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Theory of Zipf's Law and Beyond

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Preface

Zipf's law is one of the few quantitative reproducible regularities found in economics. It states that, for most countries, the size distributions of city sizes and of firms (with additional examples found in many other scientific fields) are power laws with a specific exponent: the number of cities and of firms with size greater than S is inversely proportional to S. Most explanations start with Gibrat's law of proportional growth but need to incorporate additional constraints and ingredients introducing deviations from it. Here, we present a general theoretical derivation of Zipf's law, providing a synthesis and extension of previous approaches. First, we show that combining Gibrat's law at all firm levels with random processes of firm's births and deaths yield Zipf's law under a "balance" condition between firm growth and their death rate. We find that Gibrat's law of proportionate growth does not need to be strictly satisfied. As long as the volatility of firm's sizes increases asymptotically proportionally to the size of the firm and that the instantaneous growth rate increases not faster than the volatility, the distribution of firm sizes follows Zipf's law. This suggests that the occurrence of very large firms in the distribution of firm sizes described by Zipf's law is more a consequence of random growth than systematic returns: in particular for large firms, volatility must dominate over the instantaneous growth rate. We develop the theoretical framework to take into account (i) time-varying firm creation, (ii) firm's exit resulting from both a lack of sufficient capital and sudden external shocks, (iii) the coupling between firm's birth rate and the growth of the value of the population of firms. We predict deviations from Zipf's law under a variety of circumstances, for instance when the balance between the birth rate, the non-stochastic growth rate and the death rate is not fulfilled, providing a framework for identifying the possible origin(s) of the many reports of deviations from the pure Zipf's law. Reciprocally, deviations from Zipf's law in a given economy provides a diagnostic, suggesting possible policy corrections. The results obtained here are general and provide an underpinning for understanding and quantifying Zipf's law and the power law distribution of sizes found in many fields.

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1. Introduction

One of the broadly accepted universal laws of complex systems, particularly relevant in social sciences and economics, is that proposed by Zipf (1949). Zipf's law usually refers to the fact that the probability $P(s) = \Pr\{S > s\}$ that the value S of some stochastic variable, usually a size or frequency, is greater than s, decays with the growth of s as $P(s) \sim s^{-1}$. This in turn means that the probability density functions $p(s)$ exhibits the power law dependence

$$p(s) \sim 1/s^{1+m} \quad \text{with} \quad m = 1 \ . \tag{1.1}$$

Perhaps the distribution most studied from the perspective of Zipf's law is that of firm sizes, where size is proxied by sales, income, number of employees, or total assets. Many studies have confirmed the validity of Zipf's law for firm sizes existing at current time t and estimated with these different measures (Simon and Bonini 1958, Ijri and Simon 1977, Sutton 1997, Axtell 2001, Okuyama *et al.* 1999, Gaffeo *et al.* 2003, Aoyama *et al.* 2004, Fujiwara *et al.* 2004, Fujiwara *et al.* 2004, Takayasu et al. 2008).

Initially formulated as a rank-frequency relationship quantifying the relative commonness of words in natural languages (Zipf 1949), Zipf himself recognized in his book the general relevance to this law to the distribution of city sizes, among others. Many works have since shown that Zipf's law indeed accounts well for the distribution of city sizes (see for a review (Gabaix 1999) and references therein), as well as firm sizes all over the world, as just mentioned. Zipf's law has also been found in Web access statistics and Internet traffic characteristics (Glassman 1994, Nielsen 1997, Adamic and Huberman 2000, Barabasi and Albert 2002) (and with deviations (Breslau et al. 1999)), in inbound degree distributions over Web pages (Kong et al. 2008), in weekend gross per theater for a movie (scaled by the average weekend gross over its theatrical lifespan) (Sinha and Pan 2006), in bibliometrics, informetrics, scientometrics, and library science (Adamic and Huberman 2002, and references therein) and in the distribution of incoming links to packages found in different Linux open source software releases (Maillart *et al.* 2008)). Sinha and Pan (2006) provides a rather exhaustive review of the many power laws found in the distribution of human activities. There are also suggestions for

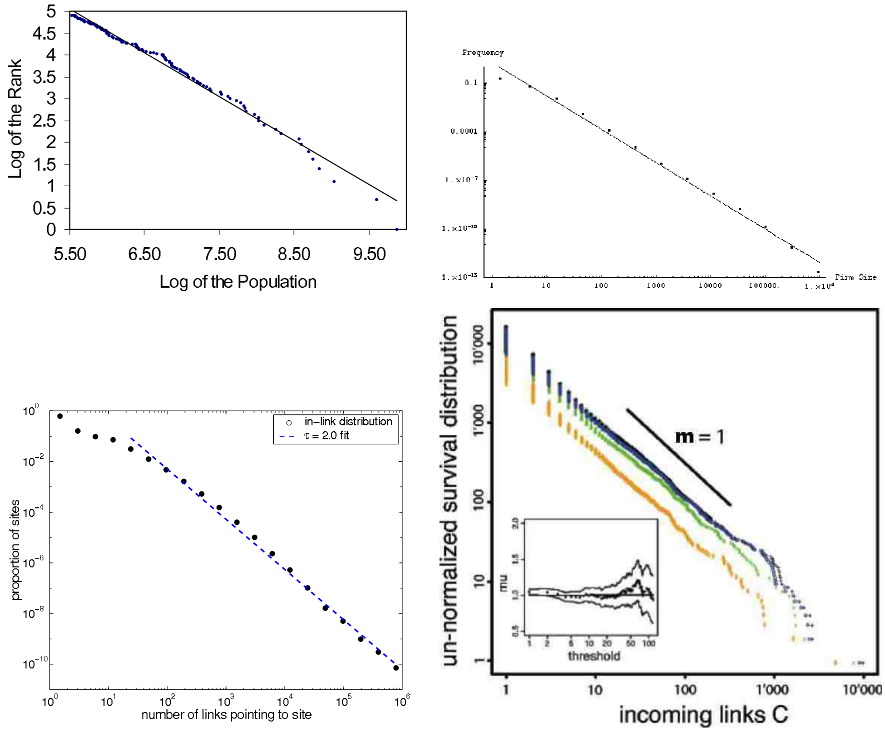

Fig. 1.1. Illustration of Zipf's law for city sizes (upper left panel, reproduced from Ioannides and Gabaix (2003)), for firm sizes (upper right panel, reproduced from Axtell (2001)), for the number of Internet links pointing to some website (lower left panel, reproduced from Adamic and Huberman (2002)) and for the number of incoming links to packages found in different Linux open source software releases (lower right panel, reproduced from Maillart *et al.* (2008)).

applications to other physical and biological, sociological and financial market processes. For instance, using data from gene expression databases on various organisms and tissues, including yeast, nematodes, human normal and cancer tissues, and embryonic stem cells, Furusawa and Kaneko (2003) found that the abundances of expressed genes obey Zipf's law. See the list of references in `http://linkage.rockefeller.edu/wli/zipf/index_ru.html`. Figure 1.1 illustrates several cases where Zipf's law holds for different fields of social and natural sciences.

We should point out that there are some dissenting notes. For instance, several works have suggested that, for the distribution of firm's sizes, the lognormal distribution may actually be a better model than Zipf's law (Stanley *et al.* 1995, Cabral and Mata 2003, Kaizoji *et al.* 2006, Duchin and Levy 2008, Schwarzkopf and Farmer 2008). The issue is confounding because often the authors are not always speaking of the same thing. Stanley *et al.*

(1995)'s result has now been understood as due to an incomplete database, missing most of the small firms and hence biasing the distribution downward towards the lognormal shape for small firms (Axtell 2001). Axtell (2001) has shown that firm' sizes measured by the number of employees, by the total sales or by the economic capital (debt + equity) are all consistently obeying Zipf's law. From an economic view point, it can indeed be expected that these three firm characteristics are globally proportional to each other in a same industry branch, or for a same business model, so that if Zipf's law holds for one of them, it should hold for the others. On the other hand, equity provides only a part of the economic capital of a firm, which depends on the financing strategies chosen by the firm, in addition to the impact of the stock market fluctuations. It is not clear that the financing strategies are stationary as a function of time, except perhaps for mature firms with no more any innovation or M&A (mergers and acquisitions) for which the financial structure of the firm (its debt/equity ratio) may be approximately constant. Therefore, the fact that Zipf's law may not be the best model for the distribution of equity sizes (Duchin and Levy 2008) is not surprising. Another issue is the possible slow convergence of the distribution to its expected asymptotic long-time shape (Schwarzkopf and Farmer 2008). Difference between countries due to the presence of specific financial constraints may be also an issue (Cabral and Mata 2003).

Kitov (2009) points out that the significant differences in the evolution of firm size distribution for various industries in the United States puts important constraints on the modelling of firm growth. This line of thought opens the road toward linking asset pricing models, investment strategies and firm growth processes. In this spirit, Malevergne and Sornette (2007) have discovered a new endogenous pricing factor resulting from the heavy-tailed distribution of firm sizes, which has empirically a similar explanatory power as the phenomenological Fama-French three-factor model (Fama and French 1993, Fama and French 1995).

Employing Census 2000 data to create the most extensive and thorough investigation to date of the distribution of city sizes in the U.S.A, Eeckhout (2004) reported that the empirical distribution follows a lognormal distribution rather than Zipf's law. Reanalyzed this data, Levy (2008) confirms that the lognormal distribution indeed provides an excellent fit to the empirical data for 99.4% of the size range. However, for the top 0.6% of largest cities, the empirical distribution is dramatically different from the lognormal, and follows a power law. Levy notes that, while this top part of the distribution involves only 0.6% of the cities, it is extremely important as it accounts for more than 30% of the sample population. This type of hybrid lognormal-power-law distribution will find a natural explanation in the framework that we develop in the following chapters, and in particular in Chapter 6. The debate is however not closed as Eeckhout (2008) argues that the de-

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viations from the lognormal model identified visually by Levy (2008) can be expected from the confidence bands generated by the Lilliefors test with five percent significance level. The problem however is that Eeckhout (2008)'s argument is based on a very weak test: the Lilliefors test, an adaptation of Kolmogorov-Smirnov test, is inadequate to identify deviations that occur in the tail, since its statistics is constructed from the maximum discrepancy between the lognormal and the empirical distribution. Anderson-Darling tests, for instance, are more adapted to the problem of distinguishing distributions in their tails (Malevergne et al. 2005, Malevergne and Sornette 2006). In a forthcoming paper, Malevergne et al. (2009) develop a more powerful test specifically designed to compare the lognormal family to the power law family, which confirms quantitatively the intuition of Levy (2008). In order to address the issues associated with the definition of a city (administrative or geographic), Rozenfeld et al. (2009) employ a recently proposed clustering algorithm Rozenfeld et al. (2008) to construct cities from the bottom-up, without administrative data, but by using geographical proximity. They find that Zipf's law holds for cities above 10'000 inhabitants in the US, and above 1'000 inhabitants in the UK.

Among the many more or less successful explanations proposed to understand the origin of Zipf's law, one of the most promising seems to be the explanation by Gabaix (1999) and Ioannides and Gabaix (2003) formulated in the context of the distribution of city sizes, based on Gibrat's law. Gabaix (1999) assumed that each city exhibits a stochastic growth rate distributed independently from its present size. Gibrat's law for city growth (together with some deviations of Gibrat's law for small sizes), normalized to the whole population of a given country, then leads to distributions of city sizes very close to Zipf's law. In general terms, Gibrat's law amounts to assume a stochastic multiplicative process. Such processes are found in many economic as well as natural systems (Sornette 2006, and references therein). As a recent illustration, Clauset and Erwin (2006) explain in this way (with the inclusion of a mechanism involving size-dependent long-term extinction risks) the evolution and distribution of biological species body sizes.

However, the derivation of Zipf's law from the pure Gibrat's rule suffer from a few problems. First, the exact scale-independent Gibrat's law leads to a log-normal distribution of sizes, which is not a power law and only slowly converges to a power law in the limit of large log-variance (and some other conditions), becoming at the same time more and more degenerate. Some additional assumptions are therefore needed in order to produce the stable non-degenerate Zipf's law. In particular, Gabaix (1999) assumed that, for cities of small sizes, there are some exogenous factors preventing further decaying of their population (see also (Levy and Solomon 1996, Malcai *et al.* 1999)). More appropriate to social and economic phenomena is to allow

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for eliminating cities or firms as they reach a small size. An example is the transition from city to rank of village as the size goes below some threshold.

More generally, it is important to take into account the continuous process with births and deaths, which plays a central role at time scales as short as a few years. This is in contrast with Gabaix's approach for instance based on the supposition, simplifying considerably the theoretical modeling, that all cities originate at the same instant t_0, and then only grow stochastically, obeying the balanced Gibrat's law mentioned above. This supposition is clearly falsified by empirical evidence, as discussed later in the book.

A goal of this book is to demonstrate that birth as well as death processes are especially important to understand the economic foundation of Zipf's law and its robustness. Yamasaki et al. (2006) have shown that a model of proportional growth of the existing firms in the presence of a steady influx of new firms leads to Zipf's law truncated by an exponential taper, without the need to modify Gibrat's law for small sizes. The exponential cut-off results from the finite life of the economy. Our general analysis encompasses these results and put them in a larger perspective. Expressed in the context of an economy of firms, we will consider two different mechanisms for the exit of a firm: (i) when the firm total asset value becomes smaller than a given minimum threshold (which can vary with time and with countries) and (ii) when an exogenous shock occurs, modeling for instance operational risks, independently of the size of the firm. Of course, these two mechanisms have their counterparts in the different fields of application where Zipf's law is discussed.

The following chapters are built on the realistic description of the behavior of the asset value of firms (which is more dynamic than the formation of cities), according to which the births of firms occur according to a random point process characterized by some intensity $\nu(t)$. Jointly, one should take into account the well-documented evidence that firms die, for instance when their size go under some low asset value level. It turns out that taking into account the random flow of firm births and deaths, in combination with Gibrat's law, leads to the pure and non-degenerate Zipf's law under a balance condition, without the need for the rather artificial modification of Zipf's law for small sizes [We note that the fact that deviation of Gibrat's law has been documented for small firms is another issue, as the documented deviations do not necessarily obey the assumptions needed in Gabaix's derivation.] As a bonus, the approach in terms of the dynamics of birth-death together with stochastic growth, that we develop here, leads to specific predictions of the conditions under which deviations from Zipf's law occur, which help rationalize the empirical evidence documented in the literature. The conditions involve either deviations from Gibrat's law in the stochastic growth process of firms or the existence of an unbalanced growth or decay of the birth intensity $\nu(t)$ of new firms, as we explain in details below.

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In the theory developed in the following chapters, we also take into account that the intensity of firm's births may increase exponentially, that the sizes of entrant firms and the minimum viable size may grow exponentially with time with additional random fluctuations, hence generalizing Blank and Solomon (2000). Putting all elements and results of our analyses together, we conclude that the explanation for the generic empirical evidence that the exponent m is close to 1 (Zipf's law) is likely due to the weak dependence of m on the different parameters of the problem. This renders unnecessary the question for why the parameters would combine to obey exactly the balance condition. The closeness of the exponent m to 1 for a large range of parameters is quantified for instance in figures 7.5 and 8.1.

For transparency of derivations and for convenience of analytic calculations, we use a continuous version of Gibrat's law, allowing us to benefit from the properties of the Wiener process and the mathematical framework of Kolmogorov's diffusion equations. We unearth new properties associated with the stochastic behavior of firm assets. We show that the death of firms at some low value level as well as the possibility of significant deviations from Gibrat's law do not affect the asymptotic validity of Zipf's law in the limit of large firm sizes. By analyzing a large class of diffusion processes modeling the behavior of firm assets with growth rates very different from Gibrat's condition, we find general conditions for the validity of Zipf's law. Specifically, we have discovered stochastic growth models with non-Gibrat properties, leading to Zipf's and related power laws for the current density of firm's asset values.

Our book does not cover the more economically based theories, in the spirit for instance of Lucas (1978), which developed a theory of size distribution of business firms based on an underlying distribution of managerial talents and the competitive process of allocation of productive factors. Similarly, we do not expand on the general equilibrium model of the distribution of firm sizes proposed by Luttmer (2007), in terms of primitives such as entry and fixed costs, and the ease with which entrant firms can imitate incumbent firms. Let us also mention Rossi-Hansberg and Wright (2007a) which develops a general equilibrium theory of economic growth in an urban environment. In this theory, variation in the urban structure through the growth, birth, and death of cities is the margin that eliminates local increasing returns to yield constant returns to scale in the aggregate. They show that scale-independent growth for a finite number of industries, combined with an empirically-based form of entry and exit and a lower bound for establishment sizes that converges to zero, is sufficient to generate an invariant distribution that satisfies Zipf's law. Rossi-Hansberg and Wright (2007b) present a theory of the establishment size dynamics based on the accumulation of industry-specific human capital that simultaneously rationalizes the economy-wide facts on establishment growth rates, exit rates, and size distributions. Using a simple model of market share dynamics with boundedly rational consumers and firms interacting with each other, Yanagita and Onozaki (2008) find that, in an oligopolistic phase associated with intermediate greediness of agents, the market-share distribution of firms follows Zipf's law and the growth-rate distribution of firms follows Gibrat's law.

The book is organized as follows. Chapter 2 presents the continuous version of Gibrat's law and some peculiarities of the stochastic behavior of the geometric Brownian motion of firm's asset values, resulting from Gibrat's law.

Chapter 3 describes the proposed model for the current density of firm's asset values, taking into account the random flow of the birth of firms. We show that, if some natural balance condition holds, while the intensity of firms is independent of time ($\nu = \text{const}$), then the exact Zipf's law holds true.

Amazingly, despite the relevance of Gibrat's law and the corresponding geometric Brownian motion in a wide range of physical, biological, sociological and other applications, many researchers do not make use of many of the interesting properties exhibited by realizations of the geometric Brownian motion, in order to derive detailed explanations of Zipf's and related power laws. Thus, in chapter 4, we gather little-known information concerning the statistical properties of realizations of the geometric Brownian motion, which play a significant role for the understanding of the roots and conditions of the validity of Zipf's law.

Chapter 5 discusses in detail the influence on the validity of Zipf's law of the occurrence of the death of firms when their value falls below some low level. In chapter 6, we derive an equation for the steady-state density of firm asset values, which enables us to explore in detail the consequences of deviations from Gibrat's law at moderate asset values on the validity of Zipf's law at higher asset values.

Chapter 7 is devoted to discussing the conditions for the existence of Zipf's law and the circumstances under which deviations from it occur, when taking into account the possibility for sudden death of firms occurring even for large sizes. Chapter 8 provides the most general treatment taking into account time dependence of birth rates, sizes at birth, and minimum firm sizes. Chapters 7 and 8 show that, with all these additional ingredients, Zipf's law holds if a generalized balance condition is valid. In particular, we discuss the robustness of Zipf's law to variations of the mean birth rate and of the rate of growth of the mean asset value of particular firms. Moreover, we find that Zipf's law is "attracting" the power laws found in absence of the strict validity of the balance condition: as the volatility of the growth of firms increases, the power law distribution of firm's sizes becomes closer to Zipf's law, and the later is obtained asymptotically for very large volatilities for all values of the other parameters.

All previous chapters have emphasized the dynamics of the statistical average of various firm properties in the limit where the number of firms in the economy grows without bounds. Chapter 9 asks if the results described in previous chapters can be used for the description of a single realization of a finite economy, an issue of great importance for the application of our theory to empirical data. For this, we derive the statistics of the number of firms, the fluctuation characteristics of the size of the global economy and the size of fluctuations decorating the asymptotic Zipf's law for finite economies. We provide a simple estimation of the expected statistical deviations from Zipf's law and its range of validity for realistic parameters. This provides a benchmark for assessing the range of validity of Zipf's law in empirical data.

Chapter 10 concludes first by stressing the importance of the balance conditions for Zipf's law to hold. Then, we provide the nucleus of what could be a more complete mathematical theory of firm sires, based on taking into account in addition the mergers between firms as well as it symmetric, the creation of spin-off firms from parents which outsource a part of their existing business as separate units. These economic events can be modeled by using the mathematics of coagulation-fragmentation processes, which are briefly described here in the context of the dynamics of firms. We provide only a preliminary introduction to encourage future works to tackle these complex and rich issues.

For clarity, consistence of language and conciseness, we will discuss the origin and conditions of the validity of Zipf's law using the terminology of financial markets and referring to the density of the firm's asset values. We use firms at the entities whose size distributions are to be explained. It should be noted, however, that most of the relations discussed in this book, especially the intimate connection between Zipf's and Gilbrat's laws, underlie Zipf's law in diverse scientific areas. The same models and variations thereof can be straightforwardly applied to any of the other domains of application.