Sustainable per capita consumption under population growth*

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Abstract

We establish two investment rules for maximal constant per capita consumption under exogenous population growth, one in terms of total stocks and the other in terms of per capita stocks. Both rules show the importance of the development of future population growth. The investment rules are illustrated in the one-sector model of capital accumulation, the DHSS model of capital accumulation and resource depletion, and the Stollery–d'Autume–Schubert model in which natural capital provides amenities. Application to recent empirical evidence indicates that actual genuine savings might be insufficient to sustain per capita consumption, when future population growth is combined with a large per capita consumption-wage gap.

Keywords: Sustainable development, population growth, intergenerational equity

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1 Introduction

Diminishing natural capital and an increasing population might both be obstacles to sustainable development that seeks to keep per capita consumption non-decreasing. Natural capital is important for maintaining a non-negative flow of resource input as well as providing needed amenity services and biological diversity. An increasing population—while contributing to an increasing labor stock—leads to a dilution effect: population growth requires that available consumption as well as the stocks of reproducible and natural capital be divided among a larger number of people.

There has been an interest in investment rules that keep consumption constant at its maximal sustainable level since the first use of the maximin criterion in the presence of resources (Solow, 1974) and the original article on Hartwick's rule (Hartwick, 1977). A series of papers (among others, Hartwick, 1978a,b; Dixit, Hammond, and Hoel, 1980; Withagen and Asheim, 1998; Mitra, 2002; Asheim, Buchholz and Withagen, 2003; Buchholz, Dasgupta and Mitra, 2005; Mitra, Asheim, Buchholz and Withagen, 2013) have contributed to our understanding of the connection between *Hartwick's investment rule*—keeping the value of net investments equal to zero—and *Solow sustainability*—sustainable development with constant consumption. This literature shows that following Hartwick's investment rule leads to Solow sustainability in a variety of models and technologies.

Much of this literature has been based on the strict assumptions that Solow (1974) and Hartwick (1977) imposed when using the so-called Dasgupta-Heal-Solow-Stiglitz (DHSS) model to analyze constant consumption paths in the presence of resource constraints. This includes assuming that both technology and population are constant.[1]

There have been contributions that incorporate exogenous population growth, including Arrow, Dasgupta and Mäler (2003) and Asheim (2004) on indicators for non-decreasing welfare, Mitra (1983) and Asheim, Buchholz, Hartwick, Mitra and Withagen (2007) on non-decreasing per capita consumption in the DHSS model when population growth has a quasi-arithmetic form, and Asheim, Hartwick and Mitra (2021) on investment rules that keep per capita net national product constant. Moreover, Dasgupta (1969, 2004, 2021) has studied the management of natural resources in the presence of population growth in a series of important contributions.

Nevertheless, none of these contributions have obtained investment rules that yield the maximal

[1]There are, however, straightforward methods to restore Hartwick's result in the presence of exogenous technological progress by including time as an additional stock.

level of per capita consumption under exogenous population growth, except for special cases like exponential population growth in the one-sector model and quasi-arithmetic growth in the DHSS model. This problem is non-trivial since the maximal level of per capita consumption does not depend only on the capital dilution effect of current population growth, but also on the capital dilution effect of future population growth. In particular, if future population growth is expected to accelerate, then this requires a higher flow of current investment to prepare for future capital dilution. Hence, investment rules for maximal sustainable per capita consumption must take into account the entire expected future development of population growth.

This is not just a theoretical but also an empirical concern, given that population dynamics in the real world are mostly non-exponential. In applied work, only simple measures of per capita capital assets are computed across countries (e.g. World Bank, 2021), and there are very few empirical studies that look into investment rules under population growth. By incorporating the capital dilution effect as analyzed by Arrow, Dasgupta and Mäler (2003) and Asheim (2004), Ferreira, Hamilton and Vincent (2008) study the correlation between per capita genuine savings and the per capita change in consumption under population change. More recently, Yamaguchi (2018) compares the value of the change in per capita capital assets between total utilitarianism and dynamic average utilitarianism, the latter being defined by Dasgupta (2004) as the sum of the discounted future utilities divided by the sum of the discounted future population.

We develop two versions of our main result in the context of a general model with multiple capital goods. Based on Samuelson (1961) we use the present value of future changes in per capita consumption as an indicator of non-decreasing per capita consumption. In Proposition 1 we show how this indicator requires that current investment in total stocks must compensate for future population growth. In Proposition 2 we develop an investment rule in terms of per capita stocks demonstrating the importance of the future development of the population growth rate.

We assume that development is efficient but does not take a stand on the choice between a total or average discounted utilitarian criterion. In particular, our analysis might be compatible with an underlying total discounted utilitarian welfare criterion where dynamic welfare increases over time even with decreasing per capita consumption, if compensated by a sufficient positive rate of population growth. Thus, it is important to stress that our results are not concerned with the question of whether some measure of dynamic welfare is non-decreasing (Arrow, Dasgupta and Mäler, 2003; Asheim, 2004). Also, we are not concerned with the question of whether per capita productive capacity is sustainable (Asheim, Hartwick and Mitra, 2021).

We apply Propositions 1 and 2 to check whether recent investment has been sufficient to support sustainable per capita consumption, using data on consumption and genuine savings (GS) by the World Bank and future population prospects till 2100 by the UN. Based on Proposition 1 we find that actual investment might have fallen short of what our investment rule suggests in some countries where population is expected to grow. Interestingly, they include both higher- and lower-income countries, contrary to what simple measures of genuine savings typically suggest for sustainability. The intuition is that, when population growth is expected, extra investment is needed to sustain per capita consumption if the cost of population growth (which is related to the consumption-wage gap) is large relative to genuine savings. This shortage in investment is amplified when a lower discount rate is used.

We introduce the intuition behind Proposition 1 in Section 2, before—in Section 3—applying this investment rule to the one-sector model of capital accumulation, the DHSS model of capital accumulation and resource depletion, and the Stollery–d'Autume–Schubert model in which natural capital provides amenities.[2] We establish Proposition 1 formally in the general model in Section 4 and develop the per capita version of our result as Proposition 2 in the following Section 5. We provide an empirical application of Proposition 1 in Section 6 to see if actual investments in the recent past have been sufficient to sustain per capita consumption. We offer concluding remarks in the final Section 7 and discuss the applicability of Proposition 2 in an appendix.

[2]Hartwick and Mitra (2020) study constant consumption in the Stollery (1998) model by allowing for the rate of growth of climate change to be proportional to a concave function of the resource extraction.

2 Presenting the main result

Consider a one consumption good/multiple capital good model with a stationary technology, as the one which we formally introduce in Section 4. If population and labor is constant, then it can be shown that short-run efficiency implies

$$p_0(t)\dot{C}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{p}(t)\mathbf{I}(t)\big)\,,$$

where C is consumption, p_0 is the consumption discount factor, $\mathbf{I} = (I_1, \ldots, I_n)$ is the vector of net investments and $\mathbf{p} = (p_1, \ldots, p_n)$ is the vector of present-value prices for the vector of net investments. This is the Dixit-Hammond-Hoel (1980) result, which implies that keeping consumption constant is equivalent to keeping the present value of net investment constant (but not necessarily zero). If the net investment transversality condition holds ($\lim_{t\to\infty} \mathbf{p}(t)\mathbf{I}(t) = 0$), then, by integration, we obtain:

$$\int_t^\infty p_0(\tau)\dot{C}(\tau)\mathrm{d}\tau = \mathbf{p}(t)\mathbf{I}(t)\,.$$

Hence, it is a basic result that keeping consumption constant at all times implies that the value of net investments are equal to zero, that is, Hartwick's (1977) rule ($\mathbf{p}(t)\mathbf{I}(t) = 0$) is followed at all times. In the one-sector model of capital accumulation, this determines the maximal sustainable level of consumption, since with a constant capital stock, consumption equals net production. In the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) of capital accumulation and resource depletion, one has to maximize the constant level of consumption subject to the initial resource stock being at least as large as the integral of resource use in order to find the efficient constant consumption path.

The question of whether the net investment transversality condition must hold has been posed in papers (see, e.g., Dixit, Hammond, and Hoel, 1980; Withagen and Asheim, 1998; Mitra, 2002) which ask whether Hartwick's rule is necessary for an efficient constant consumption path. In short, this literature shows that constant consumption with negative net investment, leading to $\lim_{t\to\infty} \mathbf{p}(t)\mathbf{I}(t) < 0$, is infeasible. Furthermore, it shows that constant consumption with positive net investment, leading to $\lim_{t\to\infty} \mathbf{p}(t)\mathbf{I}(t) > 0$, is inefficient (as in the one sector model) or infeasible (as in the DHSS model; see Buchholz, Dasgupta and Mitra, 2005). Using the one sector model as an illustration, combining constant consumption with negative net investment means that consumption exceeds net production. Furthermore, in order to maintain this unsustainable consumption level for a little longer, net investment has to become even more negative. Actually, by the Dixit-Hammond-Hoel result, the rate of growth of the absolute value of the negative investment has to equal the interest rate = net marginal productivity of capital. So even though the present value of net investments is constant and consumption is constant in the short run, this consumption level is obviously not sustainable. It all crashes when capital has been exhausted. It is like a person living beyond his means, taking up new loans at an accelerated rate, thereby keeping up the standard of living for some additional time. So clearly, this has nothing to do with sustainable behavior. On the background of the general results of Mitra (2002) we assume in this paper that an appropriate net investment transversality condition is satisfied.

Consider now the same model, but now with exogenous population growth. Then, as we show in Section 4, short-run efficiency implies

$$p_0(t)\dot{c}(t)N(t) + p_0(t)c(t)\dot{N}(t) = p_0(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(c(t)N(t)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p}(t)\mathbf{i}(t)N(t) + \left(\int_t^\infty w(\tau)\dot{N}(\tau)\mathrm{d}\tau\right)\right),$$

where N is population (and labor), $c = C/N$ is per capita consumption, $\mathbf{i} = \mathbf{I}/N$ is the vector of per capita net investments, and w is the present value of the marginal productivity of labor. The term $\int_t^\infty w(\tau)\dot{N}(\tau)\mathrm{d}\tau$ can be interpreted as the contribution to current productive capacity of future population growth. This is the Dixit-Hammond-Hoel result in the population growth setting. Note that this only requires that technology is stationary and that the path is efficient. The path might be the result of a total utilitarian or average utilitarian criterion. If, in this setting,

$$\lim_{t\to\infty} \mathbf{p}(t)\mathbf{i}(t)N(t) = 0 \tag{1}$$

holds as a net investment transversality condition and $\int_t^\infty \big(p_0(\tau)c(\tau) - w(\tau)\big)\dot{N}(\tau)\mathrm{d}\tau$ exists, then, by integration, we obtain:

$$\int_t^\infty p_0(\tau)\dot{c}(\tau)N(\tau)\mathrm{d}\tau = \mathbf{p}(t)\mathbf{i}(t)N(t) - \int_t^\infty \big(p_0(\tau)c(\tau) - w(\tau)\big)\dot{N}(\tau)\mathrm{d}\tau\,. \tag{2}$$

Remark. The left-hand side of eq. (2) can serve as an approximate indicator for sustainable per capita consumption, while the right-hand side can serve as a basis for a genuine savings indicator, as we analyze in the rest of this paper. Dasgupta (2004, eq. (A.149)) answers the question of how to generalize the genuine savings indicator to a situation with exogenous population growth, when its purpose is to serve as an approximate indicator for sustainability under total discounted utilitarianism, by considering the integral of discounted utilitarian utilities divided by the integral of discounted population:

$$V^*(t) = \frac{V(t)}{N^*(t)} = \frac{\int_t^\infty e^{-\delta(\tau-t)}U(c(\tau))N(\tau)\,d\tau}{\int_t^\infty e^{-\delta(\tau-t)}N(\tau)\,d\tau}\,.$$

As already mentioned in the introduction, this is called dynamic average utilitarianism. Arrow, Dasgupta and Mäler (2003) provide a condition in their Theorem 3 for $V^*(t)$ being non-decreasing in a setting where also capital is one-dimensional. However, based on Samuelson (1961) and in the tradition of Arrow-Dasgupta-Mäler comprehensive accounting, it would be natural to replace $\dot{V}^*(t) \geq 0$ by

$$\int_t^\infty e^{-\delta(\tau-t)}\frac{\mathrm{d}U(c(\tau))}{\mathrm{d}\tau}N(\tau)\,d\tau \geq 0\,.$$

By writing $p_0(\tau) = e^{-\delta(\tau-t)}U'(c(\tau))$, this expression becomes identical to the condition that we consider. It can be shown that this condition is equivalent to $\dot{V}^*(t) \geq 0$ under total discounted utilitarianism if the rate of population growth is constant, but not otherwise.

It follows from eq. (2) that keeping per capita consumption constant at all times when (1) holds implies that the value of net investments satisfies the following rule at all times:

$$\mathbf{p}(t)\mathbf{i}(t)N(t) = \int_t^\infty \big(p_0(\tau)c(\tau) - w(\tau)\big)\dot{N}(\tau)\mathrm{d}\tau\,. \tag{3}$$

Furthermore, the converse result holds as well: Following this rule at all times implies that per capita consumption is constant. This equivalence is the main result of this paper. Along a constant per capita consumption path, there will be a consumption-wage gap: the present value of per capita consumption, $p_0(t)c(t)$, will exceed the present value of the marginal productivity of labor, $w(t)$. This implies that with a growing population the value of net investment must be positive to compensate for the capital dilution effect of future population growth. In this manner, the sustainability of current per capita consumption depends not only on current population growth, but on the entire development of future population.

3 Illustrating the main result

The DHSS model has one produced good, which serves both as reproducible capital and as material consumption. This good is produced with a stock of reproducible capital (K), an extraction flow (R) of input from a non-renewable and exhaustible resource, and labor (N). The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfies the following two assumptions:

(F1) F is continuous, non-decreasing, concave, and homogenous of degree 1 in (K, R, N) on $\mathbb{R}^3_+$.

(F2) F is twice continuously differentiable in (K, R, N) on $\mathbb{R}^3_{++}$, with $F_K(K, R, N) > 0$, $F_R(K, R, N) > 0$, and $F_N(K, R, N) > 0$ for all $(K, R, N) \in \mathbb{R}^3_{++}$.

In the case where F is of the Cobb-Douglas form:

$$F(K, R, N) = K^\alpha R^\beta N^{1-\alpha-\beta} \quad \text{where } \alpha > 0\,,\ \beta > 0\,,\ \text{and } 1 - \alpha - \beta > 0\,, \tag{4}$$

the function F also satisfies that $F(0, R, N) = F(K, 0, N) = F(K, R, 0) = 0$ for all $(K, R, N) \in \mathbb{R}^3_+$. In particular, the resource is essential in the sense that there is no production without a positive flow of resource input. However, we need not make this assumption in our general analysis of the DHSS model. If (F2) is replaced by

(F2′) F is twice continuously differentiable in (K, R, N) on $\mathbb{R}^3_{++}$, with $F_K(K, R, N) > 0$, $F_R(K, R, N) = 0$, and $F_N(K, R, N) > 0$ for all $(K, R, N) \in \mathbb{R}^3_{++}$,

then the DHSS model is reduced to the ordinary one-sector model where the resource plays no role, and the production function can be written as $F(K, N)$.

The model is closed by letting production be split into material consumption (M) and net investment in reproducible capital ($I = \dot{K}$) and letting resource input be drawn from the stock (S) of the non-renewable and exhaustible resource.

Stollery (1998) and d'Autume and Schubert (2008) study a variant of the DHSS model with amenities where consumption (C) is a composite good that depends on material consumption and the remaining stock of the resource. The model has a natural interpretation in terms of climate change if the exhaustible resource is identified with fossil fuels. Then a large remaining resource stock corresponds to low accumulated CO_2 emissions and thus to a climate that offers high amenity value.

With population growth, we let per capita consumption ($c = C/N$) depend on per capita material consumption ($m = M/N$) and the per capita resource stock ($s = S/N$), implying that both material consumption and the amenity are private goods. This formulation can be made compatible with the climate change interpretation if a congestion effect reduces the amenity received by any one person when population increases. When the resource stock yields amenities the function $u : \mathbb{R}^2_+ \to \mathbb{R}_+$ that turns per capita material consumption and the per capita resource stock into per capita consumption is assumed to satisfy the following two assumptions:

(u1) u is continuous, non-decreasing, concave, and homogeneous of degree 1 in (m, s) on $\mathbb{R}^2_+$.

(u2) u is twice continuously differentiable in (m, s) on $\mathbb{R}^2_{++}$, with $u_m(m, s) > 0$ and $u_s(m, s) > 0$ for all $(m, s) \in \mathbb{R}^2_{++}$.

Note that $C = Nc = Nu(m, s) = Nu(M/N, S/N) = u(M, S)$ by the assumption that u is homogeneous of degree 1. If (u2) is replaced by

(u2′) u is twice continuously differentiable in (m, s) on $\mathbb{R}^2_{++}$, with $u_m(m, s) > 0$ and $u_s(m, s) = 0$ for all $(m, s) \in \mathbb{R}^2_{++}$,

then amenities play no role and it follows from (u1) that $C = Nc = Nu(m, s) = Nm = M$. So in this case, consumption C coincides with material consumption M.

Labor is throughout taken to be equal to the population (N) and assumed to be exogenously given. We assume that $N(t)$ is positive and a continuously differentiable function of t:

$$\dot{N}(t) = g(t)N(t) \quad \text{for } t \geq 0 , \quad N(0) = N_0 > 0 , \tag{5}$$

where $g(t)$ represents the exogenous rate of population growth at time $t \geq 0$.

A *path* from initial stocks $(K_0, S_0) \in \mathbb{R}^2_+$ of capital and resource is described by the functions $(C(t), M(t), I(t), R(t), K(t), S(t), N(t))$, where $C : [0, \infty) \to \mathbb{R}_+$, $M : [0, \infty) \to \mathbb{R}_+$, $I : [0, \infty) \to \mathbb{R}$, $R : [0, \infty) \to \mathbb{R}_+$, $K : [0, \infty) \to \mathbb{R}_+$, and $S : [0, \infty) \to \mathbb{R}_+$ are continuously differentiable functions of t satisfying, at all $t \in [0, \infty)$,

$$\begin{aligned} C(t) &= u(M(t), S(t)) \\ I(t) &= \dot{K}(t) = F(K(t), R(t), N(t)) - M(t) , \end{aligned}$$

$$R(t) = -\dot{S}(t)\,,$$
$$K(0) = K_0\,,$$
$$S(0) = S_0\,.$$

Under the assumptions of (F1), (F2), (u1), and (u2), it follows from the analysis of Asheim, Hartwick and Mitra (2021, eqs. (22) and (23)) that the Keynes-Ramsey rule becomes

$$-\frac{\dot{p}_0}{p_0} = F_K + \frac{\dot{u}_m}{u_m}\,, \tag{6}$$

where the left-hand side—the rate at which the consumption discount factor p_0 decreases—is the consumption discount rate. The right-hand side simplifies to F_K if (u2) is replaced by (u2′) since then $u_m \equiv 1$. Furthermore, the condition for short-run efficiency, a modified version of Hotelling's no-arbitrage condition, becomes:

$$\dot{F}_R = F_K F_R - \frac{u_s}{u_m}\,, \tag{7}$$

which simplifies to $\dot{F}_R = F_K F_R$ if (u2) is replaced by (u2′) (and to the trivial equation $0 = 0$ if, in addition, (F2) is replaced by (F2′)).

Illustration in the one-sector model. The one-sector model corresponds to assumptions (F1), (F2′), (u1), and (u2′). It follows from $-\dot{p}_0/p_0 = F_K$ that the consumption discount factor develops as follows:

$$p_0(t) = \int_0^\infty e^{-\int_0^t F_K \mathrm{d}\tau}\,.$$

Furthermore, since production is split between consumption and net investment, $p_0(t)$ is also the present-value price of investment. Hence, in this model, the investment rule (3) becomes:

$$\dot{K}(t) = \int_t^\infty e^{-\int_t^\tau F_K \mathrm{d}s}\big(c - F_N\big)\dot{N}\mathrm{d}\tau\,. \tag{8}$$

We can check that following this rule actually leads to constant per capita consumption, so that

$$0 = \dot{c} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{C}{N}\right) = \frac{1}{N}\left(\dot{C} - C\frac{\dot{N}}{N}\right)\,.$$

To check this, by differentiating the investment rule (8) it follows that

$$\ddot{K} = -\left(\frac{C}{N} - F_N\right)\dot{N} + F_K\dot{K}\,.$$

From the fact that production is split between consumption and net investment ($C + \dot{K} = F(K, N)$), we obtain:

$$\ddot{K} = F_K\dot{K} + F_N\dot{N} - \dot{C}\,.$$

Since the two right-hand sides are equal, we have $\dot{C} = C\dot{N}/N$ and $\dot{c} = 0$. The efficient constant per capita consumption level c is found by combining the investment rule at the initial time 0 with the following equation that must be satisfied at the initial time 0: $cN(0) + \dot{K}(0) = F(K(0), N(0))$.

An interesting special case is where the right-hand side of the investment rule equals $K\dot{N}/N$. Then, by (8), net capital investment $\dot{K}$ exactly compensates for the capital dilution $K\dot{N}/N$ that population causes, implying that the per capita capital stock $k = K/N$ is constant. Furthermore, it follows from $C + \dot{K} = F(K, N)$ and (F1) that $c + k\dot{N}/N = F(k, 1)$. Hence, since both per capita consumption c and the per capita capital stock k are constant, we obtain that the population growth rate $\dot{N}/N$ is constant, requiring that population growth is exponential in this case. Finally, since it follows from (F1) that $F(k, 1) = F_K k + F_N$, we have that the term $c - F_N$ on the right-hand side of the investment rule (8) is constant and equals $(F_K - \dot{N}/N)k$. Such a balanced growth path is efficient if and only if $c - F_N = (F_K - \dot{N}/N)k \geq 0$.

Illustration in the DHSS *model.* The DHSS model corresponds to assumptions (F1), (F2), (u1), and (u2′). The relative price of the flows of resource input and net investment of reproducible capital equals $F_R(K(t), R(t), N(t))$. Hence, in this model, the investment rule (3) becomes:

$$\dot{K}(t) - F_R(K(t), R(t), N(t))R(t) = \int_t^\infty e^{-\int_t^\tau F_K \mathrm{d}s}\big(c - F_N\big)\dot{N}\mathrm{d}\tau\,. \tag{9}$$

We can check that following this rule actually leads to constant per capita consumption. By differentiating the investment rule it follows that

$$\ddot{K} - F_R\dot{R} - F_K F_R R = \ddot{K} - F_R\dot{R} - \dot{F}_R R = -\left(\frac{C}{N} - F_N\right)\dot{N} + F_K\big(\dot{K} - F_R R\big)\,,$$

also using Hotelling's rule ($\dot{F}_R = F_K F_R$) as a condition for short-run efficiency. From the fact that production is split between consumption and investment ($C + \dot{K} = F(K, R, N)$), we obtain:

$$\ddot{K} = F_K \dot{K} + F_R \dot{R} + F_N \dot{N} - \dot{C} \,.$$

From these two equations, we have $\dot{C} = C\dot{N}/N$ and $\dot{c} = 0$. The efficient constant per capita consumption level is found by maximizing c subject to feasibility.

Also in this model the case where the right-hand side of the investment rule equals $K\dot{N}/N$ is of interest. Then, by (9), investment $\dot{K}$ in reproducible capital compensates for resource depletion $F_R R$ and the dilution $K\dot{N}/N$ of reproducible capital. It now follows from $C + \dot{K} = F(K, R, N)$ and (F1) that $c + F_R r + k\dot{N}/N = F(k, r, 1) = F_K k + F_R r + F_N$, where $r = R/N$ denotes the per capita flow of resource extraction. Hence, also in the DHSS model the term $c - F_N$ on the right-hand side of the investment rule (9) equals $(F_K - \dot{N}/N)k$. When the production function takes on the Cobb-Douglas form (4) with $\alpha > \beta$ it follows from Asheim, Buchholz, Hartwick, Mitra and Withagen (2007, Theorem 9) and Asheim, Hartwick and Mitra (2021, Corollary 2) that the exogenous population growth function, $N(t)$, must be of the following quasi-arithmetic form:

$$N(t) = N_0 \left(1 + \mu t\right)^{\varphi} \,, \tag{10}$$

where the gross of population growth savings rate $a = I/F(K, R, N)$ is a constant satisfying $\alpha > a > \beta$ and

$$\begin{aligned} \mu &= \beta \left[(\alpha - a)^{\beta} K_0^{\alpha-1} S_0^{\beta} N_0^{1-\alpha-\beta} \right]^{\frac{1}{1-\beta}} \,, \\ \varphi &= \frac{a - \beta}{\beta} \,. \end{aligned}$$

Even though the per capita stock k of reproducible capital is increasing to compensate for the decreasing per capita flow r of resource input, the population growth rate $\dot{N}/N$ is decreasing to make the per capita dilution $k\dot{N}/N$ of reproducible capital constant. Furthermore, it is a consequence of the Cobb-Douglas form that $F(k, r, 1)$, $F_K k$, and F_N are constant, implying that $c - F_N = (F_K - \dot{N}/N)k$ is constant. Finally, this term being positive corresponds to the condition $\alpha > a$, which is the limit on population growth (through upper bounds on μ and φ) discussed by Mitra (1983) and Asheim, Buchholz, Hartwick, Mitra and Withagen (2007).

Illustration in the Stollery–d'Autume–Schubert model. The Stollery–d'Autume–Schubert model corresponds to assumptions (F1), (F2), (u1), and (u2). It follows from the Keynes-Ramsey rule

(6) that the consumption discount factor develops as follows:

$$p_0(t) = \int_0^\infty e^{-\int_0^t \left(F_K + \frac{\dot{u}_m}{u_m}\right)\mathrm{d}\tau} .$$

Furthermore, $p_0(t)u_m(M(t)/N(t), S(t)/N(t))$ is the present-value price of investment and the relative price of the flows of resource input and net investment of reproducible capital equals $F_R(K(t), R(t), N(t))$. Hence, in this model, the investment rule (3) becomes:

$$u_m\Big(\tfrac{M(t)}{N(t)}, \tfrac{S(t)}{N(t)}\Big) \cdot \big(\dot{K}(t) - F_R(K(t), R(t), N(t))R(t)\big) = \int_t^\infty e^{-\int_t^\tau \left(F_K + \frac{\dot{u}_m}{u_m}\right)\mathrm{d}s}\big(u - u_m F_N\big)\dot{N}\mathrm{d}\tau . \quad (11)$$

We can check that following this rule actually leads to constant per capita utility by differentiating both sides of this equation. Differentiating the left-hand side of (11) and using the Hotelling rule (7) and the fact that $\dot{K} = F(K, R, N) - M$ (so that $\ddot{K} = F_K\dot{K} + F_R\dot{R} + F_N\dot{N} - \dot{M}$), we get:

$$\begin{aligned}
&\dot{u}_m \cdot \big(\dot{K} - F_R R\big) + u_m \cdot \big(\ddot{K} - F_R\dot{R} - \dot{F}_R R\big) \\
&= \dot{u}_m \cdot \big(\dot{K} - F_R R\big) + u_m \cdot \big(\ddot{K} - F_R\dot{R} - F_K F_R R\big) + u_s R \\
&= \dot{u}_m \cdot \big(\dot{K} - F_R R\big) + u_m \cdot \big(F_K\dot{K} + F_N\dot{N} - F_K F_R R\big) - \big(u_m\dot{M} + u_s\dot{S}\big) .
\end{aligned}$$

Differentiating the right-hand side of (11) yields

$$\begin{aligned}
&-\big(u - u_m F_N\big)\dot{N} + \left(F_K + \frac{\dot{u}_m}{u_m}\right) u_m \cdot \big(\dot{K} - F_R R\big) \\
&= \dot{u}_m \cdot \big(\dot{K} - F_R R\big) + u_m \cdot \big(F_K\dot{K} + F_N\dot{N} - F_K F_R R\big) - u\dot{N} .
\end{aligned}$$

Since they are equal to each other by the investment rule, we obtain $u_m\dot{M} + u_s\dot{S} = u\dot{N}$. This implies that $c = u(M/N, S/N) = u(M, S)/N$ is constant since $\dot{c} = \big(u_m\dot{M} + u_s\dot{S}\big)/N - u(M/N, S/N)\dot{N}/N$. The efficient constant per capita consumption level is found by maximizing c subject to feasibility.

Finally, note that the investment rule (11) can be restated as

$$\dot{K}(t) - F_R(K(t), R(t), N(t))R(t) = \int_t^\infty e^{-\int_t^\tau F_K \mathrm{d}s}\big(\tfrac{u}{u_m} - F_N\big)\dot{N}\mathrm{d}\tau ,$$

by writing $u_m(t) = u_m(M(t)/N(t), S(t)/N(t))$ and using the fact that

$$u_m(t) = u_m(\tau)e^{-\int_t^\tau \frac{\dot{u}_m}{u_m}\mathrm{d}s} .$$

4 Analysis in a general model

We derive the investment rule (2) in a general model with multiple capital goods which has been previously analyzed in Asheim, Hartwick and Mitra (2021, Section 3) and which in turn builds on the framework of Asheim (2004).

Denote by $\mathbf{K} = (K_1, \ldots, K_n)$ the non-negative vector of capital goods. This vector includes not only the usual kinds of man-made capital stocks, but also stocks of natural resources, environmental assets, human capital, and other durable productive assets. Corresponding to the stock of capital of type j, K_j, there is a net investment flow: $I_j = \dot{K}_j$. Hence, $\mathbf{I} = (I_1, \ldots, I_n) = \dot{\mathbf{K}}$ denotes the vector of net investments.

The quadruple $(C, \mathbf{I}, \mathbf{K}, N)$ is *feasible* if $(C, \mathbf{I}, \mathbf{K}, N) \in \mathcal{Y}$, where $\mathcal{Y}$ is a convex set, with free disposal of consumption flows. The set of feasible quadruples does not depend directly on time. Thus, current productive capacity depends solely on the vector of capital stocks and labor. As before, labor equals population, which is an exogenously given function satisfying (5). The Stollery–d'Autume–Schubert model analyzed in Section 3 is a special case of this general model by letting by letting $(I_1, I_2) = (I, -R)$, $(K_1, K_2) = (K, S)$ and allowing for free disposal of consumption and net investment flows:

$$\begin{aligned}\mathcal{Y} = &\{(C, I_1, I_2, K_1, K_2, N) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}^3_+ : \\ &\quad \text{there exists } M \in \mathbb{R}_+ \text{ such that } C \le Nu\big(\tfrac{M}{N}, \tfrac{K_2}{N}\big) \text{ and } M + I_1 \le F(K_1, -I_2, N)\}\,,\end{aligned}$$

where $\mathcal{Y}$ is convex since u and F are concave. In fact, $\mathcal{Y}$ is cone in the model of Section 3 since F is homogeneous of degree 1, implying that the technology exhibits constant returns to scale.

Society makes decisions according to a *resource allocation mechanism*, $C^* : \mathbb{R}^{n+1}_+ \to \mathbb{R}_+$ and $\mathbf{I}^* : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n$, that assigns to any vector of capital stocks $\mathbf{K}$ and any population N a pair $(C^*(\mathbf{K}, N), \mathbf{I}^*(\mathbf{K}, N))$ satisfying that $(C^*(\mathbf{K}, N), \mathbf{I}^*(\mathbf{K}, N), \mathbf{K}, N)$ is feasible. We assume that there is a continuously differentiable function $\mathbf{K} : [0, \infty) \to \mathbb{R}^n_+$ being the unique solution to the differential equations $\dot{\mathbf{K}}(t) = \mathbf{I}^*(\mathbf{K}(t), N(t))$ when $\mathbf{K}(0)$ equal the exogenously given initial stocks $\mathbf{K}_0 \in \mathbb{R}^n_+$. Hence, $\mathbf{K}(t)$ is the capital path that the resource allocation mechanism implements.

A *path* from initial stocks $\mathbf{K}_0 \in \mathbb{R}^n_+$ is described by the functions $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$, where $C : [0, \infty) \to \mathbb{R}_+$ and $\mathbf{I} : [0, \infty) \to \mathbb{R}$ are determined by $C(t) = C^*(\mathbf{K}(t), N(t))$ and $\mathbf{I}(t) = \mathbf{I}^*(\mathbf{K}(t), N(t))$ for all $t \ge 0$. A path $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ from initial stocks $\mathbf{K}_0 \in \mathbb{R}^n_+$

is called *interior* if $C^* : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ and $\mathbf{I}^* : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ are continuously differentiable at all $(\mathbf{K}', N')$ such that $(\mathbf{K}', N') = (\mathbf{K}(t), N(t))$ for some $t \geq 0$. This implies that also $C : [0, \infty) \to \mathbb{R}_+$ and $\mathbf{I} : [0, \infty) \to \mathbb{R}^n$ are continuously differentiable functions. A path $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ from initial stocks $\mathbf{K}_0 \in \mathbb{R}_+^n$ is called *competitive* if

(C) for all $t \geq 0$, there exist present-value prices of the flows of consumption, labor input, and investment, $p_0(t)$, $w(t)$, and $\mathbf{p}(t)$, with $\mathbf{p}(t) \geq 0$, such that $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ maximizes profits $p_0(t)C' - w(t)N' + \mathbf{p}(t)\mathbf{I}' + \dot{\mathbf{p}}(t)\mathbf{K}'$ over all $(C', \mathbf{I}', \mathbf{K}', N') \in \mathcal{Y}$.

A competitive path is short-run efficient. By differentiating $p_0(t)C' - w(t)N' + \mathbf{p}(t)\mathbf{I}' + \dot{\mathbf{p}}(t)\mathbf{K}'$ with respect to N' and the components of $\mathbf{K}'$ and recalling that C^* and $\mathbf{I}^*$ are continuously differentiable at all $(\mathbf{K}(t), N(t))$ along an interior path, it follows from (C) that, for all $t \geq 0$,

$$w(t) = p_0(t)\frac{\partial C^*(\mathbf{K}(t), N(t))}{\partial N} + \mathbf{p}(t)\frac{\partial \mathbf{I}^*(\mathbf{K}(t), N(t))}{\partial N}\,, \tag{12}$$

$$-\dot{\mathbf{p}}(t) = p_0(t)\nabla_{\mathbf{K}} C^*(\mathbf{K}(t), N(t)) + \mathbf{p}(t)\nabla_{\mathbf{K}} \mathbf{I}^*(\mathbf{K}(t), N(t)) \tag{13}$$

if a competitive path is interior. Since

$$\frac{\partial C^*}{\partial N} \cdot \dot{N} + \nabla_{\mathbf{K}} C^* \cdot \dot{\mathbf{K}} = \dot{C} \quad \text{and} \quad \frac{\partial \mathbf{I}^*}{\partial N} \cdot \dot{N} + \nabla_{\mathbf{K}} \mathbf{I}^* \cdot \dot{\mathbf{K}} = \dot{\mathbf{I}}\,,$$

we have that eqs. (12) and (13) imply that, at each t,

$$w(t)\dot{N}(t) - \dot{\mathbf{p}}(t)\mathbf{I}(t) = p_0(t)\dot{C}(t) + \mathbf{p}(t)\dot{\mathbf{I}}(t)\,. \tag{14}$$

Since $C = cN$ and $\mathbf{I} = \mathbf{i}N$, eq. (14) implies that

$$p_0(t)\dot{c}(t)N(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{p}(t)\mathbf{i}(t)N(t)\big) - \big(c(t) - w(t)\big)\dot{N}(t)\,.$$

Hence, along a short-run efficient path constant per capita consumption is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{p}(t)\mathbf{i}(t)N(t)\big) = \big(c(t) - w(t)\big)\dot{N}(t)\,. \tag{15}$$

Furthermore, as stated in the following proposition, if the net investment transversality condition (1) is combined with competitiveness and interiority as a requirement of short-run efficiency, then a path with constant per capita consumption throughout is equivalent to the investment rule (3) holding throughout.

Proposition 1. *Consider an interior and competitive path* $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ *which satisfies the net investment transversality condition (1) in the general model with multiple capital goods. Then per capita consumption* $c(t) = C(t)/N(t)$ *is constant at all* $t \in [0, \infty)$ *if and only if*

$$\mathbf{p}(t)\mathbf{i}(t)N(t) = \int_t^\infty \big(p_0(\tau)c(\tau) - w(\tau)\big)\dot{N}(\tau)\,d\tau \tag{3}$$

holds at all $t \in [0, \infty)$.

Proof. Let $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ be an interior and competitive path which satisfies the the net investment transversality condition (1) in the general model with multiple capital goods.

Only if. Assume that per capita consumption $c(t) = C(t)/N(t)$ is constant at all $t \in [0, \infty)$. Then eq. (15) holds at all $t \in [0, \infty)$ and the investment rule (3) follows through integration.

If. Assume the investment rule (3) holds at all $t \in [0, \infty)$. Differentiating both sides of (3) yields eq. (15) which in turn implies that per capita consumption $c(t) = C(t)/N(t)$ is constant. □

5 Per capita investment rules

If $\mathcal{Y}$ is a cone—as exemplified in the Stollery–d'Autume–Schubert model of Section 3—then the technology exhibits constant returns to scale, and the competitiveness condition (C) implies that, for all $t \geq 0$, maximized profits must be zero:

$$p_0(t)C(t) - w(t)N(t) + \mathbf{p}(t)\mathbf{I}(t) + \dot{\mathbf{p}}(t)\mathbf{K}(t) = 0\,, \tag{16}$$

where $p_0(t)C(t) + \mathbf{p}(t)\mathbf{I}(t)$ is the present value of outputs, and $w(t)N(t) - \dot{\mathbf{p}}(t)\mathbf{K}(t)$ is the present cost of inputs, as $-\dot{p}_j$ can be interpreted as the cost of holding one unit of capital of type j. Under this additional assumption we derive per capita investment rules for constant per capita consumption based on the analyses of Arrow, Dasgupta and Mäler (2003) and Asheim (2004) by investigating conditions imposed on the development of the per capita capital vector $\mathbf{k} = \mathbf{K}/N$.

If the exogenous population development as a function of time is monotone, then we can—following Arrow, Dasgupta and Mäler (2003)—assume that population is determined by the initial condition $N(0) = N_0$ and the growth function:

$$\dot{N} = \phi(N)\,.$$

Let $\nu(N)$ denote the rate of population growth as a function of population size:

$$\nu(N) = \frac{\phi(N)}{N}\,.$$

Let $\psi(t)$ denote the present value of population growth for per capita consumption.[3] Then $\psi(t)\phi'(N(t))$ is the gain in terms of the value of population growth of increased population, while $p_0(\tau)c(\tau) - w(\tau)$ is the cost in terms of per capita consumption of increased population. No-arbitrage implies that the decrease in the present-value price must equal the value of increased population:

$$-\dot{\psi}(t) = -\big(p_0(\tau)c(\tau) - w(\tau)\big) + \psi(t)\phi'(N(t))\,.$$

Integration yields:

$$\psi(t) = -\frac{1}{\phi(N(t))}\left(\int_t^\infty \big(p_0 c - w\big)\phi(N)\mathrm{d}\tau\right)\,. \tag{17}$$

Setting $\mathbf{p}(t)\dot{\mathbf{K}}(t) + \psi(t)\dot{N}(t) = 0$ corresponds to the investment rule (3) that we have already discussed and involves invoking the net investment transversality condition (1). By using $\dot{\mathbf{k}}(t) = \dot{\mathbf{K}}(t)/N(t) - \nu(N(t))\mathbf{k}(t)$ we can express this investment rule in terms of per capita variables:

$$\frac{\mathbf{p}\dot{\mathbf{K}} + \psi\dot{N}}{N} = \mathbf{p}\dot{\mathbf{k}} + \nu(N)\cdot\big(\mathbf{pk} + \psi\big) = 0\,. \tag{18}$$

To obtain expressions for $\mathbf{pk} + \psi$, note that under constant returns to scale it follows from (16) that $-\mathrm{d}\big(\mathbf{pK}\big)/\mathrm{d}t = p_0 C - wN$. By combining this equation with eq. (17) we have:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi N\big) = -\big(\dot{\psi}N + \psi\dot{N}\big) = \big(-(p_0 c - w) + \psi\phi'(N)\big)N - \psi\nu(N)N = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{pK}\big) + \nu'(N)N\psi N\,,$$

using that $\phi'(N) = \mathrm{d}\big(\nu(N)N\big)/\mathrm{d}t = \nu'(N)N + \nu(N)$. Hence:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{pK} + \psi N\big) = \nu'(N)N\psi N\,.$$

[3] The following analysis is based on Asheim (2004, Section 7), which in turn builds on results in Arrow, Dasgupta & Mäler (2003). The symbol ψ as used here corresponds to $\tilde{\psi}$ in Asheim (2004).

By integrating and imposing a capital value transversality condition, it follows that:

$$\mathbf{p}(t)\mathbf{K}(t) + \psi(t)N(t) = \int_t^\infty \nu'(N)N\psi N \mathrm{d}\tau$$

and, by dividing through by $N(t)$, we obtain a first expression for $\mathbf{pk} + \psi$:

$$\mathbf{p}(t)\mathbf{k}(t) + \psi(t) = \frac{1}{N(t)} \left(\int_t^\infty \nu'(N)\psi N^2 \mathrm{d}\tau \right) . \tag{19}$$

By differentiating this equation we can obtain a second expression for $\mathbf{pk} + \psi$, again using that $\phi'(N) = \mathrm{d}\big(\nu(N)N\big)/\mathrm{d}t = \nu'(N)N + \nu(N)$:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{pk} + \psi\big) = \nu'(N)\psi N + \nu(N)\big(\mathbf{pk} + \psi\big) = -\nu'(N)\mathbf{pK} + \phi'(N)\big(\mathbf{pk} + \psi\big) .$$

By integrating and imposing a per capita capital value transversality condition, we obtain:

$$\mathbf{p}(t)\mathbf{k}(t) + \psi(t) = -\frac{1}{\phi(N(t))} \left(\int_t^\infty \nu'(N)\mathbf{pK}\phi(N) \mathrm{d}\tau \right) . \tag{20}$$

The results on per capita investment rules for constant per capita consumption are summarized in the following proposition.

Proposition 2. *Consider an interior and competitive path* $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ *which satisfies the net investment transversality condition (1) as well as capital value transversality conditions in total and per capita terms, in the general model with multiple capital goods where the technology exhibits constant returns to scale. Then per capita consumption* $c(t) = C(t)/N(t)$ *is constant at all* $t \in [0, \infty)$ *if and only if the following per capita investment rule holds at all* $t \in [0, \infty)$*:*

$$\mathbf{p}(t)\dot{\mathbf{k}}(t) = -\nu(N(t)) \cdot \big(\mathbf{p}(t)\mathbf{k}(t) + \psi(t)\big) ,$$

where

$$\mathbf{p}(t)\mathbf{k}(t) + \psi(t) \ = \ \frac{1}{N(t)} \left(\int_t^\infty \nu'(N)\psi N^2 d\tau \right) \ = \ -\frac{1}{\phi(N(t))} \left(\int_t^\infty \nu'(N)\mathbf{pK}\phi(N) d\tau \right) .$$

If population growth is exponential, so that $\nu(N)$ is constant and $\nu'(N) = 0$, this simplifies to:

$$\mathbf{p}(t)\dot{\mathbf{k}}(t) = 0 .$$

Following this investment rule in the one-sector model leads to a balanced growth path, where the per capita capital stock is held constant, as we have already discussed in Section 3. In the DHSS model, exponential population growth cannot be combined with non-decreasing and positive per capita consumption. As we have discussed in Section 3 based on the analysis in Asheim, Buchholz, Hartwick, Mitra and Withagen (2007) and Asheim, Hartwick and Mitra (2021), quasi-arithmetic growth can be combined with constant per capita consumption leading to the investment rule $\dot{K} - F_R R = K\dot{N}/N$. By using $\dot{k} = \dot{K}/N - k\dot{N}/N$, this investment rule has the following per capita form:

$$\dot{k} - F_R r = 0\,.$$

Since $-r = -R/S = \dot{S}/N = \dot{s} + s\dot{N}/N > \dot{s}$ with positive population growth, this implies that

$$\dot{k} + F_R \dot{s} = -\nu(N) F_R s < 0\,,$$

where straightforward calculations imply that

$$\frac{\dot{N}}{N} = \nu(N) = \varphi\mu \left(\frac{N_0}{N}\right)^{\frac{1}{\varphi}}$$

in the case where population growth is governed by the quasi-arithmetic growth function (10). We see that the deviation from keeping the value of the changes of per capita capital stocks equal to zero is consistent with our results. The reason is that, with quasi-arithmetic growth where

$$\nu'(N) = -\tfrac{1}{\varphi} \cdot \frac{\nu(N)}{N} < 0\,,$$

the investment rule of Proposition 2 expressed in terms of per capita variables implies that $\mathbf{p}(t)\dot{\mathbf{k}}(t) < 0$. In this manner the investment rule in terms of per capita variables is illustrated by the special models considered in Section 3, provided that constant returns to scale are imposed.

6 Empirical application

We provide an empirical application of our theoretical results. We start by rewriting the investment rules in Propositions 1 and 2. The right-hand side of Proposition 1 provides a requirement for the value of per capita investments in terms of the difference between the present value of future additional consumption needed to sustain per capita consumption and the contribution to current productive capacity of future population growth. Per capita consumption $c(t)$ is constant at all

$t \in [0,\infty)$ if and only if:

$$\mathbf{p}(t)\mathbf{i}(t) = \frac{1}{N(t)}\int_t^{\infty}\big(p_0(\tau)c(\tau) - w(\tau)\big)\dot{N}(\tau)\mathrm{d}\tau$$

at all $t \in [0,\infty)$. The right-hand side of Proposition 2 provides a requirement for the value of per capita capital accumulation that depends on whether future population growth accelerates or not. Per capita consumption $c(t)$ is constant at all $t \in [0,\infty)$ if and only if:

$$\mathbf{p}(t)\dot{\mathbf{k}}(t) = \frac{1}{N(t)}\int_t^{\infty}\nu'(N)\mathbf{p}\mathbf{K}\phi(N)\mathrm{d}\tau \ ,$$

at all $t \in [0,\infty)$.

In either proposition, empirical application requires population prospects for the entire future. Note that Proposition 1 allows for any development of population as a function of time, while Proposition 2 requires that population growth remain positive or negative. In Proposition 2, a problem arises if a country experiences a demographic change from positive to negative population growth. Such a cross-cutting case, as well as the underlying population dynamics being dependent on time, is not allowed in a time-invariant Markovian population function on which Proposition 2 builds, as introduced by Arrow, Dasgupta and Mäler (2003). Hence, in the following we concentrate on Proposition 1 and relegate further analysis of Proposition 2 to Appendix A.

6.1 Data and assumptions

Since prices $\mathbf{p}, p_0$, and w are given in present-value terms, the consumption discount rate, $-\dot{p}_0/p_0$ plays an important role. The consumption discount rate equals the marginal productivity of capital—the rental price of a unit of K—which we assume to be constant at 4% in the base case. This is a departure from the time-varying discount rate F_K that supports efficient paths. To compensate, as well as to check sensitivity, we also consider discount rates equal to 1% and 7%.

The current monetary value of the change in capital assets has been referred to as genuine savings (GS) or adjusted net savings by the World Bank. We use the recent 10–year average of GS (for 2008–2017) for various combinations of produced, natural, and human capital stocks, with or without the damage from CO_2 and particulate matter (PM) emissions. For the models we have seen in the previous sections, the DHSS model contains produced and (nonrenewable) natural capital, while Stollery-d'Autume-Schubert model also includes the natural capital amenity value of not letting CO_2 and PM be emitted. To obtain GS in per capita form $\mathbf{p}(t)\mathbf{i}(t)$, we simply divide

GS by the number of population at t reported in the UN's *World Population Prospects 2019*.

Population prospects from 2021–2100 for the middle fertility scenario, also from the UN's *World Population Prospects 2019*, is used to derive future population change $\dot{N}$ on the right-hand side of Proposition 1. Population is assumed to be constant from 2100 on.

Per capita consumption is taken from final consumption expenditure contained in the *World Development Indicators* database by the World Bank, divided by population. For the treatment of the real wage, which equals marginal labor productivity in a competitive market, we follow Yamaguchi (2014) to use the labor income share of output in the macroeconomy reported by International Labour Organization (ILO). Proposition 1 looks at the consumption-wage gap in per capita terms, which in the empirical analysis is assumed to be unchanged in the future; that is, $c(\tau) - w(\tau)/p_0(\tau)$ is constant in the integral. The constancy of the consumption-wage gap may be empirically rationalized, as we are centrally interested in sustainable per capita consumption, while the labor income share is known to be relatively stable across time as one of the stylized facts by Kaldor (1957).[4] Moreover, as we discussed in Section 3, the constant consumption-wage gap is consistent with a special case of a balanced growth path of the one-sector model.

Finally, we have to restrict our analysis to the current investment in 2008–2017, although Propositions 1 and 2 suggest that our investment rules be applied to all the time periods t in $[0, \infty)$. Data sources and assumptions are summarized in Table 1.

Table 1: Data sources

Data	Description	Source
c	final consumption expenditure divided by population	*WDI*
w	labour share of GDP (wages & social protection transfers) per population	ILOSTAT and *WDI*
$\mathbf{i}$	adjusted net savings (GS) divided by population	*WDI*
$\mathbf{k}$	per capita produced, human, and natural capital, and net foreign assets	World Bank (2021)
$\mathbf{K}$	produced, human, natural capital, and net foreign assets	World Bank (2021)
N	total population for 1971–2020	*World Population Prospects 2019*
N	total population for 2021–2100, Medium fertility variant	*World Population Prospects 2019*

Note: The unit for c, w and i is constant 2015 USD, and the unit for k and K is in constant 2018 USD. *WDI* stands for *World Development Indicators*. Adjusted net savings is calculated as GS = net national savings + educational expenditure - energy depletion - mineral depletion - CO_2 damage - PM damage.

[4] We should also note that the labor income share is known to be on the decline in recent decades in many parts of the world (Karabarbounis and Neiman, 2014). In addition, the labor income share tends to be lower in lower-income countries, where self employment is more common. Given all this, our assumed constancy of the consumption-wage gap might tend to be under/over-estimates in high/lower-income countries.

6.2 Results

Table 2 and Figure 1 show required investment as stipulated by Proposition 1, as opposed to actual genuine savings per population averaged over the period 2008–2017 in select countries. In Table 2, the second to fifth columns show actual genuine savings divided by population for combinations of investment in capital assets, starting from only produced and natural capital, incrementally incorporating human capital and CO_2 and PM damage. The last column shows the levels of required investment according to Proposition 1.

We first note that genuine savings have been recently positive for all the studied countries, if we use the measure inclusive of all capital assets (see the fifth "ALL" column). There are a few exceptions, depending on which capital assets are captured by GS. Even if we look at the value of per capita investments in only produced and natural capital—in correspondence with the DHSS model—only Kenya and the United Kingdom have not experienced sustainable development, according to the conventional interpretation of GS.

In view of Proposition 1 to sustain per capita consumption, however, this optimistic tendency is overturned: actual per capita investments are less than required in countries where population growth is expected to be high in the future *and* the consumption-wage gap is large vis-à-vis per capita GS. In the three sub-Saharan countries, Nigeria, Kenya, and South Africa, per capita GS (in its most enhanced form) do not meet the investment that is required for sustainable per capita consumption. Also, our results show that just because population is expected to increase does not mean per capita GS is insufficient to support per capita consumption, as in Botswana, Brazil, Indonesia, and India. Moreover, if population is forecast to decrease, required investment even turns negative, as in China, Germany, Japan, and Russia.

Although discounting does not play as critical a role as in dynamic average utilitarianism (Yamaguchi, 2018), it still changes the absolute value of required investment. Table 3 and Figure 2 show sensitivity analysis with regard to discounting. Changing the discount rate from 4% to 7% lowers the bar of required investment for most countries, except where population is expected to decline, as in Brazil, China, Germany, Japan, and Russia.

The lower discounting case might be more plausible and interesting in our particular context, because with per capita consumption unchanged, the consumption discount rate would be reduced to the pure rate of time preference (i.e., the utility discount rate) in the Ramsey formula. The discount rate set at 1% presents a sterner result than the 4% case, as expected. On top of the

Table 2: Actual and required per capita genuine savings

	Actual				Required
	K&S	*H*	*X*	ALL	Prop.1
Australia	2,093	2,630	-531	4,192	2,503
Botswana	962	572	-110	1,424	710
Brazil	580	486	-78	988	105
Canada	1,564	2,056	-463	3,156	1,339
Chile	550	559	-151	958	116
China	1,946	120	-268	1,797	-4
Ecuador	430	218	-102	547	396
France	1,519	1,909	-172	3,256	149
Germany	3,671	1,835	-293	5,214	-85
Indonesia	296	90	-77	309	138
India	307	43	-69	282	33
Japan	777	1,022	-269	1,530	-809
Nigeria	147	20	-101	66	496
Kenya	-32	64	-26	7	374
Norway	10,398	4,808	-265	14,941	2,052
Russia	699	330	-325	703	-79
Saudi Arabia	3,585	1,446	-608	4,422	1,945
United States	808	2,549	-591	2,766	1,579
United Kingdom	-1,061	2,264	-246	957	838
South Africa	7	315	-283	39	418

Note: Unit: constant 2015 USD. In this table, $K\&S$, and H stand for produced, natural, and human capital, respectively, while X means the inclusion of CO_2 and particulate matter (PM) damage. Thus, ALL corresponds to net national savings *plus* educational expenditure *minus* energy and mineral depletion *minus* CO_2 and PM damage.

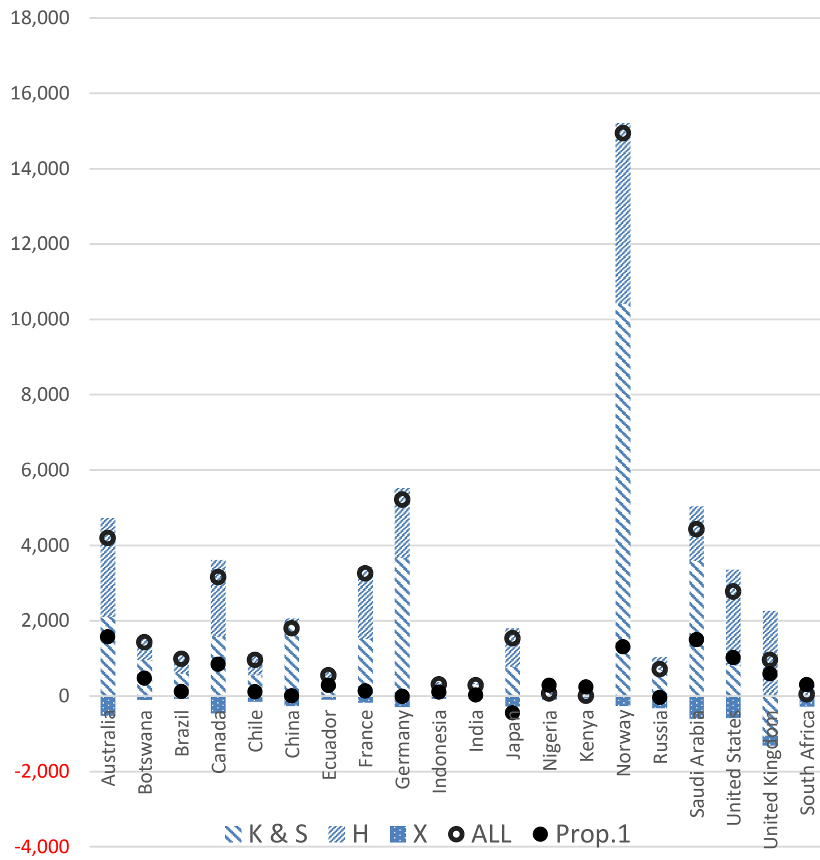

Figure 1: Actual and required per capita genuine savings

Note: Unit: constant 2015 USD. The bars show the value of per capita investments in produced and natural ($K\&S$), and human capital (H), and CO_2 and PM damage (X). The dots represent the total genuine savings per person (ALL), and the required genuine savings per person suggested by Prop.1.

Table 3: Actual and required per capita genuine savings: sensitivity analysis

	Actual	Required by Prop.1		
	ALL	1%	4%	7%
Australia	4,192	5,408	2,503	1,566
Botswana	1,424	1,269	710	474
Brazil	988	-68	105	114
Canada	3,156	2,891	1,339	845
Chile	958	6	116	113
China	1,797	-31	-4	2
Ecuador	547	621	396	278
France	3,256	118	149	130
Germany	5,214	-312	-85	-11
Indonesia	309	183	138	104
India	282	32	33	27
Japan	1,530	-1,929	-809	-450
Nigeria	66	1,196	496	281
Kenya	7	749	374	234
Norway	14,941	4,254	2,052	1,302
Russia	703	-181	-79	-41
Saudi Arabia	4,422	2,473	1,945	1,489
United States	2,766	3,215	1,579	1,013
United Kingdom	957	1,461	838	589
South Africa	39	658	418	296

Note: Unit: constant 2015 USD. "ALL" means per capita GS inclusive of produced, natural, and human capital, as well as CO_2 and PM damage.

three sub-Saharan countries with insufficient GS in view of Proposition 1 with 4% discounting, Australia, the United States, and the United Kingdom join the club of insufficient investors. This surprising result about the high-income countries can be interpreted as: if a relatively large portion of (already high) per capita consumption is financed by non-labor income, then it is necessary to save more for the future, as an additional population is expected to require more in consumption than it contributes to labor.

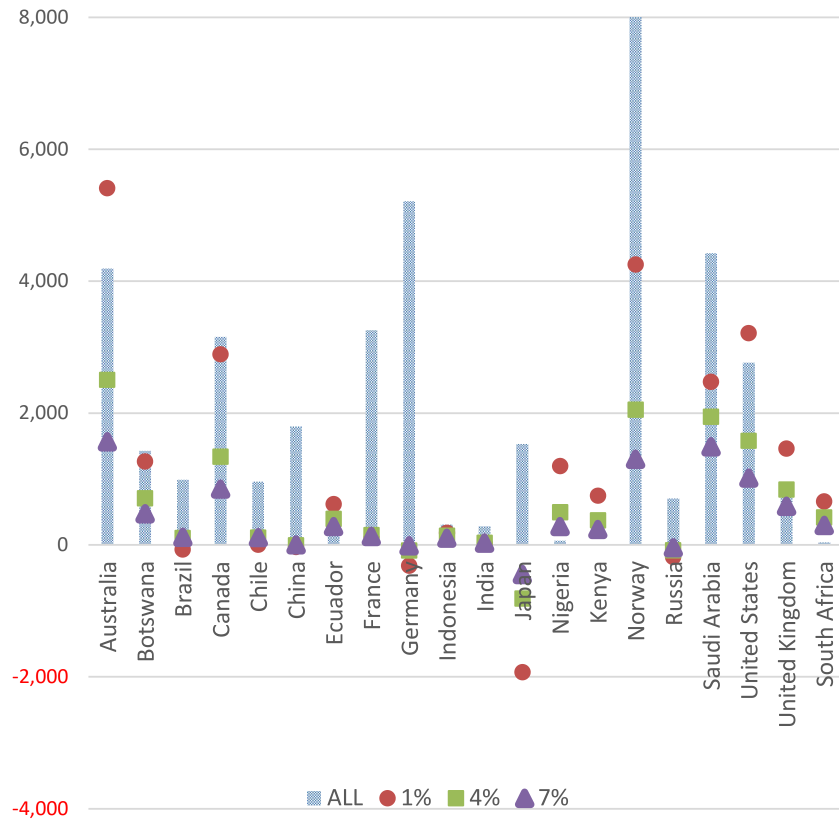

Figure 2: Actual and required per capita genuine savings: sensitivity analysis

Note: Unit: constant 2015 USD. The bars show the actual value of per capita GS. The dots represent required per capita GS under different discount rates: round shape (1%), square shape (4%), and triangle shape (7%). Norway's actual per capita GS is over 8,000 USD. See Table 3.

7 Concluding remarks

Propositions 1 and 2 lay two different foundations for an empirical strategy for estimating the maximal per capita consumption in real economies with population growth. Proposition 1 provides a requirement for the value of per capita investments in terms of the difference between the present value of future additional consumption needed to sustain per capita consumption and the contribution to current productive capacity of future population growth. Proposition 2 provides a requirement for the value of per capita capital accumulation that depends on whether future population growth accelerates or not. In both cases, the sustainability of current per capita consumption depends also on the development of future population growth. With afterthought, this makes sense. Our empirical application to recent genuine savings and consumption data raises some concerns about sustaining per capita consumption when future population growth is combined with a large per capita consumption-wage gap. Despite serious data challenges, we hope that our forward-looking exercise, utilizing future population prospects, might provide a useful indication of the requirements for genuine savings needed to sustain per capita consumption.

The models we have considered have exogenous population growth. Hence, no issues of population ethics are involved. Also, with an infinite horizon and a population that is non-decreasing or does not decrease too fast, total (i.e., current and future) population is infinite and will remain so, when the length of life is finite. Hence, the total population does not change over time. The total discounted utilitarian criterion trades off consumption for some individual now with consumption for some individual in the future, but where the utility of the future individual is discounted. This might make more sense than using an average discounted utilitarian criterion where the weight of the utility of a future individual is further reduced by dividing by the greater temporal population at that point in time. However, such arguments for total discounted utilitarianism do not contradict that we might be interested in the question of whether current per capita consumption can be sustained indefinitely. Hence, we believe that the investment rules we discuss in our paper are also of interest under a total discounted utilitarian criterion. The rules—if used at one point in time—are generalizations of the genuine savings indicator as an indicator of sustainability, while the rules—if imposed at all future times—are generalizations of Hartwick's rule as a characterization of the maximal constant per capita consumption path. The question of whether dynamic welfare increases over time (as posed by Arrow, Dasgupta and Mäler, 2003; Asheim, 2004) concerns whether population growth can compensate for decreasing per capita consumption, which is different from—as we do here—establishing investment rules for non-decreasing per capita consumption.

Finally, we note that the investment rules considered in this paper do not rely on an assumption of a constant utility discount rate as in the case where the underlying welfare criterion is total utilitarian. Rather, it is sufficient that the path considered is efficient and thus price supported. This minimum requirement also justifies the application of our theoretical results to population estimates in suboptimal economies in the real world.

A Some empirics for Proposition 2

Proposition 2 describes the investment rule as the value of the change in per capita capital assets being equal to a portion of the value of capital assets in the future in case of non-exponential population growth. For the right-hand side of Proposition 2, we use the sum of the value of produced, natural, human, and net foreign capital reported by the World Bank (2021). For the left-hand side, two strategies might be considered. One strategy is to rely on the relationship $\dot{\mathbf{k}} = \mathbf{i} - \mathbf{k}\dot{N}/N$ where GS divided by population is used for $\mathbf{i}$ and the value of capital assets is used for $\mathbf{k}$. The other strategy is to use the value of the change in per capita capital assets directly in $\dot{\mathbf{k}}$. We only report the results according to the first methodology, although in principle, the same set of capital assets should be recorded on both sides of the equation.[5]

Proposition 2 also requires that individual time-invariant Markovian population growth functions be estimated to compute the extra investment. In the notation of Arrow, Dasgupta and Mäler (2003), we assume a simple logistic curve for the population growth function:

$$\dot{N} = \phi(N) = \nu(N)N = AN(N^* - N)\,,$$

where N^* denotes a stable steady-state level of population. $A > 0$ is sometimes called the intrinsic growth rate in the bioeconomics literature. The derivative of the rate of population growth simply becomes $\nu'(N) = -A < 0$.

[5]In theory, genuine savings and the monetary value of the change in capital assets are identical when there is no population change. In practical accounting, they have slightly different categories, as GS is an extension of conventional national accounting in its spirit. For instance, the change in agricultural land is not considered in genuine savings, while carbon emissions do not enter capital accounting. See World Bank (2021) for details. Moreover, within the second methodology, as it turns out, investment required by Proposition 2 is typically negative, due to the specification of $\nu(N)$, so using different classes of capital assets do not change our qualitative results, unless net foreign assets become largely negative.

Using the past population in 1971-2020 as well as medium fertility forecasts for population in 2021-2100, we approximate parameters of population growth function for individual countries. Parameters and coefficients of determination (of somewhat varying degree) are reported in Table 4 below. The coefficients of determination (R^2) tend to be higher in emerging economies and the U.S., and lower in western Europe and Japan.

Table 4: Population growth functions estimates for the medium fertility case

Country	$A \times 10^9$	R^2	N^*	$N(2020)$
Australia	0.398	0.343	51,778	25,500
Botswana	8.988	0.841	4,355	2,352
Brazil	0.193	0.371	217,693	212,559
Canada	0.280	0.428	67,267	37,742
Chile	2.015	0.437	19,622	19,116
China	0.015	0.082	1,368,494	1,439,324
Ecuador	1.591	0.859	25,682	17,643
France	0.471	0.602	67,627	65,274
Germany	-0.150	0.034	81,985	83,784
Indonesia	0.126	0.822	338,516	273,524
India	0.028	0.634	1,606,036	1,380,004
Japan	-0.174	0.185	124,183	126,476
Nigeria	0.035	0.997	926,092	206,140
Kenya	0.291	0.976	134,266	53,771
Norway	1.154	0.275	10,840	5,421
Russia	-0.057	0.016	142,391	145,934
Saudi Arabia	1.388	0.788	44,734	34,814
Unites States of America	0.045	0.798	468,058	331,003
United Kingdom	0.133	0.120	88,751	67,886
Venezuela	1.294	0.469	36,902	28,436
South Africa	0.497	0.896	81,922	59,309

Note: Unit: thousands of population for N^* and $N(2020)$.

As mentioned, a tricky but realistic scenario is the population growth turning from positive to negative, which is experienced or expected in many countries, but which is not consistent with a time-invariant Markovian population growth function. Thus, we focus on those countries where such cross-cutting change is not expected for some time in the future and the goodness of fit is high ($R^2 > .75$).[6] This reduces our sample to only eight countries.

As Figure 3 shows, the right-hand side of Proposition 2 (indicated by square dots) is negative

[6]If there were no cross-cutting cases from positive to negative growth, and if the dynamics were not time-dependent, then a negative population growth could have also been achieved by assuming $N > N^*$.

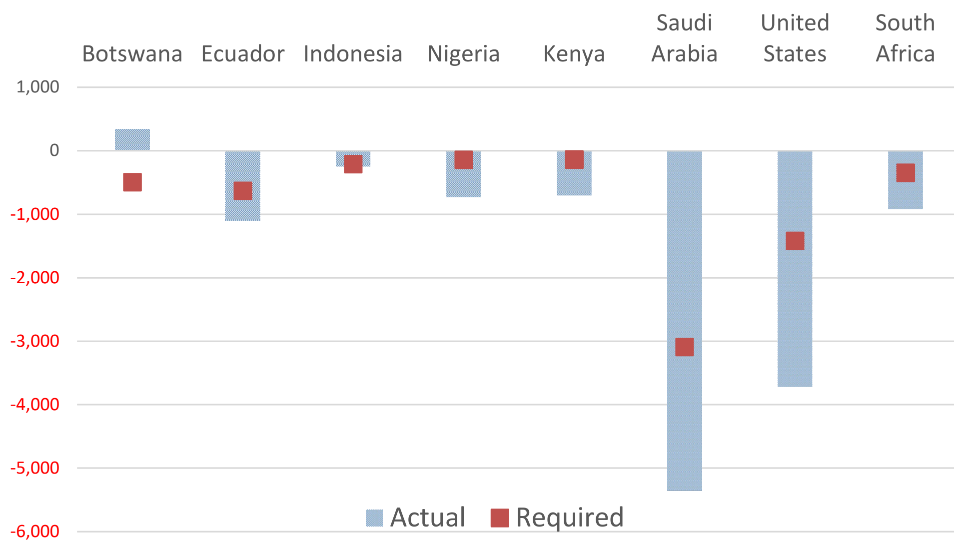

Figure 3: Actual and required change in the value of per capita capital stocks for Proposition 2

Note: Unit: USD. The bars show the actual value of the change in per capita capital stocks (i.e., per capita GS net of per capita capital stocks multiplied by population change). The dots represent required value of the change in per capita capital stocks. In computing per capita capital stocks, produced, human, and natural capital by World Bank (2021) are included.

for all these countries. This is due to the fact that in most countries population growth rate is estimated to decrease as population increases (i.e., $\nu'(N) < 0$). Nonetheless, actual investment in terms of the change in per capita capital stocks (indicated by bars) has been insufficient in view of Proposition 2, with the only exception of Botswana. Indeed, the magnitude of the effect of population deceleration turns out to be non-negligible in most of these countries.

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