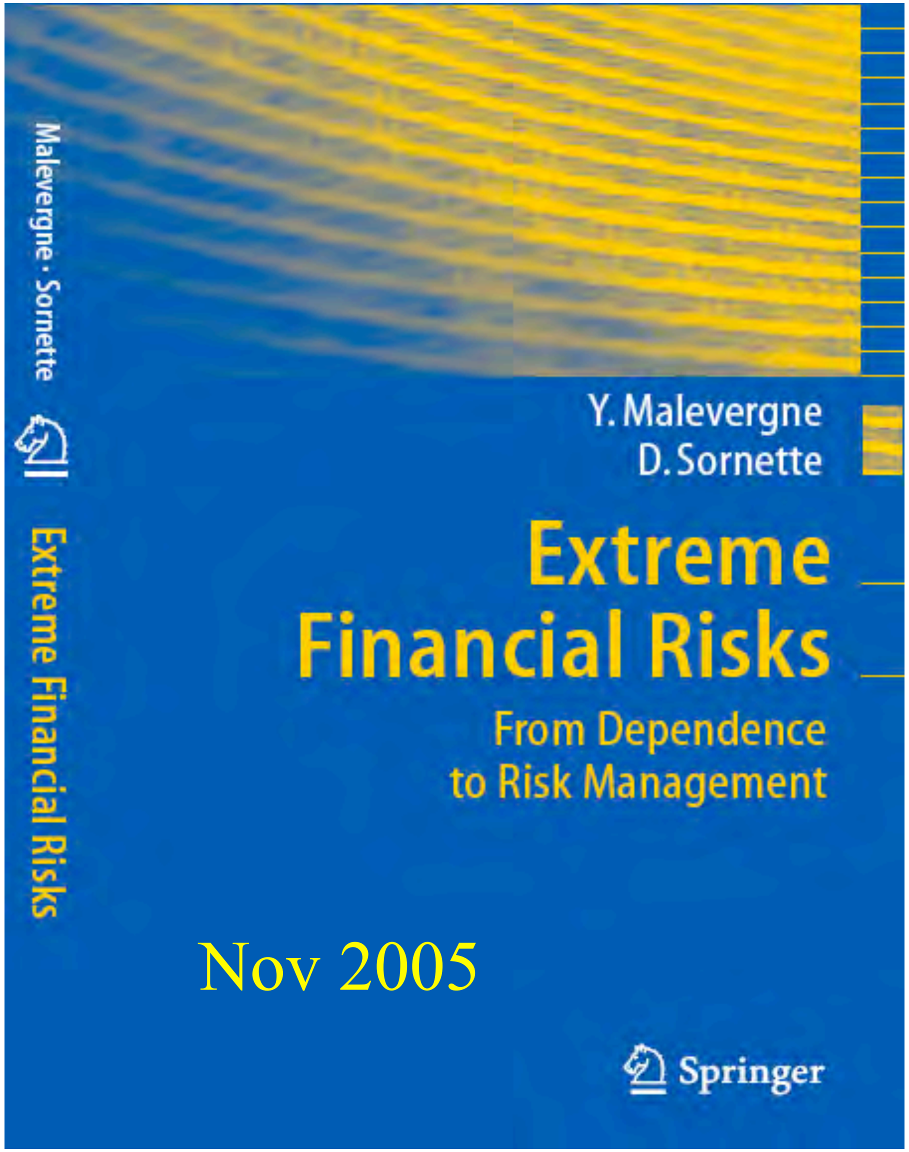
Copulas, Higher-Moments and Tail Risks



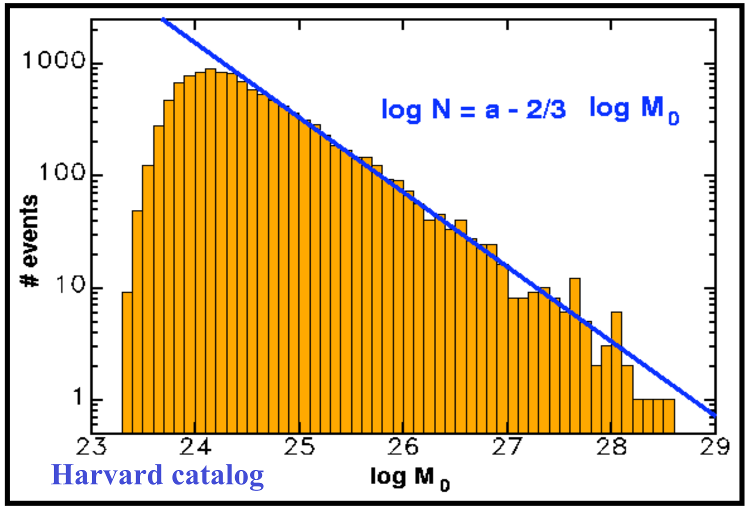
ETH-Zurich
Chair of Entrepreneurial Risks
Department of Management, Technology
and Economics (D-MTEC)
Zurich, Switzerland
http://www.mtec.ethz.ch/

Optimal "orthogonal" decomposition of multivariate risks in terms of

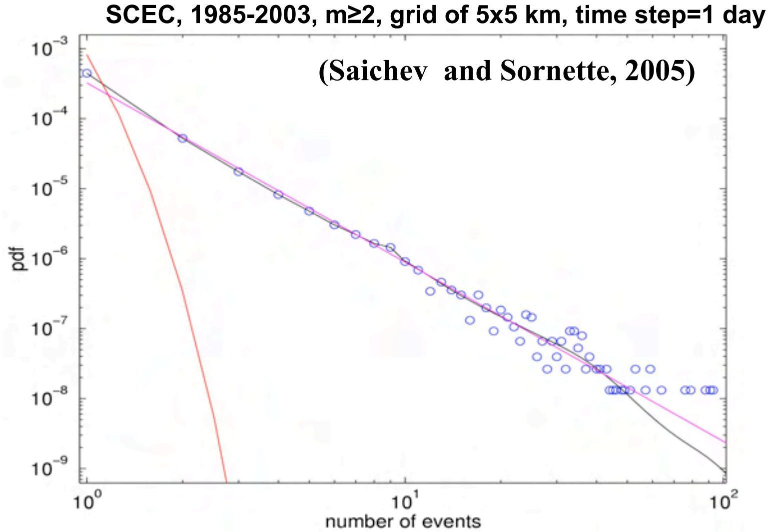
-marginal distributions

-intrinsic dependence

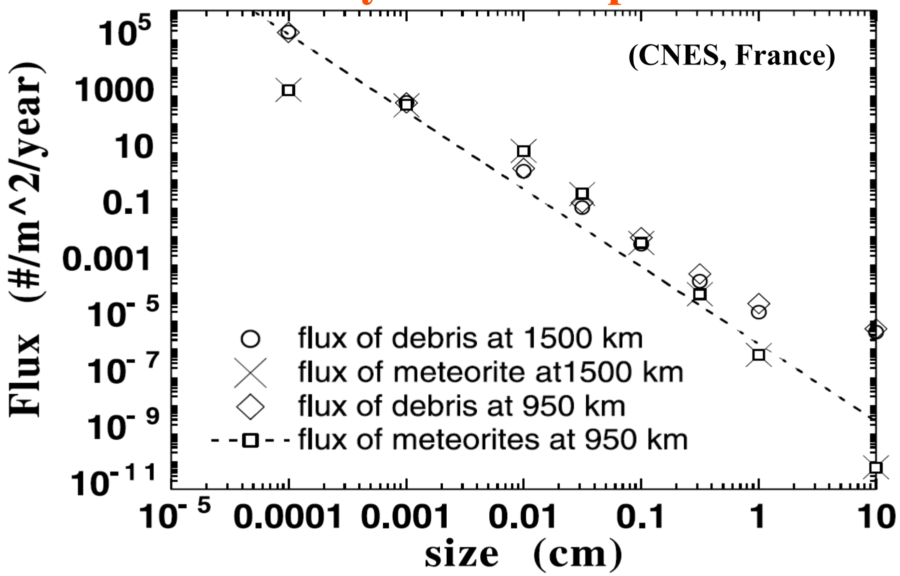
Heavy tails in pdf of earthquakes



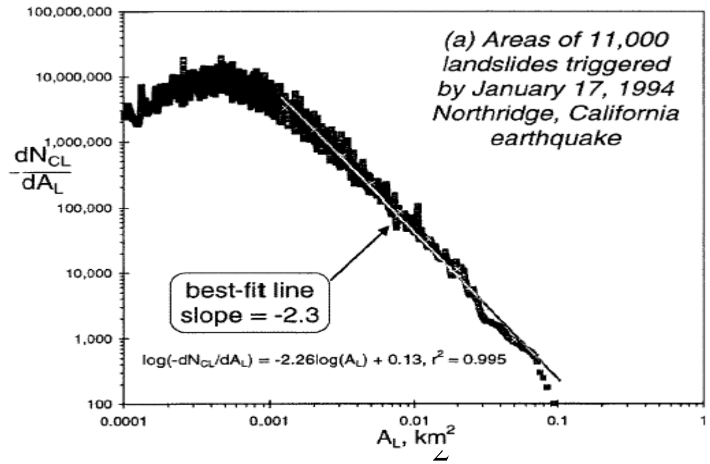
Heavy tails in pdf of seismic rates



Heavy tails in ruptures



Heavy tails in pdf of rock falls, Landslides, mountain collapses



Turcotte (1999)

Heavy tails in pdf of forest fires

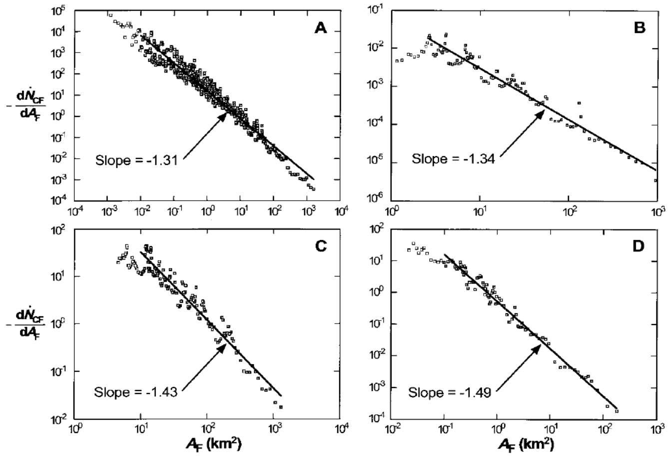
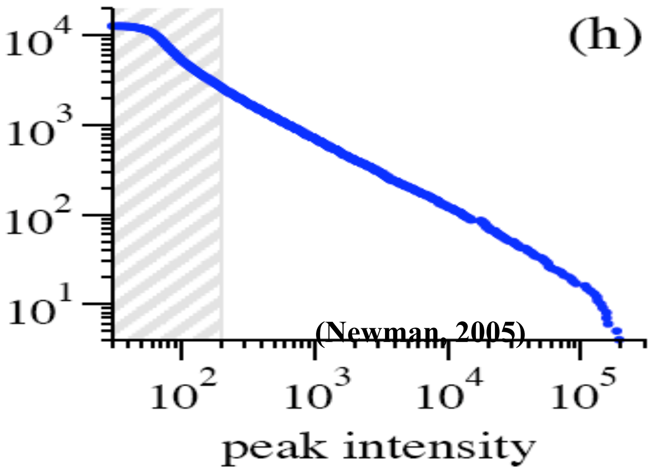


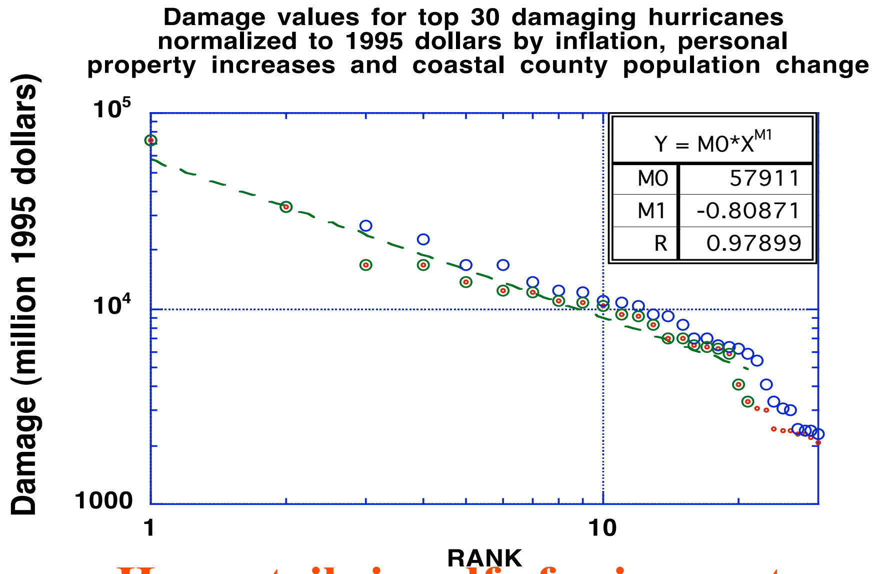
Fig. 2. Noncumulative frequency-area distributions for actual forest fires and wildfires in the United States and Australia: (**A**) 4284 fires on U.S. Fish and Wildlife Service lands (1986–1995) (*9*), (**B**) 120 fires in the western United States (1150–1960) (*10*), (**C**) 164 fires in Alaskan boreal forests (1990–1991) (*11*), and (**D**) 298 fires in the ACT (1926–1991) (*12*). For each data set, the noncumulative number of fires per year ($-d\dot{N}_{CF}/dA_F$) with area (A_F) is given as a function of A_F (*13*). In each case, a reasonably good correlation over many decades of A_F is obtained by using the power-law relation (Eq. 1) with $\alpha = 1.31$ to 1.49; $-\alpha$ is the slope of the best-fit line in log-log space and is shown for each data set.

Malamud et al., Science 281 (1998)

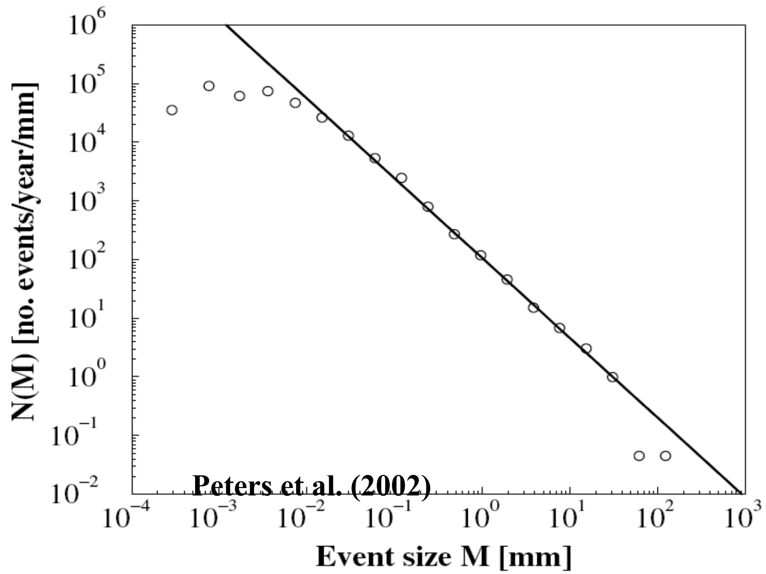
Heavy tails in pdf of Solar flares



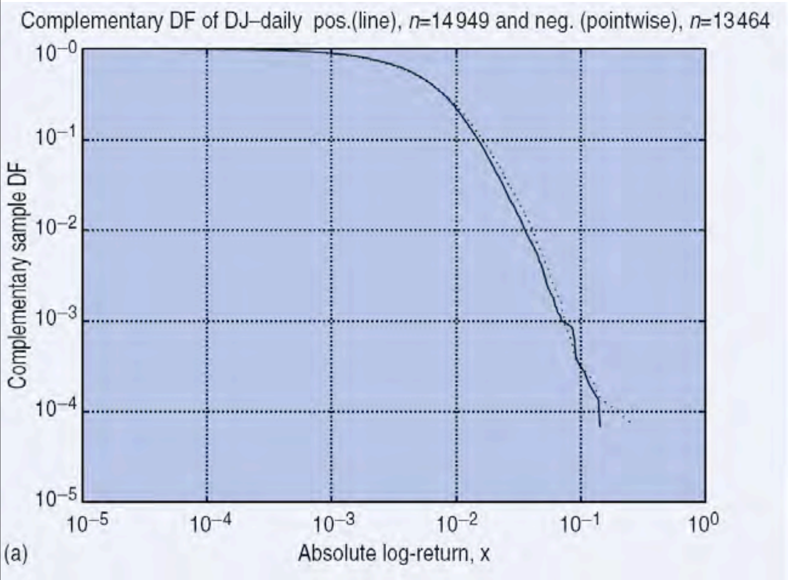
Heavy tails in pdf of Hurricane losses

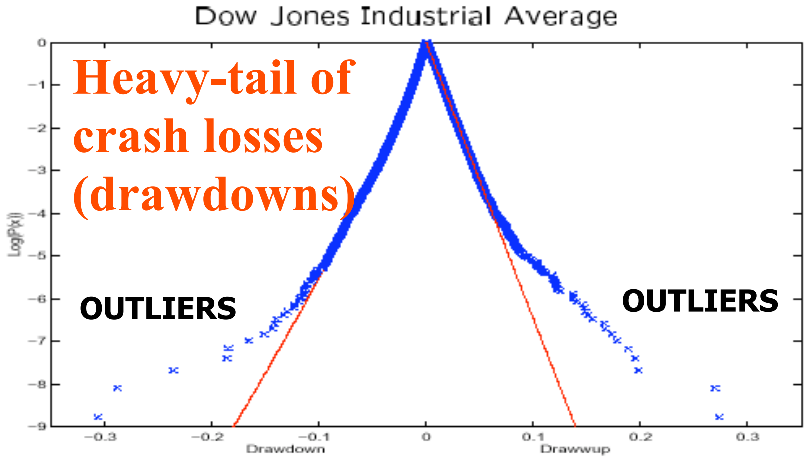


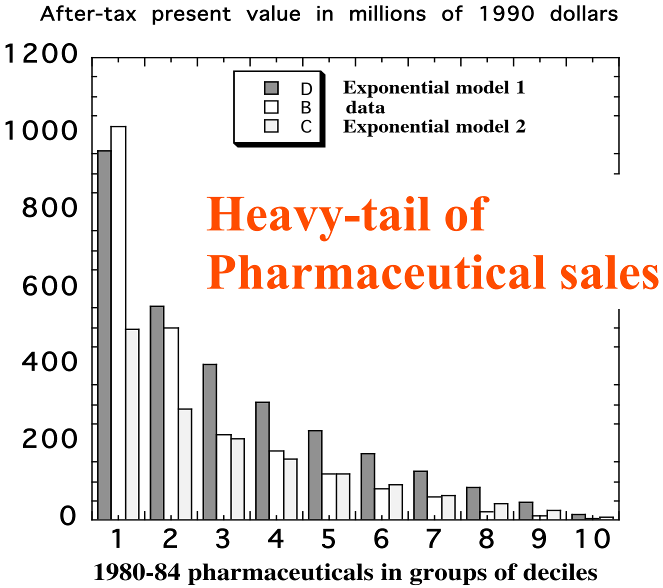
Heavy tails in pdf of rain events



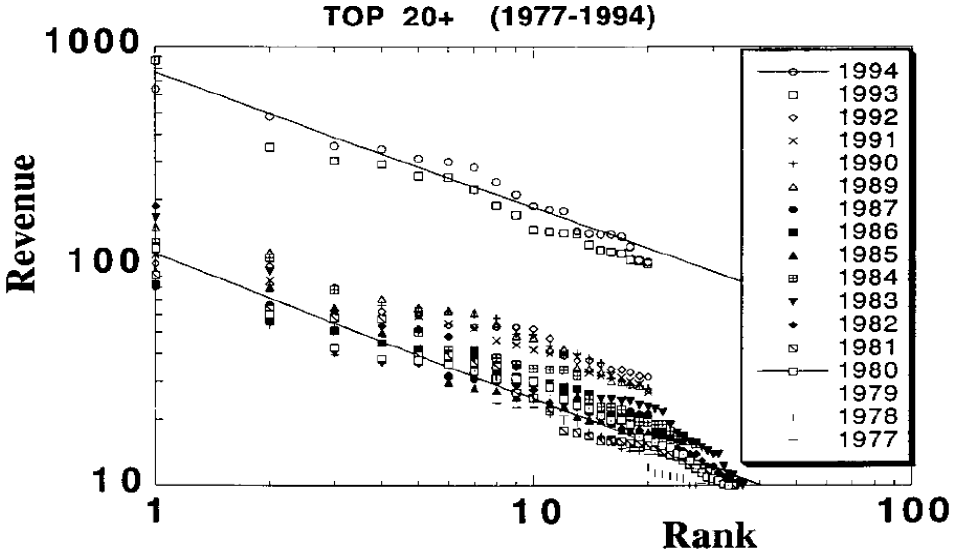
Heavy-tail of price changes

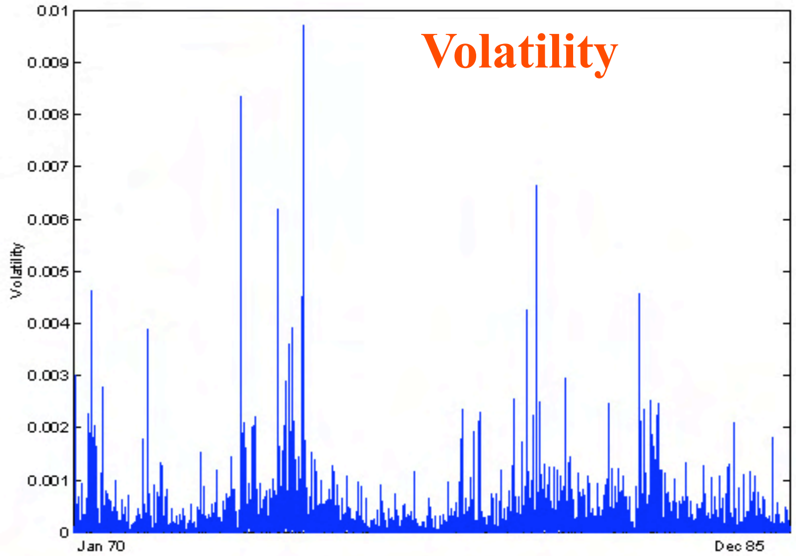


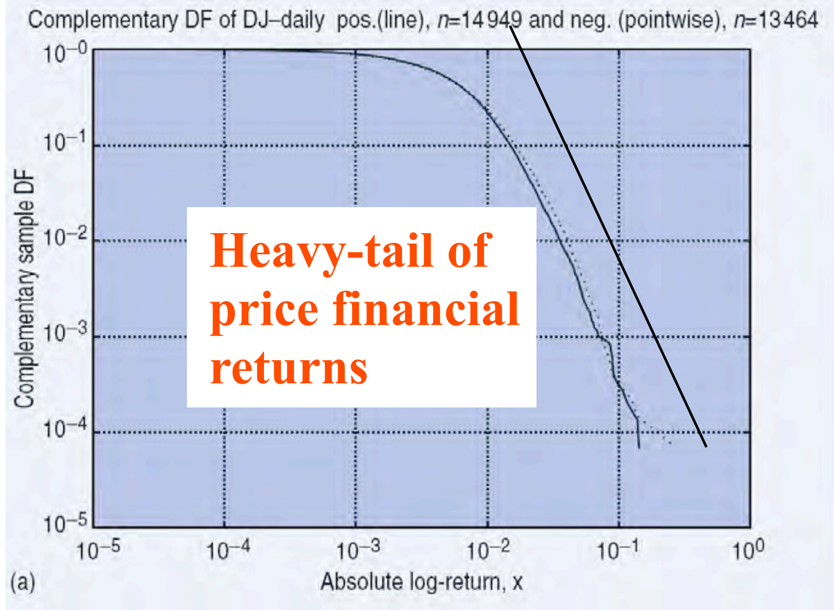


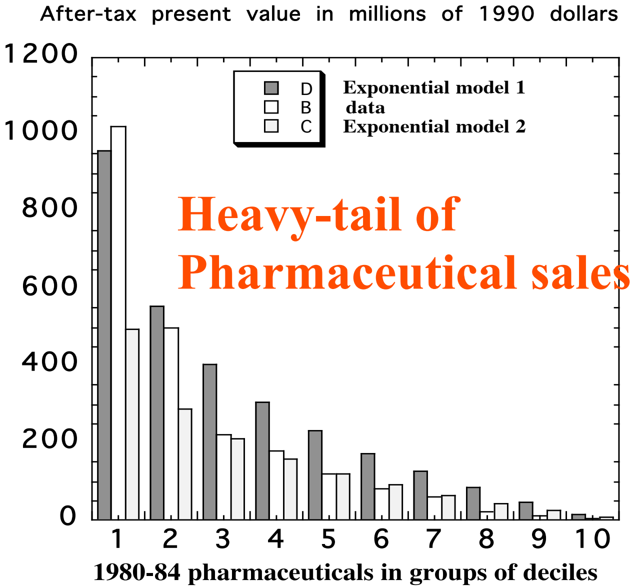


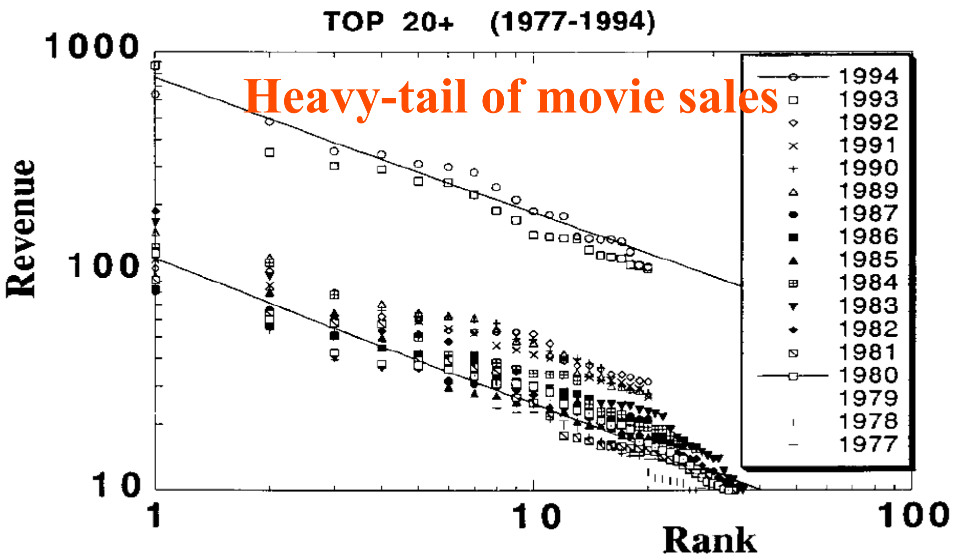
Heavy-tail of movie sales



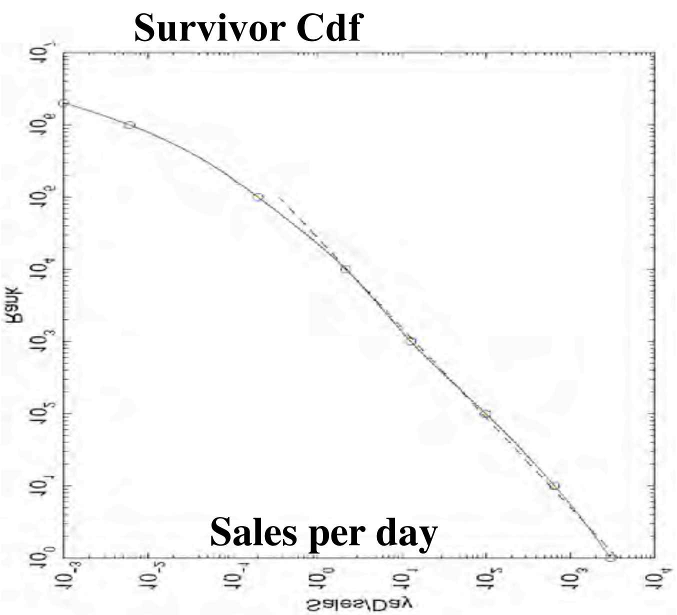

Volatility
Volatility
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
Jan 70
Dec 85


Complementary DF of DJ–daily pos.(line), n=14949 and neg. (pointwise), n=13464
Heavy-tail of price financial returns
Complementary sample DF
Absolute log-return, x
(a)

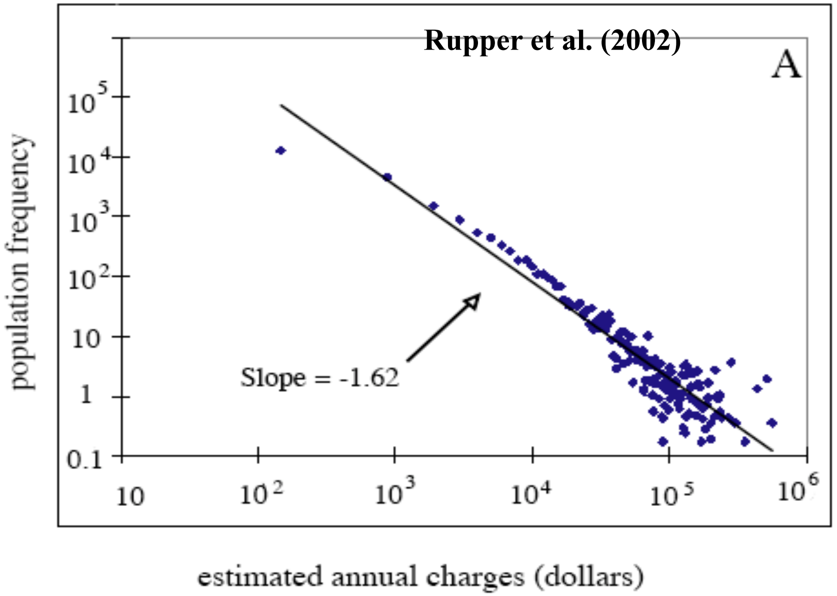

After-tax present value in millions of 1990 dollars
1200
1000
800
600
400
200
0
D Exponential model 1
B data
C Exponential model 2
Heavy-tail of Pharmaceutical sales
1 2 3 4 5 6 7 8 9 10
1980-84 pharmaceuticals in groups of deciles


TOP 20+ (1977-1994)
Heavy-tail of movie sales
Revenue
1000
100
10
1
10
100
Rank
1994
1993
1992
1991
1990
1989
1987
1986
1985
1984
1983
1982
1981
1980
1979
1978
1977

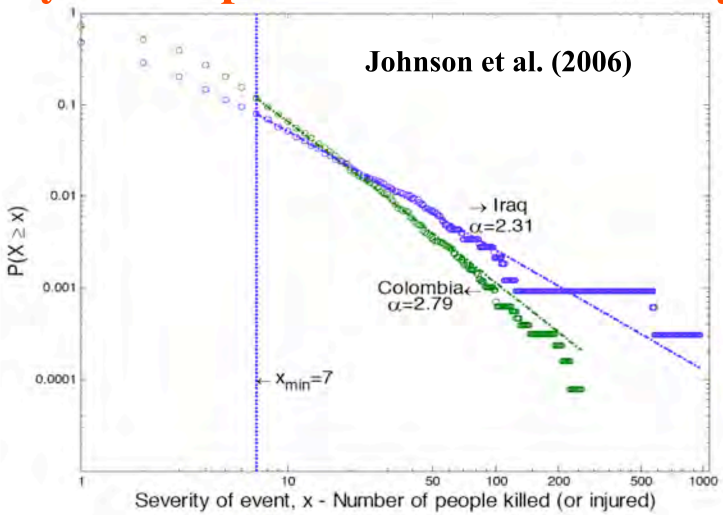
Heavy-tail of pdf of book sales



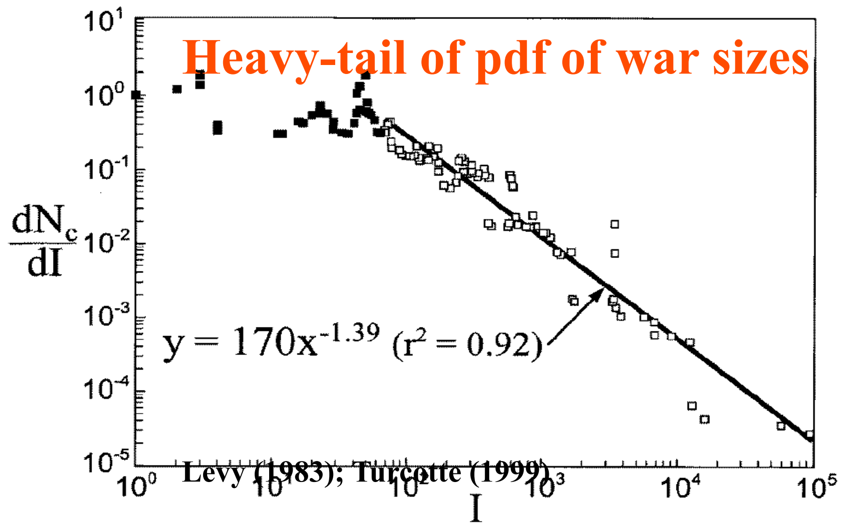
Heavy-tail of pdf of health care costs



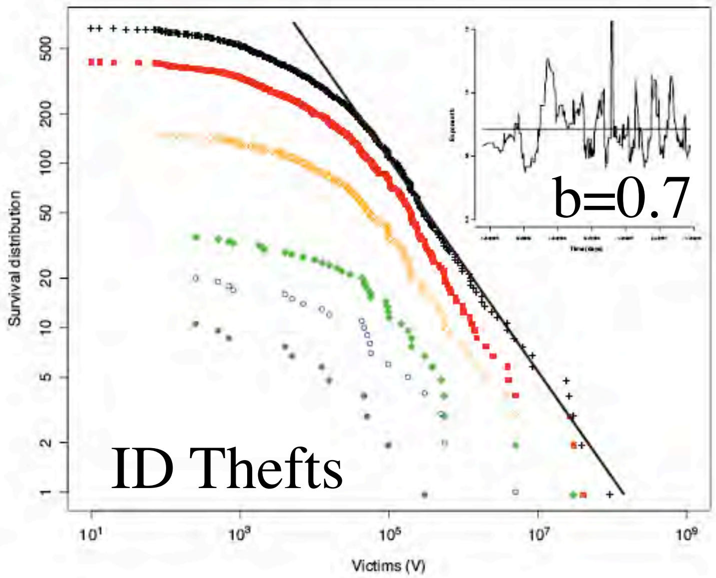
Heavy-tail of pdf of terrorist intensity



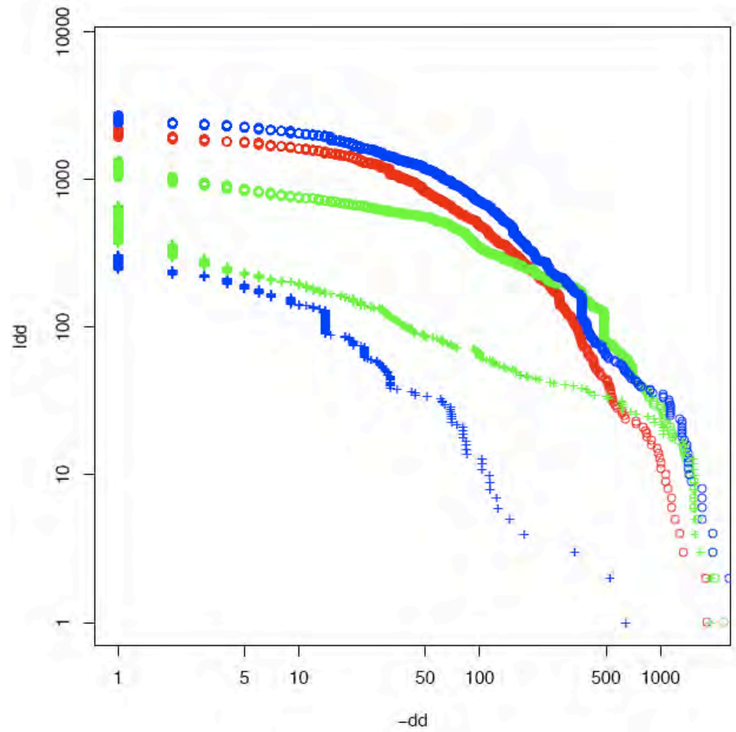
Heavy-tail of pdf of war sizes



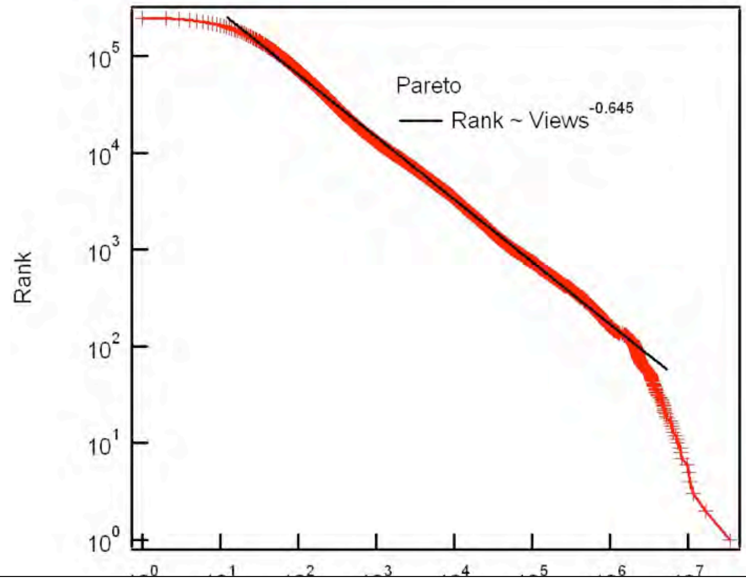
Heavy-tail of pdf of cyber risks



Software vulnerabilities



Heavy-tail of YouTube view counts



Standard measure of dependence: the correlation coefficient

$$\rho_{12} = \quad \rho(X, Y) = \frac{\mathrm{Cov}\,[X,\ Y]}{\sqrt{\mathrm{Var}\,[X] \cdot \mathrm{Var}\,[Y]}}$$

Pearson estimator:

$$\hat{\rho}_T = \frac{\frac{1}{T}\sum_{i=1}^{T}(x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\frac{1}{T}\sum_{i=1}^{T}(x_i - \bar{x})^2 \cdot \frac{1}{T}\sum_{i=1}^{T}(y_i - \bar{y})^2}}$$

ρ is a linear measure of dependence

$$Y = \beta X + \epsilon \quad \Leftrightarrow \quad \rho = \beta\sqrt{\frac{\mathrm{Var}[X]}{\mathrm{Var}[Y]}}$$

The correlation coefficient is invariant under an increasing affine change of variable

$$X' = a \cdot X + b, \quad a > 0$$
$$Y' = c \cdot Y + d, \quad c > 0$$

But lack of invariance with respect to NONLINEAR change of variables

- local correlation and generalized correlation for N>2 variables
- Kendall's tau

$$\tau = \Pr\left[(X_1 - X_2) \cdot (Y_1 - Y_2) > 0\right] - \Pr\left[(X_1 - X_2) \cdot (Y_1 - Y_2) < 0\right]$$

$$\tau(C) = 4 \int \int C(u,v)\ dC(u,v) - 1$$

- Spearman's rho $\rho_s(C) = 12 \int \int_{[0,1]^2} C(u,v)\ dudv - 3$

$$\rho_s = 3\left(\Pr[(X_1 - X_2)(Y_1 - Y_3) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_3) < 0]\right)$$

- Gini's gamma $\gamma(C) = 4\left[\int_0^1 C(u,u)\ du + \int_0^1 C(u, 1-u)\ du - \frac{1}{2}\right]$

Concordance measures of dependence

Kendall's tau, Spearman's rho, Gini's gamma share the following properties

1. they are defined for any pair of continuous random variables X and Y,
2. they are symmetric: for any pair X and Y, $\tau(X,Y) = \tau(Y,X)$, for instance,
3. they range from -1 to $+1$, and reach these bounds when X and Y are countermonotonic and comonotonic respectively,
4. they equal zero for independent random variables,
5. if the pair of random variables (X_1, X_2) is more dependent than the pair (Y_1, Y_2) in the following sense:

   $$C_X(u,v) \geq C_Y(u,v), \quad \forall u,v \in [0,1],$$

   then the same ranking holds for any of these three measures; for instance, $\tau(X_1, X_1) \geq \tau(Y_1, Y_2)$.

- Gaussianization of multivariate distributions
- Copulas
- Test of the Gaussian copula hypothesis
- Extreme conditional dependence measures
- Tail dependence for factor models

Multivariate representation of the joint distribution

using Gaussianization by nonlinear change of variable

Information Theory (Rao, 1973)

(Maximum entropy principle)

"Best" representation of the multivariable distribution:

$$\hat{P}(\boldsymbol{y}) = (2\pi)^{-N/2}|V|^{-1/2}\exp(-\tfrac{1}{2}\boldsymbol{y}' V^{-1}\boldsymbol{y})$$

$$P(\boldsymbol{x}) = |V|^{-1/2}\exp(-\tfrac{1}{2}\boldsymbol{y}'(V^{-1} - I)\boldsymbol{y})\prod_{j=1}^{N} P_j(x^{(j)})$$

V is again the covariance matrix for $\boldsymbol{y}$ and I is the identity matrix.

(amounts to using the Gaussian copula)

D. Sornette, J. V. Andersen and P. Simonetti, Portfolio Theory for Fat Tails, International Journal of Theoretical and Applied Finance 3 (3), 523-535 (2000)

D. Sornette, P. Simonetti and J. V. Andersen, phi^q field theory for Portfolio optimization: ``fat tails'' and non-linear correlations, Physics Report 335 (2), 19-92 (2000)

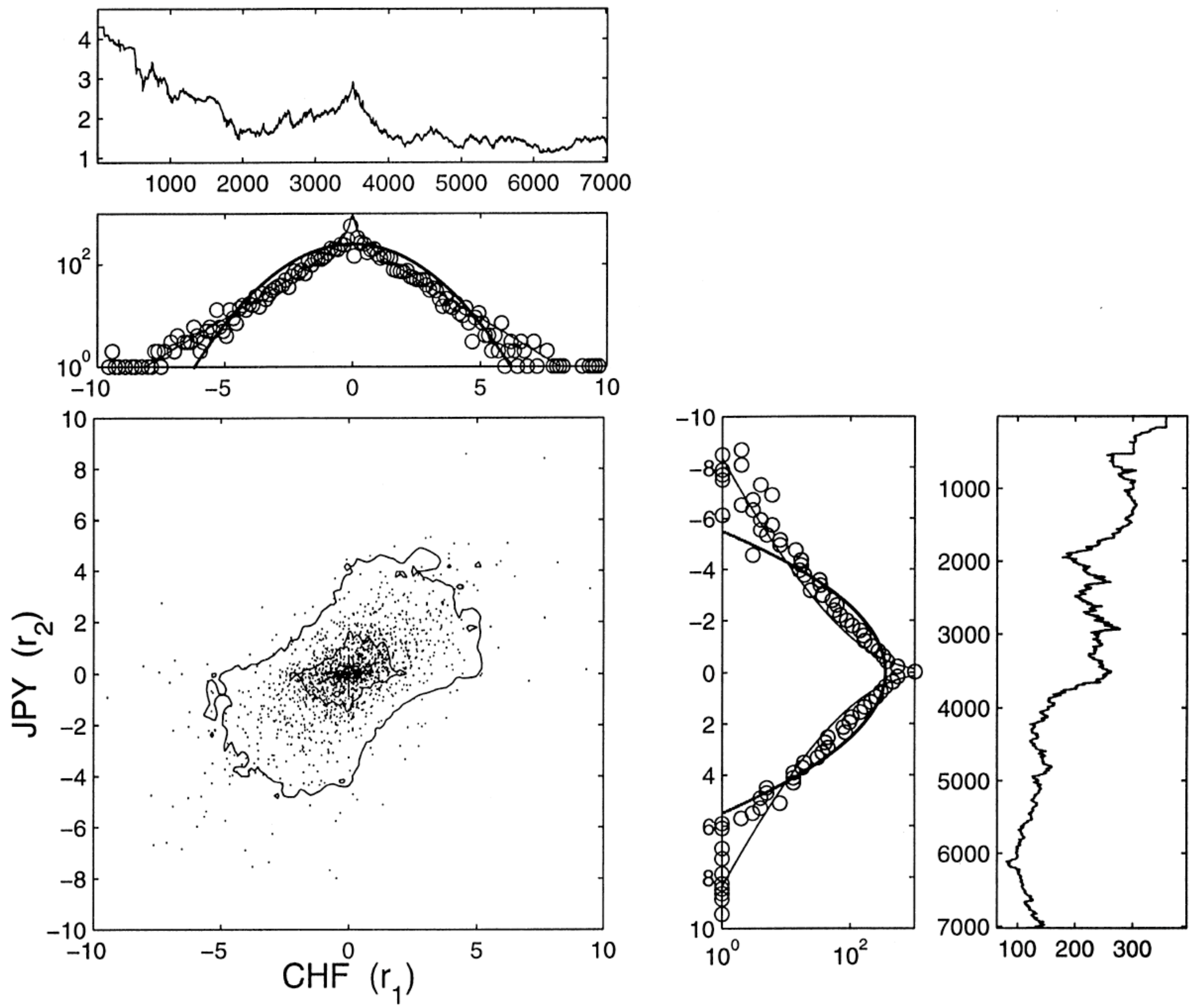
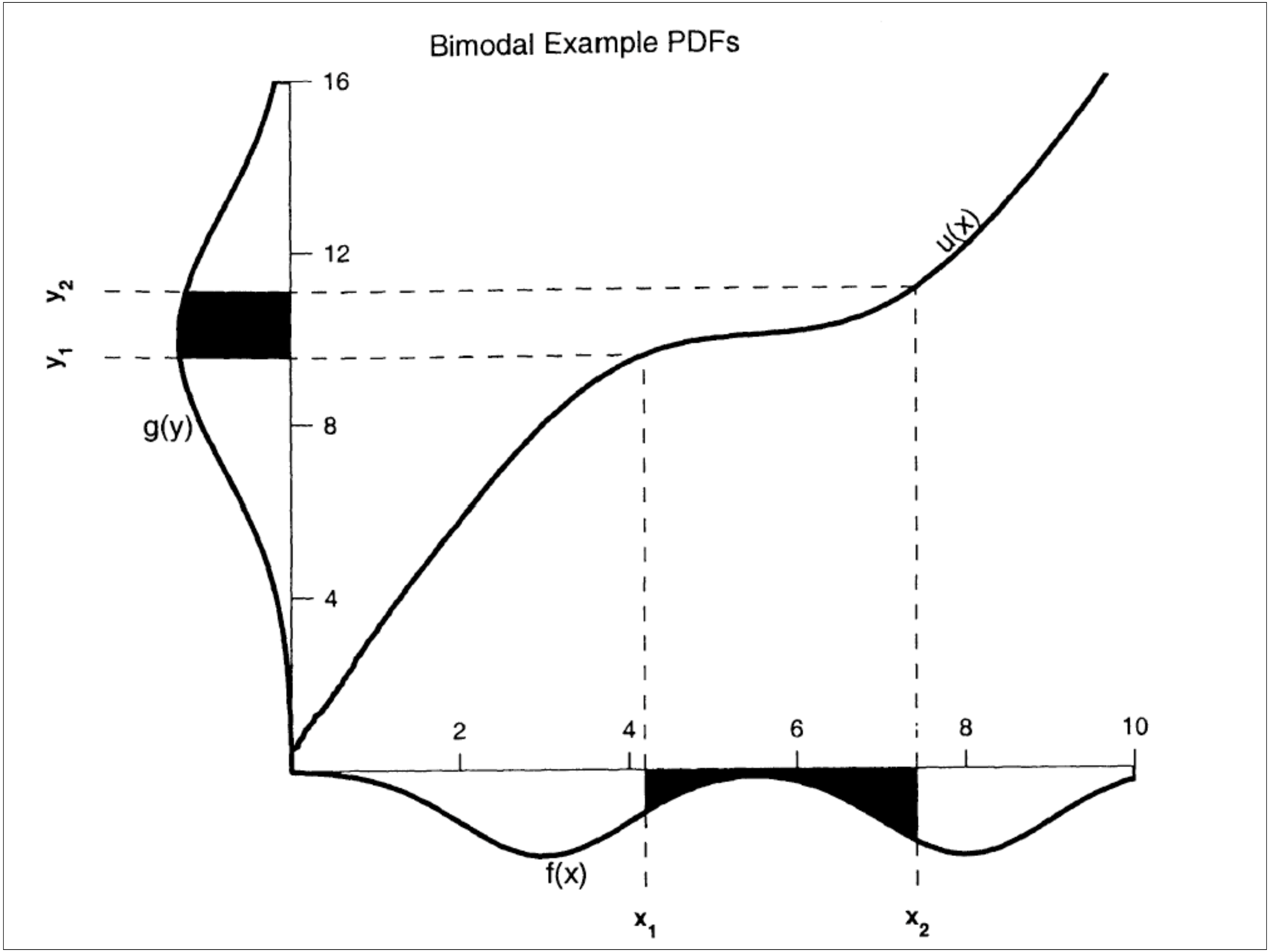
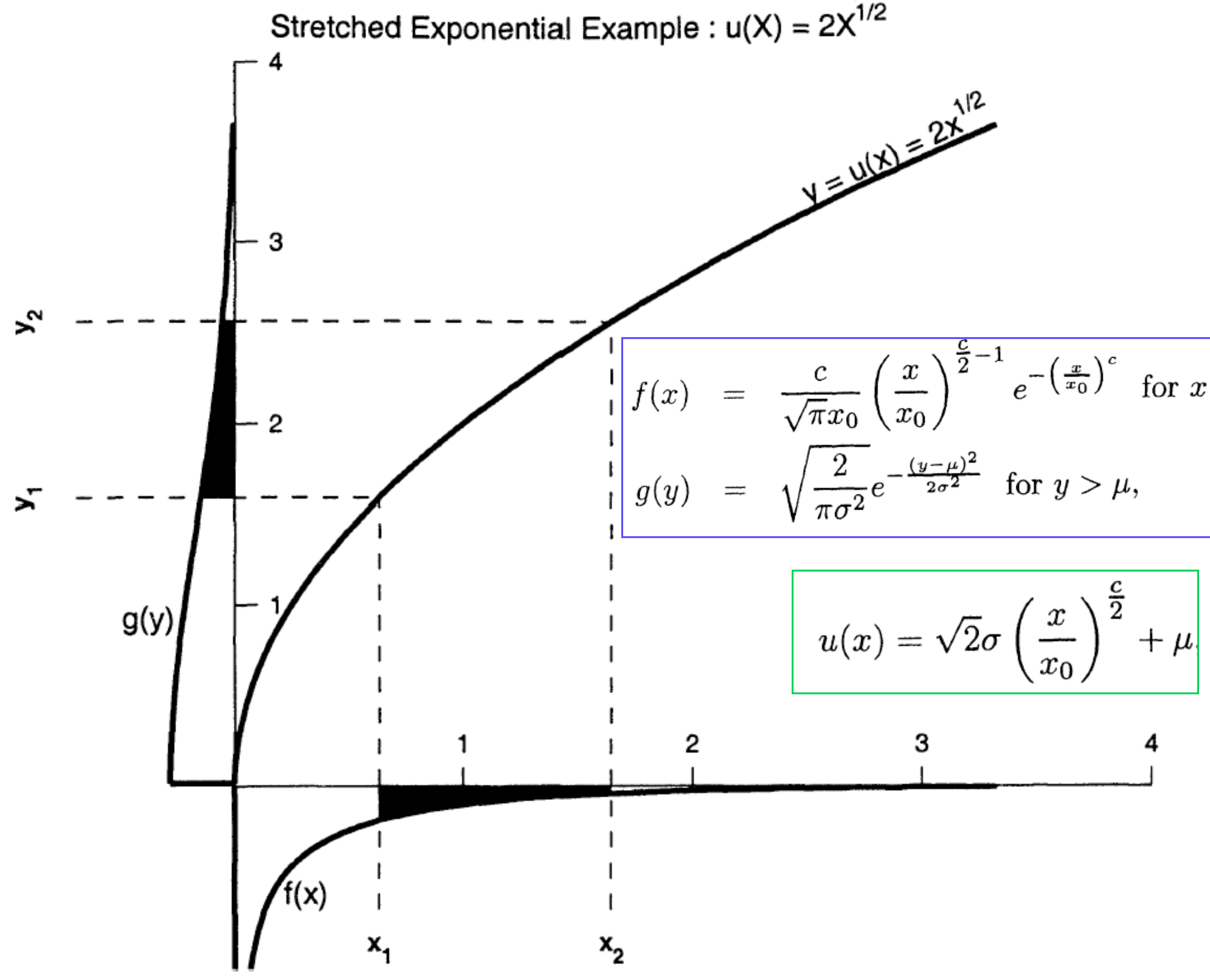


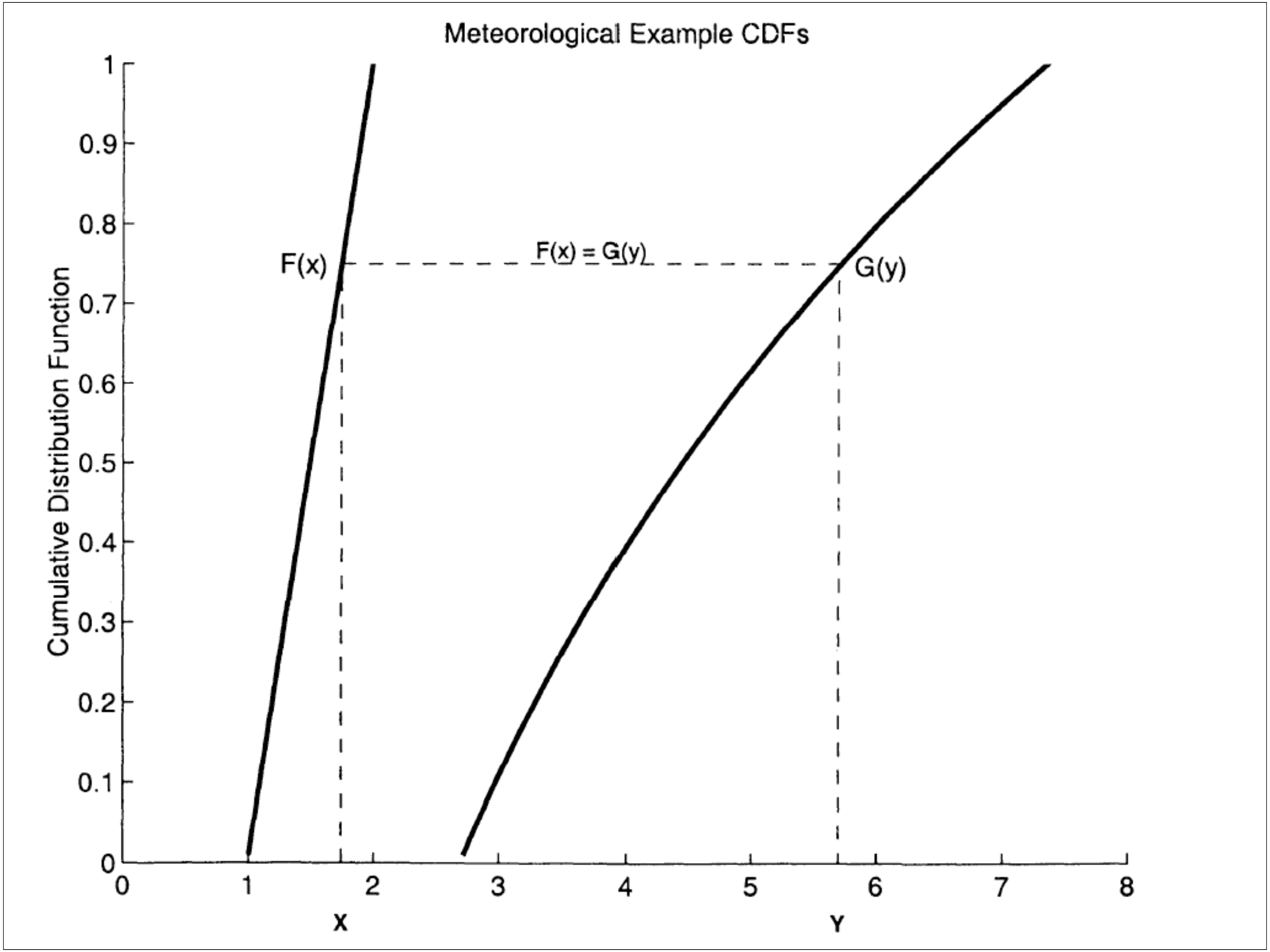
Fig. 1. Bivariate distribution of the daily annualized returns of the CHF in US \$ ($i = 1$) and of JPY in US \$ ($i = 2$) for the time interval from Jan. 1971 to Oct. 1998. One-fourth of the data points are represented for clarity of the figure. The contour lines define the probability confidence level of 90% (outer line), 50% and 10%. Also shown are the time series and the marginal distributions in the panels at the top and on the side. The parameters for the fit of the marginal pdf's are: CHF in US \$: $A_1 = 250, c_1 = 1.14, r_{01} = 2.13$ and JPY in US \$: $A_2 = 350, c_2 = 0.8, r_{02} = 1.25$.

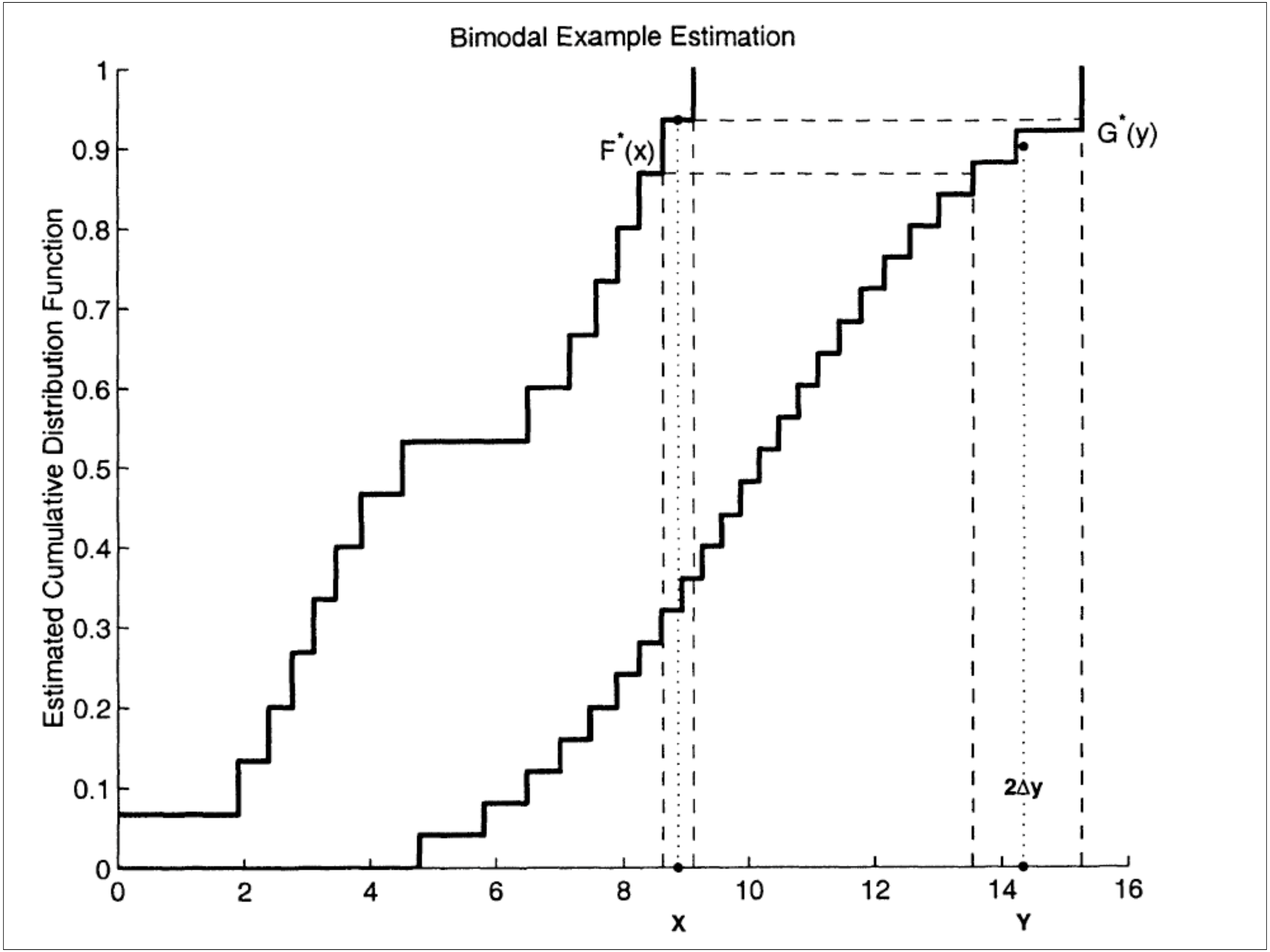

Bimodal Example PDFs
16
12
8
4
y2
y1
g(y)
u(x)
2
4
6
8
10
f(x)
x1
x2



$$
\begin{aligned}
f(x) &= \frac{c}{\sqrt{\pi}x_0}\left(\frac{x}{x_0}\right)^{\frac{c}{2}-1} e^{-\left(\frac{x}{x_0}\right)^c} \quad \text{for } x > 0 \\
g(y) &= \sqrt{\frac{2}{\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \text{for } y > \mu,
\end{aligned}
$$

$$
u(x) = \sqrt{2}\sigma\left(\frac{x}{x_0}\right)^{\frac{c}{2}} + \mu.
$$


Meteorological Example CDFs
Cumulative Distribution Function
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
F(x)
F(x) = G(y)
G(y)
X
Y



Bimodal Example Estimation
Estimated Cumulative Distribution Function
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
F*(x)
G*(y)
2Δy
X
Y

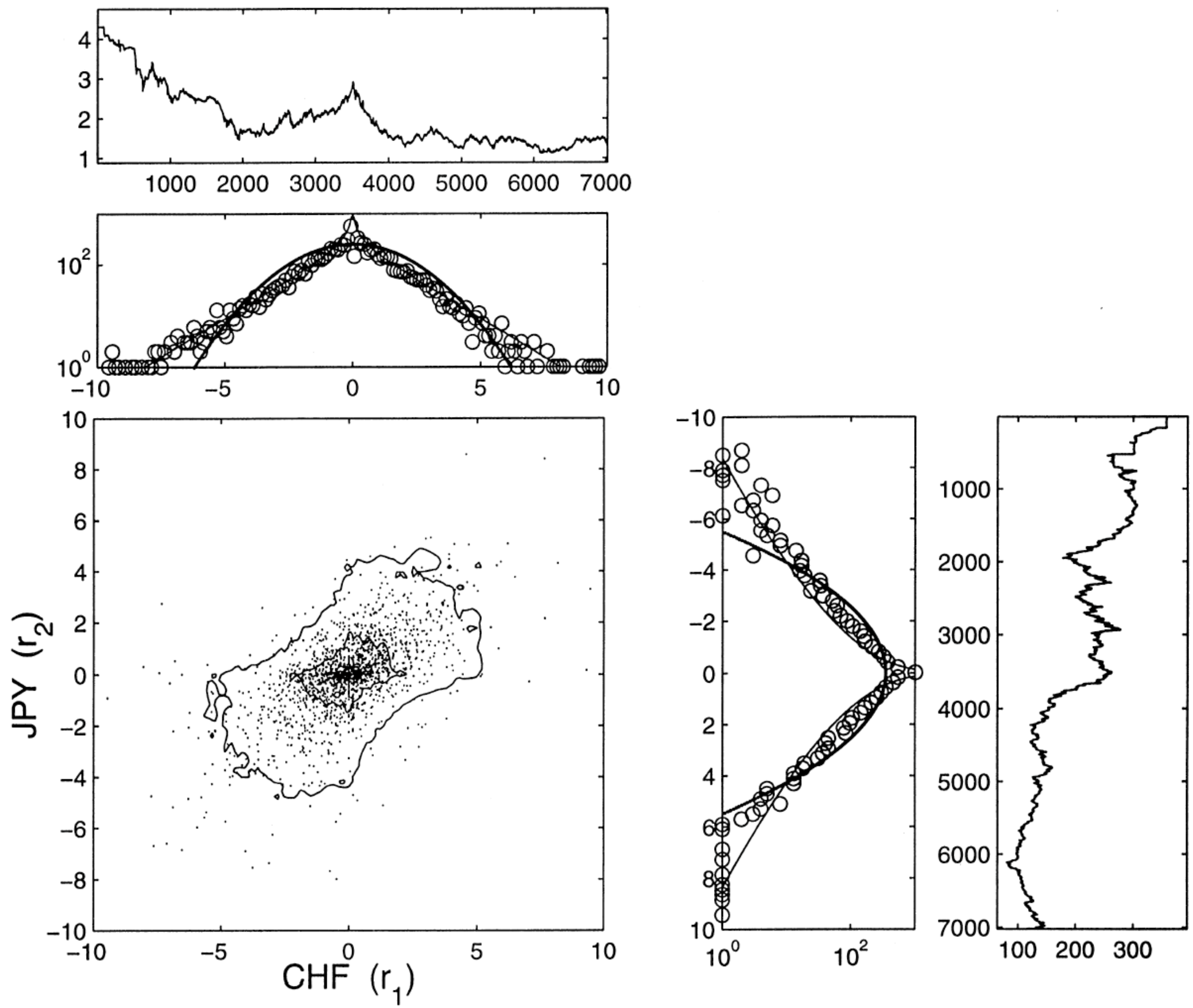


Fig. 1. Bivariate distribution of the daily annualized returns of the CHF in US \$ ($i = 1$) and of JPY in US \$ ($i = 2$) for the time interval from Jan. 1971 to Oct. 1998. One-fourth of the data points are represented for clarity of the figure. The contour lines define the probability confidence level of 90% (outer line), 50% and 10%. Also shown are the time series and the marginal distributions in the panels at the top and on the side. The parameters for the fit of the marginal pdf's are: CHF in US \$: $A_1 = 250, c_1 = 1.14, r_{01} = 2.13$ and JPY in US \$: $A_2 = 350, c_2 = 0.8, r_{02} = 1.25$.

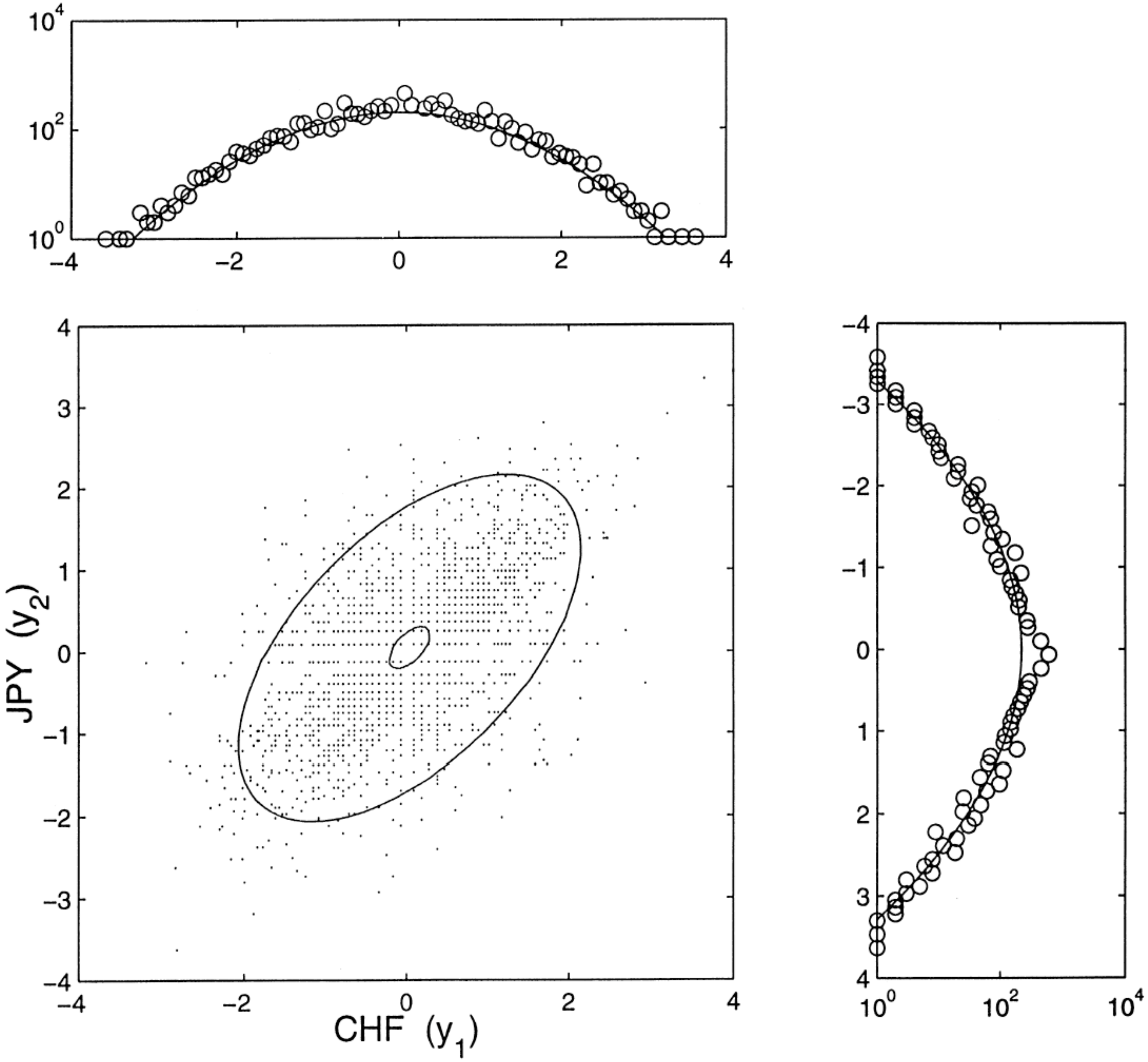
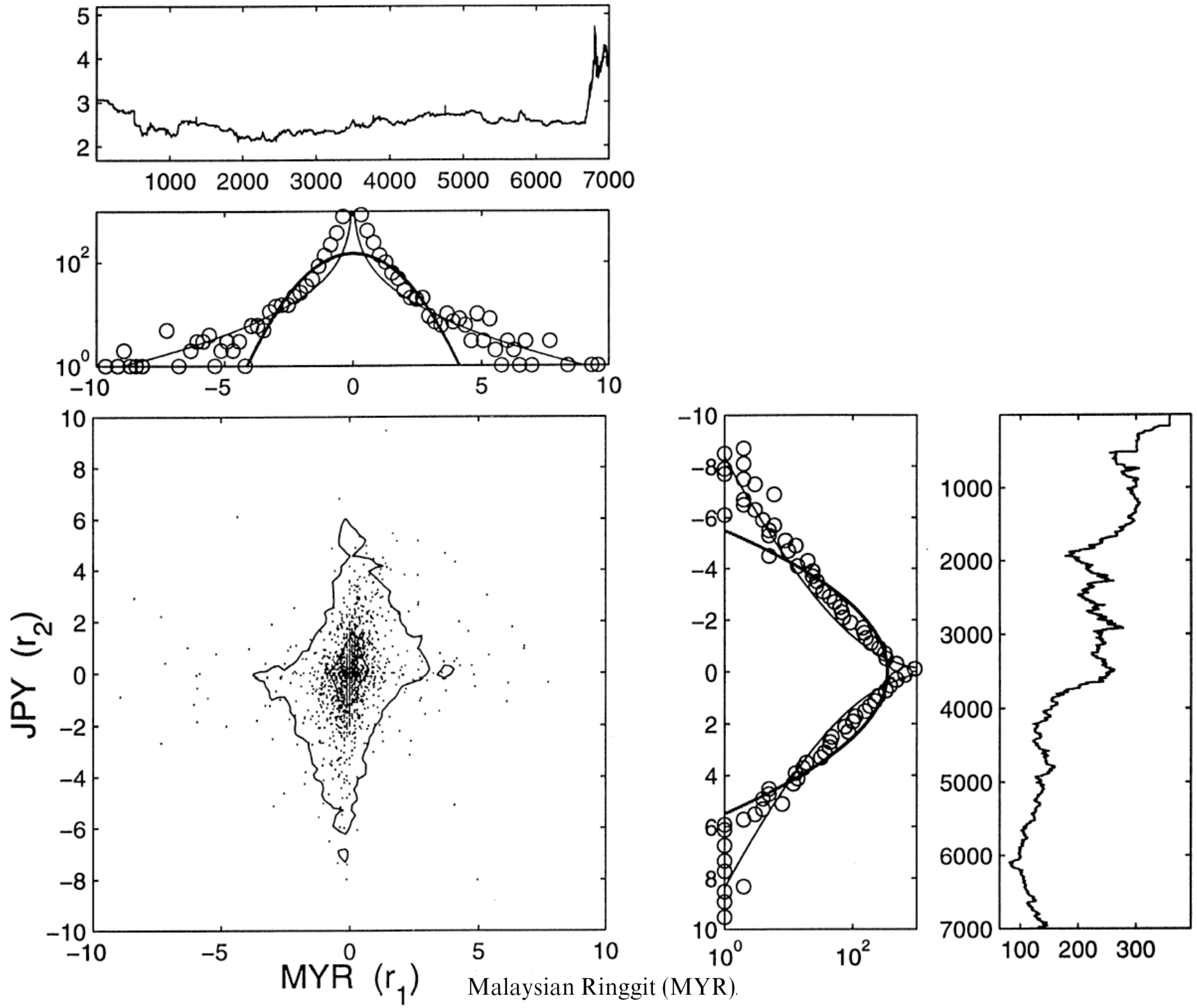


Fig. 7. Bivariate distribution $\hat{P}(\boldsymbol{y})$ obtained from Fig. 1 using the transformation equation (16). The contour lines are defined as in Fig. 1. The upper and right diagrams show the corresponding projected marginal distributions, which are Gaussian by construction of the change of variable, Eq. (16). The solid lines are fits of the form $A\exp(-|y|^2/2)$ with $A_1 = 200, A_2 = 220$.



Malaysian Ringgit (MYR).

Fig. 5. Bivariate distribution of the daily annualized returns of the MYR in US \$ ($i = 1$) and of JPY in US \$ ($i = 2$) for the time interval from Jan. 1971 to Oct. 1998. One-fourth of the data points are represented for clarity of the figure. The contour lines define the probability confidence level of 90% (outer line), 50% and 10%. Also shown are the time series and the marginal distributions in the panels at the top and on the side. The parameters for the fit of the marginal pdf's are: MYR in US \$: $A_1 = 150, c_1 = 0.56, r_{01} = 1.00$ and JPY in US \$: $A_2 = 350, c_2 = 0.8, r_{02} = 1.25$.

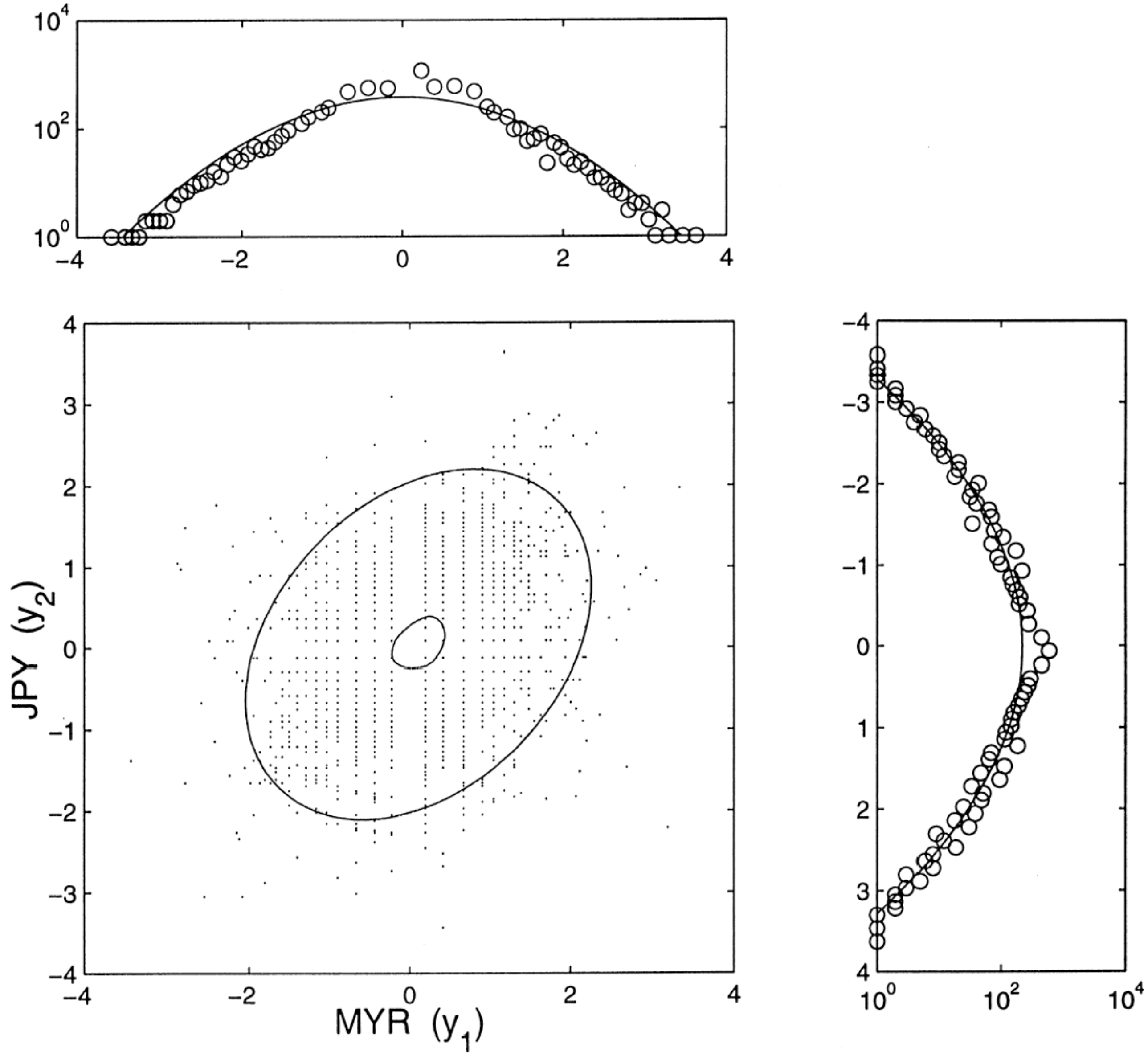


Fig. 11. Bivariate distribution $\hat{P}(\boldsymbol{y})$ obtained from Fig. 5 using the transformation equation (16). The contour lines are defined as in Fig. 5. The upper and right diagrams show the corresponding projected marginal distributions, which are Gaussian by construction of the change of variable, Eq. (16). The solid lines are fits of the form $A\exp(-|y|^2/2)$ with $A_1 = 380, A_2 = 220$.

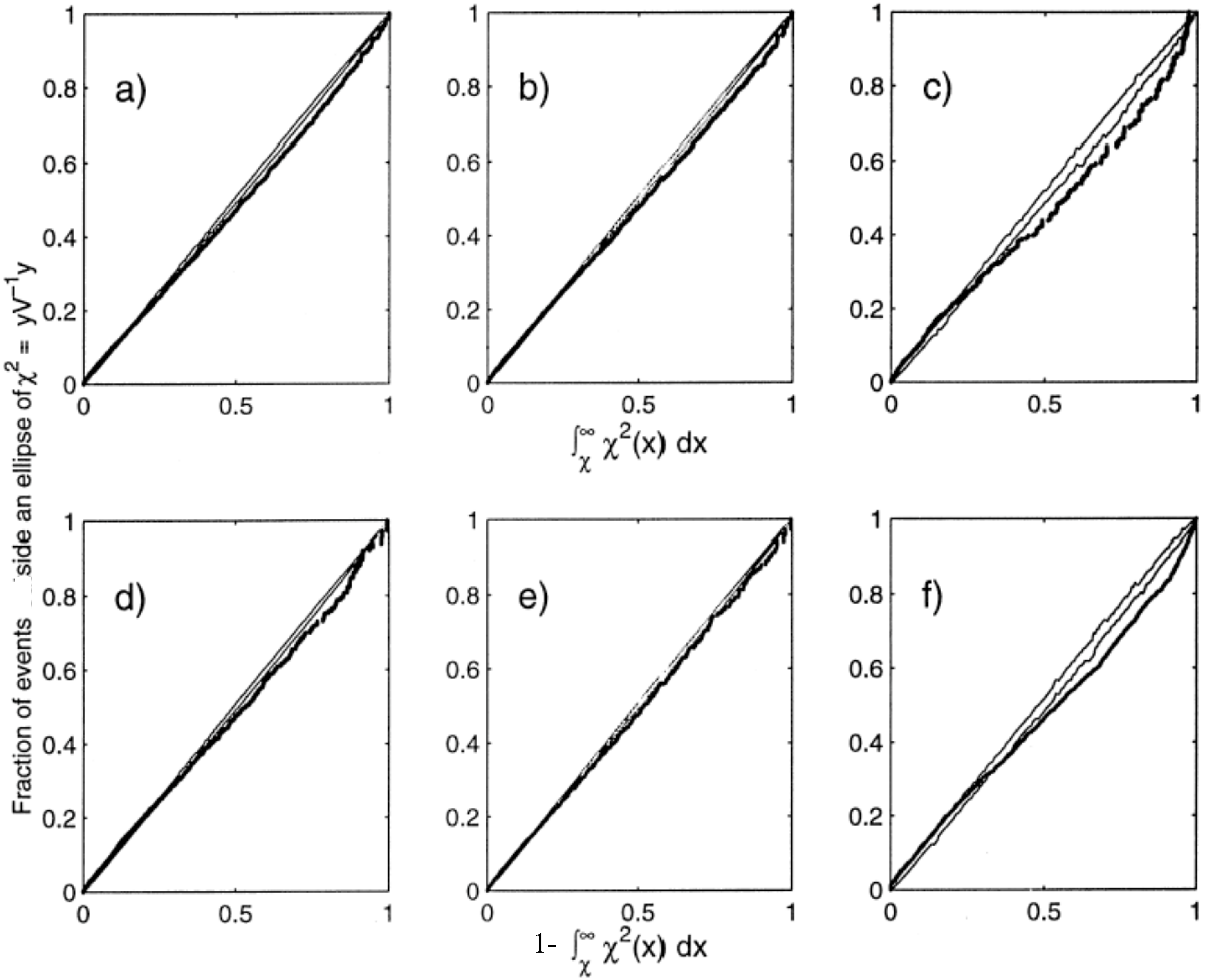


Fig. 13a–e. χ^2 cumulative distribution for $N = 2$ degrees of freedom versus the fraction of events shown in Figs. 7–11 inside an ellipse of equation $\chi^2 = (\boldsymbol{y}' V^{-1})\boldsymbol{y}$. (a) CHF-JPY, (b) UKP-JPY, (c) RUR-JPY, (d) CAD-JPY, (e) MYR-JPY; (f) same plot as (a)–(e) but for $N = 6$ degrees of freedom for the multivariate data set CHF–UKP–RUR–CAD–MYR–JPY.

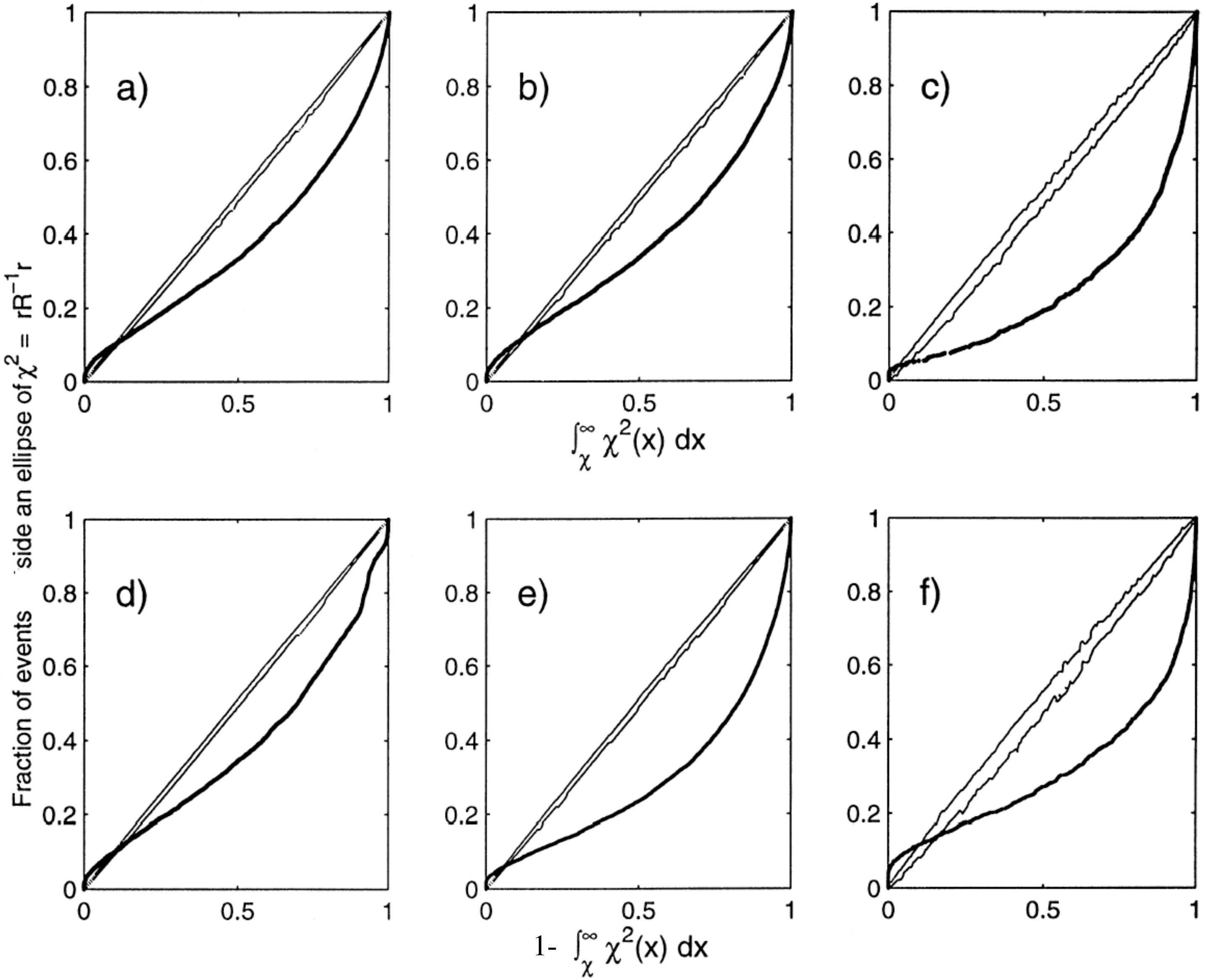


Fig. 14. Same as Fig. 13 but for the returns r. a–e: χ^2 cumulative distribution for $N = 2$ degrees of freedom versus the fraction of events shown in Figs. 1–5 outside an ellipse of equation $\chi^2 = (\boldsymbol{r}'\mathscr{V}^{-1}\boldsymbol{r})$. (a) CHF-JPY, (b) UKP-JPY, (c) RUR-JPY, (d) CAD-JPY, (e) MYR-JPY; (f) same plot as (a)–(e) but for $N = 6$ degrees of freedom for the data set CHF-UKP-RUR-CAD-MYR-JPY.

Modified-Weibull distributions

Definition: A random variable X is said to follow a modified Weibull distribution with exponent c and scale factor χ, if and only if the random variable

$$Y = \operatorname{sgn}(X)\sqrt{2}\left(\frac{|X|}{\chi}\right)^{c/2}$$

follows a normal distribution.

Its density is:

$$p(x) = \frac{1}{2\sqrt{\pi}}\frac{c}{\chi^{c/2}}|x|^{c/2-1}\exp\left[-\left(\frac{|x|}{\chi}\right)^{c}\right]$$

Modified-Weibull distributions

For a modified-Weibull distribution:

$$Y = \operatorname{sgn}(X)\sqrt{2}\left(\frac{|X|}{\chi}\right)^{c/2}$$

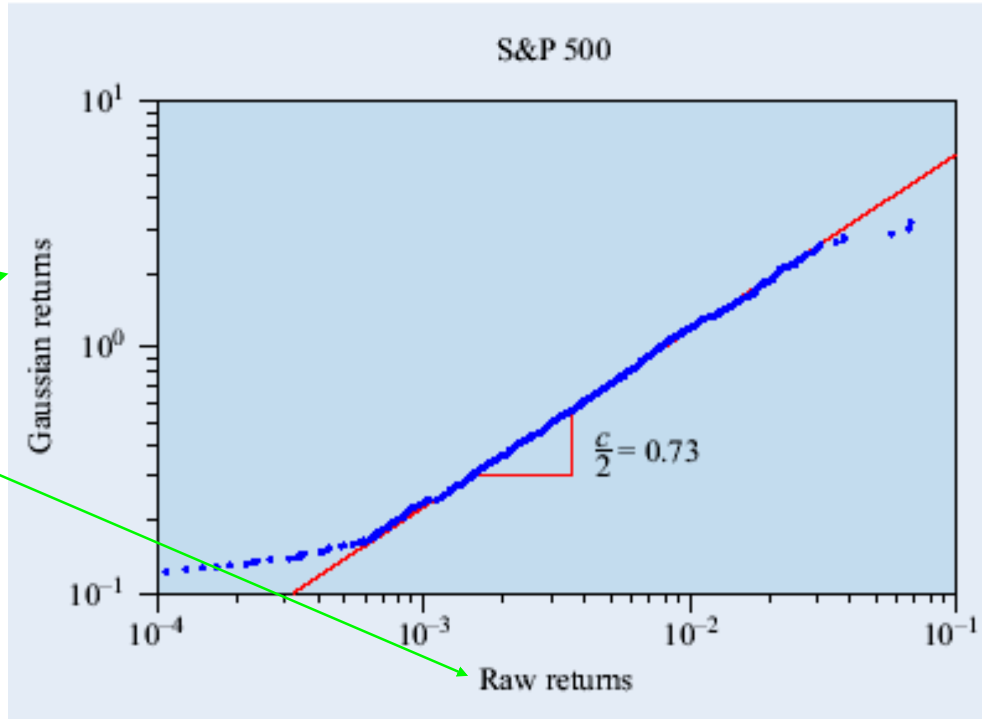


Figure 2. A graph of the returns of the Gaussianized Standard and Poor's 500 index versus its raw returns, from 3 January 1995 to 29 December 2000 for the negative tail of the distribution.

Empirical Results about the Distributions of Returns

- Models in terms of Regularly varying distributions:

$$\Pr\left[r_t \geq x\right] = \mathcal{L}(x) \cdot x^{-\mu} \qquad (\mu \approx 3-4)$$

Longin (1996) , Lux (1996-2000), Pagan (1996), Gopikrishnan et *al.* (1998)…

- Models in terms of Weibull-like distributions:

$$\Pr\left[r_t \geq x\right] = \exp\left[-\mathcal{L}(x) \cdot x^{c}\right] \qquad (c < 1)$$

Mantegna and Stanley (1994), Ebernlein et *al.*(1998), Gouriéroux and Jasiak (1998), Laherrère and Sornette (1999)…

VaR: Value-at-Risk

- VaR(p) at confidence level p: maximum loss not exceeded with a given probability level p, defined as the confidence level

$$\Pr[\text{Return}(t) < -\text{Var}(p)] = 1\text{-}p$$

$$VaR_\alpha = \inf\{l \in \Re : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \Re : F_L(l) \geq \alpha\}$$

- VaR at 95% confidence level: p=0.95
- VaR at 99% confidence level: p=0.99

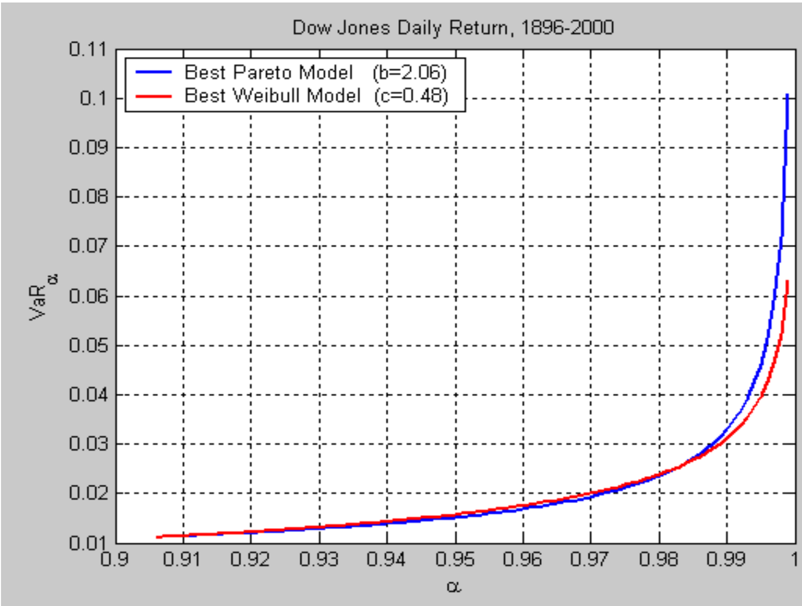
In probabilistic terms, VaR(p) is the p-quantile of the loss distribution

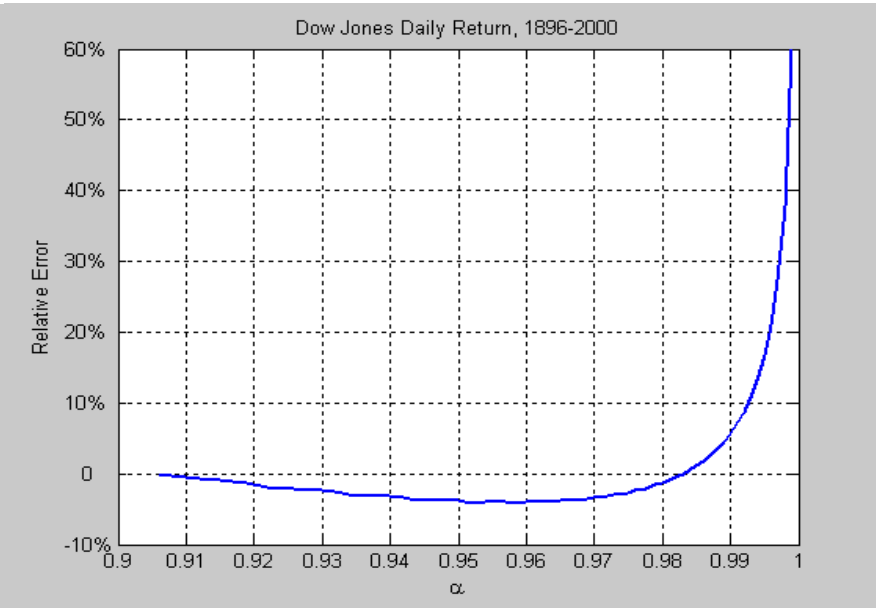
Implications of the two models

Practical consequences :

Extreme risk assessment,

Multi-moment asset pricing methods.





	Positive Tail				Negative Tail			
	$<\chi_+>$	$<c_+>$	χ_+	c_+	$<\chi_->$	$<c_->$	χ_-	c_-
CHF	2.45	1.61	2.33	1.26	2.34	1.53	1.72	0.93
DEM	2.09	1.65	1.74	1.03	2.01	1.58	1.45	0.91
JPY	2.10	1.28	1.30	0.76	1.89	1.47	0.99	0.76
MAL	1.00	1.22	1.25	0.41	1.01	1.25	0.44	0.48
POL	1.55	1.02	1.30	0.73	1.60	2.13	1.25	0.62
THA	0.78	0.75	0.75	0.54	0.82	0.73	0.30	0.38
UKP	1.89	1.52	1.38	0.92	2.00	1.41	1.82	1.09

Table 3: Table of the exponents c and the scale parameters χ for different currencies. The subscript "+" or "-" denotes the positive or negative part of the distribution of returns and the terms between brackets refer to parameters estimated in the bulk of the distribution while naked parameters refer to the tails of the distribution.

$$p(x) = \frac{1}{2\sqrt{\pi}} \frac{c}{\chi^{c/2}} |x|^{c/2-1} \exp\left[-\left(\frac{|x|}{\chi}\right)^{c}\right]$$

	Positive Tail				Negative Tail			
	$<\chi_+>$	$<c_+>$	χ_+	c_+	$<\chi_->$	$<c_->$	χ_-	c_-
Applied Material	12.47	1.82	8.75	0.99	11.94	1.66	8.11	0.98
Coca-Cola	5.38	1.88	4.46	1.04	5.06	1.74	2.98	0.78
EMC	13.53	1.63	13.18	1.55	11.44	1.61	3.05	0.57
General Electric	5.21	1.89	1.81	1.28	4.80	1.81	4.31	1.16
General Motors	5.78	1.71	0.63	0.48	5.32	1.89	2.80	0.79
Hewlett Packart	7.51	1.93	4.20	0.84	7.26	1.76	1.66	0.52
IBM	5.46	1.71	3.85	0.87	5.07	1.90	0.18	0.33
Intel	8.93	2.31	2.79	0.64	9.14	1.60	3.56	0.62
MCI WorldCom	9.80	1.74	11.01	1.56	9.09	1.56	2.86	0.58
Medtronic	6.82	1.95	6.09	1.11	6.49	1.54	2.55	0.67
Merck	5.36	1.91	4.56	1.16	5.00	1.73	1.32	0.59
Pfi zer	6.41	2.01	5.84	1.27	6.04	1.70	0.26	0.35
Procter & Gambel	4.86	1.83	3.53	0.96	4.55	1.74	2.96	0.82
SBC Communication	5.21	1.97	1.26	0.59	4.89	1.59	1.56	0.60
Texas Instrument	9.06	1.78	4.07	0.72	8.24	1.84	2.18	0.54
Wall Mart	7.41	1.83	5.81	1.01	6.80	1.64	3.75	0.78

Table 4: Table of the exponents c and the scale parameters χ for different stocks. The subscript "+" or "-" denotes the positive or negative part of the distribution and the terms between brackets refer to parameters estimated in the bulk of the distribution while naked parameters refer to the tails of the distribution.

	Mean (10^{-3})	Variance (10^{-3})	Skewness	Kurtosis	min	max
Applied Material	2.11	1.62	0.41	4.68	-14%	21%
Coca-Cola	0.81	0.36	0.13	5.71	-11%	10%
EMC	2.76	1.13	0.23	4.79	-18%	15%
Exxon-Mobil	0.92	0.25	0.30	5.26	-7%	11%
General Electric	1.38	0.30	0.08	4.46	-7%	8%
General Motors	0.64	0.39	0.12	4.35	-11%	8%
Hewlett Packard	1.17	0.81	0.16	6.58	-14%	21%
IBM	1.32	0.54	0.08	8.43	-16%	13%
Intel	1.71	0.85	-0.31	6.88	-22%	14%
MCI WorldCom	0.87	0.85	-0.18	6.88	-20%	13%
Medtronic	1.70	0.55	0.23	5.52	-12%	12%
Merck	1.32	0.35	0.18	5.29	-9%	10%
Pfi zer	1.57	0.46	0.01	4.28	-10%	10%
Procter&Gambel	0.90	0.41	-2.57	42.75	-31%	10%
SBC Communication	0.86	0.39	0.06	5.86	-13%	9%
Texas Instrument	2.20	1.23	0.50	5.26	-12%	24%
Wall Mart	1.35	0.52	0.16	4.79	-10%	9%

Table 5: This table presents the main statistical features of the daily returns of the set of seventeen assets studied here over the time interval from the end of January 1995 to the end of December 2000.

Asymptotic embedding

The Pareto distribution:

$$F_u(x) = 1 - (u/x)^b,$$

The Weibull distribution:

$$F_u(x) = 1 - \exp\left[-\left(\frac{x}{d}\right)^c + \left(\frac{u}{d}\right)^c\right]$$

x>u (lower threshold)

$$c \cdot \left(\frac{u}{d}\right)^c \to \beta, \quad \text{as } c \to 0$$

$$\begin{aligned}
\frac{c}{d^c} \cdot x^{c-1} \cdot \exp\left(-\frac{x^c - u^c}{d^c}\right) &= c\left(\frac{u}{d}\right)^c \cdot \frac{x^{c-1}}{u^c} \exp\left[-\left(\frac{u}{d}\right)^c \cdot \left(\left(\frac{x}{u}\right)^c - 1\right)\right] , \\
&\simeq \beta \cdot x^{-1} \exp\left[-c\left(\frac{u}{d}\right)^c \cdot \ln\frac{x}{u}\right] , \quad \text{as } c \to 0 \\
&\simeq \beta \cdot x^{-1} \exp\left[-\beta \cdot \ln\frac{x}{u}\right] , \\
&\simeq \beta \frac{u^\beta}{x^{\beta+1}} ,
\end{aligned}$$

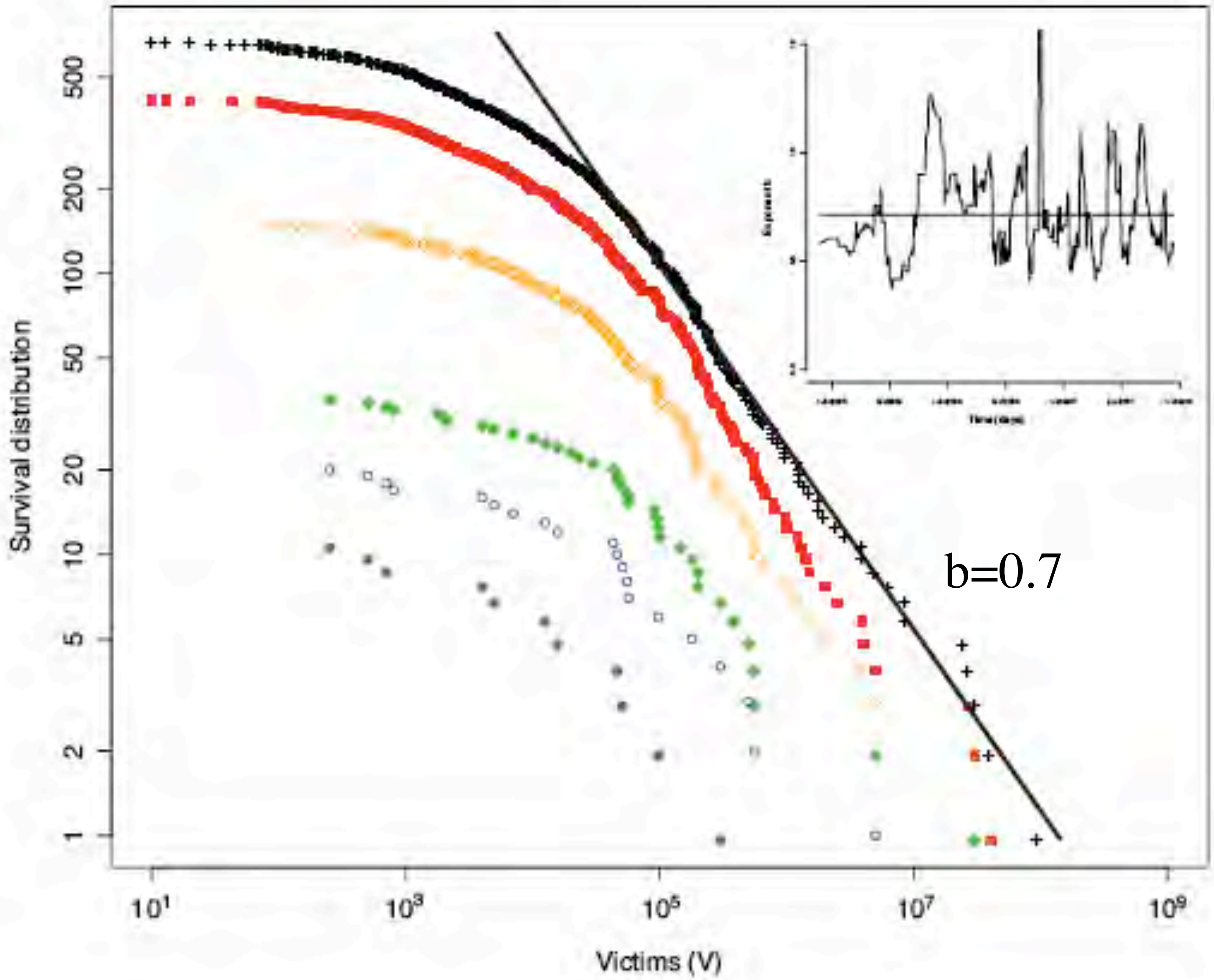
The statistic $W = 2\log \frac{\max_{b,c} \mathcal{L}_{SE}}{\max_b \mathcal{L}_{PD}}$ follows a

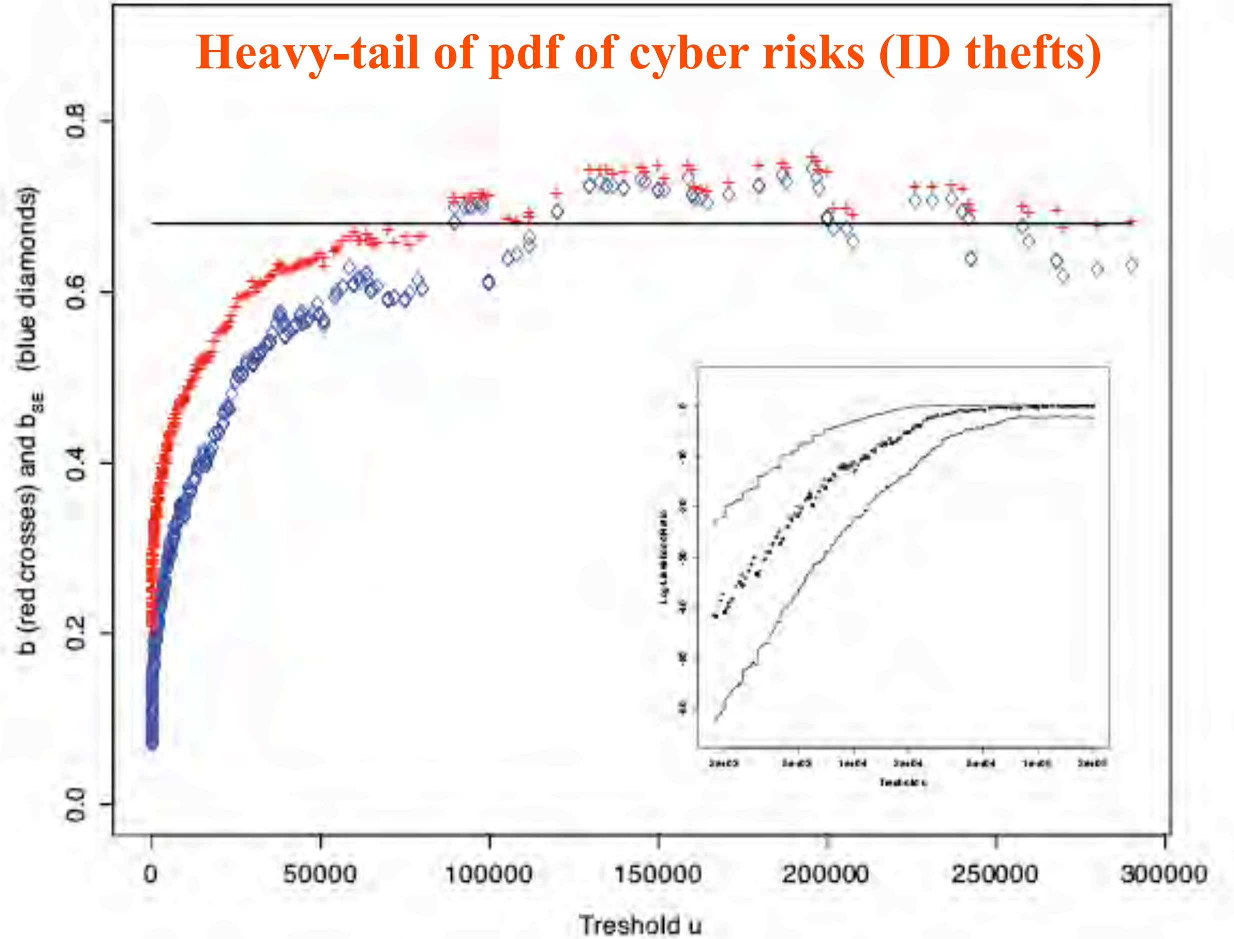
χ^2- distribution with one degree of freedom (Wilks' statistic holds due to the asymptotic embedding of power laws into stretched exponentials)

Sample	c	d	$c(u(q_{12})/d)^c$	b
DJ pos. returns	0.274 (0.111)	$4.81 \cdot 10^{-6}$	2.68	2.79 (0.10)
DJ neg. returns	0.362 (0.119)	$1.02 \cdot 10^{-4}$	2.57	2.77 (0.11)
ND pos. returns	0.039 (0.138)	$4.54 \cdot 10^{-52}$	3.03	3.23 (0.14)
ND neg. returns	0.273 (0.155)	$1.90 \cdot 10^{-7}$	3.10	3.35 (0.15)
SP pos. returns (1min)	-	-	3.01	3.02 (0.02)
SP neg returns (1min)	-	-	2.97	2.97 (0.02)
SP pos. returns (5min)	0.033 (0.031)	$3.06 \cdot 10^{-59}$	2.95	2.95 (0.03)
SP neg. returns (5min)	0.033 (0.031)	$3.26 \cdot 10^{-56}$	2.87	2.86 (0.03)

Best parameters c and d of the Stretched Exponential model and best parameter b of the Pareto model estimated up to quantile $q_{12} = 95\%$. The apparent Pareto exponent $c(u(q_{12})/d)^c$ (see expression (2.30)) is also shown

Heavy-tail of pdf of cyber risks ID Thefts





Dependence of the index b obtained directly from the MLE of the exponent of the power law (1) (crosses) and indirectly from the MLE of the parameters c, d of the stretched exponential (SE) law (3) using the correspondence $b_{\text{SE}} = c(u/d)^c$ (diamonds). The horizontal line is $b = 0.68$. The inset shows the logarithm of the likelihood ratio LLR of the power law versus the SE fits converging to 0 when u increases. The two grey lines show the 95% confidence interval obtained by bootstrap.

Multivariate representation of the joint distribution

Information Theory (Rao, 1973)

(Maximum entropy principle)

"Best" representation of the multivariable distribution:

$$\hat{P}(\boldsymbol{y}) = (2\pi)^{-N/2}|V|^{-1/2}\exp\left(-\tfrac{1}{2}\boldsymbol{y}'V^{-1}\boldsymbol{y}\right)$$

$$P(\boldsymbol{x}) = |V|^{-1/2}\exp\left(-\tfrac{1}{2}\boldsymbol{y}'(V^{-1}-I)\boldsymbol{y}\right)\prod_{j=1}^{N} P_j(x^{(j)})$$

V is again the covariance matrix for $\boldsymbol{y}$ and I is the identity matrix.

(amounts to using the Gaussian copula)

D. Sornette, J. V. Andersen and P. Simonetti, Portfolio Theory for Fat Tails, International Journal of Theoretical and Applied Finance 3 (3), 523-535 (2000)

D. Sornette, P. Simonetti and J. V. Andersen, phi^q field theory for Portfolio optimization: ``fat tails'' and non-linear correlations, Physics Report 335 (2), 19-92 (2000)

COPULAS

DEFINITION 1 (COPULA)

A function $C : [0,1]^n \longrightarrow [0,1]$ is a n-copula if it enjoys the following properties :

- $\forall u \in [0,1]$, $C(1,\cdots,1,u,1\cdots,1) = u$,
- $\forall u_i \in [0,1]$, $C(u_1,\cdots,u_n) = 0$ if at least one of the u_i equals zero ,
- C is grounded and n-increasing, i.e., the C-volume of every boxes whose vertices lie in $[0,1]^n$ is positive.

THEOREM 1 (SKLAR'S THEOREM)

Given an n-dimensional distribution function F with *continuous* marginal (cumulative) distributions $F_1, \cdots, F_n$, there exists a *unique* n-copula $C : [0,1]^n \longrightarrow [0,1]$ such that :

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)) \ . \tag{1}$$

$$C(u_1, \cdots, u_n) = F\left(F_1^{-1}(u_1), \cdots, F_n^{-1}(u_n)\right) \tag{2}$$

is automatically a copula

THEOREM 2 (INVARIANCE THEOREM)

Consider n *continuous* random variables $X_1, \cdots, X_n$ with copula C. Then, if $g_1(X_1), \cdots, g_n(X_n)$ are strictly increasing on the ranges of $X_1, \cdots, X_n$, the random variables $Y_1 = g_1(X_1), \cdots, Y_n = g_n(X_n)$ have exactly the same copula C.

The Gaussian copula

the Gaussian n-copula with correlation matrix ρ is

$$C_\rho(u_1, \cdots, u_n) = \Phi_{\rho,n}\left(\Phi^{-1}(u_1), \cdots, \Phi^{-1}(u_n)\right) \ , \tag{8}$$

whose density

$$c_\rho(u_1, \cdots, u_n) = \frac{\partial C_\rho(u_1, \cdots, u_n)}{\partial u_1 \cdots \partial u_n} \tag{9}$$

reads

$$c_\rho(u_1, \cdots, u_n) = \frac{1}{\sqrt{\det \rho}} \exp\left(-\frac{1}{2} y^t_{(u)} (\rho^{-1} - \mathrm{Id}) y_{(u)}\right) \tag{10}$$

with $y_k(u) = \Phi^{-1}(u_k)$. Note that theorem 1 and equation (2) ensure that $C_\rho(u_1, \cdots, u_n)$ in equation (8) is a copula.

Its main advantages

From a practical point of view, a copula must:

- Be easy to handle even in high dimension,
- Account for non-exchangeable risks,
- Involve only a few parameters,
- Allow for a robust estimation of the parameters

Its main advantages

From a theoretical point of view:

- Traditional financial theory relies on the Gaussian copula,
- Default modeling: KMV, CreditMetrics, Basle II,...
- Options on basket

Its drawbacks

Student copula

- Weak dependence in the tails:

$$\lambda_+ = \lim_{u \to 1^-} \Pr\left\{X > F_X^{-1}(u) \middle| Y > F_Y^{-1}(u)\right\}$$
$$= \lim_{u \to 1^-} \Pr\left\{X > VaR_u(X) \middle| Y > VaR_u(Y)\right\}$$

- For a Gaussian copula, λ=0.

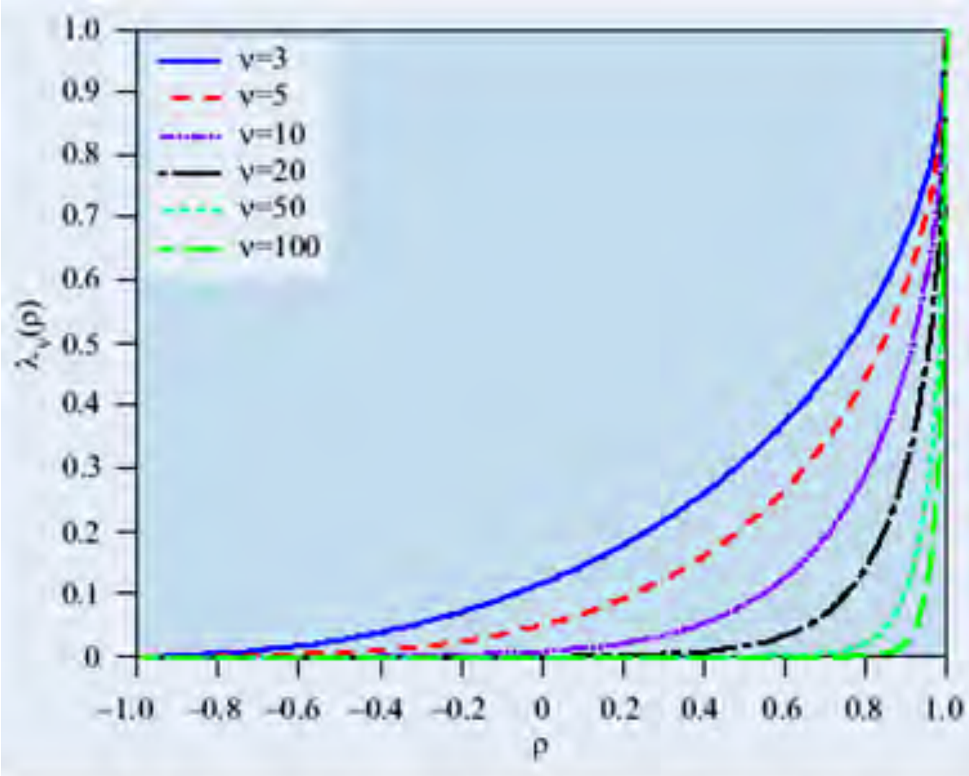


Figure 1. The upper tail dependence coefficient $\lambda_\nu(\rho)$ for the Student copula with ν degrees of freedom as a function of the correlation coefficient ρ, for different values of ν.

$$\lambda_\nu(\rho) = \lim_{u \to 1} \frac{\bar{C}_{\rho,\nu}(u,u)}{1-u} = 2\bar{t}_{\nu+1}\left(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) ,$$

where $\bar{t}_{\nu+1}$ is the complementary cumulative univariate Student's distribution with $\nu + 1$ degrees

The Student's copula

Student's distribution $T_{\rho,\nu}$ with ν degrees of freedom and a correlation matrix ρ

$$T_{\rho,\nu}(\mathbf{x}) = \frac{1}{\sqrt{\det \rho}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{N/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \frac{d\mathbf{x}}{\left(1+\frac{x^t \rho x}{\nu}\right)^{\frac{\nu+n}{2}}} , \tag{16}$$

the corresponding Student's copula reads :

$$C_{\rho,\nu}(u_1, \cdots, u_n) = T_{\rho,\nu}\left(t_\nu^{-1}(u_1), \cdots, t_\nu^{-1}(u_n)\right) , \tag{17}$$

where t_ν is the univariate Student's distribution with ν degrees of freedom. The density of the Student's copula is thus

$$c_{\rho,\nu}(u_1, \cdots, u_n) = \frac{1}{\sqrt{\det \rho}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{n-1}}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n} \frac{\prod_{k=1}^n \left(1+\frac{y_k^2}{\nu}\right)^{\frac{\nu+1}{2}}}{\left(1+\frac{y^t \rho y}{\nu}\right)^{\frac{\nu+n}{2}}} , \tag{18}$$

where $y_k = t_\nu^{-1}(u_k)$.

Its drawbacks

•For n=2-3, the dependence structure is correctly capture by the Gaussian copula

•For n>3, the Gaussian copula underestimates the true dependence

	$100 \times \Pr(R < -n \cdot \sigma)$							
	$n = 2$		$n = 3$		$n = 4$		$n = 5$	
	P_r	P_g	P_r	P_g	P_r	P_g	P_r	P_g
Abbott Labs	2.07	2.06	0.58	0.51	0.21	0.15	0.09	0.07
American Home Products Corp	1.98	2.07	0.51	0.56	0.30	0.24	0.17	0.13
Boeing Co	2.03	1.96	0.53	0.51	0.21	0.18	0.13	0.09
Bristol-Myers Squibb Co	1.56	1.81	0.55	0.48	0.26	0.22	0.11	0.10
Chevron Corp	1.94	1.99	0.40	0.42	0.13	0.15	0.08	0.07
Du Pont (E.I.) de Nemours & Co	2.13	2.02	0.51	0.47	0.21	0.19	0.09	0.07
Disney (Walt) Co	1.83	1.87	0.47	0.53	0.24	0.22	0.15	0.12
General Motors Corp	1.73	1.95	0.45	0.42	0.21	0.13	0.08	0.06
Hewlett-Packard Co	1.77	2.08	0.53	0.51	0.21	0.19	0.08	0.09
Coca-Cola Co	1.60	1.83	0.45	0.50	0.19	0.18	0.09	0.07
Minnesota Mining & MFG Co	1.85	2.01	0.57	0.49	0.19	0.19	0.08	0.09
Philip Morris Cos Inc	2.00	2.07	0.45	0.50	0.21	0.19	0.13	0.12
Pepsico Inc	1.92	2.08	0.51	0.49	0.15	0.18	0.15	0.07
Procter & Gamble Co	1.51	1.67	0.45	0.48	0.24	0.21	0.13	0.09
Pharmacia Corp	1.81	1.94	0.53	0.54	0.23	0.25	0.11	0.12
Schering-Plough Corp	1.85	1.94	0.49	0.44	0.11	0.14	0.08	0.06
Texaco Inc	1.90	1.94	0.55	0.55	0.28	0.23	0.11	0.11
Texas Instruments Inc	1.87	2.02	0.49	0.50	0.21	0.15	0.06	0.07
United Technologies Corp	2.17	2.10	0.47	0.45	0.17	0.14	0.11	0.06
Walgreen Co.	1.81	1.96	0.47	0.41	0.23	0.14	0.09	0.08
Mean ratio P_r/P_g		0.95		1.02		1.15		1.24

A portfolio made of 50% of S&P500 and 50% of one stock (whose name is indicated in the first column) is considered. We estimate the probability P_r that this portfolio incurs a loss, R, larger than n times its standard deviation ($n = 2, \ldots, 5$). For the same portfolio, we estimate the probability P_g that it incurs the same loss (ie, n times its standard deviation) when the dependence between the index and the stock is given by a Gaussian copula. The mean ratio P_r/P_g gives the average value of P_r/P_g over the 20 portfolios

Test statistics

H_0: The dependence between N random variables X_1, ...,X_N can be described by the Gaussian copula.

Proposition: Assuming that the N-dimensional random vector $X=(X_1,...,X_N)$, with marginal distribution F_i, satisfies H_0 then, the variable:

$$z^2 = \sum_{j,i=1}^{N} \Phi^{-1}(F_i(x_i))(\rho^{-1})_{ij}\Phi^{-1}(F_j(x_j)),$$

where the matrix r is given by

$$\rho_{ij} = \mathrm{cov}[\Phi^{-1}(F_i(x_i)), \Phi^{-1}(F_j(x_j))],$$

follows a χ^2-distribution with N degrees of freedom.

Y. Malevergne and D. Sornette, Testing the Gaussian copula hypothesis for financial assets dependences, Quantitative Finance, 3, 231–250 (2003)

Testing procedure (1)

We consider two financial time series ($N=2$) of size T: $\{x_1(1),\ldots,x_1(t),\ldots,x_1(T)\}$ and $\{x_2(1),\ldots,x_2(t),\ldots,x_2(T)\}$.

The cumulative distribution of each variable x_i, which is estimated empirically, is given by:

$$\hat{F}_i(x_i) = \frac{1}{T}\sum_{k=1}^{T} \mathbf{1}_{\{x_i(k)\leqslant x_i\}},$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, which equals one if its argument is true and zero otherwise. We use these estimated cumulative distributions to obtain the Gaussian variables $\hat{y}_i$ as

$$\hat{y}_i(k) = \Phi^{-1}(\hat{F}_i(x_i(k))), \qquad k \in \{1, \ldots, T\}.$$

Testing procedure (2)

The sample covariance matrix $\hat{\rho}$ is estimated by the expression

$$\hat{\rho} = \frac{1}{T}\sum_{i=1}^{T} \hat{\boldsymbol{y}}(i) \cdot \hat{\boldsymbol{y}}(i)^{\mathrm{t}}$$

which allows us to calculate the variable

$$\hat{z}^2(k) = \sum_{i,j=1}^{2} \hat{y}_i(k)(\hat{\rho}^{-1})_{ij}\hat{y}_j(k),$$

Testing procedure (3)

Comparison of the distribution of $\hat{z}^2$ with the χ^2-distribution:

Kolmogorov: $d_1 = \max_z |F_{z^2}(z^2) - F_{\chi^2}(z^2)|$;

average Kolmogorov:

$$d_2 = \int |F_{z^2}(z^2) - F_{\chi^2}(z^2)| \, \mathrm{d}F_{\chi^2}(z^2);$$

Anderson–Darling: $d_3 = \max_z \dfrac{|F_{z^2}(z^2) - F_{\chi^2}(z^2)|}{\sqrt{F_{\chi^2}(z^2)[1 - F_{\chi^2}(z^2)]}}$;

average Anderson–Darling:

$$d_4 = \int \frac{|F_{z^2}(z^2) - F_{\chi^2}(z^2)|}{\sqrt{F_{\chi^2}(z^2)[1 - F_{\chi^2}(z^2)]}} \, \mathrm{d}F_{\chi^2}(z^2).$$

Power of the test

Can we distinguish between a Gaussian copula and a Student copula ?

The values $p_{95\%}(\nu,\rho)$ shown in the table give the minimum values that the significance p should take in order to be able to reject the hypothesis that a Student's copula with ν degrees and correlation ρ is undistinguishable from a Gaussian copula at the 95% confidence level. p is the probability that pairs of Gaussian random variables with the correlation coefficient ρ have a distance (between the distribution of z^2 and the theoretical χ^2 distribution) equal to or larger than the corresponding distance obtained for the Student's vector time series. A small p corresponds to a clear distinction between Student's and Gaussian vectors, as it is improbable that Gaussian vectors exhibit a distance larger than found for the Student's

ν	ρ	0.1	0.3	0.5	0.7	0.9
3	d_1	0.07	0.08	0.07	0.04	0.07
	d_2	0.03	0.03	0.07	0.04	0.06
	d_3	0.22	0.17	0.08	0.03	0.01
	d_4	0.03	0.03	0.08	0.03	0.04
4	d_1	0.28	0.26	0.32	0.30	0.29
	d_2	0.18	0.17	0.21	0.21	0.24
	d_3	0.36	0.33	0.26	0.15	0.03
	d_4	0.18	0.17	0.23	0.21	0.21
5	d_1	0.46	0.47	0.46	0.52	0.52
	d_2	0.36	0.34	0.39	0.44	0.43
	d_3	0.52	0.54	0.47	0.30	0.14
	d_4	0.37	0.36	0.43	0.45	0.45
7	d_1	0.78	0.81	0.81	0.81	0.86
	d_2	0.71	0.78	0.76	0.77	0.82
	d_3	0.80	0.81	0.82	0.73	0.52
	d_4	0.75	0.81	0.79	0.80	0.83
8	d_1	0.85	0.86	0.87	0.88	0.89
	d_2	0.85	0.84	0.86	0.87	0.88
	d_3	0.91	0.91	0.91	0.81	0.70
	d_4	0.86	0.85	0.90	0.89	0.90
10	d_1	0.92	0.93	0.96	0.95	0.94
	d_2	0.93	0.92	0.95	0.96	0.94
	d_3	0.96	0.96	0.96	0.95	0.88
	d_4	0.94	0.94	0.96	0.97	0.95
20	d_1	0.97	0.99	0.97	0.99	0.99
	d_2	0.99	0.99	0.97	0.99	0.99
	d_3	0.99	0.99	0.98	0.99	0.97
	d_4	0.99	0.99	0.98	0.99	0.99
50	d_1	0.99	0.99	0.99	0.99	0.99
	d_2	0.99	0.99	0.99	0.99	0.99
	d_3	0.99	0.99	0.99	0.99	0.99
	d_4	0.99	0.99	0.99	0.99	0.99

Results

- Currencies (1989-1998)
  - Swiss Franc, German Mark, Japanese Yen, Malaysian Ringgit, Thai Bath, British Pound.
- Commodities (1989-1997)
  - aluminum, copper, lead, nickel, tin, zinc.
- Stocks (1991-2000)
  - Appl. Materials, AT&T, Citigroup, Coca Cola, EMC, Exxon-Mobil, Ford, General Electric, General Motors, Hewlett-Packard, IBM, Intel, MCI WorldCom, Medtronic, Merck, Microsoft, Pfizer, Procter&Gamble, SBC Communication, Sun Microsystem, Texas Instruments, Wal-Mart.

Results: commodities

- The Gaussian copula is strongly rejected

Results: currencies

Fraction of pairs compatible with the Gaussian copula hypothesis

- 40% of the pairs of currencies compatible, over a ten-year time interval (due to non-stationary data),
- 67% of the pairs of currencies compatible, over the first five-year time interval,
- 73% of the pairs of currencies compatible, over the second five-year time interval.

However: p-values are about 30-40%:

Student copula with 5 to 7 degrees of freedom cannot be rejected.

In line with *Breymann et al.(2003)* : Student copula with six degrees of freedom of German Mark/Japanese Yen

Results: stocks

- 75% of the pairs of stocks compatible, over a ten-year time interval,
- 93% of the pairs of stocks compatible, over the first five-year time interval
- 92% of the pairs of stocks compatible, over the second five-year time

Mashal & Zeevi (2002) have found that the Student's copula with 11-12 degrees of freedom provides a better description

Conditional measures of dependence (I)

- Conditional correlation coefficient on one variable

$$\rho_{\mathcal{A}} = \frac{\mathrm{Cov}(X, Y \mid Y \in \mathcal{A})}{\sqrt{\mathrm{Var}(X \mid Y \in \mathcal{A}) \cdot \mathrm{Var}(Y \mid Y \in \mathcal{A})}}$$

For bivariate Gaussian rv: $\rho_{\mathcal{A}} = \dfrac{\rho}{\sqrt{\rho^2 + (1-\rho^2)\dfrac{\mathrm{Var}(Y)}{\mathrm{Var}(Y \mid Y \in \mathcal{A})}}}$

- o $\mathcal{A} = [v, +\infty)$, with $v \in \mathbb{R}_+$ => ρ_v^+
- o $\mathcal{A} = (-\infty, -v] \cup [v, +\infty)$, with $v \in \mathbb{R}_+$ => ρ_v^s

Conditional measures of dependence (II)

- Conditional correlation coefficient on two variables

$$\rho_{\mathcal{A},\mathcal{B}} = \frac{\mathrm{Cov}(X, Y \mid X \in \mathcal{A}, Y \in \mathcal{B})}{\sqrt{\mathrm{Var}(X \mid X \in \mathcal{A}, Y \in \mathcal{B}) \cdot \mathrm{Var}(Y \mid X \in \mathcal{A}, Y \in \mathcal{B})}}$$

subsets $\mathcal{A}$ and $\mathcal{B}$ are both chosen equal to $[u, +\infty)$ $u \in \mathbb{R}_+$

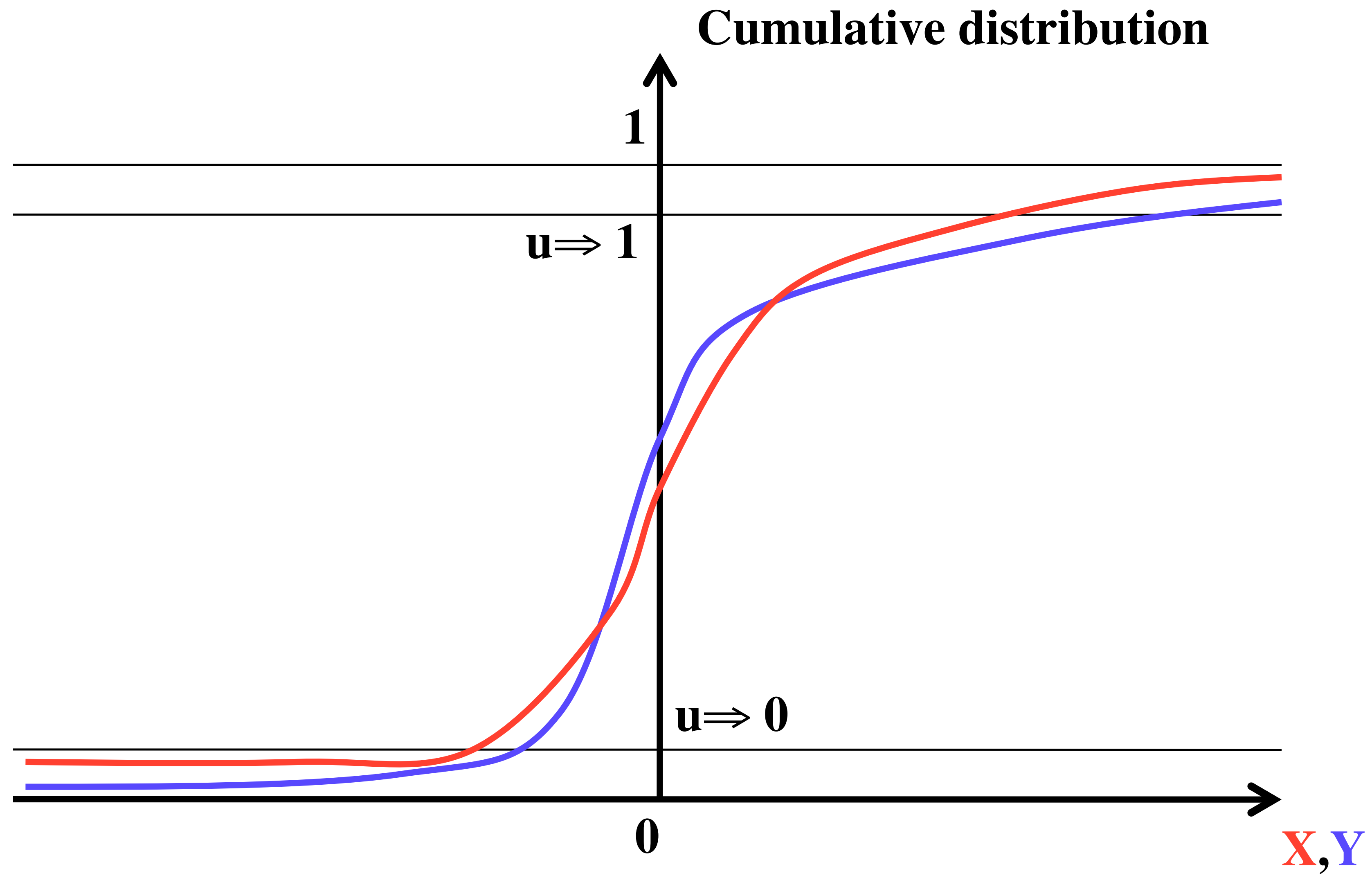
=> ρ_u

Extreme dependence

$$\lambda_{ij}^{+} = \lim_{u \to 1} \Pr\left\{ X_i > F_i^{-1}(u) \middle| X_j > F_j^{-1}(u) \right\}$$

$$\lim_{u \to 1} \frac{\bar{C}(u,u)}{1-u} = \lambda \qquad \bar{C}(u,u) = 1 - 2u - C(u,u)$$

$$\begin{aligned} \bar{\lambda} &= \lim_{u \to 1} \frac{2 \log \Pr\{X > F_X{}^{-1}(u)\}}{\log \Pr\{X > F_X{}^{-1}(u), Y > F_Y{}^{-1}(u)\}} - 1 \\ &= \lim_{u \to 1} \frac{2 \log(1-u)}{\log[1 - 2u + C(u,u)]} - 1 \,. \end{aligned}$$


Cumulative distribution
1
u⇒ 1
u⇒ 0
0
X,Y

Investigating Extreme Dependences: Conditioning Effect Versus Contagion in Latin-American Crises

Y. Malevergne and D. Sornette

Minimizing extremes, *Risk* **15**(November), 129–132 (2002)

How to account for extreme co-movements between individual stocks and the market, *The Journal of Risk* **6**(3), 71–116 (2004)

	ρ_v^+	ρ_v^s	ρ_u
Bivariate Gaussian	$\frac{\rho}{\sqrt{1-\rho^2}} \cdot \frac{1}{v}$ (3)	$1 - \frac{1}{2}\, \frac{1-\rho^2}{\rho^2}\, \frac{1}{v^2}$ (4)	$\rho \frac{1+\rho}{1-\rho} \cdot \frac{1}{u^2}$ (13)
Bivariate Student's	$\frac{\rho}{\sqrt{\rho^2+(\nu-1)\sqrt{\frac{\nu-2}{\nu}}\,(1-\rho^2)}}$ (6)	$\frac{\rho}{\sqrt{\rho^2+\frac{1}{(\nu-1)}\sqrt{\frac{\nu-2}{\nu}}\,(1-\rho^2)}}$ (7)	-
Gaussian Factor Model	same as (3)	same as (4)	same as (13)
Student's Factor Model	$1 - \frac{K}{v^2}$ (11)	$1 - \frac{K}{v^2}$ (11)	-

Table 3: Large v and u dependence of the conditional correlations ρ_v^+ (signed condition), ρ_v^s (unsigned condition) and ρ_u (on both variables) for the different models studied in the present paper, described in the first column. The numbers in parentheses give the equation numbers from which the formulas are derived. The factor model is defined by (8), i.e., $X = \alpha Y + \epsilon$. ρ is the unconditional correlation coefficient.

	$\rho^+_{v=\infty}$	$\rho^s_{v=\infty}$	$\rho_{u=\infty}$	λ	$\bar{\lambda}$
Bivariate Gaussian	0	1	0	0	ρ
Bivariate Student's	see Table 3	see Table 3	-	$2 \cdot T_{\nu+1}\left(\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$	1
Gaussian Factor Model	0	1	0	0	ρ
Student's Factor Model	1	1	-	$\frac{\rho^\nu}{\rho^\nu+(1-\rho^2)^{\nu/2}}$	1

Table 4: Asymptotic values of ρ^+_v, ρ^s_v and ρ_u for $v \rightarrow +\infty$ and $u \rightarrow \infty$ and comparison with the tail-dependence λ and $\bar{\lambda}$ for the four models indicated in the first column. The factor model is defined by (8), i.e., $X = \alpha Y + \epsilon$. ρ is the unconditional correlation coefficient. For the Student's factor model, Y and ϵ have centered Student's distributions with the same number ν of degrees of freedom and their scale factors are respectively equal to 1 and σ, so that $\rho = (1 + \frac{\sigma^2}{\alpha^2})^{-1/2}$. For the Bivariate Student's distribution, we refer to Table 1 for the constant values of $\rho^+_{v=\infty}$ and $\rho^s_{v=\infty}$.

- first, the conditional correlation coefficients put much less weight on the extreme tails that the tail-dependence parameter λ. In other words, $\rho^+_{\upsilon=\infty}$ and $\rho^s_{\upsilon=\infty}$ are sensitive to the marginals, i.e., there are determined by the full bivariate distribution, while, as we said, λ is a pure copula property independent of the marginals. Since $\rho^+_{\upsilon=\infty}$ and $\rho^s_{\upsilon=\infty}$ are measures of tail dependence weighted by the specific shapes of the marginals, it is natural that they may behave differently.

- Secondly, the tail dependence λ probes the extreme dependence property of the original copula of the random variables X and Y. On the contrary, when conditioning on Y, one changes the copula of X and Y, so that the extreme dependence properties investigated by the conditional correlations are not exactly those of the original copula. This last remark explains clearly why we observe what [Boyer et al. (1997)] call a "bias" in the conditional correlations. Indeed, changing the dependence between two random variables obviously leads to changing their correlations.

coefficient of lower tail dependence between the two assets X_i and X_j:

$$\lambda_{ij}^{-} = \lim_{u \to 0} \Pr\left\{X_i < F_i^{-1}(u) \middle| X_j < F_j^{-1}(u)\right\}$$

$$\lambda_{+} = \lim_{u \to 1^{-}} \frac{1 - 2u + C(u,u)}{1 - u}$$

$$F(x,y) = C(F_X(x), F_Y(y))$$

Definition of copula

coefficient of upper tail dependence

$$\lambda_{ij}^{+} = \lim_{u \to 1} \Pr\left\{X_i > F_i^{-1}(u) \middle| X_j > F_j^{-1}(u)\right\}$$

Hult & Lindskog (2002)

Multiplicative model:

$$\mathbf{X} = \sigma \mathbf{Y}$$

$$\lambda_{ij}^{\pm} = \frac{\int_{(\pi/2 - \arcsin \rho_{ij})/2}^{\pi/2} dt \cos^{\nu} t}{\int_0^{\pi/2} dt \cos^{\nu} t} = 2 I_{\frac{1+\rho_{ij}}{2}}\left(\frac{\nu+1}{2}, \frac{1}{2}\right)$$

where the function:

$$I_x(z,w) = \frac{1}{B(z,w)} \int_0^x dt\ t^{z-1} (1-t)^{w-1}$$

for regularly varying multivariate elliptic distribution

Extreme dependence

- Currencies:
  - ρ =0.7 - 0.8
  - Student copula, ν =5 - 7
  - λ = 30%
- Stocks:
  - ρ =0.4
  - Student copula, ν =11 - 12
  - λ = 2.5%

the factor model reads:

$$\mathbf{X} = \beta Y + \varepsilon$$

Theorem:

$$\lambda_i^+ = \frac{1}{\max\left\{1, \frac{l}{\beta_i}\right\}^{\nu}}$$

with $$l = \lim_{u \to 1} \frac{F_X^{-1}(u)}{F_Y^{-1}(u)}$$

$$\lambda_i = 0 \quad \text{if} \quad \nu_Y > \nu_{\varepsilon_i}$$

$$\lambda_i = \frac{1}{1 + \left(\frac{\sigma_{\varepsilon_i}}{\beta_i \sigma_Y}\right)^{\nu}} \quad \text{if} \quad \nu_Y = \nu_{\varepsilon_i} = \nu$$

$$\lambda_i = 1 \quad \text{if} \quad \nu_Y < \nu_{\varepsilon_i}$$

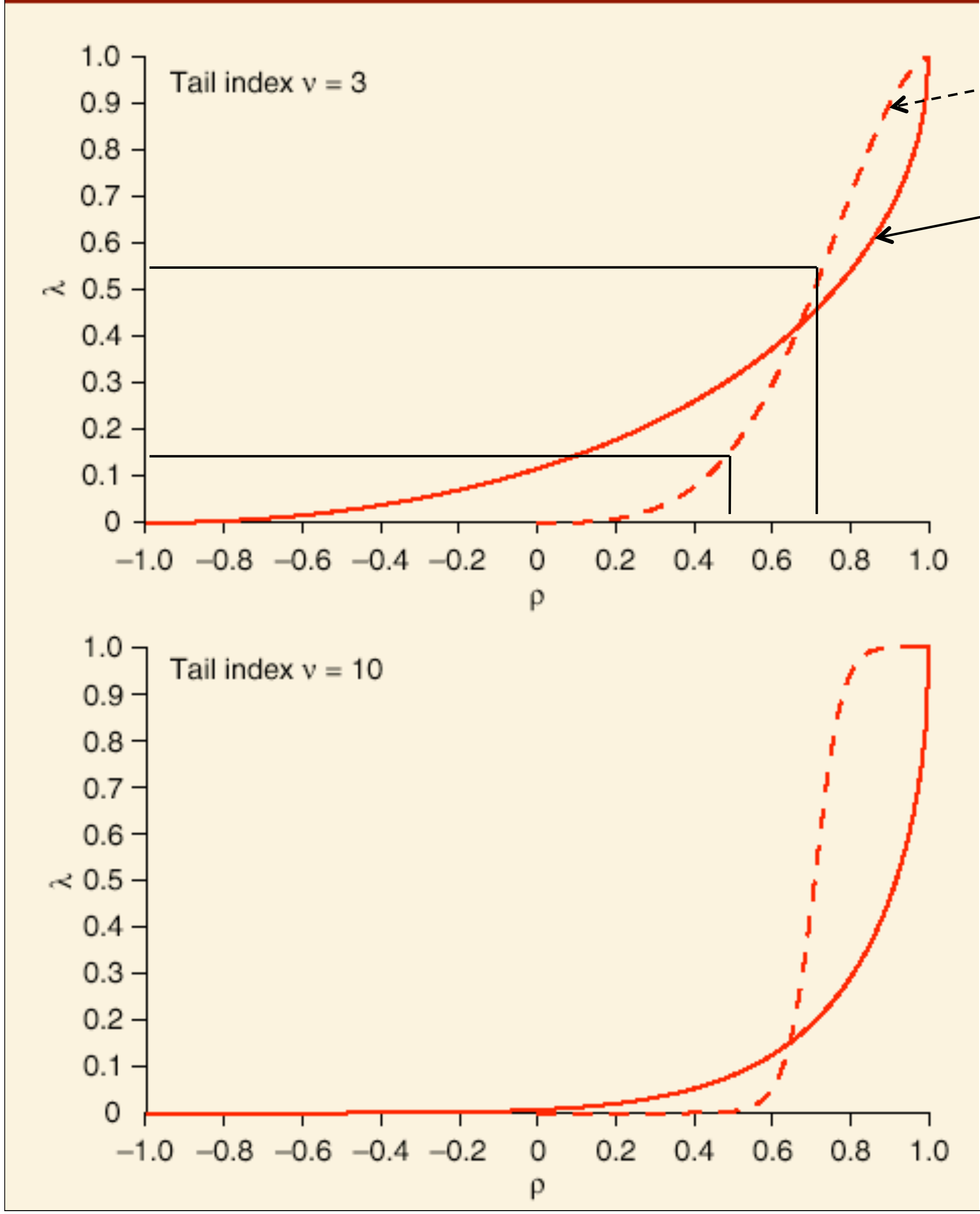
for Student distributions with scale factors σ

between two assets

$$\lambda_{ij} = \min\left\{\lambda_i, \lambda_j\right\}$$

Absence of tail dependence for rapidly varying factors

1. Tail dependence versus correlation



Student factor

Elliptic bi-pdf

Evolution as a function of the correlation coefficient ρ of the coefficient of tail dependence λ for an elliptical bivariate student distribution (solid line) and for the additive factor model with Student factor and noise (dashed line)

$$\mathbf{X} = \beta Y + \varepsilon$$

Given a sample of N realisations $\{X_1, X_2, \ldots, X_N\}$ and $\{Y_1, Y_2, \ldots, Y_N\}$ of X and Y, we first estimate the coefficient β using the ordinary least square estimator. Let $\hat{\beta}$ denote its estimate. Then, using Hill's estimator, we obtain the tail index $\hat{\nu}$ of the factor Y:

$$\hat{\nu}_k = \left[\frac{1}{k} \sum_{j=1}^{k} \log Y_{j,N} - \log Y_{k,N} \right]$$

where $Y_{1,N} \geq Y_{2,N} \geq \ldots \geq Y_{N,N}$ are the order statistics of the N realisations of Y. The constant l is non-parametrically estimated with the formula:

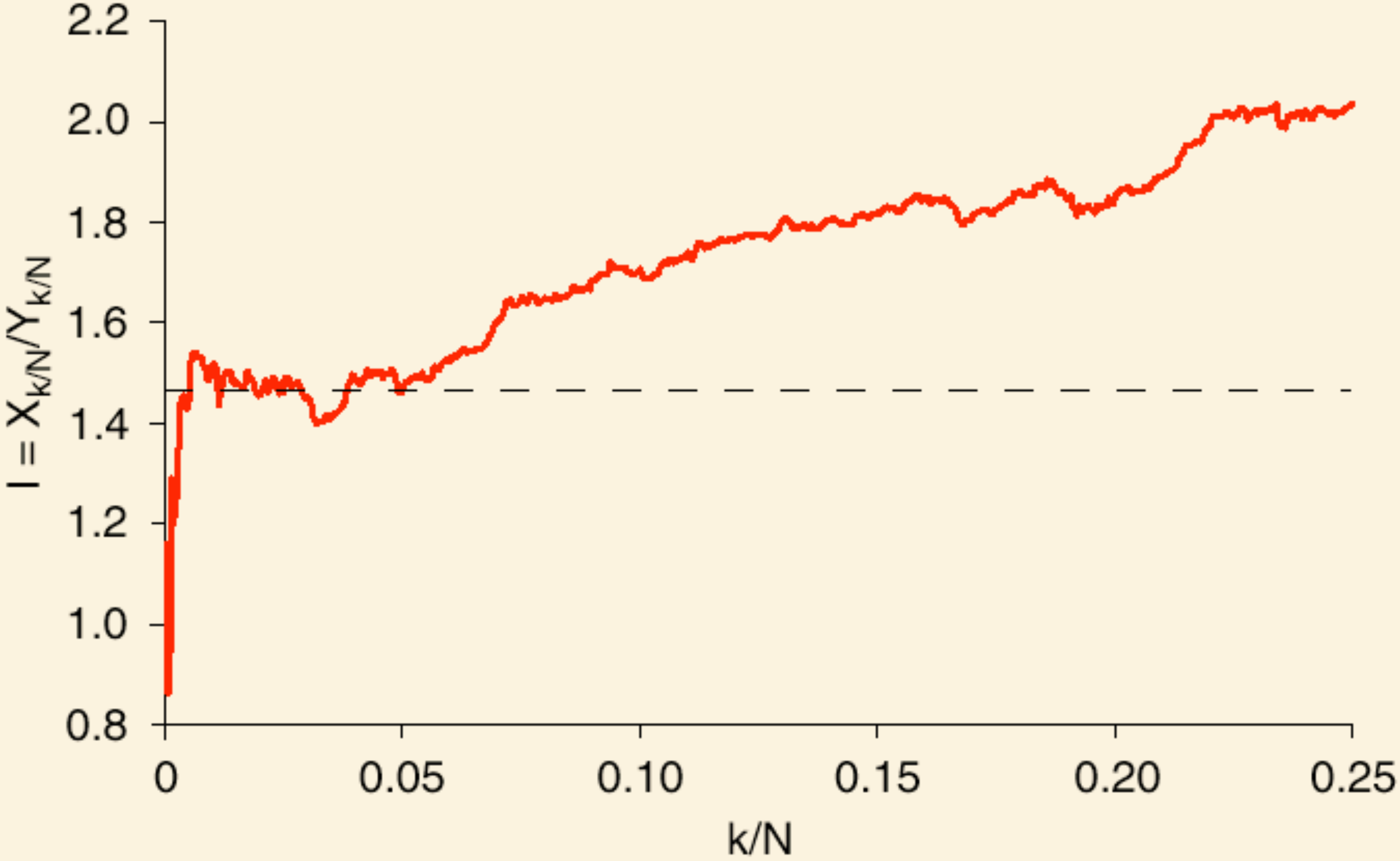
$$l = \lim_{u \to 1} \frac{F_X^{-1}(u)}{F_Y^{-1}(u)} \simeq \frac{X_{k,N}}{Y_{k,N}}$$

for $k = o(N)$, which means that k must remain very small with respect to N but large enough to ensure an accurate determination of l. Figure 2 presents $\hat{l}$ as a function of k/N.

Finally, using equation (7), the estimated $\hat{\lambda}$ is:

$$\hat{\lambda}^+ = \frac{1}{\max\left\{1, \frac{\hat{l}}{\hat{\beta}}\right\}^{\hat{\nu}}}$$

2. Quantile ratio



Empirical estimate $\hat{l}$ of the quantile ratio l in (7) versus the empirical quantile k/N. We observe a very good stability of $\hat{l}$ for quantiles ranging between 0.005 and 0.05

	July 1962 - December 1979				January 1980 - December 2000				July 1962 - December 2000			
	Mean	Std.	Skew.	Kurt.	Mean	Std.	Skew.	Kurt.	Mean	Std.	Skew.	Kurt.
Abbott Labs	0.6677	0.0154	0.2235	2.192	0.9217	0.0174	-0.0434	2.248	0.8066	0.0165	0.0570	2.300
American Home Products Corp.	0.4755	0.0136	0.2985	3.632	0.8486	0.0166	0.1007	8.519	0.6803	0.0154	0.1717	7.557
Boeing Co.	0.8460	0.0228	0.6753	4.629	0.7752	0.0193	0.1311	4.785	0.8068	0.0209	0.4495	4.901
Bristol-Myers Squibb Co.	0.5342	0.0152	-0.0811	2.808	0.9353	0.0175	-0.3437	16.733	0.7546	0.0165	-0.2485	12.573
Chevron Corp.	0.4916	0.0134	0.2144	2.442	0.6693	0.0169	0.0491	4.355	0.5885	0.0154	0.1033	4.209
Du Pont (E.I.) de Nemours & Co.	0.2193	0.0126	0.3493	2.754	0.6792	0.0172	-0.1021	4.731	0.4715	0.0153	0.0231	4.937
Disney (Walt) Co.	0.9272	0.0215	0.2420	2.762	0.8759	0.0195	-0.6661	17.655	0.8997	0.0204	-0.1881	9.568
General Motors Corp.	0.3547	0.0126	0.4138	4.302	0.5338	0.0183	-0.0128	5.373	0.4538	0.0160	0.0872	6.164
Hewlett-Packard Co.	0.7823	0.0199	0.0212	3.063	0.8913	0.0238	0.0254	4.921	0.8420	0.0221	0.0256	4.624
Coca-Cola Co.	0.4829	0.0138	0.0342	5.436	0.9674	0.0170	-0.1012	14.377	0.7483	0.0157	-0.0513	12.611
Minnesota Mining & MFG Co.	0.3459	0.0139	0.3016	2.997	0.6885	0.0150	-0.7861	20.609	0.5333	0.0145	-0.3550	14.066
Philip Morris Cos Inc.	0.7930	0.0153	0.2751	2.799	0.9664	0.0180	-0.2602	10.954	0.8863	0.0169	-0.0784	8.790
Pepsico Inc.	0.4982	0.0147	0.2380	2.867	0.9443	0.0180	0.1372	4.594	0.7431	0.0166	0.1786	4.413
Procter & Gamble Co.	0.3569	0.0115	0.3911	4.343	0.7916	0.0164	-1.6610	46.916	0.5947	0.0144	-1.2408	44.363
Pharmacia Corp.	0.3801	0.0145	0.2699	3.508	0.9027	0.0191	-0.6133	13.587	0.6666	0.0172	-0.3773	12.378
Schering-Plough Corp.	0.6328	0.0163	0.2619	3.112	1.0663	0.0192	0.1781	7.9979	0.8703	0.0179	0.2139	6.757
Texaco Inc.	0.3416	0.0134	0.2656	2.596	0.6644	0.0166	0.1192	6.477	0.5197	0.0152	0.1725	5.829
Texas Instruments Inc.	0.6839	0.0198	0.2076	3.174	1.0299	0.0268	0.1595	7.848	0.8726	0.0239	0.1831	7.737
United Technologies Corp	0.5801	0.0185	0.3397	2.826	0.7752	0.0170	0.0396	3.190	0.6876	0.0177	0.1933	3.034
Walgreen Co.	0.5851	0.0165	0.3530	3.030	1.1996	0.0185	0.1412	3.316	0.9217	0.0176	0.2260	3.295
Standart & Poor's 500	0.1783	0.0075	0.2554	3.131	0.5237	0.0101	-1.6974	36.657	0.3674	0.0090	-1.2236	32.406

Table 1: This table gives the main statistical features of the three samples we have considered. The columns *Mean, Std., Skew.* and *Kurt.* respectively give the average return multiplied by one thousand, the standard deviation, the skewness and the excess kurtosis of each asset over the time intervals form July 1962 to December 1979, January 1980 to Decemeber 2000 and July 1962 to December 2000. The excess kurtosis is given as indicative of the relative weight of large return amplitudes, and can always be calculated over a fi nite time series even if it may not be asymptotically defi ned for power tails with exponents less than 4.

	Negative Tail			Positive Tail		
	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
Abbott Labs	0.12	0.09	0.06	0.11	0.08	0.06
American Home Products Corp.	0.22	0.18	0.15	0.25	0.22	0.19
Boeing Co.	0.16	0.13	0.10	0.13	0.10	0.07
Bristol-Myers Squibb Co.	0.22	0.19	0.16	0.28	0.25	0.23
Chevron Corp.	0.21	0.17	0.14	0.26	0.23	0.20
Du Pont (E.I.) de Nemours & Co.	0.38	0.37	0.35	0.37	0.35	0.33
Disney (Walt) Co.	0.24	0.20	0.17	0.23	0.19	0.16
General Motors Corp.	0.39	0.37	0.35	0.48	0.47	0.47
Hewlett-Packard Co.	0.15	0.12	0.09	0.23	0.20	0.17
Coca-Cola Co.	0.26	0.22	0.19	0.26	0.23	0.20
Minnesota Mining & MFG Co.	0.35	0.32	0.30	0.35	0.33	0.31
Philip Morris Cos Inc.	0.25	0.22	0.19	0.20	0.17	0.14
Pepsico Inc.	0.15	0.12	0.09	0.17	0.14	0.11
Procter & Gamble Co.	0.23	0.19	0.16	0.24	0.21	0.18
Pharmacia Corp.	0.23	0.19	0.16	0.26	0.23	0.20
Schering-Plough Corp.	0.21	0.18	0.15	0.20	0.17	0.14
Texaco Inc.	0.47	0.46	0.46	0.49	0.49	0.49
Texas Instruments Inc.	0.06	0.04	0.03	0.07	0.05	0.03
United Technologies Corp	0.13	0.10	0.07	0.13	0.10	0.07
Walgreen Co.	0.03	0.02	0.01	0.02	0.01	0.01

Table 7: This table summarizes the mean values over the fi rst centile of the distribution of the coeffi cients of (upper or lower) tail dependence for the positive and negative tails during the time interval from July 1962 to December 1979, for three values of the tail index $\alpha = 3, 3.5, 4$.

	Negative Tail			Positive Tail		
	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
Abbott Labs	0.20	0.17	0.14	0.16	0.13	0.10
American Home Products Corp.	0.12	0.09	0.06	0.10	0.08	0.05
Boeing Co.	0.14	0.11	0.08	0.10	0.07	0.05
Bristol-Myers Squibb Co.	0.32	0.29	0.26	0.25	0.21	0.19
Chevron Corp.	0.18	0.14	0.11	0.13	0.09	0.07
Du Pont (E.I.) de Nemours & Co.	0.23	0.20	0.17	0.16	0.13	0.10
Disney (Walt) Co.	0.16	0.13	0.10	0.15	0.12	0.09
General Motors Corp.	0.26	0.22	0.19	0.20	0.16	0.13
Hewlett-Packard Co.	0.19	0.15	0.13	0.21	0.18	0.15
Coca-Cola Co.	0.24	0.20	0.18	0.20	0.17	0.14
Minnesota Mining & MFG Co.	0.26	0.23	0.20	0.20	0.17	0.14
Philip Morris Cos Inc.	0.11	0.08	0.06	0.11	0.08	0.06
Pepsico Inc.	0.17	0.14	0.11	0.14	0.11	0.09
Procter & Gamble Co.	0.24	0.21	0.18	0.20	0.16	0.13
Pharmacia Corp.	0.10	0.08	0.05	0.10	0.07	0.05
Schering-Plough Corp.	0.23	0.20	0.17	0.16	0.13	0.10
Texaco Inc.	0.43	0.42	0.41	0.31	0.28	0.26
Texas Instruments Inc.	0.02	0.01	0.01	0.02	0.01	0.01
United Technologies Corp	0.20	0.16	0.14	0.18	0.14	0.11
Walgreen Co.	0.15	0.12	0.09	0.09	0.07	0.05

Table 8: This table summarizes the mean values over the first centile of the distribution of the coefficients of (upper or lower) tail dependence for the positive and negative tails during the time interval from January 1980 to December 2000, for three values of the tail index $\alpha = 3, 3.5, 4$.

	July 1962 - Dec. 1979			Jan.1980 - Dec. 2000		
	Extremes	λ_-	p-value	Extremes	λ_-	p-value
Abbott Labs	0	0.12	0.2937	4	0.20	0.0904
American Home Products Corp.	1	0.22	0.2432	2	0.12	0.2247
Boeing Co.	0	0.16	0.1667	3	0.14	0.1176
Bristol-Myers Squibb Co.	2	0.22	0.2987	4	0.32	0.2144
Chevron Corp.	3	0.21	0.2112	4	0.18	0.0644
Du Pont (E.I.) de Nemours & Co.	0	0.38	0.0078	4	0.23	0.1224
Disney (Walt) Co.	2	0.24	0.2901	2	0.16	0.2873
General Motors Corp.	2	0.39	0.1345	4	0.26	0.1522
Hewlett-Packard Co.	0	0.15	0.1909	2	0.19	0.3007
Coca-Cola Co.	2	0.26	0.2765	5	0.24	0.0494
Minnesota Mining & MFG Co.	2	0.35	0.1784	4	0.26	0.1571
Philip Morris Cos Inc.	1	0.25	0.1841	2	0.11	0.2142
Pepsico Inc.	2	0.15	0.2795	5	0.17	0.0141
Procter & Gamble Co.	1	0.23	0.2245	3	0.24	0.2447
Pharmacia Corp.	2	0.23	0.2956	4	0.10	0.0128
Schering-Plough Corp.	0	0.21	0.0946	4	0.23	0.1224
Texaco Inc.	1	0.47	0.0161	3	0.43	0.1862
Texas Instruments Inc.	0	0.06	0.5222	2	0.02	0.0212
United Technologies Corp	1	0.13	0.3728	4	0.20	0.0870
Walgreen Co.	1	0.03	0.2303	3	0.15	0.1373

Table 10: This table gives, for the time intervals from July 1962 to December 1979 and from January 1980 to December 2000, the number of losses within the ten largest losses incured by an asset which have occured together with one of the ten largest losses of the Standard & Poor's 500 index during the same time interval. The probabilty of occurence of such a realisation is given by the p-value derived from the binomial law (27) with parameter λ .

A. Coefficients of tail dependence

	Lower tail dependence	Upper tail dependence
Bristol-Myers Squibb	0.16 (0.03)	0.14 (0.01)
Chevron	0.05 (0.01)	0.03 (0.01)
Hewlett-Packard	0.13 (0.01)	0.12 (0.01)
Coca-Cola	0.12 (0.01)	0.09 (0.01)
Minnesota Mining & MFG	0.07 (0.01)	0.06 (0.01)
Philip Morris	0.04 (0.01)	0.04 (0.01)
Procter & Gamble	0.12 (0.02)	0.09 (0.01)
Pharmacia	0.06 (0.01)	0.04 (0.01)
Schering-Plough	0.12 (0.01)	0.11 (0.01)
Texaco	0.04 (0.01)	0.03 (0.01)
Texas Instruments	0.17 (0.02)	0.12 (0.01)
Walgreen	0.11 (0.01)	0.09 (0.01)

This table presents the coefficients of lower and upper tail dependence with the S&P 500 index for a set of 12 major stocks traded on the New York Stock Exchange from January 1991 to December 2000. The numbers in brackets give the estimated standard deviation of the empirical coefficients of tail dependence

$$\text{portfolio } X = \sum w_i X_i,$$

$X_i = \beta_i \cdot Y + \varepsilon_i$; with independent noises ε_i, whose scale factors are C_{ε_i}.

Portfolio beta: $\beta = \sum w_i \beta_i$

Portfolio scale factor: $C_\varepsilon = \sum |w_i|^\alpha \cdot C_{\varepsilon_i}$.

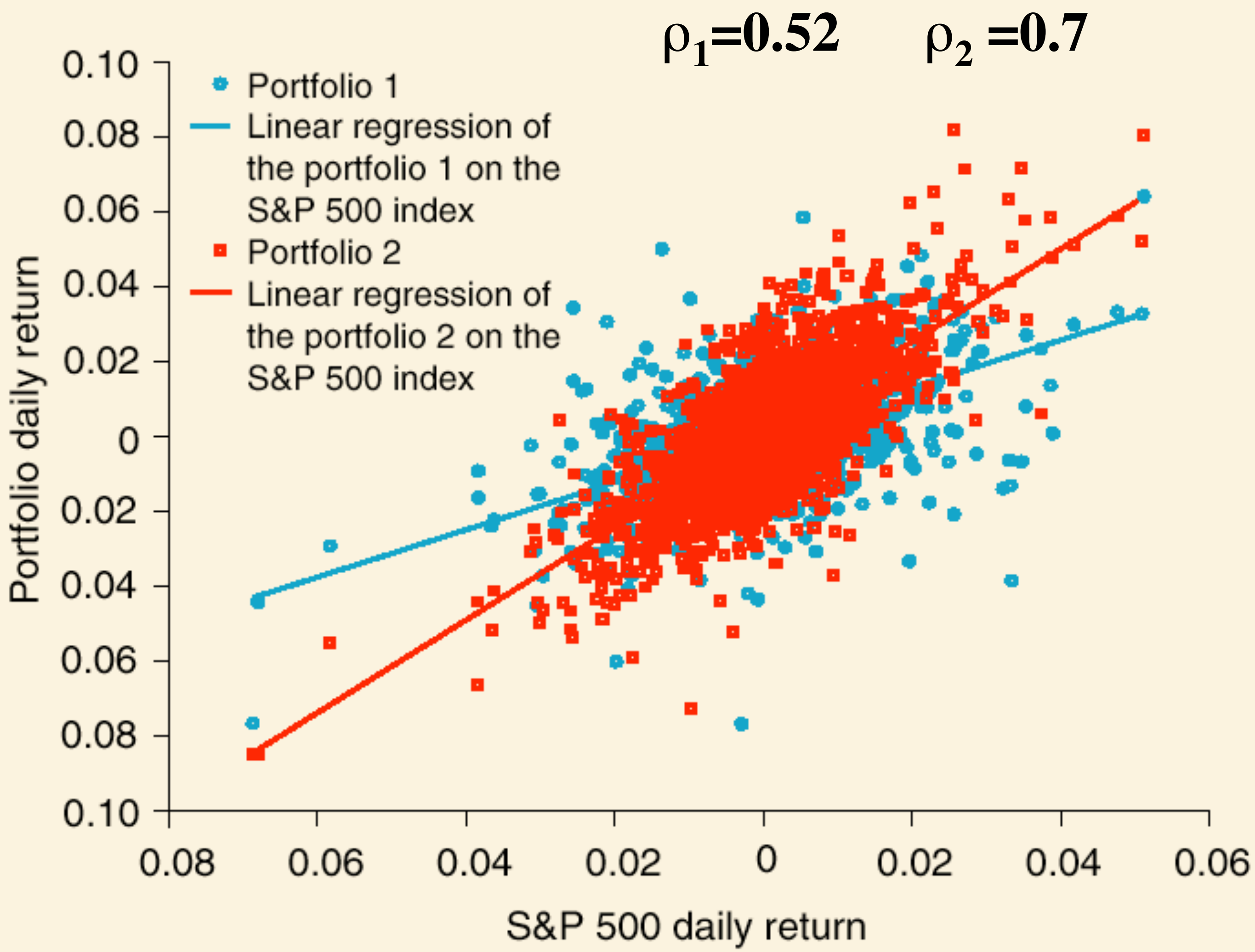
tail dependence between the portfolio and the factor

$$\lambda = \left[1 + \frac{\sum |w_i|^\alpha \cdot C_{\varepsilon_i}}{(\sum w_i \beta_i)^\alpha \cdot C_Y}\right]^{-1}.$$

compare with

$$\rho = \left[1 + \frac{\sum w_i^2 \cdot Var(\varepsilon_i)}{(\sum w_i \beta_i)^2 Var(Y)}\right]^{-1/2}$$

3. Portfolios versus market



Daily returns of two equally weighted portfolios P_1 (made of four stocks with small $\lambda \leq 0.06$) and P_2 (made of four stocks with large $\lambda \geq 0.12$) as a function of the daily returns of the S&P 500 from Jan 1991–Dec 2000

-provide a completely general analytical formula for the extreme dependence between any two assets, which holds for any distribution of returns and of their common factor

-provide a novel and robust method for estimating empirically the extreme dependence

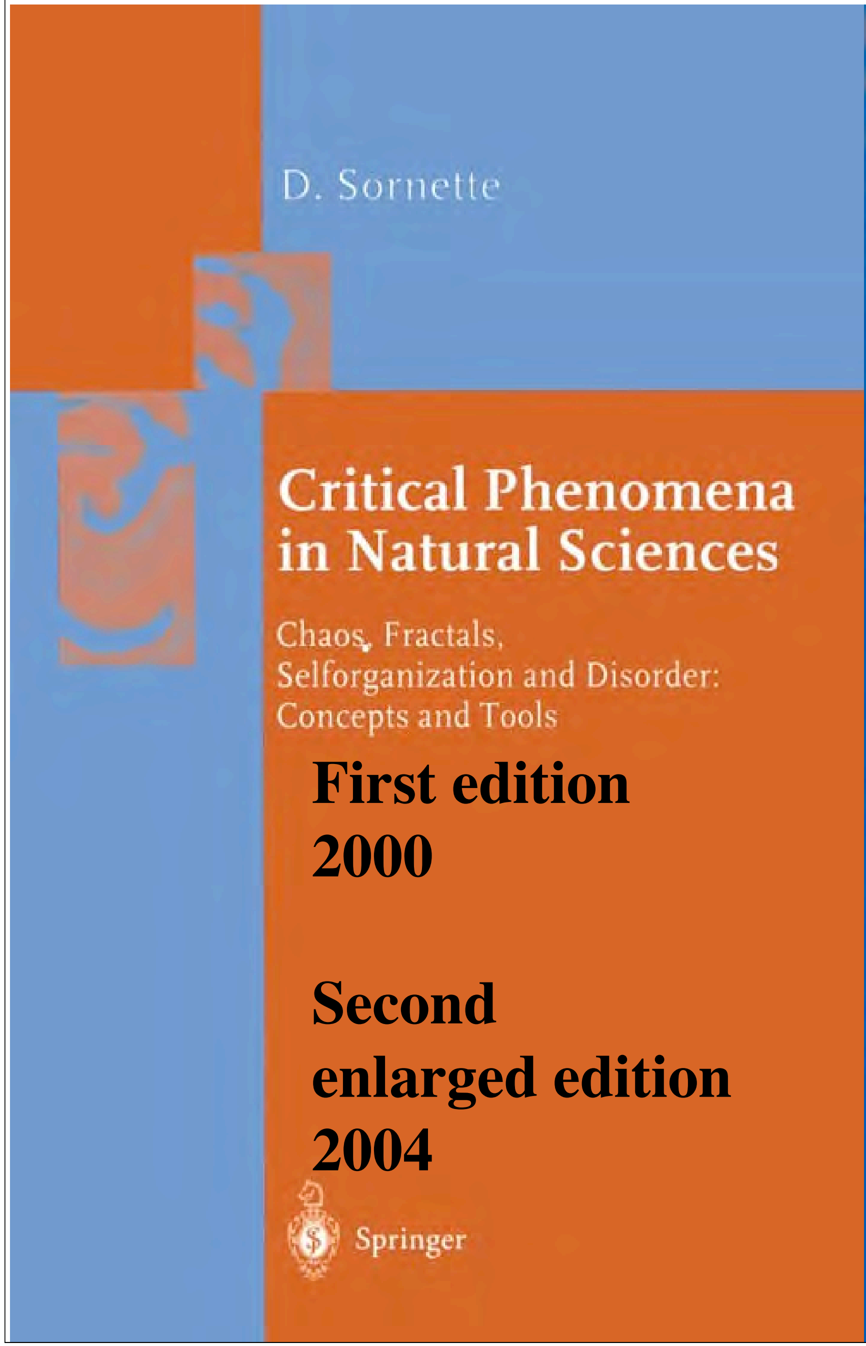
-tests on twenty majors stocks of the NYSE.

-comparing with historical co-movements in the last forty years, our prediction is validated out-of-sample and thus provide an ex-ante method to quantify futur stressful periods

-directly use to construct a portfolio aiming at minimizing the impact of extreme events.

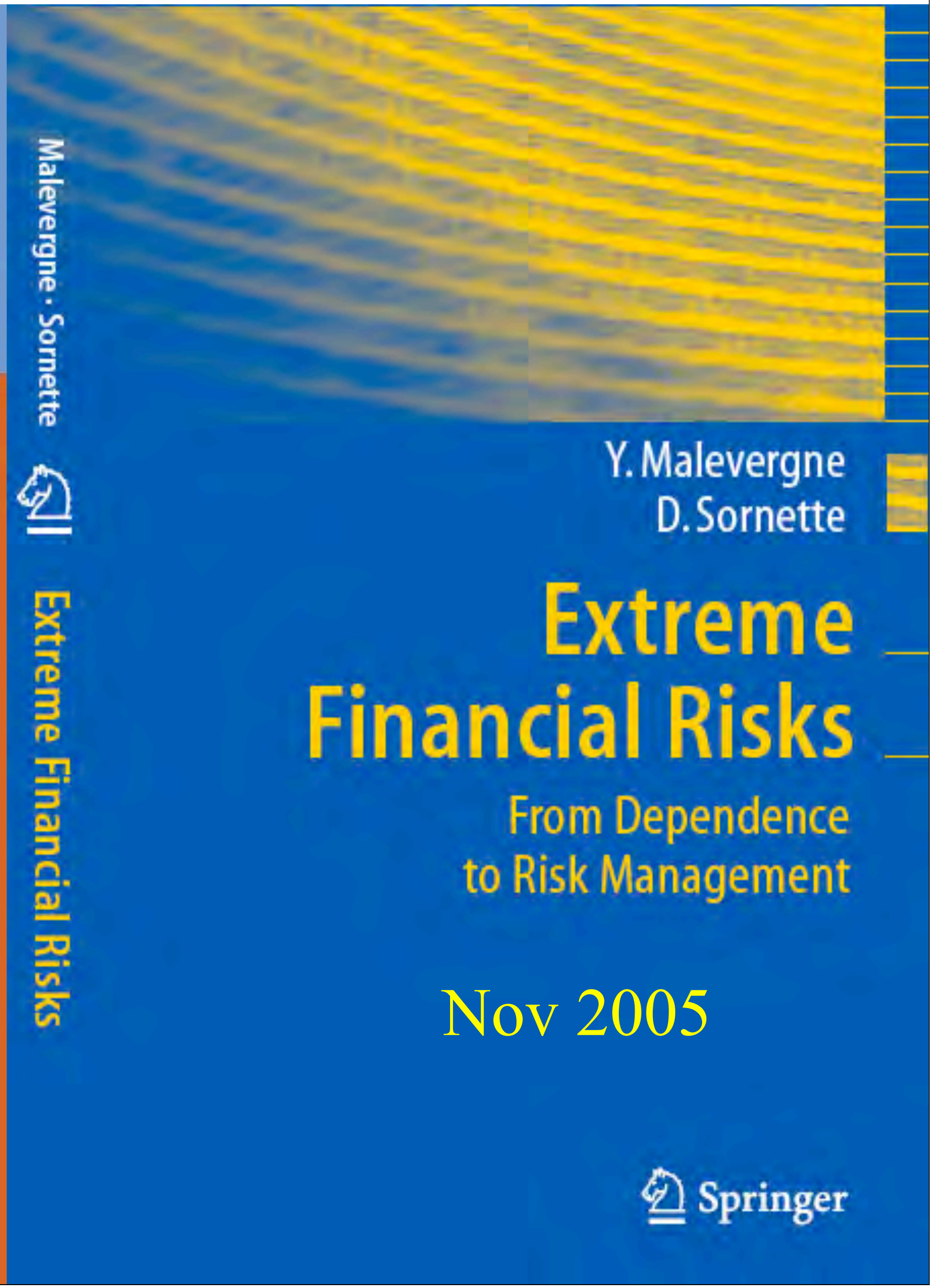
-anomalous co-monoticity associated with the October 1987 crash.

- Gaussianization of multivariate distributions
- Copulas
- Test of the Gaussian copula hypothesis
- Extreme conditional dependence measures
- Tail dependence for factor models



First edition 2000

Second enlarged edition 2004



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An Approach Using Copulas

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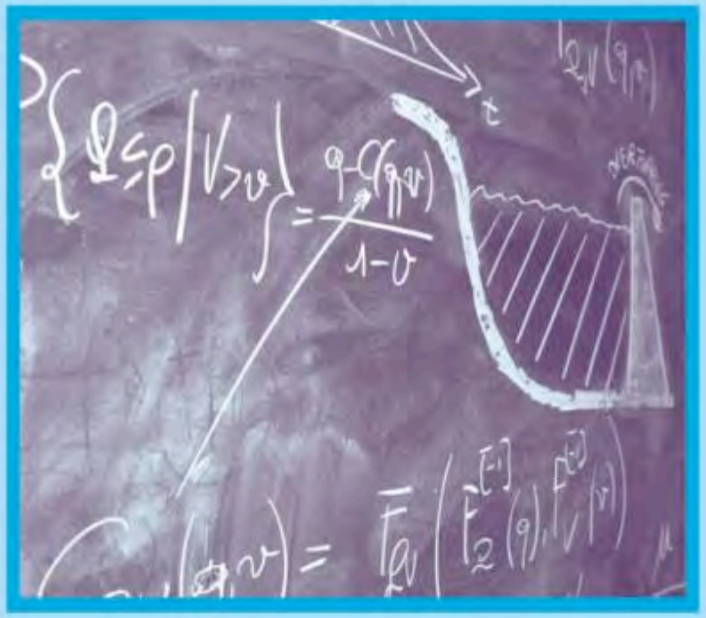

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