb{R}(\cdot)$ denotes the real part of the argument and $\Gamma(\cdot)$ the Gamma function (Abramowitz and Stegun, 1972, 15.3.1). In general, ${}_2F_1(a,b;c;z)$ defined as a Gauss series (Abramowitz and Stegun, 1972, 15.1.1) converges if $|z| < 1$. It also converges if additionally $\mathbb{R}(c-b-a) > 0$ for $|z| \leq 1$ and if $-1 < \mathbb{R}(c-b-a) \leq 0$ for $|z| \leq 1$ but $z \neq 1$. Comparing the integral on the right hand side of (40) with the integrals in (39) reveals that the present case can be seen as a special case with $c-b-1 = 0$ or equivalent $c = b+1$. And hence, $\mathbb{R}(c) > \mathbb{R}(b)$ holds. We will see shortly that $\mathbb{R}(b) > 0$ poses no problem for the model's parametrization. If we apply the representation (40) to our problem, ζ will play the role of z. We already saw above that $\zeta < 1$ holds. If $\lambda_{1,0}$ is sufficiently small and/or $\lambda_{2,0}$ or $\lambda_{3,0}$ are sufficiently large, it might turn out that $\zeta \leq -1$. In this case, one has to think about how to compute the integrals in (39) or other integrals of the same type that appear further down below. This is because the integral representation (40) is an analytic continuation of the Gaussian hypergeometric function defined by a Gauss series (Abramowitz and Stegun, 1972, 15.3.1). Only for the restrictions on z and $\mathbb{R}(c-b-a)$ laid out above, both are identical. In general, for $z \leq -1$ and $\mathbb{R}(c) > \mathbb{R}(b) > 0$, the integral (40) exists but the Gauss series that defines the hypergeometric function is not converging and hence, is not identical to the integrals that we aim to compute. In such cases, it is necessary to use analytic continuation formulas for ${}_2F_1(a,b;c;z)$ (see Abramowitz and Stegun, 1972, 15.3.3 through 15.3.9).[41]

We can therefore make use of

$$\begin{aligned} {}_2F_1(a,b;b+1;z) &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt \\ &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1}(1-zt)^{-a} dt \\ &= b\int_0^1 t^{b-1}(1-zt)^{-a} dt, \end{aligned}$$

where we applied the gamma function's continuation $\Gamma(b+1) = b\Gamma(b)$ and that $\Gamma(1) = 1$ (Abramowitz and Stegun, 1972, 6.1.15). Note that

[41] For a general discussion about this situation see Section 3.1 in Boucekkine and Ruiz-Tamarit (2008).

we need to keep in mind that $z \leq -1$ needs special attention. Inspecting (39) shows that we can apply this special case of the Gaussian hypergeometric function to both integrals. Through a suitable change in the variable of integration, the integrals ranging from x_t to 1 can be split up into two separate integrals each running from 0 to 1 and each representable by the hypergeometric function. This results in

$$\begin{aligned} K_t =& K_0 e^{-\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \\ & - e^{-\delta_1 t} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) \right. \\ & \left. - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)\right] \\ & - \underline{c} e^{-\delta_1 t} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \frac{1}{\psi}\frac{1}{\tilde{b}_2} \\ & \times L_0 \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)\right], \end{aligned} \tag{41}$$

with

$$\begin{aligned} \tilde{a}_1 &= \frac{\eta-\alpha}{\eta(1-\alpha)}, \\ \tilde{b}_1 &= \frac{1}{\psi}\left(\frac{\rho}{\eta} + \frac{\eta-\alpha}{(1-\alpha)\eta}\psi + \frac{1-\eta}{\eta}\delta_1 - n\right) \\ &= 1 + \frac{\alpha}{\eta(1-\alpha)} \frac{(1-\alpha)(\rho-n) + (\eta-1)\left[\beta(B-\delta_2+n) - (1-\alpha)n\right]}{\beta(B-\delta_2+n) + (1-\alpha)\delta_1}, \\ \tilde{a}_2 &= \frac{1}{1-\alpha} > 1, \\ \tilde{b}_2 &= \frac{1}{1-\alpha} - \frac{\delta_1}{\psi} - \frac{n}{\psi} = 1 + \frac{\beta(B-\delta_2+n) - (1-\alpha)n}{(1-\alpha)\psi}. \end{aligned}$$

where we require that $\tilde{b}_1, \tilde{b}_2 > 0$. In case these inequalities do not hold, the integrals in (39) are not finite and the economy's needs for accumulating physical capital to solve the problem would be indefinitely large and, hence, no solution would exist.

Transversality condition K_t. We have to show that $\lim_{t\to\infty} \lambda_{1,t} K_t = 0$. Using $\lambda_{1,t}$ given by (35) and K_t given by (41), $\lambda_{1,t} K_t$ reads as

$$\begin{aligned} \lambda_{1,t} K_t =& \lambda_{1,0} K_0 (1-\zeta)^{-1} \frac{1-\zeta x_t}{x_t} \\ & - \lambda_{1,0}^{1-\frac{1}{\eta}} (1-\zeta)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}} \frac{1-\zeta x_t}{x_t} \frac{1}{\psi}\frac{1}{\tilde{b}_1} \\ & \times L_0 \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)\right] \\ & - \lambda_{1,0}\underline{c} (1-\zeta)^{\frac{\alpha}{1-\alpha}} \frac{1-\zeta x_t}{x_t} \frac{1}{\psi}\frac{1}{\tilde{b}_2} \\ & \times L_0 \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)\right]. \end{aligned}$$

As $t \to \infty$ we see $x_t \to 0$. and $x_t^{-1} \to \infty$. Rewriting $\lambda_t K_t$ as $\frac{x_t \lambda_t K_t}{x_t}$ and applying L'Hospital's rule as $x_t \to 0$ requires $\lim_{x_t \to 0} \frac{\partial x_t \lambda_t K_t}{\partial x_t} = 0$. $\frac{\partial x_t \lambda_t K_t}{\partial x_t}$ using the above expression is given by

$$\begin{aligned} \frac{\partial x_t \lambda_t K_t}{\partial x_t} =& -\lambda_{1,0} K_0 (1-\zeta)^{-1} \zeta \\ & + \lambda_{1,0}^{1-\frac{1}{\eta}} (1-\zeta)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}} \zeta \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \\ & \times \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)\right] \\ & + \lambda_{1,0}^{1-\frac{1}{\eta}} (1-\zeta)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}} (1-\zeta x_t) \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \\ & \times \left[\tilde{b}_1 x_t^{\tilde{b}_1 - 1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)\right] \\ & + \lambda_{1,0}^{1-\frac{1}{\eta}} (1-\zeta)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}} (1-\zeta x_t) \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \\ & \times \left[x_t^{\tilde{b}_1} \frac{\partial {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)}{\partial x_t}\right] \\ & + \lambda_{1,0}\underline{c} (1-\zeta)^{\frac{\alpha}{1-\alpha}} \zeta \frac{1}{\psi}\frac{1}{\tilde{b}_2} L_0 \\ & \times \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)\right] \\ & + \lambda_{1,0}\underline{c} (1-\zeta)^{\frac{\alpha}{1-\alpha}} (1-\zeta x_t) \frac{1}{\psi}\frac{1}{\tilde{b}_2} L_0 \\ & \times \left[\tilde{b}_2 x_t^{\tilde{b}_2 - 1} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)\right] \\ & + \lambda_{1,0}\underline{c} (1-\zeta)^{\frac{\alpha}{1-\alpha}} (1-\zeta x_t) \frac{1}{\psi}\frac{1}{\tilde{b}_2} L_0 \\ & \times \left[x_t^{\tilde{b}_2} \frac{\partial {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)}{\partial x_t}\right]. \end{aligned}$$

Evaluating $\frac{\partial x_t \lambda_t K_t}{\partial x_t}$ at $x_t = 0$ gives as long as $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$ and because $\frac{{}_2F_1(a,b;b+1;0)}{b} = 1$

$$\begin{aligned} \left.\frac{\partial x_t \lambda_t K_t}{\partial x_t}\right|_{x_t=0} =& -\lambda_{1,0} K_0 (1-\zeta)^{-1} \zeta \\ & + \lambda_{1,0}^{1-\frac{1}{\eta}} (1-\zeta)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}} \zeta \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)\right] \\ & + \lambda_{1,0}\underline{c} (1-\zeta)^{\frac{\alpha}{1-\alpha}} \zeta \frac{1}{\psi}\frac{1}{\tilde{b}_2} L_0 \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)\right]. \end{aligned}$$

For transversality to hold, it is then required additionally that $\frac{\partial x_t \lambda_t K_t}{\partial x_t} \Big|_{x_t=0} = 0$ which implies

$$\begin{aligned} K_0 = \frac{L_0}{\psi} & \left[\lambda_{1,0}^{-\frac{1}{\eta}} (1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{\tilde{b}_1} \right. \\ & \left. \times + \underline{c} (1-\zeta)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)}{\tilde{b}_2}\right]. \end{aligned} \tag{42}$$

Inserting this into (41) gives K_t as

$$\begin{aligned} K_t =& e^{-\delta_1 t} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \frac{L_0}{\psi} \\ & \times \left[\lambda_{1,0}^{-\frac{1}{\eta}} (1-\zeta)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}} \frac{x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)}{\tilde{b}_1} \right. \\ & \left. + \underline{c} \frac{x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)}{\tilde{b}_2}\right]. \end{aligned}$$

Dividing by $L_t = L_0 e^{nt}$ and evaluating at $\zeta = \zeta^*$ gives the result in Lemma 2.

In the limit, $t \to \infty$ and $x_t \to 0$, $\lim_{t\to\infty} k_t$ can behave in different ways depending on the models parameters.[42] We find the same three cases that need to be differentiated as in case of consumption.

$$\lim_{t\to\infty} k_t = \begin{cases} \frac{\underline{c}}{\psi \tilde{b}_2} & \text{for} \quad \beta(B-\delta_2+n) - (1-\alpha)\rho < 0, \\ \frac{\underline{c}}{\psi \tilde{b}_2} + \frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{\psi} \frac{(1-\zeta)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1} & \text{for} \quad \beta(B-\delta_2+n) - (1-\alpha)\rho = 0, \\ \frac{\underline{c}}{\psi \tilde{b}_2} + \frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{\psi} \frac{(1-\zeta)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1} \times & \text{for} \quad \beta(B-\delta_2+n) - (1-\alpha)\rho > 0. \\ \quad \lim_{t\to\infty} x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}} \to \infty & \end{cases}$$

Appendix D. Derivations related to H_t

Using (32) and (35) gives

$$\begin{aligned} BH_t u_t =& A^{\frac{1}{\alpha}} \lambda_{1,t}^{\frac{1}{\alpha}} \left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{1-\gamma}{\alpha}} \left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}} \left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}} K_t \\ =& A^{\frac{1}{\alpha}} \left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{1-\gamma}{\alpha}} \left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}} \left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}} \lambda_{1,t}^{\frac{1}{\alpha}} e^{\frac{(1-\gamma)(B-\delta_2)+\beta n}{\alpha} t} K_t \\ =& e^{[(B-\delta_2)+\delta_1]t} (1-\zeta)^{\frac{1}{1-\alpha}} \left(\frac{\lambda_{2,0}}{\beta}\right)^{-1} \frac{\lambda_{1,0}}{\varphi_1} x_t^{\frac{\alpha}{1-\alpha}} (1-\zeta x_t)^{-\frac{1}{1-\alpha}} K_t. \end{aligned} \tag{43}$$

[42] Note that ${}_2F_1(\tilde{a}, \tilde{b}; \tilde{b}+1; 0) = 1$.

(4) implies $\dot{H}_t = B(1-u_t)H_t - \delta_2 H_t = (B-\delta_2)H_t - Bu_t H_t$. Proceeding analogous to (37) gives

$$H_t = H_0 e^{-\int_0^t f_2(z)dz} + \int_0^t g_2(z) e^{-\int_z^t f_2(s)ds} dz,$$

with

$$f_2(z) = -(B-\delta_2), \quad g_2(z) = -BH_z u_z.$$

This delivers H_t as $H_t = H_0 e^{(B-\delta_2)t} - \int_0^t Bu_z H_z e^{(B-\delta_2)(t-z)}dz$. Using (43) gives

$$\int_0^t Bu_z H_z e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\int_0^t e^{\delta_1 z}(1-\zeta)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1} \times x_z^{\frac{\alpha}{1-\alpha}}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}K_z dz.$$

Inserting (41) for the physical capital stock yields

$$\begin{aligned}\int_0^t Bu_z\ H_z\ e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\Bigg\{ K_0\int_0^t x_z^{-1}dz \\ -\int_0^t \lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0 x_z^{-1} \\ \times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) - x_z^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)\right]dz \\ -\int_0^t \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0 x_z^{-1} \\ \times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_z^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)\right]dz\Bigg\}.\end{aligned}$$

Using $\zeta x_z = \zeta e^{-\psi z}$, $d\zeta x_z = -\zeta\psi e^{-\psi z}dz$, the integration rule

$$\int z^{b-2}{}_2F_1(a,b;c;z)dz = \frac{z^{b-1}}{b-1}{}_2F_1(a,b-1;c;z) + \text{constant} \quad \text{for} \quad b>1 \tag{44}$$

and adjusting the direction of integration delivers

$$\begin{aligned}\int_0^t Bu_z\ H_z\ e^{(B-\delta_2)(t-z)}dz &= e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1} \\ &\times\Bigg\{\int_{\zeta x_t}^{\zeta} K_0\frac{\zeta}{\psi}(\zeta x_z)^{-2}d\zeta x_z \\ &-\int_{x_t}^{1}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0(\zeta x_z)^{-2} \\ &\times{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)d\zeta x_z \\ &+\int_{\zeta x_t}^{\zeta}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0(\zeta x_z)^{\tilde{b}_1-2} \\ &\times{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)d\zeta x_z \\ &-\int_{\zeta x_t}^{\zeta}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_1}}{\psi^2}\frac{1}{\tilde{b}_2}L_0(\zeta x_z)^{-2} \\ &\times{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)d\zeta x_z \\ &+\int_{\zeta x_t}^{\zeta}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_2}}{\psi^2}\frac{1}{\tilde{b}_2}L_0(\zeta x_z)^{\tilde{b}_2-2} \\ &\times{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)d\zeta x_z\Bigg\}, \\ &= e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\Bigg\{K_0\frac{1}{\psi}(1-x_t^{-1}) \\ &-\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_{02}F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)(1-x_t^{-1}) \\ &+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0 \\ &\times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] \\ &-\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2}L_{02}F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)(1-x_t^{-1}) \\ &+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0 \\ &\times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\Bigg\},\end{aligned} \tag{45}$$

where we need to impose the parameter restriction

$$\tilde{b}_1 > 1 \quad \text{and} \quad \tilde{b}_2 > 1 \tag{46}$$

for the solution to exist. Using (45), H_t can now be computed as $H_t = e^{(B-\delta_2)t}H_0 - \int_0^t Bu_z H_z e^{(B-\delta_2)(t-z)}dz$

$$\begin{aligned}H_t &= e^{(B-\delta_2)t}H_0 - e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\Bigg\{K_0\frac{1}{\psi}(1-x_t^{-1}) \\ &-\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_{02}F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)(1-x_t^{-1}) \\ &+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0 \\ &\times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] \\ &-\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2}L_{02}F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)(1-x_t^{-1}) \\ &+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0 \\ &\times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\Bigg\}.\end{aligned}$$

Effective human capital $H_t u_t$ employed in final goods production follows next. Dividing both sides of (43) by B gives

$$\begin{aligned}H_t u_t &= e^{[(B-\delta_2)+\delta_1]t}(1-\zeta)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{1}{B}x_t^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)^{-\frac{1}{1-\alpha}}K_t \\ &= e^{[(B-\delta_2+n)+\delta_1]t}(1-\zeta)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{L_0}{B}x_t^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)^{-\frac{1}{1-\alpha}}k_t.\end{aligned} \tag{47}$$

Effective human capital in units of final output is $\tilde{H}_t u_t = p_{H,t}H_t u_t = \frac{\lambda_{2,t}}{\lambda_{1,t}}H_t u_t$. Using (30) and (35) in (47) gives

$$\tilde{H}_t u_t = e^{(B-\delta_2+n)t}\frac{\beta\psi}{1-\alpha}\frac{L_0}{B}(1-\zeta x_t)^{-1}k_t,$$

where it is clear that $\tilde{H}_t u_t$ even in case of a stationary k_t grows without bound as $t\to\infty$ for $B-\delta_2+n>0$. Inserting (41) for K_t and using the transversality condition for K_t gives effective human capital as a function of time via x_t

$$\begin{aligned}H_t u_t &= e^{[(B-\delta_2)]t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{1}{B}x_t^{-1}\times \\ &\times\Bigg\{K_0 - \lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{L_0}{\tilde{b}_1} \\ &\times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\ &-\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{L_0}{\tilde{b}_2} \\ &\times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\Bigg\} \\ &= e^{[(B-\delta_2)]t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{L_0}{B}\frac{x_t^{-1}}{\psi}\times \\ &\times\Bigg[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1}x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t) \\ &+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2}x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\Bigg], \\ \tilde{H}_t u_t &= e^{-\delta_1 t}x_t^{-\frac{1}{1-\alpha}}(1-\zeta)^{\frac{\alpha}{1-\alpha}}\frac{\beta}{1-\alpha}\frac{L_0}{B}\times \\ &\times\Bigg[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1}x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t) \\ &+\ \underline{c}\frac{1}{\tilde{b}_2}x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\Bigg].\end{aligned}$$

Next, we need to work out the transversality condition for H_t.

Transversality condition H_t. Transversality demands $\lim_{t\to\infty}\lambda_{2,t}H_t = 0$.

$$\lambda_{2,t}H_t = \lambda_{2,0}e^{-(B-\delta_2)t}H_0 e^{(B-\delta_2)t} - \lambda_{2,0}e^{-(B-\delta_2)t}\int_0^t Bu_z H_z e^{(B-\delta_2)(t-z)}dz$$

$$= \lambda_{2,0}\left(H_0 - \int_0^t Bu_z H_z e^{-(B-\delta_2)z} dz\right).$$

Using $\int_0^t Bu_z H_z dz$ given in (43) together with the transversality condition for K_t in (42) gives

$$\lambda_{2,t}H_t = \lambda_{2,0}H_0 - \lambda_{2,0}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)} \times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}.$$

As $t\to\infty$, $x_t\to 0$. With $\tilde{b}_1,\tilde{b}_2 > 1$ and by noting that $\lim_{z\to 0}{}_2F_1(\tilde{a},b-1;b+1;z)$ is finite, we find

$$\lim_{x_t\to 0}\lambda_{2,t}H_t = \lambda_{2,0}H_0 - \lambda_{2,0}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}.$$

Transversality consequently demands

$$H_0 = \left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \times{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}. \quad (48)$$

Imposing the transversality condition for K_t(42) onto (45) yields

$$\int_0^t Bu_z\ H_z\ e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)} \times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}.$$

Inserting (48) into this expression gives

$$\int_0^t Bu_z\ H_z\ e^{(B-\delta_2)(t-z)}dz = H_0e^{(B-\delta_2)t} - e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t) + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\},$$

and finally H_t and $\tilde{H}_t$ as

$$H_t = H_0e^{(B-\delta_2)t} - \int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t) + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\},$$

$$\tilde{H}_t = e^{-\delta_1 t}\frac{\beta}{1-\alpha}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{\alpha}{1-\alpha}}L_0 \times \times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t) \quad (49) + \underline{c}\frac{1}{\psi}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\}.$$

To arrive at the expression for $\tilde{H}_t$, we have to multiply H_t by $p_{H,t} = \frac{\lambda_{2,t}}{\lambda_{1,t}}$ where $\lambda_{1,t}$ and $\lambda_{2,t}$ are given by (30) and (35).

H_t tends to infinity as $t\to\infty$. This is because of the second term in curly brackets in the representation of H_t. It is easy to verify that $e^{(B-\delta_2)t}x_t^{\tilde{b}_2-1} = e^{\frac{\gamma}{1-\alpha}(B-\delta_2+n)t}$.[43] Hence, human capital necessary to cover subsistence consumption grows asymptotically at a positive rate. The first term in the curly brackets represents human capital necessary to cover excess consumption. This part of human capital might tend to zero or infinity depending on the model's parameters and consequently whether consumption tends to infinity. The responsible term $e^{(B-\delta_2)t}x_t^{\tilde{b}_1-1} = e^{-\frac{1}{(1-\alpha)\eta}\{(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]-\gamma\eta(B-\delta_2+n)\}t}$ tends to infinity as long as $-[\beta(B-\delta_2+n)-(1-\alpha)\rho]-\gamma\eta(B-\delta_2+n) < 0$. If this condition is not met, this part of human capital tends to zero. However, total human capital will always grow without bounds.

We proceed by analyzing the behavior of R_t and S_t.

Derivations involving R_t and S_t. Reformulating (31) using $L_t = L_0e^{nt}$ and the time path for the co-state $\lambda_{2,t}$ in (30) gives

$$R_t = L_tH_tu_t\frac{\lambda_{2,0}}{\beta}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left(\frac{L_0}{B}\right)^{-1}e^{-(B-\delta_2+n)t},$$

which yields together with (47) the resource use depending on the initial values of the co-states

$$R_t = \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}(1-\zeta)^{-\frac{1}{1-\alpha}}x_t^{\frac{\alpha}{1-\alpha}}K_t. \quad (50)$$

Expressing R_t as a function of t via x_t can be done by substituting K_t by (41) and using the transversality condition (42)

$$R_t = \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}x_t^{-1}\left\{K_0 - \lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0 \times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] - \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\right\} = \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}x_t^{-1}\frac{L_0}{\psi} \times \times\left[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1}x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t) + \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2}x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]. \quad (51)$$

We turn to $S_t = S_0 - \int_0^t R_sds$. Integration over R_t given by (51) gives

$$\int_0^t R_sds = \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left\{K_0\int_0^t x_s^{-1}ds - \lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\int_0^t x_s^{-1} \times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) - x_s^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_s)\right]ds - \underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\int_0^t x_s^{-1} \times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta) - x_s^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_s)\right]ds\right\}.$$

Using again the integration rule (44), $\zeta x_s = \zeta e^{-\psi s}$, $d\zeta x_s = -\zeta\psi e^{-\psi s}ds = -\psi\zeta x_sds$ (implying $ds = -\frac{1}{\psi}\frac{1}{\zeta x_s}d\zeta x_s$) and adjusting the direction of

[43] It is helpful to note that $\lim_{z\to 0}{}_2F_1(a,b-1;b+1,z) = 1$.

integration delivers

$$\begin{aligned}
\int_0^t R_s ds =& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\Bigg\{K_0\frac{\zeta}{\psi}\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s \\
&-\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s \\
&+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta^{1-\tilde{b}_1}}{\psi^2}\frac{1}{\tilde{b}_1}L_0\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{\tilde{b}_1-2}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_s)d\zeta x_s \\
&-\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_2}}{\psi^2}\frac{1}{\tilde{b}_2}L_0\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{\tilde{b}_2-2}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_s)d\zeta x_s\Bigg\}, \\
=& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\Bigg\{-K_0\frac{1}{\psi}\left[1-x_t^{-1}\right] \\
&+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0 \\
&\times\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\left[1-x_t^{-1}\right] \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0 \\
&\times\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\Bigg\}.
\end{aligned} \tag{52}$$

We proceed by working the transversality condition for S_t.

Transversality condition S_t. Transversality demands that $\lim_{t\to\infty}\lambda_{3,t}S_t = 0$. As $\lambda_{3,t}=\lambda_{3,0}$, this is equivalent to $\lim_{t\to\infty}S_t=0$ or $\int_0^t R_s ds = S_0$. Rearranging (52) yields

$$\begin{aligned}
\int_0^t R_s ds =& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left[1-x_t^{-1}\right]\Bigg\{-K_0\frac{1}{\psi} \\
&+\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\Bigg\} \\
&+\frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \\
&\times L_0\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\
&-x_t^{\tilde{b}_1-1}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\Big] \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\Big[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) \\
&-x_t^{\tilde{b}_2-1}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\Big]\Bigg\},
\end{aligned} \tag{53}$$

where we note that the first term in curly brackets is zero due to the transversality condition for K_t given by (42). As $\tilde{b}_1$ and $\tilde{b}_2$ need to be larger than one by (46), $x_t \to 0$ for $t\to\infty$ and we find[44]

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t R_s ds = S_0 =& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\frac{1}{\psi^2}L_0 \\
&\times\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\Bigg\},
\end{aligned} \tag{54}$$

as the transversality condition for S_t.

Inserting (42) and (54) into (53) gives S_t as

$$\begin{aligned}
S_t =& S_0-\int_0^t R_s ds \\
=& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\frac{1}{\psi^2}L_0 \\
&\times\Bigg[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t) \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\Bigg].
\end{aligned} \tag{55}$$

Derivation of results in Lemmas 4 and 5. Dividing the transversality condition (48) by (54) and using the definition of φ_1 from (15) by each other gives

$$\frac{H_0}{S_0}=\frac{\beta}{\gamma}\frac{\lambda_{3,0}}{\lambda_{2,0}}. \tag{56}$$

It follows from the definitions of φ_1, φ_2 and ζ in (15) that

$$\frac{\lambda_{1,0}}{\lambda_{2,0}}\beta = A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{\alpha}{1-\alpha}}(1-\zeta)^{-\frac{\alpha}{1-\alpha}}. \tag{57}$$

Using this again in the definition of φ_1 gives

$$\frac{\lambda_{1,0}}{\varphi_1}\frac{\beta}{\lambda_{2,0}} = A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}. \tag{58}$$

Using (58) in (47) together with the definitions in (15), $k_t=\frac{K_t}{L_t}$, and $L_t = L_0 e^{nt}$ gives the first result in Proposition 3. The development of $\tilde{H}_t$ follows after using (57) in (49) and multiplying by $p_{H,t}=\frac{\lambda_{2,t}}{\lambda_{1,t}}$. Evaluating this at $\zeta=\zeta^*$ gives the result in Lemma 4.

Using (50), dividing by L_t and using (56) together with (57) gives the result in Proposition 4 for $r_t=\frac{R_t}{L_t}$. The result for the per capita resource stock follows by dividing S_t given in (55) by L_t and using (56) together with (57). The per capita stock in final output equivalents in Lemma 5 is derived after multiplying by $p_{R,t}=\frac{\lambda_{3,t}}{\lambda_{1,t}}$ and evaluation at $\zeta=\zeta^*$.

Appendix E. Uniqueness of the solution ζ^*

The results (56), (57) and (58) used in the three transversality conditions for K_t, H_t and S_t (42), (48), (54) give

$$\begin{aligned}
K_0 =& \frac{L_0}{\psi}\Bigg[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1} \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}\Bigg],
\end{aligned} \tag{59}$$

$$\begin{aligned}
H_0 =& A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\times \\
&\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\Bigg\},
\end{aligned} \tag{60}$$

$$\begin{aligned}
S_0 =& A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\times \\
&\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\,{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\
&+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\Bigg\}.
\end{aligned} \tag{61}$$

The system of reformulated transversality conditions (59) through (61) is a system in just two variables, i.e. ζ and $\lambda_{1,0}$, that condenses the initial conditions $\lambda_{i,0}, i=1,2,3$. Further, (61) together with (56) implies (60), and hence, it is sufficient to concentrate either on (59) and (60) or (59) and (61) in solving for ζ in place of $\lambda_{i,0}, i=1,2,3$.

To arrive at a system of equations that summarizes initial conditions in only one variable, i.e. ζ, we need to define some additional quantities. We split up each state variable, i.e. each capital stock, into

[44] It is helpful to note that $\lim_{z\to 0}{}_2F_1(a,b-1;b+1,z)=1$.

a component used to cover subsistence consumption and a second component available for excess consumption. A solution can only exist if available stocks are able to cover both components.

$$\begin{aligned}
K_0^+ &= \frac{L_0}{\psi}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta)}{\tilde b_1},\\
\underline{K}_0 &= K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta)}{\tilde b_2}\\
&= K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\tilde a_2}\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta)}{\tilde b_2},\\
H_0^+ &= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\\
&\quad\times\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta)}{\tilde b_1(\tilde b_1-1)}\\
&= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\\
&\quad\times\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta)}{\tilde b_1(\tilde b_1-1)},\\
\underline{H}_0 &= H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\\
&\quad\times\frac{L_0}{\psi^2}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta)}{\tilde b_2(\tilde b_2-1)}\\
&= H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\\
&\quad\times\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta)}{\tilde b_2(\tilde b_2-1)},\\
S_0^+ &= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\\
&\quad\times\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta)}{\tilde b_1(\tilde b_1-1)},\\
\underline{S}_0 &= S_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\\
&\quad\times\underline{c}\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta)}{\tilde b_2(\tilde b_2-1)}.
\end{aligned}$$

K_0^+, H_0^+ and S_0^+ are the parts of the initial capital stocks that are required to allow for future consumption in excess of $\underline{c}$. $\underline{K}_0$, $\underline{H}_0$ and $\underline{S}_0$ are the parts of the initial capital stocks left for excess consumption after covering the needs for subsistence consumption. Taking ratios gives

$$\begin{aligned}
\frac{K_0^+}{H_0^+} &= A^{\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{1-\alpha}}\varphi_2^{\frac{1}{1-\alpha}}\psi(\tilde b_1-1)(1-\zeta)^{\frac{1}{1-\alpha}}\\
&\quad\times\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta)}{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta)},\\
\frac{\underline{K}_0}{\underline{H}_0} &= \frac{K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta)}{\tilde b_2}}{H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta)}{\tilde b_2(\tilde b_2-1)}}.
\end{aligned}$$

Equilibrium requires $\frac{K_0^+}{H_0^+}=\frac{\underline{K}_0}{\underline{H}_0}$. This defines one non-linear equation in ζ given initial values K_0, H_0 and S_0. Once a solution ζ^* is found, it will pin down $\lambda_{1,0}^*$ through e.g. (59). This pins down $\lambda_{2,0}^*$ via (57). $\lambda_{3,0}^*$ can then be computed via e.g. (56). The equilibrium value ζ^* satisfies $\frac{K_0^+}{H_0^+}=\frac{\underline{K}_0}{\underline{H}_0}$, with

$$\begin{aligned}
\frac{K_0^+}{H_0^+} &= A^{\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{1-\alpha}}\varphi_2^{\frac{1}{1-\alpha}}\psi(\tilde b_1-1)(1-\zeta^*)^{\frac{1}{1-\alpha}}\\
&\quad\times\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta^*)},
\end{aligned}\tag{62}$$

$$\frac{\underline{K}_0}{\underline{H}_0} = \frac{K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta^*)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{\tilde b_2}}{H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)}{\tilde b_2(\tilde b_2-1)}}.\tag{63}$$

We first notice that (62) and (63) demand $\zeta<1$.

We show first that $\frac{K_0^+}{S_0^+}$ given by (62) is decreasing in ζ^*. Second, we show that $\frac{\underline{K}_0}{\underline{S}_0}$ given by (63) is increasing in ζ^*. This implies that there can be at most one solution to $\frac{K_0^+}{H_0^+}=\frac{\underline{K}_0}{\underline{H}_0}$.

Investigating $\frac{K_0^+}{S_0^+}$, we have to distinguish three cases, i.e. $\tilde a_1<0, \tilde a_1=0, \tilde a_1>0$.

Case 1: $\tilde a_1<0$: Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that $\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta^*)}$ is decreasing in ζ^* in case $\tilde a_1<0$. It is obvious that $(1-\zeta^*)^{\tilde a_2}$ is decreasing in ζ^* as well because $\tilde a_2=\frac{1}{1-\alpha}>0$. Therefore, $\frac{K_0^+}{S_0^+}$ is in this case decreasing in ζ^*.

Case 2: $\tilde a_1=0$: This case prevails if it happens to be that $\eta=\alpha$. Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that in this case $\frac{\partial\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta^*)}}{\partial\zeta^*}=0$ applies. As $(1-\zeta^*)^{\tilde a_2}$ is decreasing in ζ^*, $\frac{K_0^+}{S_0^+}$ is in this case again decreasing in ζ^*.

Case 3: $\tilde a_1>0$: The denominator in $\frac{K_0^+}{S_0^+}$ is increasing in ζ^* as $\frac{\partial{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta^*)}{\partial\zeta^*}=\frac{\tilde a_1(\tilde b_1-1)}{\tilde b_1+1}{}_2F_1(\tilde a_1+1,\tilde b_1;\tilde b_1+2;\zeta^*)>0$ (Abramowitz and Stegun, 1972, 15.2.1) because $\tilde b_1-1>0$ is required by (46). There are opposing forces at work in the nominator as ${}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)$ increases and $(1-\zeta^*)^{\tilde a_2}$ decreases in ζ^*. To find out which is stronger, we define $h(\zeta^*)$ as

$$\begin{aligned}
h(\zeta^*) &= (1-\zeta^*)^{\tilde a_2}{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)=(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\\
&\quad\times{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)\\
&\quad\text{with}\\
&\quad\tilde a_2-\tilde a_1=\frac{1}{1-\alpha}-\frac{\eta-\alpha}{\eta(1-\alpha)}=\frac{\alpha}{\eta(1-\alpha)}>0,\\
&\quad{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)=\tilde b_1\int_0^1 x^{\tilde b_1-1}(1-\zeta^*x)^{-\tilde a_1}\,dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial h(\zeta^*)}{\partial\zeta^*} &= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}-\tilde a_1\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\\
&\quad\times\frac{\partial{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{\partial\zeta^*}\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}-\tilde a_1\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\tilde a_1\tilde b_1\\
&\quad\times\int_0^1 x^{\tilde b_1}(1-\zeta^*x)^{-\tilde a_1-1}dx\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\tilde a_1\tilde b_1\\
&\quad\times\int_0^1\left(x^{\tilde b_1}(1-\zeta^*x)^{-\tilde a_1-1}-x^{\tilde b_1-1}\frac{(1-\zeta^*x)^{-\tilde a_1}}{1-\zeta^*}\right)dx\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\tilde a_1\tilde b_1\\
&\quad\times\int_0^1\left(x^{\tilde b_1}(1-\zeta^*x)^{-\tilde a_1-1}-x^{\tilde b_1-1}\frac{(1-\zeta^*x)^{-\tilde a_1}}{1-\zeta^*}\right)dx\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\tilde a_1\tilde b_1\\
&\quad\times\int_0^1\left(x^{\tilde b_1-1}(1-\zeta^*x)^{-\tilde a_1-1}\left(x-\frac{1-\zeta^*x}{1-\zeta^*}\right)\right)dx\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}+(1-\zeta^*)^{\tilde a_2-\tilde a_1}(1-\zeta^*)^{\tilde a_1}\tilde a_1\tilde b_1\\
&\quad\times\int_0^1\left(x^{\tilde b_1-1}(1-\zeta^*x)^{-\tilde a_1-1}\frac{x-1}{1-\zeta^*}\right)dx\\
&= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*}-\tilde a_1(1-\zeta^*)^{\tilde a_2-1}\tilde b_1
\end{aligned}$$

$$\times \int_0^1 x^{\tilde b_1-1}(1-x)(1-\zeta^* x)^{-\tilde a_1-1}dx$$
$$= -(\tilde a_2-\tilde a_1)\frac{h(\zeta^*)}{1-\zeta^*} - \tilde a_1(1-\zeta^*)^{\tilde a_2-1}\frac{{}_2F_1(\tilde a_1+1,\tilde b_1;\tilde b_1+2;\zeta^*)}{\tilde b_1+1}.$$

As $\tilde a_2-\tilde a_1>0$ and $\tilde a_1>0$ in this case, we find $\frac{\partial h(\zeta^*)}{\partial \zeta^*}<0$. Summing up case 3, the denominator in $\frac{K_0^+}{H_0^+}$ is increasing while the nominator is decreasing in ζ^*. Hence, $\frac{K_0^+}{S_0^+}$ is again decreasing in ζ^*.

We turn to $\frac{K_0}{H_0}$ given by (63). Its denominator is obviously decreasing in ζ^* as $\tilde a_2=\frac{1}{1-\alpha}>0$ and $\frac{\partial\, {}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)}{\partial \zeta^*}=\frac{\tilde a_2(\tilde b_2-1)}{\tilde b_2+1}{}_2F_1(\tilde a_2+1,\tilde b_2;\tilde b_2+2;\zeta^*)$ with $\tilde b_2-1>0$ due to the conditions (46).

The nominator in $\frac{K_0}{H_0}$ is increasing in ζ^*. To see this, define

$$k(\zeta^*) = (1-\zeta^*)^{\tilde a_2}{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)$$
$$= (1-\zeta^*)^{\tilde a_2}\tilde b_2\int_0^1 x^{\tilde b_2-1}(1-\zeta^* x)^{-\tilde a_2}dx.$$

Therefore,

$$\frac{\partial k(\zeta^*)}{\partial \zeta^*} = \tilde a_2(1-\zeta^*)^{\tilde a_2}\tilde b_2$$
$$\times\left[\int_0^1 x^{\tilde b 2}(1-\zeta^* x)^{-\tilde a_2-1}dx - \int_0^1 x^{\tilde b 2-1}\frac{(1-\zeta^* x)^{-\tilde a_2}}{1-\zeta^*}dx\right]$$
$$= \tilde a_2(1-\zeta^*)^{\tilde a_2}\tilde b_2\int_0^1 x^{\tilde b_2-1}(1-\zeta^* x)^{-\tilde a_2-1}\left[x-\frac{1-\zeta^* x}{1-\zeta^*}\right]dx$$
$$= -\tilde a_2(1-\zeta^*)^{\tilde a_2-1}\tilde b_2\int_0^1 x^{\tilde b_2-1}(1-x)(1-\zeta^* x)^{-\tilde a_2-1}dx$$
$$= -\tilde a_2(1-\zeta^*)^{\tilde a_2-1}\frac{{}_2F_1(\tilde a_2+1,\tilde b_2;\tilde b_2+2;\zeta^*)}{\tilde b_2+1}$$

which is negative for $\zeta^*<1$.

Summing up, we have shown that $\frac{K_0}{H_0}$ is increasing while $\frac{K_0^+}{H_0^+}$ is decreasing in ζ. If an equilibrium $\frac{K_0^+}{H_0^+}=\frac{K_0}{H_0}$ exists, it is unique.

Properties of $\frac{K_0^+}{S_0^+}$. To work out conditions for existence, we focus first on $\frac{K_0^+}{S_0^+}$ given by (62). Any solution ζ^* needs to fulfill $\zeta^*<1$; we know that $\frac{K_0^+}{S_0^+}$ is decreasing in ζ^*. We show first that $\frac{K_0^+}{S_0^+}$ is unbounded from above for $\zeta^*\to-\infty$. Let ε_1 be an arbitrarily large but finite real number. The critical term in $\frac{K_0^+}{S_0^+}$ is $(1-\zeta^*)^{\tilde a_2}\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}$. Now suppose that

$$\lim_{\zeta^*\to-\infty}(1-\zeta^*)^{\tilde a_2}\frac{{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)}{{}_2F_1(\tilde a_1,\tilde b_1-1;\tilde b_1+1;\zeta^*)}<\varepsilon_1$$

would be true. As $\frac{K_0^+}{S_0^+}$ decreases with ζ^*. This would imply that for any finite $\zeta^*<1$ and for $\zeta^*\to-\infty$ it would be true that

$${}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)<\varepsilon_1(1-\zeta^*)^{-\tilde a_2}{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*),$$
$$\tilde b_1\int_0^1 x^{\tilde b_1-1}(1-\zeta^* x)^{-\tilde a_1}dx-\varepsilon_1(1-\zeta^*)^{-\tilde a_2}\tilde b_1(\tilde b_1-1)$$
$$\times\int_0^1 x^{\tilde b_1-2}(1-x)(1-\zeta^* x)^{-\tilde a_1}dx<0,$$
$$\int_0^1 x^{\tilde b_1-2}(1-\zeta^* x)^{-\tilde a_1}\left[x-\varepsilon_1(\tilde b_1-1)(1-\zeta^*)^{-\tilde a_2}(1-x)\right]dx<0,$$
$$\int_0^1 x^{\tilde b_1-2}(1-\zeta^* x)^{-\tilde a_1}\kappa(x;\varepsilon_1)dx<0, \tag{64}$$

with

$$\kappa(x;\varepsilon_1)=\left[x-\varepsilon_1(\tilde b_1-1)(1-\zeta^*)^{-\tilde a_2}(1-x)\right],$$

where $\kappa(x;\varepsilon)$ is an affine function of x. $\kappa(x;\varepsilon)$ is zero for $x=\bar x(\varepsilon_1)$ with

$$\bar x(\varepsilon_1)=\frac{\varepsilon_1(\tilde b_1-1)(1-\zeta^*)^{-\tilde a_2}}{1+\varepsilon_1(\tilde b_1-1)(1-\zeta^*)^{-\tilde a_2}}.$$

Therefore, $\kappa(x;\varepsilon_1)<0$ for $x<\bar x(\varepsilon_1)$ and $\kappa(x;\varepsilon_1)>0$ for $x>\bar x(\varepsilon_1)$. For any finite ε_1, $\bar x(\varepsilon_1)\to 0$ for $\zeta^*\to-\infty$ as $\tilde a_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_1)$ becomes positive for $0\le x\le 1$ as $\zeta^*\to-\infty$ and inequality (64) cannot be fulfilled. Hence, $\frac{K_0^+}{H_0^+}$ cannot be bounded from above as $\zeta^*\to-\infty$ and $\lim_{\zeta^*\to-\infty}\frac{K_0^+}{H_0^+}=\infty$. Next, turn to the case $\zeta^*\to 1$. Suppose that $\frac{K_0^+}{H_0^+}$ would be bounded from below by some $\varepsilon_2>0$. By the same logic as above, this would imply for any $\zeta^*<1$ and $\zeta^*\to 1$ that

$${}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*)>\varepsilon_2(1-\zeta^*)^{-\tilde a_2}{}_2F_1(\tilde a_1,\tilde b_1;\tilde b_1+1;\zeta^*),$$
$$\int_0^1 x^{\tilde b_1-2}(1-\zeta^* x)^{-\tilde a_1}\kappa(x;\varepsilon_2)dx>0. \tag{65}$$

For any finite $\varepsilon_2>0$, $\bar x(\varepsilon_2)\to 1$ for $\zeta^*\to 1$ as $\tilde a_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_2)$ becomes negative for $0\le x\le 1$ as $\zeta^*\to 1$ and inequality (65) cannot be fulfilled. Hence, $\frac{K_0^+}{H_0^+}$ cannot be bounded from below by any finite $\varepsilon_2>0$ and $\lim_{\zeta^*\to 1}\frac{K_0^+}{H_0^+}=0$.

Properties of $\frac{K_0}{H_0}$*:*. We turn to $\frac{K_0}{H_0}$ which we know is increasing in ζ^* for $\zeta^*<1$. If a maximum exists, it must be reached as $\zeta^*\to 1$. The critical term in the nominator is $(1-\zeta^*)^{\tilde a_2}{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)$ which can be written as $(1-\zeta^*)\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde a_2}}$. We are interested in

$$\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde a_2}}$$

as $\lim_{\zeta^*\to 1}(1-\zeta^*)$ is finite and equal to zero, we can rewrite this expression as

$$\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde a_2}}$$
$$=\left[\lim_{\zeta^*\to 1}(1-\zeta^*)\right]\left[\lim_{\zeta^*\to 1}\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde a_2}}\right]$$

if the second limit on the right hand side in the above equation is finite. 15.4.23 in DLMF (2010) states that

$$\lim_{\zeta^*\to 1}\frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}}=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

if $\mathbb{R}(c-a-b)<0$. Applied to our case, $c-a-b=1+\tilde b_2-\tilde a_2-\tilde b_2=1-\tilde a_2=-\frac{\alpha}{1-\alpha}<0$. Furthermore, $\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}=\frac{\Gamma(\tilde b_2+1)\Gamma(\tilde a_2-1)}{\Gamma(\tilde a_2)\Gamma(\tilde b_2)}=\frac{\tilde b_2}{\tilde a_2-1}$ which is finite. Hence, $\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde a_2}}=0$ and $\lim_{\zeta^*\to 1}\underline{K}_0=K_0$.

The critical term in the denominator of $\frac{K_0}{H_0}$ is ${}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)$. As $\frac{\partial\,{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)}{\partial\zeta^*}=\frac{\tilde a_2(\tilde b_2-1)}{\tilde b_2+1}{}_2F_1(\tilde a_2+1,\tilde b_2;\tilde b_2+2;\zeta^*)>0$ for $\zeta^*<1$, $\underline{H}_0$ declines with ζ^* in this range. 15.3.6 in Abramowitz and Stegun (1972) implies that $\lim_{\zeta^*\to 1}{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)=\frac{\Gamma(\tilde b_2+1)\Gamma(2-\tilde a_2)}{\Gamma(\tilde b_2+1-\tilde a_2)\Gamma(2)}$ if $2-\tilde a_2=\frac{1-2\alpha}{1-\alpha}>0$ which is the case for $\alpha<\frac{1}{2}$. In case $\alpha>\frac{1}{2}$ we find ${}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)\to\infty$ as $\zeta^*\to 1$. In both cases, it is possible that $\underline{H}_0$ turns negative as ζ^* grows for $\zeta^*<1$. Define $\bar\zeta$ as

$$\bar\zeta=\underset{\zeta^*\le 1}{\operatorname{argmin}}\,|H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}$$
$$\times\frac{{}_2F_1(\tilde a_2,\tilde b_2-1;\tilde b_2+1;\zeta^*)}{\tilde b_2(\tilde b_2-1)}|,$$

As $\underline{H}_0$ is decreasing in ζ^* for $\zeta^*<1$, the admissible range for a solution to the present problem has the upper bound $\bar\zeta$. Therefore, if $\bar\zeta<1$ ($\bar\zeta=1$) we find $\underline{H}_0|_{\zeta^*=\bar\zeta}=0$ ($\underline{H}_0|_{\zeta^*=\bar\zeta}\ge 0$).

Lastly, we turn to $\frac{K_0}{H_0}$ as $\zeta^*\to-\infty$. Again, we start with the nominator $\underline{K}_0$. We know already that $(1-\zeta^*)^{\tilde a_2}{}_2F_1(\tilde a_2,\tilde b_2;\tilde b_2+1;\zeta^*)$ is decreasing in ζ^* for $\zeta^*<1$. Obviously, $\underline{K}_0$ then declines as $\zeta^*\to-\infty$. 15.3.4 in Abramowitz and Stegun (1972) states that

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1})$$

which implies for the present case

$$(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*) = {}_2F_1(\tilde{a}_2,1;\tilde{b}_2+1;\frac{\zeta^*}{\zeta^*-1}).$$

As $\tilde{a}_2, \tilde{b}_2+1>0$ and $\lim_{\zeta^*\to-\infty}\frac{\zeta^*}{\zeta^*-1}=1$, $\lim_{\zeta^*\to-\infty}(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)=\infty$. This implies that $\underline{K}_0$ becomes necessarily negative if ζ^* becomes too small. The range for admissible values for ζ^* is therefore bounded from below at $\underline{\zeta}$ which satisfies the condition

$$K_0 = \frac{L_0}{\psi}\underline{c}\left(1-\underline{\zeta}\right)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\underline{\zeta})}{\tilde{b}_2}.$$

We observe $\lim_{\zeta^*\to\underline{\zeta}}\frac{\underline{K}_0}{\underline{H}_0}=0$.

Taken together, if $\underline{\zeta}<\bar{\zeta}$ and $\bar{\zeta}<1$, $\lim_{\zeta^*\to\bar{\zeta}}\frac{\underline{K}_0}{\underline{H}_0}\to\infty$. If $\underline{\zeta}<\bar{\zeta}$, $\bar{\zeta}=1$, $\lim_{\zeta^*\to\bar{\zeta}}\frac{\underline{K}_0}{\underline{H}_0}$ either diverges to infinity or a strictly positive constant. The latter occurs if $\underline{H}_0\neq 0$ for $\zeta^*\leq 1$. In all possible cases we therefore observe $\lim_{\zeta^*\to\bar{\zeta}}\frac{\underline{K}_0}{\underline{H}_0}>\lim_{\zeta^*\to\bar{\zeta}}\frac{K_0^+}{H_0^+}$.

Furthermore, if $\underline{\zeta}<\bar{\zeta}$ we know that $\lim_{\zeta^*\to\underline{\zeta}}\frac{\underline{K}_0}{\underline{H}_0}=0$ and $\lim_{\zeta^*\to\underline{\zeta}}\frac{K_0^+}{h_0^+}>0$ as $\frac{K_0^+}{h_0^+}$ is decreasing in ζ^* for $\zeta^*<1$ and approaches 0 as $\zeta^*\to 1$.

If it happens that $\underline{\zeta}=\bar{\zeta}$, this value is the unique solution to the initial value problem. If we find $\underline{\zeta}>\bar{\zeta}$, there is no solution to the initial value problem because initial endowments K_0, H_0 are too low to allow for subsistence consumption $\underline{c}$.

This proves that a unique solution always exists if and only if $\underline{\zeta}\leq\bar{\zeta}<1$.

Appendix F. Sustainability present values

$PV[X_s]_t$ denotes the present value of X_s at time t for s running from t to ∞ and $PV[X_t]_t$ denotes the present value of X_t constant from t to ∞. Discounting uses the net interest rate $i_s-\delta_1=\frac{\alpha}{1-\alpha}\psi(1-\zeta^*x_s)^{-1}-\delta_1$, $s\in[t,\infty)$. We start by evaluating the present value of a consumption stream $(\bar{c}_t-\underline{c})L_t$ where c_t and $\underline{c}$ are constant from t onward:

$$PV[(\bar{c}_t-\underline{c})L_s]_t=\int_t^\infty(\bar{c}_t-\underline{c})L_0e^{-\int_t^s(i_\tau-\delta_1-n)d\tau}ds$$

With $x_\tau=e^{-\psi\tau}$, and ψ given by (15)

$$\begin{aligned}-\int_t^s(i_\tau-\delta_1-n)d\tau &= -\int_t^s\frac{\alpha}{1-\alpha}\psi(1-\zeta^*x_\tau)^{-1}-(\delta_1+n)d\tau\\ &=\frac{\alpha}{1-\alpha}\psi\int_t^s\frac{1}{\psi}(1-\zeta^*x_\tau)^{-1}x_\tau^{-1}dx_\tau+(\delta_1+n)(s-t)\\ &=\frac{\alpha}{1-\alpha}\int_{x_t}^{x_s}(1-\zeta^*x_\tau)^{-1}x_\tau^{-1}dx_\tau+(\delta_1+n)(s-t)\\ &=\frac{\alpha}{1-\alpha}\left[\ln\frac{x_\tau}{1-x_\tau}\right]_{x_t}^{x_s}+(\delta_1+n)(s-t).\end{aligned}$$

Therefore,

$$e^{-\int_t^s(i_\tau-\delta_1-n)d\tau}=\left(\frac{x_s}{x_t}\frac{1-\zeta^*x_t}{1-\zeta^*x_s}\right)^{\frac{\alpha}{1-\alpha}}e^{(\delta_1+n)(s-t)} \tag{66}$$

and

$$\begin{aligned}PV[(\bar{c}_t-\underline{c})L_s]_t &=(\bar{c}_t-\underline{c})L_te^{-(\delta_1+n)t}x_t^{-\frac{\alpha}{1-\alpha}}(1-\zeta^*x_t)^{\frac{\alpha}{1-\alpha}}\\ &\quad\times\int_t^\infty x_s^{\frac{\alpha}{1-\alpha}}(1-\zeta^*x_s)^{-\frac{\alpha}{1-\alpha}}e^{(\delta_1+n)s}ds.\end{aligned}$$

Using $x_\tau=e^{-\psi\tau}$, defining $x=\frac{x_s}{x_t}$ and noting that $dx=x_t^{-1}dx_s=-\psi x_t^{-1}x_sds$ gives

$$\begin{aligned}PV[(\bar{c}_t-\underline{c})L_s]_t &=(\bar{c}_t-\underline{c})L_t(1-\zeta^*x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\psi}\\ &\quad\times\int_0^1x^{\frac{\alpha}{1-\alpha}-\frac{\delta_1+n}{\psi}-1}(1-\zeta^*x_tx)^{-\frac{\alpha}{1-\alpha}}dx\\ &=(\bar{c}_t-\underline{c})L_t(1-\zeta^*x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\psi(\tilde{b}_2-1)}\\ &\quad\times{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta^*x_t),\end{aligned}$$

where $\tilde{a}_2$ and $\tilde{b}_2$ are defined as in (41). The present value of welfare maximizing total consumption in excess of $\underline{c}$ is

$$PV[(c_s-\underline{c})L_s]_t=\int_t^\infty(c_s-\underline{c})L_te^{-\int_t^s(i_\tau-\delta_1-n)d\tau}ds,$$

where $c_s-\underline{c}$ is given by Proposition 1. Using (66) and applying again the definitions from above gives

$$\begin{aligned}PV[(c_s-\underline{c})L_s]_t &=(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}L_t\\ &\quad\times\int_t^\infty x_s^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}(1-\zeta^*x_s)^{\frac{\alpha}{1-\alpha}\frac{1}{\eta}}\\ &\quad\times\left(\frac{x_s}{x_t}\frac{1-\zeta^*x_t}{1-\zeta^*x_s}\right)^{\frac{\alpha}{1-\alpha}}e^{(\delta_1+n)(s-t)}ds\\ &=(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}L_tx_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}\\ &\quad\times(1-\zeta^*x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\psi}\\ &\quad\times\int_0^1x^{(\tilde{b}_1-1)-1}(1-\zeta^*x_tx)^{-(\tilde{a}_1-1)}dx\\ &=(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}L_tx_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}\\ &\quad\times(1-\zeta^*x_t)^{\frac{\alpha}{1-\alpha}}\frac{1}{\psi}\\ &\quad\times\frac{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta^*x_t)}{\tilde{b}_1-1},\end{aligned}$$

where $\tilde{a}_1$ and $\tilde{b}_1$ are defined as in (41).

Equating $PV(\bar{c}_t-\underline{c})_t$ and $PV(c_s-\underline{c})_t$ gives $\bar{c}_t-\underline{c}$ as

$$\begin{aligned}\bar{c}_t-\underline{c} &=(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}\frac{\tilde{b}_2-1}{\tilde{b}_1-1}\\ &\quad\times\frac{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta^*x_t)}{{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta^*x_t)}\\ &=(c_0-\underline{c})(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}\\ &\quad\times\frac{\tilde{b}_2-1}{\tilde{b}_1-1}\frac{{}_2F_1(\tilde{a}_1-1,\tilde{b}_1-1;\tilde{b}_1;\zeta^*x_t)}{{}_2F_1(\tilde{a}_2-1,\tilde{b}_2-1;\tilde{b}_2;\zeta^*x_t)}\end{aligned}$$

where we used the first order condition for c_0 from (8) at $t=0$. We note that $PV(\bar{c}_t-\underline{c})_t$ and $PV(c_s-\underline{c})_t$ are in general depending on t.

For $t\to\infty$ and, hence, $x_t\to 0$ we arrive at the steady-state. In steady-state, $\lim_{t\to\infty}\bar{c}_t-\underline{c}$ behaves differently depending on the models parameters. In the subsistence consumption case, i.e. $\beta(B-\delta_2+n)-(1-\alpha)\rho<0$ and $\lim_{t\to\infty}c_t=\underline{c}$, we find $\lim_{t\to\infty}\bar{c}_t-\underline{c}=0$. This becomes obvious from as ${}_2F_1(a,b;b+1;0)=1$ and $\lim_{t\to\infty}x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}=0$.

For the intermediate case $\beta(B-\delta_2+n)-(1-\alpha)\rho=0$, we find $\lim_{t\to\infty}\bar{c}_t-\underline{c}=\lim_{t\to\infty}c_t-\underline{c}$. This is true as ${}_2F_1(a,b;b+1;0)=1$ and additionally in this special case $\lim_{t\to\infty}x_t^{-\frac{1}{\psi}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{(1-\alpha)\eta}}=1$ and $\frac{\tilde{b}_2-1}{\tilde{b}_1-1}=1$.

For the permanent positive growth case, i.e. $\beta(B-\delta_2+n)-(1-\alpha)\rho>0$, we find $\lim_{t\to\infty}\frac{\bar{c}_t-\underline{c}}{c_t-\underline{c}}=\frac{\tilde{b}_2-1}{\tilde{b}_1-1}>0$ as the existence of a solution to the problem demands $\tilde{b}_1,\tilde{b}_2>1$. This follows from using the expression for $c_t-\underline{c}$ from Proposition 1 with $(\lambda_{1,0}^*)^{-\frac{1}{\eta}}=c_0-\underline{c}$ and ${}_2F_1(a,b;b+1;0)=1$. Inspecting the definition of $\tilde{b}_1$ and $\tilde{b}_2$ in (41) reveals that $\frac{\tilde{b}_2-1}{\tilde{b}_1-1}=\left[1-\frac{1}{\eta}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{\beta(B-\delta_2+n)-(1-\alpha)n}\right]^{-1}>1$ as $\beta(B-\delta_2+n)-(1-\alpha)\rho>0$. This guarantees asymptotically sustainability.

Appendix G. Factor substitution

We consider a generalization of the production technology (2) that captures an elasticity of substitution between physical capital and

Table H.1
Calibration values for individual countries 2014.

Country	$\tilde{b}_1, \tilde{b}_2 > 0$	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{c}^{\max}$	$\lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$	I_{cg}	I_{gs}	I_W
High income countries																			
ARG	yes	42.98	11,994	38,901	7,033	5,144	0.0103	0.0377	0.0076	0.5443	0.0303	14,947	4,619	82,837	8,353	0.0100	1.00	0.92	1.00
AUS	yes	23.48	60,767	280,077	168,785	11,125	0.0149	0.0378	0.0066	0.5966	0.0734	221,057	148,928	1,210,076	52,795	0.0110	1.00	0.98	1.00
AUT	yes	8.55	51,786	256,361	10,353	8,842	0.0078	0.0432	0.0092	0.5781	0.0020	200,370	10,015	2,930,020	40,485	0.0092	1.00	1.00	1.00
BEL	yes	11.21	47,682	236,425	1,398	8,858	0.0044	0.0419	0.0093	0.6321	0.0002	184,458	1,362	4,271,091	40,299	0.0076	1.00	1.00	1.00
BHR	yes	1.34	20,745	100,488	13,722	4,500	0.0158	0.0347	0.0023	0.2936	0.0861	65,592	9,078	30,944	5,049	0.0095	1.00	0.80	1.00
CAN	yes	35.54	49,813	233,383	44,420	9,281	0.0108	0.0356	0.0073	0.6418	0.0211	183,759	40,301	1,227,601	43,647	0.0109	1.00	0.99	1.00
CHE	yes	8.19	86,978	435,276	3,366	11,673	0.0122	0.0515	0.0078	0.6442	0.0001	369,774	3,331	16,018,929	77,570	0.0122	0.61	1.00	1.00
CHL	yes	17.61	14,280	43,453	51,354	5,372	0.0086	0.0430	0.0060	0.4396	0.1468	18,721	35,694	106,885	9,438	0.0047	1.00	0.87	1.00
DEU	yes	80.98	49,085	253,695	3,906	8,512	0.0042	0.0373	0.0107	0.6189	0.0009	200,532	3,747	2,532,906	40,618	0.0067	1.00	1.00	1.00
DNK	yes	5.64	64,824	299,123	9,402	10,895	0.0051	0.0402	0.0091	0.6137	0.0102	237,215	7,207	434,275	29,115	0.0076	1.00	0.96	1.00
ESP	yes	46.48	29,525	116,579	5,164	7,332	−0.0030	0.0374	0.0085	0.5756	0.0008	77,318	5,078	3,640,052	21,771	0.0042	1.00	1.00	1.00
EST	yes	1.31	19,671	83,770	015,487	5,832	−0.0026	0.0455	0.0118	0.5846	0.0143	50,749	13,904	568,762	14,796	0.0024	1.00	0.89	1.00
FIN	yes	5.46	50,368	248,303	13,111	10,042	0.0041	0.0404	0.0096	0.6087	0.0059	187,585	11,929	1,229,367	41,068	0.0071	1.00	0.96	1.00
FRA	yes	66.32	43,941	214,595	5,326	8,939	0.0048	0.0372	0.0085	0.6327	0.0005	162,802	5,241	6,766,450	37,039	0.0081	1.00	1.00	1.00
GBR	yes	64.61	46,004	182,879	4,625	9,589	0.0075	0.0384	0.0088	0.5862	0.0060	132,009	3,630	352,220	26,387	0.0091	1.00	1.00	1.00
GRC	yes	10.89	21,908	109,716	5,611	6,765	−0.0067	0.0301	0.0104	0.4930	0.0018	63,510	5,469	1,534,619	16,063	0.0014	1.00	1.00	1.00
HRV	yes	4.24	13,337	47,597	6,719	5,142	−0.0041	0.0434	0.0120	0.5960	0.0091	22,189	5,914	388,834	9,633	0.0017	1.00	0.90	1.00
HUN	yes	9.87	13,606	56,339	3,013	4,642	−0.0027	0.0428	0.0128	0.5926	0.0040	30,426	2,669	392,506	10,093	0.0021	1.00	0.91	1.00
IRL	yes	4.66	46,979	136,491	2,996	9,066	0.0073	0.0530	0.0063	0.4320	0.0009	95,124	2,925	1,428,837	29,915	0.0105	1.00	1.00	1.00
ISL	yes	0.33	51,612	83,266	3,390	9,900	0.0111	0.0390	0.0063	0.6082	0.0000	57,702	3,389	590,192,538	32,245	0.0124	1.00	1.00	1.00
ITA	yes	60.79	35,398	177,497	3,803	8,188	0.0092	0.0364	0.0098	0.5223	0.0014	121,296	3,609	1,312,410	25,858	0.0096	1.00	1.00	1.00
JPN	yes	127.28	39,481	202,798	2,286	8,116	−0.0013	0.0448	0.0101	0.5621	0.0003	148,485	2,242	4,183,485	30,303	0.0043	1.00	1.00	1.00
KOR	yes	50.75	27,899	127,476	1,161	6,909	0.0063	0.0510	0.0053	0.5176	0.0005	83,891	1,108	1,177,813	20,206	0.0105	1.00	1.00	1.00
KWT	no	3.78	47,131	258,269	585,432	4,793	0.0499	0.0475	0.0026	0.2495	0.5515	232,143	-	-	-	0.0001	-	-	-
LUX	yes	0.56	78,931	398,040	5,011	9,787	0.0236	0.0487	0.0069	0.3675	0.0005	316,481	4,952	3,938,495	47,765	0.0183	1.00	1.00	1.00
LVA	yes	1.99	15,660	104,696	13,948	5,508	−0.0094	0.0328	0.0143	0.5446	0.0104	58,363	12,911	674,992	12,445	0.0000	0.45	0.73	0.00
NLD	yes	16.87	52,699	266,325	4,713	8,953	0.0036	0.0395	0.0083	0.5877	0.0080	210,091	3,369	246,719	18,707	0.0073	1.00	1.00	1.00
NOR	yes	5.14	100,268	563,923	98,381	12,285	0.0113	0.0386	0.0079	0.5083	0.0794	476,107	75,103	480,618	30,514	0.0081	1.00	0.91	1.00
OMN	no	3.96	19,361	61,021	98,895	4,004	0.0650	0.0536	0.0026	0.3027	0.3966	41,985	-	-	-	0.0093	-	-	-
POL	yes	38.01	13,858	31,173	7,185	4,673	−0.0007	0.0500	0.0099	0.5582	0.0094	14,795	6,631	394,877	8,894	0.0044	1.00	0.95	1.00
PRT	yes	10.40	21,700	93,101	5,079	6,408	−0.0054	0.0331	0.0101	0.5786	0.0029	56,355	4,798	955,193	16,235	0.0022	1.00	0.95	1.00
QAT	no	2.37	82,936	342,282	603,424	5,674	0.0536	0.1004	0.0015	0.1767	0.3055	286,146	-	-	-	0.0037	-	-	-
RUS	yes	143.82	13,876	50,982	43,517	4,569	0.0022	0.0335	0.0131	0.6303	0.1327	31,865	25,807	122,564	8,566	0.0011	1.00	0.99	1.00
SAU	yes	30.78	25,016	104,184	250,438	3,884	0.0274	0.0554	0.0035	0.2796	0.4124	82,029	−440,228	−298,461	975	0.0000	-	-	-
SGP	yes	5.47	55,597	309,021	22	5,559	0.0130	0.0531	0.0047	0.4395	0.0000	263,184	22	1,777,791	37,476	0.0141	1.00	1.00	1.00
SVK	yes	5.42	18,395	58,445	5,205	5,373	0.0010	0.0492	0.0095	0.5496	0.0044	33,458	4,860	602,647	12,568	0.0056	1.00	1.00	1.00
SVN	yes	2.06	23,975	112,045	11,317	6,545	0.0010	0.0417	0.0092	0.6384	0.0026	75,521	10,978	2,701,517	20,077	0.0058	1.00	0.95	1.00
SWE	yes	9.70	60,506	281,718	23,802	10,613	0.0099	0.0431	0.0092	0.5616	0.0057	216,264	22,660	2,214,136	45,679	0.0101	1.00	1.00	1.00
URY	yes	3.42	16,172	61,290	4,756	6,665	0.0034	0.0386	0.0093	0.4741	0.0152	23,936	3,514	109,312	10,937	0.0063	1.00	0.99	1.00
USA	yes	318.39	56,731	193,502	16,766	8,343	0.0073	0.0478	0.0082	0.5944	0.0067	154,690	15,805	1,410,912	41,598	0.0093	1.00	1.00	1.00
VEN	yes	30.74	15,516	74,968	36,313	7,400	0.0138	0.0365	0.0055	0.4882	0.1182	28,385	15,108	62,414	7,683	0.0084	1.00	0.93	1.00

(*continued on next page*)

Table H.1 (*continued*).

Country	$\tilde{b}_1, \tilde{b}_2 > 0$	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$	I_{cg}	I_{gs}	I_W
Upper-middle income countries																			
AZE	yes	9.54	7,665	27,445	38,001	945	0.0125	0.0687	0.0058	0.2506	0.2308	21,415	30,400	33,005	2,273	0.0000	0.99	0.80	0.91
BGR	yes	7.22	7,797	17,602	10,331	949	−0.0057	0.0531	0.0151	0.5309	0.0158	14,149	10,136	340,123	4,836	0.0000	0.97	0.90	0.88
BIH	yes	3.57	5,245	12,044	4,495	986	−0.0109	0.0500	0.0107	0.6711	0.0138	8,818	4,256	206,980	3,681	0.0000	0.94	0.95	0.69
BLR	yes	9.47	8,064	28,800	5,935	928	0.0009	0.0465	0.0128	0.6059	0.0134	24,281	5,692	258,211	5,917	0.0036	1.00	0.94	1.00
BRA	yes	204.21	11,789	28,466	24,911	1,566	0.0089	0.0468	0.0060	0.5681	0.0414	22,879	23,912	327,860	7,978	0.0096	1.00	0.97	1.00
BWA	yes	2.17	7,334	23,105	16,519	963	0.0186	0.0574	0.0074	0.2780	0.0252	17,998	16,371	180,841	4,355	0.0131	1.00	1.00	1.00
CHN	yes	1364.27	7,693	29,670	5,953	1,205	0.0051	0.0546	0.0072	0.5780	0.0245	23,547	5,391	127,279	5,840	0.0080	1.00	0.99	1.00
COL	yes	47.79	7,712	25,682	9,618	1,260	0.0094	0.0432	0.0059	0.4696	0.0662	19,416	8,352	59,201	5,130	0.0085	1.00	0.87	1.00
CRI	yes	4.76	10,180	20,186	8,596	1,480	0.0108	0.0533	0.0048	0.5877	0.0139	15,712	8,322	352,547	6,680	0.0123	1.00	1.00	1.00
DOM	yes	10.41	6,037	17,984	2,931	1,003	0.0120	0.0306	0.0061	0.4535	0.0162	13,437	2,781	77,717	3,968	0.0120	1.00	1.00	1.00
ECU	yes	15.90	6,299	19,748	23,419	1,174	0.0153	0.0420	0.0051	0.6670	0.1166	15,990	19,750	112,986	5,873	0.0106	0.84	1.00	1.00
GAB	yes	1.88	9,188	37,977	75,725	1,163	0.0317	0.0782	0.0080	0.2754	0.2712	29,557	17,560	17,830	828	0.0035	1.00	0.94	1.00
IRQ	no	35.01	6,629	14,824	69,087	904	0.0326	0.0512	0.0052	0.2963	0.4563	11,837	-	-	-	0.0002	-	-	-
JAM	yes	2.86	4,712	22,442	2,432	1,202	0.0036	0.0325	0.0069	0.6059	0.0120	15,441	2,177	109,608	3,854	0.0079	1.00	0.93	1.00
JOR	yes	8.81	4,020	11,027	2,150	944	0.0460	0.0368	0.0038	0.4943	0.0103	7,004	2,022	96,549	2,587	0.0301	1.00	1.00	1.00
KAZ	yes	17.29	11,498	37,691	53,914	1,087	0.0147	0.0404	0.0077	0.4022	0.1842	32,177	48,671	106,271	6,553	0.0046	1.00	0.87	1.00
LBN	yes	5.60	8,557	18,894	124	1,253	0.0602	0.0394	0.0046	0.4445	0.0000	14,287	123	2,476,275	5,140	0.0378	1.00	1.00	1.00
MEX	yes	124.22	10,337	36,850	9,050	1,280	0.0137	0.0366	0.0048	0.3742	0.0501	29,745	8,408	62,774	6,638	0.0110	1.00	1.00	1.00
MKD	yes	2.08	5,367	16,009	4,932	865	0.0008	0.0344	0.0097	0.5021	0.0300	12,111	4,624	77,501	3,553	0.0044	1.00	0.94	1.00
MNG	yes	2.92	3,842	8,282	27,593	737	0.0189	0.0608	0.0063	0.4076	0.2730	5,874	19,176	28,628	1,645	0.0037	1.00	0.98	1.00
Upper-middle income countries																			
MUS	yes	1.26	9,897	42,519	137	1,113	0.0018	0.0453	0.0077	0.4261	0.0000	35,379	137	4,874,670	6,629	0.0071	1.00	1.00	1.00
MYS	yes	30.23	10,814	29,495	20,053	925	0.0174	0.0574	0.0048	0.3804	0.0907	25,309	18,886	79,193	6,519	0.0103	1.00	0.83	1.00
NAM	yes	2.37	5,382	13,620	10,615	1,122	0.0232	0.0556	0.0078	0.5214	0.0256	9,223	10,168	206,836	3,476	0.0162	1.00	1.00	1.00
PAN	yes	3.90	11,522	19,916	8,125	1,278	0.0169	0.0485	0.0050	0.2957	0.0030	16,551	8,116	806,021	7,567	0.0157	1.00	1.00	1.00
PER	yes	30.97	6,199	17,515	19,892	1,125	0.0132	0.0394	0.0056	0.4603	0.0744	12,597	18,716	115,746	4,006	0.0098	1.00	0.85	1.00
PRY	yes	6.55	5,935	10,190	3,630	1,124	0.0133	0.0453	0.0056	0.5649	0.0154	7,099	3,432	125,580	3,704	0.0131	1.00	1.00	1.00
ROU	yes	19.91	9,900	35,742	6,736	1,028	−0.0037	0.0517	0.0128	0.4568	0.0144	30,030	6,551	207,220	6,433	0.0012	1.00	0.94	1.00
SUR	yes	0.55	9,426	43,553	80,516	1,249	0.0099	0.0757	0.0072	0.4522	0.2356	37,065	67,512	129,574	5,319	0.0023	0.06	0.99	0.98
THA	yes	68.42	5,647	18,691	2,414	802	0.0040	0.0627	0.0076	0.3928	0.0286	14,413	2,167	29,817	3,302	0.0066	1.00	0.99	1.00
TUN	yes	11.14	4,121	10,164	4,126	794	0.0117	0.0443	0.0064	0.5021	0.0487	7,124	3,610	37,255	2,657	0.0102	1.00	0.90	1.00
TUR	yes	77.03	12,021	21,197	1,655	1,067	0.0163	0.0557	0.0058	0.4276	0.0042	18,105	1,627	167,080	7,316	0.0149	1.00	1.00	1.00
ZAF	yes	54.54	6,257	18,550	8,673	1,038	0.0143	0.0520	0.0105	0.5523	0.0579	14,120	7,411	70,728	4,349	0.0093	1.00	0.97	1.00

(*continued on next page*)

Table H.1 (*continued*).

Country	$\tilde{b}_1, \tilde{b}_2 > 0$	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$	I_{cg}	I_{gs}	I_W
Lower-middle income countries																			
ARM	yes	2.91	4,181	13,305	5,222	584	0.0044	0.0296	0.0097	0.5797	0.0339	10,673	4,905	83,768	2,950	0.0061	1.00	0.89	1.00
BOL	yes	10.56	2,963	6,903	9,924	575	0.0154	0.0609	0.0074	0.4724	0.1127	4,795	8,629	36,155	1,883	0.0084	1.00	0.94	1.00
CIV	yes	22.53	1,528	4,409	3,402	598	0.0254	0.0378	0.0129	0.3319	0.0618	1,481	3,023	16,250	900	0.0114	1.00	0.92	1.00
CMR	yes	22.24	1,558	3,509	5,591	572	0.0266	0.0514	0.0106	0.5027	0.0798	1,454	4,666	29,387	977	0.0135	1.00	0.95	1.00
EGY	yes	91.81	3,249	4,537	5,210	392	0.0221	0.0695	0.0060	0.3537	0.0866	3,576	4,944	20,183	1,853	0.0115	1.00	0.87	1.00
GEO	yes	3.72	4,398	15,764	2,504	590	0.0005	0.0377	0.0132	0.4335	0.0083	12,536	2,455	128,122	2,879	0.0033	1.00	1.00	1.00
GTM	yes	15.92	3,600	8,743	3,689	601	0.0208	0.0459	0.0049	0.4164	0.0315	6,380	3,516	46,407	2,223	0.0156	1.00	1.00	1.00
HND	yes	8.81	2,059	6,153	4,591	617	0.0173	0.0560	0.0048	0.5965	0.0271	3,675	4,280	94,048	1,532	0.0149	1.00	0.99	1.00
IDN	yes	255.13	3,375	13,738	4,823	406	0.0122	0.0369	0.0071	0.4638	0.0516	11,318	4,495	40,365	2,307	0.0098	1.00	0.90	1.00
IND	yes	1293.86	1,557	4,722	1,276	341	0.0119	0.0574	0.0073	0.5163	0.0281	3,176	1,124	20,613	1,041	0.0109	1.00	0.97	1.00
KEN	yes	46.02	1,316	3,007	1,202	565	0.0264	0.0528	0.0060	0.6384	0.0277	1,254	900	20,729	968	0.0187	1.00	1.00	1.00
KGZ	yes	5.84	1,227	5,130	2,583	468	0.0201	0.0363	0.0061	0.5278	0.0738	2,523	1,832	13,097	922	0.0131	1.00	0.94	1.00
LAO	yes	6.58	1,929	3,005	14,887	432	0.0125	0.0624	0.0069	0.3976	0.1220	1,834	14,243	46,404	1,109	0.0063	1.00	0.82	1.00
LKA	yes	20.78	3,732	9,309	971	418	0.0093	0.0374	0.0067	0.3104	0.0011	7,750	969	279,360	2,493	0.0112	1.00	1.00	1.00
MAR	yes	34.32	3,133	11,492	5,245	520	0.0145	0.0520	0.0052	0.4974	0.0206	8,699	5,080	122,453	2,139	0.0135	1.00	1.00	1.00
MDA	yes	3.56	2,477	12,630	415	549	−0.0006	0.0309	0.0114	0.5713	0.0036	9,000	381	59,749	1,909	0.0039	1.00	1.00	1.00
MRT	yes	4.06	1,264	2,381	10,270	423	0.0294	0.0613	0.0081	0.4542	0.3103	1,310	1,051	1,539	360	0.0062	1.00	1.00	1.00
NGA	yes	176.46	3,114	3,531	8,423	660	0.0266	0.0496	0.0131	0.4888	0.1032	2,175	7,326	34,692	1,718	0.0112	1.00	0.95	1.00
Lower-middle income countries																			
NIC	yes	6.01	1,923	6,882	6,763	481	0.0114	0.0397	0.0048	0.5526	0.0442	4,545	6,401	80,100	1,416	0.0112	1.00	0.93	1.00
PHL	yes	100.10	3,445	7,318	1,788	503	0.0163	0.0489	0.0065	0.3565	0.0317	5,579	1,694	19,031	2,115	0.0125	1.00	1.00	1.00
SEN	yes	14.55	1,333	3,068	1,840	560	0.0297	0.0453	0.0063	0.4045	0.0375	928	1,639	17,676	802	0.0186	1.00	1.00	1.00
SWZ	yes	1.30	3,388	18,359	1,112	500	0.0184	0.0470	0.0102	0.6121	0.0355	15,319	626	10,798	741	0.0125	0.57	0.98	1.00
TJK	yes	8.36	1,366	29,601	2,407	505	0.0224	0.0237	0.0052	0.4635	0.0186	18,279	2,074	51,747	1,320	0.0173	0.23	1.00	1.00
UKR	yes	45.27	2,914	23,756	5,972	439	−0.0048	0.0281	0.0147	0.5584	0.0566	19,718	5,376	52,992	2,580	0.0000	0.00	0.85	0.00
Low-income countries																			
BDI	yes	9.89	273	415	219	247	0.0299	0.0384	0.0113	0.6062	0.1703	−119	−1,259	−4,482	28	0.0118	0.00	0.00	0.00
BFA	yes	17.59	683	1,599	2,394	308	0.0296	0.0596	0.0091	0.5730	0.1698	710	412	1,391	276	0.0122	1.00	1.00	1.00
CAF	yes	4.52	379	2,374	9,271	437	0.0035	0.0299	0.0145	0.1643	0.1236	−2,068	8,742	11,621	233	0.0000	0.00	0.00	0.00
GIN	yes	11.81	719	1,069	2,451	296	0.0230	0.0600	0.0099	0.4835	0.1712	337	1,094	3,091	351	0.0083	1.00	0.98	1.00
MOZ	yes	27.21	616	95	1,752	399	0.0290	0.0624	0.0107	0.4148	0.1356	−34	1,543	4,719	294	0.0107	0.00	0.00	0.00
NER	yes	19.15	422	2,088	2,701	329	0.0384	0.0363	0.0102	0.4322	0.1377	−317	727	2,283	256	0.0147	0.00	0.00	0.00
RWA	yes	11.35	691	1,320	751	299	0.0250	0.0457	0.0063	0.7410	0.0640	683	283	3,282	387	0.0166	1.00	1.00	1.00
SLE	yes	7.08	692	769	2,991	289	0.0224	0.0784	0.0134	0.5450	0.3646	470	−5,314	−7,945	95	0.0027	0.00	0.00	0.00
TCD	yes	13.57	982	1,006	5,390	342	0.0326	0.0635	0.0135	0.4524	0.2005	365	2,944	6,642	491	0.0089	1.00	0.99	1.00
TZA	yes	52.23	941	2,684	3,112	281	0.0311	0.0450	0.0073	0.4980	0.0622	1,459	2,778	22,232	615	0.0178	1.00	0.96	1.00
ZWE	yes	15.41	1,145	1,675	3,763	396	0.0234	0.0357	0.0089	0.5507	0.0700	702	3,347	26,338	681	0.0137	1.00	0.94	1.00

Note: Calibration values and results as explained in Section 4.5. Parameters $\rho = 0.03$, $\eta = 2$ and $B = 0.05$ uniformly calibrated for all countries. All nominal quantities in 2014 US$ per capita except population L_0 for 2014 in mln.

resource input unequal to one.

$$Y_t = \left[\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \left((H_t u_t L_t)^{\frac{\beta}{1-\alpha}} R_t^{\frac{\gamma}{1-\alpha}} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}},$$

where we focus on the case $\sigma < 1$. For $\sigma = 1$, we arrive back at the production technology (2). The Hamiltonian associated with the present problem, the first order and the transversality conditions read exactly as in Section 3.1. Proceeding as in Appendix A, shows that $\lambda_{2,t}$ and $\lambda_{3,t}$ behave as in case $\sigma = 1$; $\lambda_{2,t} = \lambda_{2,0} e^{-(B-\delta_2)t}$ and $\lambda_{3,t} = \lambda_{3,0}$. $\lambda_{1,t}$, however, develops differently. Following the same steps as in Appendix A gives

$$\dot{\lambda}_{1,t} = - \left[\alpha^{-\sigma} \lambda_{1,t}^{\sigma-1} - (1-\alpha)^{-\sigma} \left(\left(\frac{\lambda_{2,t} \frac{B}{L_t}}{\frac{\beta}{1-\alpha}} \right)^{\frac{\beta}{1-\alpha}} \left(\frac{\lambda_{3,0}}{\frac{\gamma}{1-\alpha}} \right)^{\frac{\gamma}{1-\alpha}} \right)^{\sigma-1} \right]^{-\frac{1}{\sigma-1}} + \lambda_{1,t}\delta_1. \tag{67}$$

To the best of our knowledge, the differential Eq. (67) does not allow for an explicit solution. We are therefore restricted to analyze the steady-state of the economy characterized by a constant $\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}}$ that provides a constant interest rate. (67) allows for this if $\lambda_{1,t}$ grows at the same rate as $\left(\lambda_{2,t} \frac{B}{L_t}\right)^{\frac{\beta}{1-\alpha}}$. This implies $\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} = \frac{\beta}{1-\alpha}(B - \delta_2 + n)$ which equals the steady-state net interest as can be seen from (8).

As the net interest rate governs the behavior of excess consumption, similar to (18) and depending on whether $\frac{\beta}{1-\alpha}(B-\delta_2+n)-\rho \lesseqgtr 0$, three cases for steady-state consumption emerge. The corresponding growth rate of excess consumption is $\frac{1}{\eta}\frac{\beta(B-\delta_2+n)-(1-\alpha)\rho}{1-\alpha}$ which is identical to the case $\sigma = 1$ in the main text.

Also with $\sigma < 1$, the existence of a solution is not always guaranteed. Next, we derive the conditions corresponding to (22) and (23) for $\sigma \neq 1$. Evaluating the accumulation Eq. (3) at a constant per capita stock of physical capital in steady-state gives

$$\lambda_{1,t} = \left[k_t - \left(\left(\frac{\psi}{1-\alpha} \right)^{\sigma} - (\delta_1 + n) \right)^{-1} \underline{c} \right]^{-\eta} \left(\left(\frac{\psi}{1-\alpha} \right)^{\sigma} - (\delta_1 + n) \right)^{-\eta}.$$

For a solution $\lambda_{1,t}^*$ to exist, we need

$$k_t > \left(\left(\frac{\psi}{1-\alpha} \right)^{\sigma} - (\delta_1 + n) \right)^{-1} \underline{c}$$

which is identical to the existence condition (22) for $t = 0$, $\zeta^* = 0$ and $\sigma = 1$. If $\sigma < 1$, this existence condition becomes more stringent and the economy requires more physical capital to guarantee minimum subsistence consumption. The equivalence to the existence condition (23) can be obtained by computing per capita production that is sufficient to cover subsistence consumption and physical capital depreciation. This quantity needs then to be compared with the implied resource requirements. The former is given by

$$\bar{y} = \underline{c} \left. \frac{\frac{Y_t}{K_t}}{\frac{Y_t}{K_t} - (\delta_1 + n)} \right|_{\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} = \frac{\beta}{1-\alpha}(B-\delta_2+n)} = \underline{c} \frac{\left(\frac{\psi}{1-\alpha}\right)^{\sigma}}{\left(\frac{\psi}{1-\alpha}\right)^{\sigma} - (\delta_1 + n)},$$

which is obtained from solving the accumulation Eq. (3) evaluated at $\dot{k}_t = 0$ and $c_t = \underline{c}$ for y_t. The resource requirement for $\bar{y}$ is $R_t = \frac{\lambda_{1,t}}{\lambda_{3,0}} \gamma \bar{y} L_t$. Consequently, the implied requirement for the resource stock is

$$\bar{S}_t = \int_{s=t}^{\infty} R_s ds = \frac{\lambda_{1,t}}{\lambda_{3,0}} \gamma \bar{y} L_0 \int_{s=t}^{\infty} e^{-[\frac{\beta}{1-\alpha}(B-\delta_2+n)-n]s} ds = \frac{\lambda_{1,t}}{\lambda_{3,0}} \gamma \frac{\bar{y} L_t}{\frac{\beta}{1-\alpha}(B-\delta_2+n)-n}$$

and hence in per capita terms and final output equivalents

$$\tilde{\bar{s}}_t = \gamma \frac{\bar{y}}{\frac{\beta}{1-\alpha}(B-\delta_2+n)-n} = \gamma \frac{\left(\frac{\psi}{1-\alpha}\right)^{\sigma}}{\left(\frac{\psi}{1-\alpha}\right)^{\sigma} - (\delta_1+n)} \frac{\underline{c}}{\frac{\beta}{1-\alpha}(B-\delta_2+n)-n}.$$

To guarantee minimum subsistence consumption, we need $\tilde{s}_t > \tilde{\bar{s}}_t$. For $\sigma = 1$, this resembles the existence condition (23) at $t = 0$ and $\zeta^* = 0$.[45] Once again, the existing condition gets more stringent and $\sigma < 1$ requires a higher initial resource endowment compared with $\sigma = 1$.

Appendix H

See Table H.1.

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[45] This can be seen from (23) as $\lim_{z \to 0} {}_2F_1(a, b-1; b+1, z) = 1$.

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