Capacity Expansion and Dynamic Monopoly Pricing

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Abstract

We substitute to the plant size problem, as investigated by Chenery (1952), a new version in which a profit-maximising monopolist may combine its investment policy with a price policy adjusting demand upwards or downwards over time. We characterize the optimal price and investment policies. The optimal price policy determines an investment pattern either with constant increments of capacity over time, or becoming constant after a finite time. The existing capacity is either fully used at each instant between two investment dates; or the monopolist first quotes the instantaneous monopoly price and, thereafter, the price dampening instantaneous demand at the optimal installed capacity level.

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1 Introduction

The so-called plant-size problem introduced in the fifties by Chenery (1952) studies the optimal capacity to install in order to serve an exogenous demand increasing over time, with economies of scale in plant construction. This problem arose in the context of developing countries: explosive demographics forced the governments to regularly increase productive capacity over time, in order to face increase in demand resulting from such populations’ booms. Increasing returns to scale in plant construction opened the door to the possibility of anticipating the increments of capacity required to meet future demand levels, and exploiting thereby the resulting economies of scale. In this context, it was reasonable to assume that demand expansion was pushed exogenously by the population growth.

Today, in developed countries, similar demand booms are still observed in several industries, but for other reasons than an exogenous population growth. These booms now often follow from the globalisation of trade: while markets in developed countries expanded initially around local or national demands, they are now progressively concerned by larger and larger geographical areas. In some sense, firms operating in these industries are faced today in developed countries with a problem akin to the problem met by governments in developing countries in the past: they also expect progressive expansion of demand consecutive to trade globalisation, while benefiting as well from economies of scale in plant construction. However, in this new context, it is not reasonable to view demand increases addressed to firms operating in such industries as exogenous. These firms are no longer public institutions, as in the original situation, but rather profit maximizing institutions. As such, they benefit from specific instruments which were not at the disposal of the governments. In particular, they can now accompany their investment policy by a price policy through which they can manipulate the levels of demand through time.

In order to capture this new strategic possibility, we study in the present paper a similar plant size problem as in Chenery (1952) and Manne (1961), but now allowing the profit-maximising firm producing the good to combine its investment policy with a product price policy adjusting demand upwards, or downwards, over time. This extension has to cope with the difficulty that, even if the volume demanded is exogenously increasing (or decreasing) at each price over time, the firm can now decrease or increase it by instantaneous increases or decreases in the price pattern. We characterize the optimal price and investment policies
of the monopolist under the assumption that the firm while controlling the size of the investments to be undertaken, does not control the dates, which are assumed to be equally spaced through time, at which these investments have to be consented. Our findings are as follows. The optimal price policy, through which the monopolist can possibly dampen or enhance the expansion of demand through time, leads to an investment pattern with increments of capacity which are either constant over time at each investment date, or start to become constant after a finite set of investment dates. Furthermore, the optimal price pattern between two dates at which new capacity is installed falls into two categories: either the price pattern is such that total existing capacity is fully used at each instant between these dates; or the instantaneous monopoly price is used for some period within the cycle and, thereafter, until the end of the cycle, the price which dampens instantaneous demand at the level of installed capacity is adopted. Finally, we show that the optimal constant increment of capacity is smaller than the one which would have been selected by a planner facing a demand growing exogenously at a rate corresponding to the instantaneous monopoly price, as in Manne (1961). These conclusions are a priori far from being evident. Even constrained to invest at equally spaced dates, the monopolist could have as well preferred to manipulate the price, according to the value of the interest rate, in order to decrease or increase the size of the investments through time rather than using a constant-cycle investment size policy1.

Research on the plant-size problem has been pursued without discontinuity since the sixties. Manne (1961) and (1967) faced the capacity expansion problem which was met by several manufacturing facilities in India. Assuming an exogenous demand growing linearly over time and economies of scale in investment costs, he finds that the optimal investment policy is constant-cycle: successive investments are all of the same size and undertaken at equally spaced points of time. These findings have been extended afterwards in several directions. Srinivasan (1967) proves that the constant-cycle property of the optimal plan extends to the case where demand is growing geometrically over time. Gabszewicz and Vial (1972) extend the analysis to exogenous technological progress which can arise either at a deterministic date or at a random time in the future. In both cases, the constant-cycle property of the optimal policy

1An alternative assumption would be that the monopolist may decide about the dates, but not about the investment sizes. It could be proved that in this case, the optimal price policy leads the monopolist to manipulate the price pattern in order to guarantee that the dates at which the fixed investments have to be undertaken, are all equally spaced on the time-axis. This is the object of a forthcoming paper (see Tarola (2004)).
still holds. Around the same period a considerable amount of research work has been devoted to the problems of lead times and uncertain demand (see in particular Nickell (1977), Freidenfelds (1981), Bean and al. (1992), Chaouch and Buzacott (1994) and Ryan (2002)).

Although these research contributions represent important advances in the understanding of the plant-size problem, they all assume that demand for capacity increases is exogenously given, and cannot be adjusted through some price policy. Of course, this assumption was a natural entry point since past contributions were mainly concerned with finding the solution of a planning problem met in the framework of a country development program, and not with identifying the solution of a profit-maximiser monopolist. Nevertheless, as shown in the present paper, the plant-size problem can be reformulated in order to take into account this market alternative interpretation, in which the firm is allowed to manipulate instantaneous demand through price. So, departing from the above research lines, we propose to relax the assumption of exogenous demand increase. We replace it by an assumption about the time expansion of a price-quantity relation specifying how instantaneous demand varies with price at each point of time. More precisely, we suppose that, at each point of time, market demand is given by a linear function of instantaneous price, the intercept of which increases linearly with time. Moreover, we make the assumption alluded above concerning the size of the investments, namely, we fix a sequence of equally spaced points of time at which the investments for adding capacity may be undertaken and characterise the optimal policy of the monopolist in terms of his instruments: price regime and investment’s size.

The model is described in section 2. In the same section, we fully characterise both the optimal price and investment policies. We summarize our findings in the conclusion and propose some paths for further research.

2 The analysis of the optimal policy

2.1 The model

We consider a monopolist facing a demand function $D(t,p)$ defined by

$$D(t,p) = At - p$$

with $t$ denoting continuous time and $p$ instantaneous price: demand at each price is accordingly expanding through time because the intercept of the demand function increases at a rate proportional to $t$. At each
instant of time $t$, the productive capacity of the firm is bounded by the existing amount of equipment, which we denote by $X(t)$. While the existing capacity may exceed the current demand level $D(t, p)$, no undercapacity is admitted, so that $D(t, p) \leq X(t)$. The investment cost for adding new capacity $x$, which includes both fixed cost $k, k > 0$, and variable cost $ax$, exhibits economies of scale, namely

$$f(x) = k + ax;$$

This cost structure is assumed to hold for ever (no technological progress). Also we assume no capital depreciation. Time is discounted at a constant interest rate $r$. The time horizon is unbounded.

Now we assume that the monopolist is allowed to make investment decisions $x_{t_i}$ at fixed equally spaced points of time $t_i, i = 0, 1, 2, ...$. In the interval $[t_i, t_{i+1}]$, the existing capacity remains constant and is equal to $X(t_i) = \sum_{k:t_k \leq t_i} x_{t_k}$. At each $t_i$, the firm decides the increment of capacity $x_{t_i}$ to be installed and sets the price $p(t)$ for $t \in [t_i, t_{i+1}]$. Then at time $t_{i+1}$ a new investment is undertaken and a new price schedule for the period is defined, and so on. Up to a change of units we set $t_i = i$. The interval of time $[i, i + 1]$ between two dates at which a new investment is decided is called a cycle.

Formally, given the sequence of investment dates $\{i\}, i = 0, 1, 2, 3,...$, the problem of identifying the optimal policy for the monopolist consists in finding $x = (x_0; x_1; ..., x_i; ...) \in R^\infty_+$ and $p(t)$, measurable function of $t \in R_+$ so that the objective function $V(x, p(t))$ defined by

$$V(x, p(t)) = \int_0^\infty p(t) D(t, p(t)) e^{-rt} dt - \sum_{i=1}^\infty (k + ax_i) e^{-ri}$$

is maximized, subject to the capacity constraint

$$D(t, p(t)) \leq X(t).$$

The definition of optimality decomposes into two components: The first refers to the properties of the optimal price pattern trough time while the second concerns the optimal sequence of capacity increments required in order to meet demand at each instant of time.

### 2.2 The optimal price policy within a cycle

In order to find the optimal price solution we first remark that the cost function $\sum_{i=0}^\infty (k + ax_i)$ does not depend on the choice of $p(t)$. Accordingly, a sufficient condition for the optimality of $p(t)$ is that it maximizes the integrand $p(t) D(t; p(t))$ at any point $t$, given the capacity constraint.
Then we observe that, if the investment size decision \( x \) is kept fixed from one period to the other, \( V(x, p(t)) \) achieves its maximum for \( p(t) \) given by

\[
p(t) = \max \left( \frac{At}{2}, At - X_i \right)
\]

for \( i = \lfloor t \rfloor \) the integer part of \( t \). This follows simply from the maximization problem

\[
\max_{p(t)} = p(t)D(t; p(t)) \quad \text{s.t. } D(t; p(t)) \leq X_i, i = \lfloor t \rfloor.
\]

During the cycle \([i, i + 1]\), there are two possible price regimes depending on the capacity level compared with the demand level. Assume that at some date \( t, i \leq t < i + 1 \), the capacity constraint is not binding, namely \( D(t; p(t)) < X_i \). Then, at the optimal policy, \( p(t) \) should be set equal to the maximizing price \( p^M(t) = \frac{At}{2} \) as the demand does not need to be dampened. Yet, the demand expands over time while the current capacity remains fixed. When it happens that the capacity constraint turns out to be binding, namely \( D(t; p(t)) = X_i \), then the firm has to choose the price \( p^C(t) = At - X_i \) so as to contract the demand \( D(t; p(t)) \) within the limits imposed by its plant size. These two price patterns \( p^M(t) \) and \( p^C(t) \) are called, respectively, monopoly price regime and constrained price regime, and the resulting demand patterns are denoted by \( D^M(t) \) and \( D^C(t) \), respectively. Further, we denote by \( t^*_i \) the point of time when the monopolist switches from the monopoly price regime into the constrained regime (namely when \( p^M(t) \) becomes equal to \( p^C(t) \)); we label this point as the switching point \( t^*_i \). It is easy to see that \( t^*_i = \frac{2At}{X_i} \). Then, four scenarios may arise. Assume first that the switching point lies after \( i + 1 \) or that it is exactly equal to \( i + 1 \); then the optimal price pattern coincides with the monopoly price regime during the whole cycle (scenarios A and B respectively). Assume now that the switching point lies between the two investment dates of a cycle \([i, i + 1]\); then both these two regimes are used at the optimal price pattern, the first one between \( i \) and \( t^*_i \) and the second between \( t^*_i \) and \( i + 1 \) (scenario C). Assume now that the switching point lies before \( i \); then the monopolist is forced to use the constrained price regime during the whole cycle in order to meet the capacity constraint (scenario D). Figures 1 and 2 illustrate these 4 scenarios.

It is easy to see that this switching point \( t^*_i \) can never exceed the point \( i + 1 \); in this case, it would be sufficient to reduce the installed capacity up to the point where this capacity is equal to \( D^M(i + 1, p^M(i + 1)) \), say
Figure 1: The switching point $t^*_i$ lies either exceeds (scenario A) or coincides with the point of time $i+1$ (scenario B).

by $\varepsilon$, in order to gain the discounted costs $\varepsilon a \left( e^{-r_i} - e^{-r(i+1)} \right)$ without incurring any reduction of revenue. So the first scenario never arises. We prove now that the switching point $t^*_i$ can never be exactly equal to the regeneration point $i + 1$, excluding thereby also scenario B.

**Lemma 1** During any cycle $i$, the switching point $t^*_i$ belongs either to the interior of the cycle $i$, or it is strictly smaller than $i$.

**Proof.** Assume that, for some $i$, $X'_i$ would be the optimal installed capacity at date $i$ and that $X'_i = D^M(i + 1)$. Then the monopolist can use the monopoly regime $p^M_i(t)$ during the whole cycle. The present value of the discounted flow of revenues $R_i$ during the cycle $i$ obtains as

$$R_i = \int_{i}^{i+1} \left( \frac{At}{2} \right)^2 e^{-rt} dt. \quad (1)$$

Now assume that the installed capacity $X'_i$ drops by a small quantity $\varepsilon$, so that $X'_i - \varepsilon < D^M(i + 1)$ with $0 < \varepsilon < 1$. The monopolist gains the discounted cost saved by reducing the investment by $\varepsilon$, namely $\varepsilon a \left( e^{-r_i} - e^{-r(i+1)} \right)$. Yet, the demand is not completely met. This induces to switch from the monopoly price regime $p^M_i$ to the constrained price $p^C_i$. Thus, the present value of the discounted flow of revenues
during the cycle $i$ turns into
\[
R_i' = \int_{i}^{i+1-\delta} \left( \frac{At}{2} \right)^2 e^{-rt} dt + \int_{i+1-\delta}^{i+1} \left[ \left( \frac{A(i+1-\delta)}{2} \right)^2 + \frac{A^2}{2} (i+1-\delta)(t-(i+1-\delta)) \right] e^{-rt} dt,
\]
where $\delta = \frac{4\varepsilon}{2}$ and the second term\(^2\) corresponds to the revenue stemming from the use of the constrained regime between $i+1-\delta$ and $i+1$. Subtracting (2) from (1), the loss $L$ derived from the switch between the two price regimes writes as
\[
L = \int_{i+1-\delta}^{i+1} \left[ \left( \frac{At}{2} \right)^2 - \left( \frac{A(i+1-\delta)}{2} \right)^2 - \frac{A^2}{2} (i+1-\delta)(t-(i+1-\delta)) \right] e^{-rt} dt.
\]

The loss $L$ is of third order in $\varepsilon$, as it is given by the integral of a function of the order of $\delta^2$ over an interval of length $\delta$. The gain $G = \varepsilon a \left( e^{-ri} - e^{-r(i+1)} \right)$ is of first order in $\varepsilon$. Accordingly, for $\varepsilon$ small enough, the net loss should be negative, which is the desired contradiction. Q.E.D.

We deduce immediately from the above that only the two remaining scenarios can be observed at an optimal price policy. Thus, the optimal price pattern within a cycle must either consist of quoting always the instantaneous constrained price, or using first this price and then, after the switching point, the constrained regime. The first alternative should necessarily hold when the investment at time $i$ is so low that it is even not sufficient to meet the monopoly demand $D(i, p^M(i))$ at that time.

The second one corresponds to a situation where the capacity installed at time $i$ is large enough to serve the monopoly demand for some period in the beginning of cycle $i$, so that the switching point is interior to the cycle. Accordingly, the nature of the optimal price policy within a cycle is directly related to the size of the investment consented at the beginning of the cycle.

### 2.3 The optimal investment policy

Thus, it remains to identify the optimal sequence of investments in order to fully characterize the optimal policy. Clearly, the optimal policy must depend on the main parameters of the model, namely, the interest rate $r$, the intercepts $At$ of the demand functions and the unit variable cost $a$. When the unit variable investment cost $a$ is high compared with the growth parameter of demand $A$, then the revenue derived from deciding a non-negative increment of capacity at date $i$ may well not compensate the increase in investment cost that it entails: in that case,

\(^2\)In appendix A we show how this term can be computed.
Figure 2: The switching point lies either between $i$ and $i + 1$ (scenario C) or before the point of time $i + 1$ (scenario D).

no increment of capacity is consented at the beginning of the cycle so that only the constrained price regime can be adopted within the rest of the cycle, as in figure 2 D. However, as we shall see in the following, this scenario cannot last forever at the optimal investment policy: after a while the increase in demand makes profitable a positive increment of capacity and both the price regimes are then alternated. On the contrary, when the unit variable investment cost $a$ is low compared with the growth parameter of demand $A$, then the revenue derived from deciding a strictly positive increment of capacity at date $i$ compensates the increase in investment cost that it entails: then at optimum, the price policy alternates instantaneous monopoly and constrained prices, as in figure 2 C. In this case, we prove below that this optimal price policy remains identical in each cycle for ever. The two above cases displayed in figure 2C and 2D, and the ensuing optimal investment policies, are now considered successively. Let us remark the following which will be used in the proof of the next propositions. Consider an optimal $x^*$ and denote by $V(x)_{i,h}$ the value of the objective function at the vector $(x)_{i,h} = (x_0, ..., x_i - h, ..., x_i + h, ..., x_{i+1})$ for any $i$ and $h \leq x_i$. Clearly it follows from the optimality property of $x^*$ that $V(x^*) \geq V(x)_{i,h}$ for $(x)_{i,h} = (x_0, ..., x_i - h, ..., x_i + h, ..., x_{i+1})$ for any $i$ and $h \leq x_i$. Then $\frac{\partial}{\partial h} V(x)_{i,h} = 0$, $i = 0, 1, 2$, namely the marginal revenue and the marginal costs derived from decreasing by $h$ the capacity installed at time $i$ and increasing by the same amount the one installed at time $i + 1$ counterbalance each other, for any cycle $[i, i + 1]$. 
Let us start with the first one and consider the following condition

\[ a \geq A \left[ 1 - \frac{r}{e^{r(i+1)} - e^{ri}} \right] \]  

(3)

on the parameters\(^3\). It is easy to see that condition (3) can only hold for a finite number of cycles \([i, i + 1]\). Let \(i_0\) be the first cycle at which this inequality is reversed. We have

**Proposition 2** When (3) holds, the optimal policy consists, up to the cycle \([i_0-1, i_0]\), in zero increments of capacity and for \(i \geq i_0\) in a constant increment of capacity at each cycle. Moreover, the optimal price policy displays first a constrained price regime within the whole cycle, adjusted in each cycle, and then a monopoly and a constrained price regime within each cycle.

**Proof.** We consider here the case when the switching point \(t_i^*\) is outside of the interior of the cycle \([i, i + 1]\). Then it is on the left of \(i\) (see figure 2 D), and the installed capacity is not sufficient to meet the instantaneous monopoly demand at \(i\). The constrained price regime only applies. For the F.O.C. to be satisfied at the optimal solution \(x_i^*\) for any \(i\), the marginal revenue must be equal to its marginal costs, within any cycle \([i, i + 1]\). Then, the following holds:

\[
\int_i^{i+1} (At - 2X_i)e^{-rt} \, dt = ae^{-ri}(1 - e^{-r})
\]

with \(X_i = X_{i-1} + \lambda_i\). Then, the above condition can be re-expressed as follows

\[
\int_0^1 (At - 2X_{i-1})e^{-rs} \, ds - a(1 - e^{-r}) = 2\lambda_i \int_0^1 e^{-rs} \, ds
\]

(4)

where \(s = t - i\) and \(\lambda_i\) is the solution of (4) which does not depend on \(i\). Then,

\[ x_i^* = \max\{0, \lambda_i\} \]

Indeed, for

\[ i < i_0, \quad x_i^* = 0 \]

\(^3\)Details on this condition are provided in appendix B.
\( i \geq i_0, \ x^*_i > 0 \)

where \( i_0 \) is the first decision point \( i \) such that the revenue

\[
\int_i^{i+1} Ate^{-r(s-i)} \, ds
\]

and the cost \( a(1 - e^{-r}) \) resulting from a non-negative investment size exactly counterbalance each other. So, as \( x^*_i = 0 \) the constrained price regime is operated till \( i_0 \). Furthermore, when \( x^*_i > 0 \) the constrained price regime tends to decrease from cycle to cycle as the investment for adding capacity increases. At the time point when the plant size is so big as to satisfy the demand growing linearly over time, the constrained price regime is replaced by the monopolistic price regime and then we fall in the case when both price regimes apply forever. Q.E.D.

Notice that even if (3) holds, in the long run the optimal investment policy is constant cycle and the optimal price policy always consists in alternating monopoly price and constrained price regimes within a cycle. By contrast, as stated in the next proposition the same optimal policies turn out to be observed from the very beginning when the reverse of (3) holds.

**Proposition 3** When the reverse of (3) holds, the investment optimal policy is unique and stationary through all cycles.

**Proof.** First notice that the switching point \( t^*_i \) exists and it is unique for each cycle \([i, i+1]\). Accordingly, if we prove that the optimal investment policy is constant cycle, the optimal price policy repeats identically from a cycle to the other. Consider a specific cycle \([i, i+1]\) and assume that the switching point \( t^*_i \) is now in the interior of the cycle. Then, the first order conditions become

\[
\int_{t^*_i}^{i+1} (At - 2X_i)e^{-rt} \, dt = ae^{-ri}(1 - e^{-r})
\]

or

\[
Ae^{-ri} \int_{t^*_i-i}^{1} (s - (t^*_i - i))e^{-rs} \, ds = ae^{-ri}(1 - e^{-r}) \tag{5}
\]

where it is still \( s = t - i \). Then (5) can be rewritten

\[
f(t^*_i - i) = a \tag{6}
\]
where \( f(l) = A \int_{l}^{1} \frac{(s-l)e^{-rs}ds}{(1-e^{-r})} \) is a function that does not depend on \( i \).

It is immediate to see that \( t^*_i = i + \bar{l} \) where \( \bar{l} \) is the unique solution to \( f(l) = a \). Then, given the time \( i \) when the increment of capacity is installed, \( t^*_i \) is univocally determined by \( \bar{l} \), which does not depend on \( i \), for any \( i \). Finally, as \( t^*_i \) identifies the level of available capacity \( X_i \) in any cycle \([i, i+1]\), the solution is stationary, as claimed. Q.E.D.

A simple byproduct of the above results is that the optimal constant increment of capacity is smaller than the one which would have been selected by a planner facing a demand growing exogenously at a rate corresponding to the instantaneous monopoly price. If the monopolist would quote at each instant of time the monopoly price, it would generate a linear demand trajectory with slope \( A_2 \) in the demand-time space (see figure 3). We can find out what is the optimal solution of the planning problem corresponding to this expansion of demand, viewed as in Manne (1961). To this solution, there corresponds an optimal constant cycle in investment and, accordingly, a sequence of equally-spaced dates (regeneration points in the language of Manne) at which this constant increment of capacity is consented. Starting with this sequence of cycles, we can wonder whether the constant-cycle optimal investment policy of the monopolist leads at the beginning of each cycle to a capacity increment which is larger or smaller than the one resulting from the planning solution provided by Manne.

**Proposition 4** The optimal constant increment of capacity chosen by the monopolist is smaller than the one which would have been selected by a planner facing demand levels \( D(i + 1, p^M(i + 1)) \) growing exogenously at a rate corresponding to the instantaneous monopoly price.

**Proof.** The demand levels resulting from the use of the monopoly price within the whole cycle, for each cycle, correspond to the price policy in scenario A. In lemma 1, it is shown that this cannot be a price policy corresponding to the optimal pattern selected by the monopolist. Similarly, we have seen that an optimal increment of capacity decided at \( i \) cannot strictly exceed the demand level \( D^M(i + 1, p^M(i + 1)) \). Consequently, the optimal increment of capacity must be strictly smaller than \( D^M(i + 1, p^M(i + 1)) \) while, at the Manne solution, it is exactly equal to this magnitude. Q.E.D.

Thus, whether proposition 2 or 3 applies depends on condition (3), which itself depends on the values of parameters \( A, a \) and \( r \). Assume that, at \( i = 1 \), condition (3) holds. Then we know that proposition 2
Figure 3: Demand levels corresponding to the monopoly price regime.

applies: at the beginning, for low values of \( i \), no increment of capacity is profitable. However, as the demand function increases over time, at cycle \( i_0 \), the increment of capacity starts to be positive and we enter in the scenario described in proposition 3. Notice that, ceteris paribus, condition (3) cannot hold for high values of the demand growth rate \( A \). The reason is that, whatever the cost of investing, as expressed by the unit variable cost \( a \) and the interest rate \( r \), a sufficiently large value of \( A \) should necessarily reverse the inequality (3). Assume now that, at \( i = 0 \), the reverse of condition (3) holds. Then, from the very beginning, a positive increment of capacity is built, and proposition 3 applies. In the proof of this proposition, we have defined the function \( f(l) \) through which the switching point \( t^*_i \) of any cycle \( [i, i + 1] \) corresponding to the optimal policy is derived as

\[
f(t^*_i - i) = a
\]

with \( f(l) \) defined by (6)

Now consider figure 4 where the function \( f(l) \) is depicted.

Then, the switching point \( t^*_i \) obtained as \( i + \bar{l} \), as defined in the proof of proposition 3, is easily identified through figure 4. Moreover, it is immediate that, ceteris paribus, when the unit variable cost \( a \) increases, the value of \( \bar{l} \) decreases; this entails a reduction in the value of the switching point and, accordingly, in the size of the optimal increment of
capacity. Notice that when $A$ increases, the whole function $f(l)$ shifts upwards displacing the point $\bar{l}$ to the right. Thus the switching point moves to the right, entailing an increase in the optimal capacity installed at time $i$. Finally, when the interest rate $r$ increases, it is easy to see that $\bar{l}$ moves to the left, reflecting thereby the increase in investment cost, and entailing the ensuing reduction of installed capacity at the optimum.

3 Conclusion

The plant size problem introduced in the fifties was designed in order to solve planning problems raised by an exogenous expansion of demand due to the demographic trend in developing countries. In this paper we have considered an alternative version of the plant size problem, now formulated in the context of a market environment. A monopolist, facing demand expansion and increasing returns, has the opportunity to combine his investment policy with a price policy aiming at manipulating the instantaneous price in order to dampen demand whenever the existing capacity is in a bottleneck. We have characterised both the optimal policies under the assumption that increments of capacity can take place at the beginning of each cycle of a sequence of equally-spaced cycles.

This analysis calls for several generalizations. First, it would be interesting to relax the assumption according to which decision points are
fixed, and to analyze the reverse problem, namely, the one when the investment size is fixed and the optimal plan is defined on the basis of both price regime and investment timing (see footnote 1). Although this would change the frame, yet it would probably not change the main properties of the model. The stationarity property of investment plans involving two price regimes should still hold. Second, it would be natural to introduce other dynamic elements which could influence the optimal time trajectories of prices and investments, like obsolescence (see d’Aspremont et alii (1972)) or technical progress, as in Gabszewicz and Vial (1972). Finally, the extension of the model to a dynamic game with two oligopolists would enrich the market environment in which it could be applied. A natural question in this extended framework would then be: is a similar cyclical behaviour reproduced at the Nash equilibrium of the resulting game? In the longer run, the prospective analysis could even be more ambitious. Since the main outcome of the present paper is to stress the optimal property of successive periods of capacity expansion followed by periods of no investment and contained demand, perhaps a similar model could be used to explain, at least partially, the existence of business cycles (See in particular Abel and Eberly (1996), Caballero and Engel (1991) and (1999)).
References

4 Appendix

4.1 Appendix A

The term \( \int_{i+1-\delta}^{i+1} \left[ \left( \frac{A(i+1-\delta)}{2} \right)^2 + \frac{A^2}{2} (i + 1 - \delta) (t - (i + 1 - \delta)) \right] e^{-rt} dt \) denotes the discounted flow of revenues stemming from the use of the constrained regime during the interval \([i+1-\delta, 1+1]\). This can be computed as follows. Let

\[
R_L = \int_{i+1-\delta}^{i+1} ((At - X_i)X_i) e^{-rt} dt
\]

be the discounted revenue when the monopolist uses the constrained regime within the period \([i + 1 - \delta, 1 + 1]\). Also, we know that \(X_i = \frac{A(i+1-\delta)}{2}\). Then by replacing this value of \(X_i\) in (7), we easily get

\[
R_L = \int_{i+1-\delta}^{i+1} \left( At - \left( \frac{A(i+1-\delta)}{2} \right) \left( \frac{A(i+1-\delta)}{2} \right) \right) e^{-rt} dt
\]

From algebraic manipulations it derives:

\[
\int_{i+1-\delta}^{i+1} \left[ \left( \frac{A(i+1-\delta)}{2} \right)^2 + \frac{A^2}{2} (i + 1 - \delta) (t - (i + 1 - \delta)) \right] e^{-rt} dt.
\]

4.2 Appendix B

As stated in proposition 2 and proposition 3, the optimal investment policy can display two different price patterns: either at the beginning of a cycle \([i, i+1]\) the monopoly price regime is adopted and then from the switching point \(t^*_i\) up to the end of the cycle the constrained price regime is quoted, or the constrained price regime applies for the whole cycle. Namely when the optimal increment of capacity \(x^*_i\) installed at time \(i\) suffices to meet levels of demand within the interval \([i, t^*_i]\), the firm charges both the price regimes, otherwise the constrained regime only arises. Without loss of generality, let us consider the first cycle \([0, 1]\) when the plant has still to be initiated. Then, for an initial investment to be non-negative at the optimal solution, the marginal revenue stemming from a non-negative investment must be equal to the marginal cost of installing this capacity, namely:

\[
\int_0^1 Ate^{-rt} dt = a(1 - e^{-r})
\]
which can be easily solved. So, it writes:

\[
\frac{A}{r^2} \left( 1 - \frac{r}{e^r - 1} \right) = a
\]

Then, when the cost of investing is high with respect to the gain of installing a non-negative capacity or

\[
\frac{A}{r^2} \left( 1 - \frac{r}{e^r - 1} \right) < a
\]

the monopolist refrains from investing, namely \( x^* = 0 \), and constrained regime only applies; whereas when the cost of investing is low with respect to the gain of installing a non-negative capacity or

\[
\frac{A}{r^2} \left( 1 - \frac{r}{e^r - 1} \right) > a
\]

then both the regimes apply. Finally, when the growth of demand and investment costs exactly countervail each other or

\[
\frac{A}{r^2} \left( 1 - \frac{r}{e^r - 1} \right) = a
\]

the firm starts to undertake a non-null investment and the price policy tends to decompose in both the price regimes.