ETH
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Theory of earthquake recurrence times

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Statistical laws of seismicity

- Gutenberg-Richter law: $\sim 1/E^{1+\beta}$ (with $\beta \approx 2/3$)
- Omori law $\sim 1/t^p$ (with $p \approx 1$ for large earthquakes)
- Productivity law $\sim E^a$ (with $a \approx 2/3$)
- PDF of fault lengths $\sim 1/L^2$
- Fractal/multifractal structure of fault networks $\zeta(q)$, $f(\alpha)$
- PDF of seismic stress sources $\sim 1/s^{2+\delta}$ (with $\delta \geq 0$)

Is the distribution of inter-earthquake times revealing new physics?

1. **Generalized Molchan's argument in terms of independent regions**

2. **Mean-field theory of the PDF of earthquake recurrence times using ETAS model**

3. **Generating Probability Function theory**

- **Comparison with Corral's data analysis**

- **Extension to describe Kossobokov and Mazhkenov (1988) and Bak et al. (2002) data analysis**

A. Saichev and D. Sornette, Universal Distribution of Inter-Earthquake Times Explained, Phys. Rev. Letts. 97, 078501 (2006)

A. Saichev and D. Sornette, Theory of Earthquake Recurrence Times, J. Geophys. Res., 112, B04313, doi:10.1029/2006JB004536 (2007)

Is the distribution of recurrence times revealing new physics?

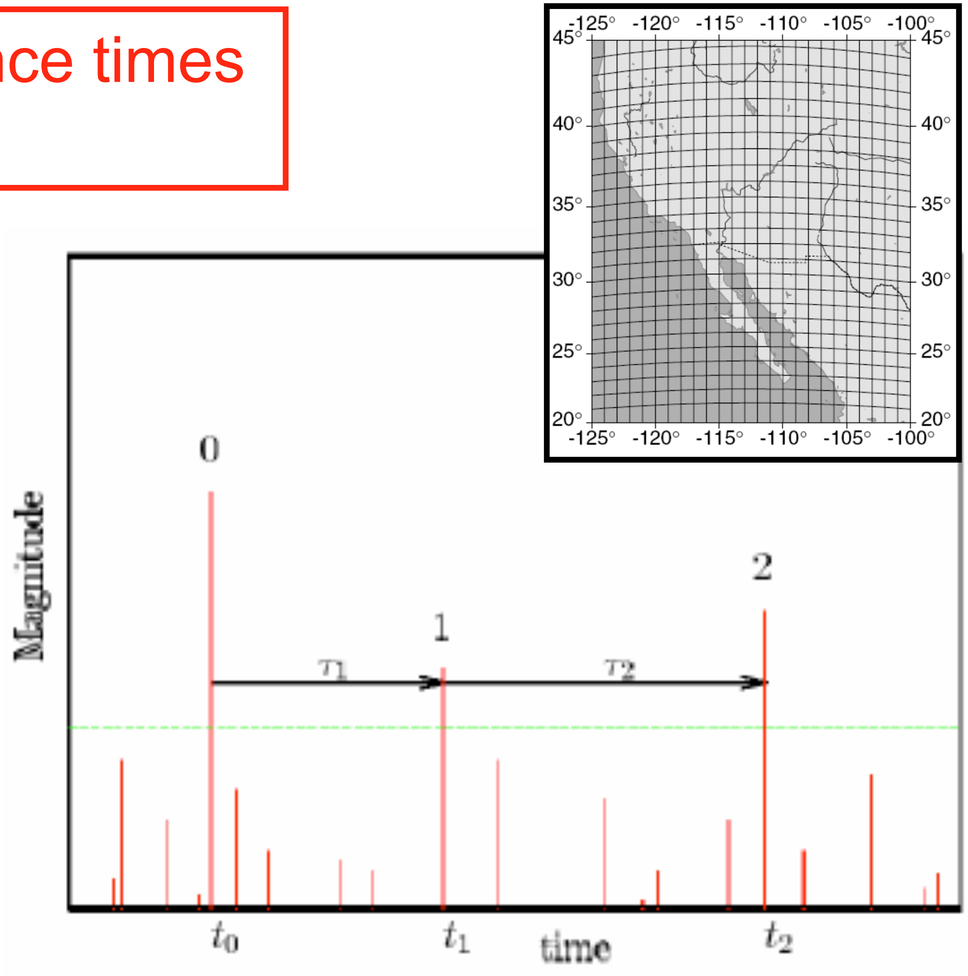
Consider a fixed spatial region

Consider earthquakes with magnitude larger than a threshold, $M \geq M_c$

Compute inter-event time as the time between consecutive earthquakes

$$\tau_i \equiv t_i - t_{i-1}$$

$$i = 1, 2, 3 \ldots$$



•Bak, P., K. Christensen, L. Danon, and T. Scanlon (2002), Unified scaling law for earthquakes, Phys. Rev. Lett., 88, 178501.
•Christensen, K., L. Danon, T. Scanlon, and P. Bak (2002), Unified scaling law for earthquakes, Proc. Natl. Acad. Sci. U.S.A., 99, 2509– 2513.
•Corral, A. (2003), Local distributions and rate fluctuations in a unified scaling law for earthquakes, Phys. Rev. E, 68, 035102(R).
•Corral, A. (2004a), Universal local versus unified global scaling laws in the statistics of seismicity, Physica A, 340, 590–597.
•Corral, A. (2004b), Long-term clustering, scaling, and universality in the temporal occurrence of earthquakes, Phys. Rev. Lett., 92, 108501.

(Alvaro Corral)

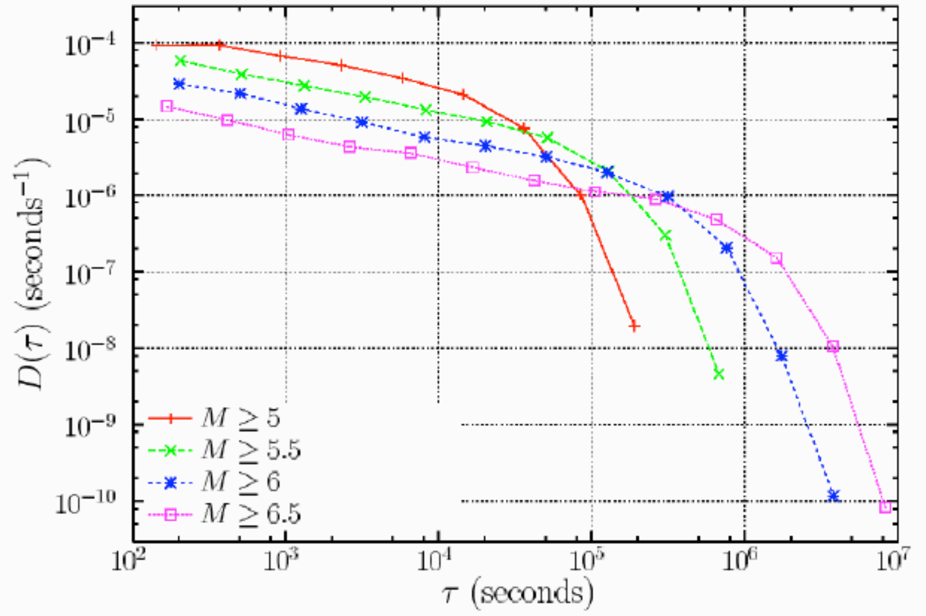
Inter-event time probability density

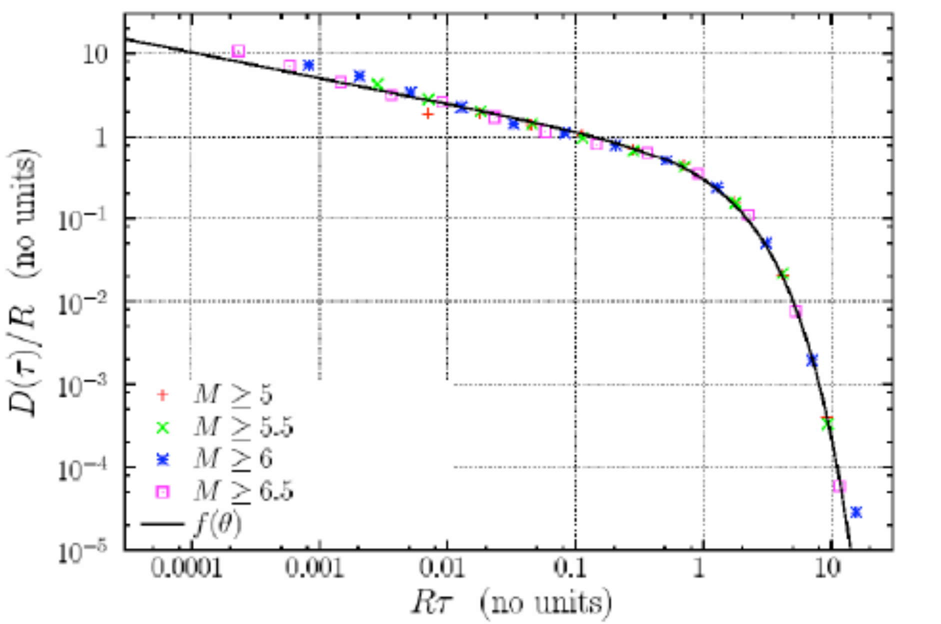
$$D(\tau) \equiv \frac{\text{Prob}[\tau \leq \text{inter-event time} < \tau + d\tau]}{d\tau}$$

$$\begin{array}{ccc} \tau & \longrightarrow & R\tau \\ D(\tau) & \longrightarrow & D(\tau)/R \end{array}$$

$\Longrightarrow$ Scaling law:

$$D(\tau) = Rf(R\tau)$$

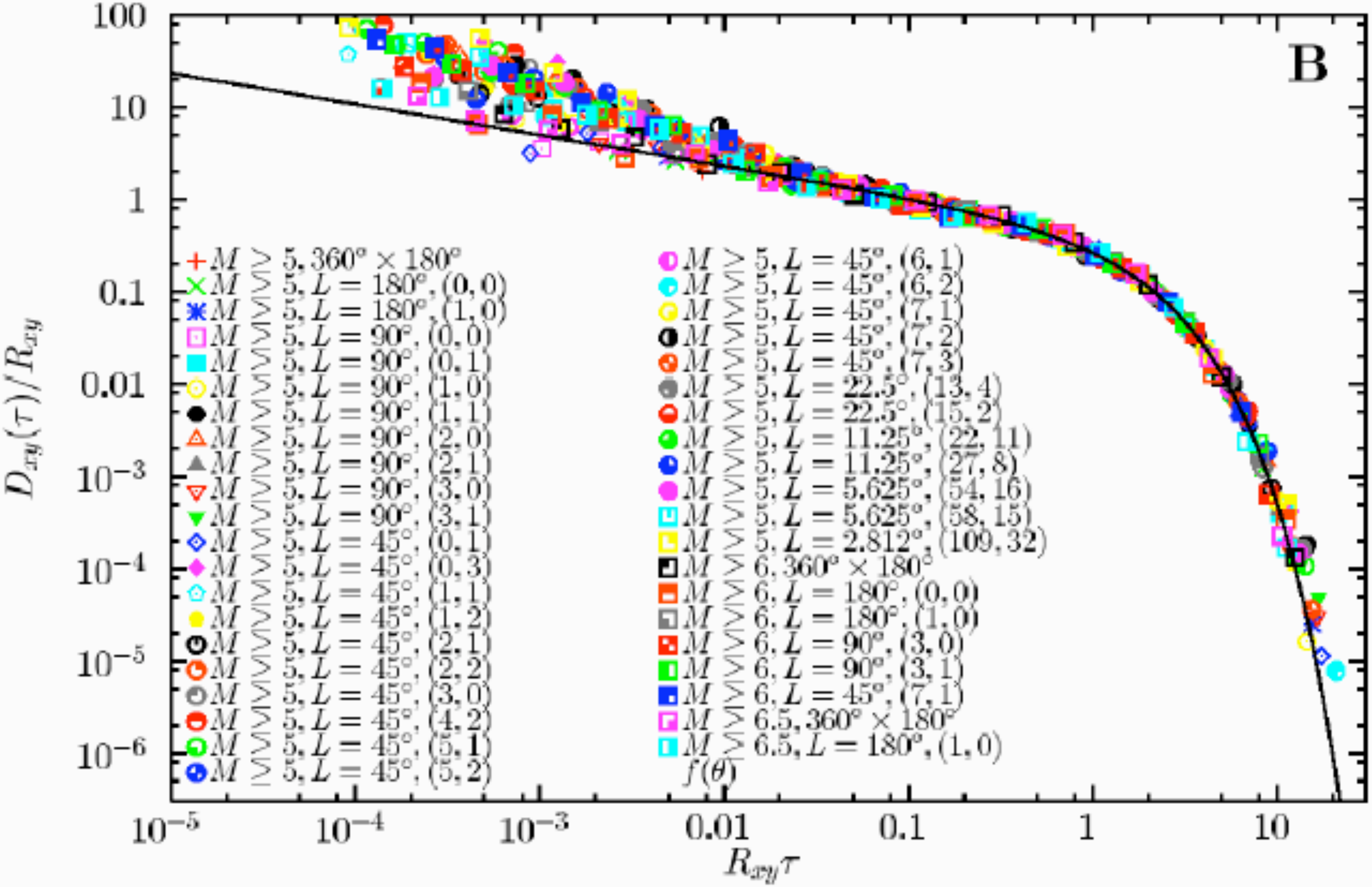




(Alvaro Corral)

Universality

Changing the spatial region: stationary seismicity up to 2.8° (300 km)



(Alvaro Corral)

Molchan's Argument

$$P(\tau) = \psi(\lambda\tau)$$

Probability of no earthquake in time τ

where λ is the regional and instantaneous seismicity rate

For two regions:

$$\psi((\lambda_1 + \lambda_2)\tau) = \psi(\lambda_1\tau)\psi(\lambda_2\tau)$$

This leads to

$$P(\tau) = \exp(-\lambda\tau)$$

(incompatible with the data)

Molchan, G. M. (2005), Interevent time distribution of seismicity: A theoretical approach, Pure Appl. Geophys., 162, 1135-1150.

Generalization

(Saichev-Sornette, JGR 2007)

$$P(\tau) = \psi(\lambda\tau, \lambda)$$

where λ is the regional and instantaneous seismicity rate

For two regions:

$$\psi((\lambda_1 + \lambda_2)\tau, \lambda_1 + \lambda_2) = \psi(\lambda_1\tau, \lambda_1)\psi(\lambda_2\tau, \lambda_2)$$

This leads to

$$P(\tau) = \exp\left(-\lambda\tau g_0\left(\frac{\tau}{c}\right)\right)$$

(where a new time scale c necessarily emerges)

A natural choice for $g_0(y)$ is $g_0(y) = a + hy^{-\theta}$ leading to

(**) $$P(\tau) = \exp\left(-ax - \epsilon^{\theta} h x^{1-\theta}\right), \quad \epsilon = \lambda c, x = \lambda\tau$$

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Epidemic Type Aftershock Sequence (ETAS)

Model proposed by Kagan and Knopoff [1981, 1987] and Ogata [1988]

- each earthquake can be both a mainshock, an aftershock and a foreshock
- each earthquake triggers aftershocks according to the Omori law, that in turn trigger their own aftershocks

$$\phi(t) = \frac{K}{(t+c)^{1+\theta}}$$

- the number of aftershocks triggered by a mainshock depends on the mainshock magnitude :

$$N(m) \sim 10^{\alpha m}$$

- aftershock magnitudes follow the Gutenberg-Richter distribution, independently of the time and of the mainshock magnitude

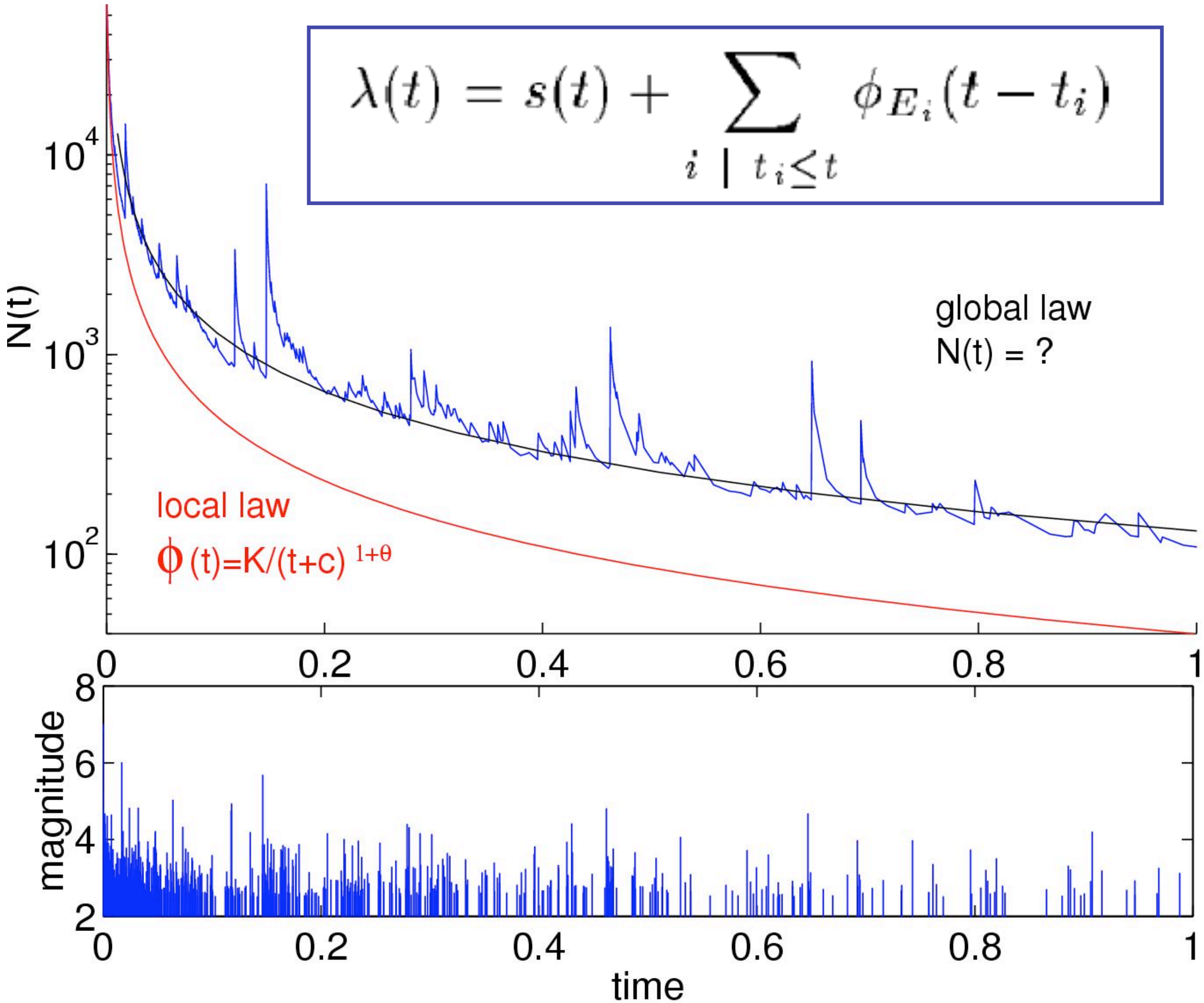
$$P(m) \sim 10^{-bm}$$

ETAS model

(self-excited Hawkes-type conditional Poisson process)

Rate of aftershocks predicted by the ETAS model

$$\lambda(t) = s(t) + \sum_{i \mid t_i \leq t} \phi_{E_i}(t - t_i)$$



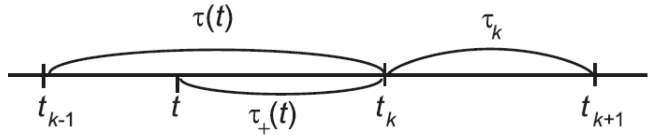
- is the global law also an Omori law ?
- p_{global} # p_{local} ?

(Helmstetter and Sornette, 2002-2004)

Simplified Model of the Impact of the Omori-Utsu Law on the PDF of Inter-event Times

Probability of absence of events in [t, t+τ]: $P(\tau) \approx \exp(-\omega\tau - \omega\Lambda(\tau))$

Inter-event time distribution : $$H(\tau) = \frac{1}{\lambda}\frac{d^2P(\tau)}{d\tau^2}$$



$$\omega\Lambda(\tau) = \int_t^{t+\tau} \lambda(t')dt', \quad \lambda(t') \approx \sum_{t_k<t'} \Phi(t'-t_k),$$

Mean-field approximation:

$$\langle\lambda\rangle(t') = \frac{\omega n}{1-n}\int_{-\infty}^{t} \Phi(t'-t'')dt'' = \frac{\omega n}{1-n} a(t'-t).$$

$$a(t) = \int_t^{\infty} \Phi(t')dt' = \frac{c^\theta}{(c+t)^\theta}$$

$$\omega\Lambda(\tau) \approx \frac{\omega n}{1-n}\frac{c}{1-\theta}\left(\frac{\tau}{c}\right)^{1-\theta}$$

$$P(\tau) \approx \exp\left(-\omega\tau - \frac{\omega n}{1-n}\frac{c}{1-\theta}\left(\frac{\tau}{c}\right)^{1-\theta}\right), \quad \tau \gg c.$$

recovers the generalized Molchan's argument

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Generating probability function (GPF)

(Saichev-Sornette, JGR 2007)

GPF of the random number R(t, τ, m) of
observable events within the time interval [t, t + τ]

$$\Omega(z;\tau,m) = \langle z^{R(t,\tau,m)} \rangle = \sum_{r=0}^{\infty} P(r;\tau,m) z^r$$

where P(r, t, m) be the probability that the number R(t, τ, m)
of observable events is equal to r.

Example of application:

$$\langle R(t,\tau,m) \rangle = \left. \frac{d\Omega(z;\tau,m)}{dz} \right|_{z=1}$$

$$\langle R(t,\tau,m) \rangle = \frac{\omega(m)}{1-n}\tau, \quad \omega(m) = \omega Q(m)$$

The GPF formalism uses optimally the independence between the different branches.

This results from the linear structure of the conditional Poisson rates.

Generation 1: random number R_1 of daughters described by a probability function associated with the GPF $G_1(z)$.

Generation 2: random number R_2 of daughters of events of generation1 also described by a probability function associated with the GPF $G_1(z)$.

The probability function of the total number of triggered events is described by

$$G_2(z) = G_1[zG_1(z)]$$

Summing over all generations, the GPF G(z) of the total number of events over all generations satisfies the functional equation

$$G(z) = G_1(zG(z))$$

Results of GPF calculation:

$$\Omega(z;\tau,m) = e^{-\omega L(z;\tau,m)}$$

$$L(z;\tau,m) = \int_0^\infty [1 - D(z;t,\tau,m)]dt + \int_0^\tau [1 - D(z;t,m) + (1-z)D_+(z;t,m)]dt$$

$$D(z;t,\tau,m) = \int_{m_0}^\infty \Theta(z;t,\tau,m',m)p(m')dm',$$

$$D(z;\tau,m) = \int_{m_0}^\infty \Theta(z;\tau,m',m)p(m')dm',$$

$$D_+(z;\tau,m) = \int_m^\infty \Theta(z;\tau,m',m)p(m')dm'.$$

$$L(z;\tau,m) = \int_0^\infty [1 - D(z;t,\tau,m)]dt + \int_0^\tau [1 - D(z;t,m) + (1-z)D_+(z;t,m)]dt$$

$$\begin{aligned} D(z;t,\tau,m) = \Psi[&b(t+\tau) - \Phi(t)\otimes D(z;t,\tau,m) \\ &- \Phi(t+\tau)\otimes D(z;\tau,m) + (1-z) \\ &\cdot \Phi(t+\tau)\otimes D_+(z;\tau,m)] \end{aligned}$$

$$\begin{aligned} D(z;\tau,m) = \Psi[&b(\tau) - \Phi(\tau)\otimes D(z;\tau,m) \\ &+ (1-z)\Phi(\tau)\otimes D_+(z;\tau,m)] \end{aligned}$$

$$\begin{aligned} D_+(z;\tau,m) = \Psi_+[&b(\tau) - \Phi(\tau)\otimes D(z;\tau,m) \\ &+ (1-z)\Phi(\tau)\otimes D_+(z;\tau,m)]. \end{aligned}$$

$$\Psi(y) = \int_{m_0}^\infty p(m')e^{-\rho(m')y}dm' = \gamma(\kappa y)^\gamma \Gamma(-\gamma,\kappa y)$$

$$\Psi_+(y) = \int_m^\infty p(m')e^{-\rho(m')y}dm' = Q(m)\Psi\left[Q^{-1/\gamma}(m)y\right]$$

$$\boxed{\Psi(y) \simeq 1 - ny + \beta y^\gamma - \eta y^2}$$

$$n = \frac{\kappa\gamma}{\gamma-1}, \quad \beta = -\left(n\frac{\gamma-1}{\gamma}\right)^\gamma \Gamma(1-\gamma), \quad \eta = \frac{n^2(\gamma-1)^2}{2\gamma(2-\gamma)}$$

Result: $$P(\tau) = \exp\left(-\frac{\omega(m)\tau}{1-\delta} - \omega(m)\Delta \int_0^\tau \frac{a(t)dt}{1-\delta+\delta a(t)}\right)$$

$$\omega(m) = \omega Q(m) \qquad \Delta = \frac{n}{1-n} - \frac{\delta}{1-\delta} \qquad \delta = n\left[1 - Q^{\frac{\gamma-1}{\gamma}}(m)\right]$$

$\gamma = b/\alpha$

$$a(t) = \int_t^\infty \Phi(t')dt' = \frac{c^\theta}{(c+t)^\theta}$$

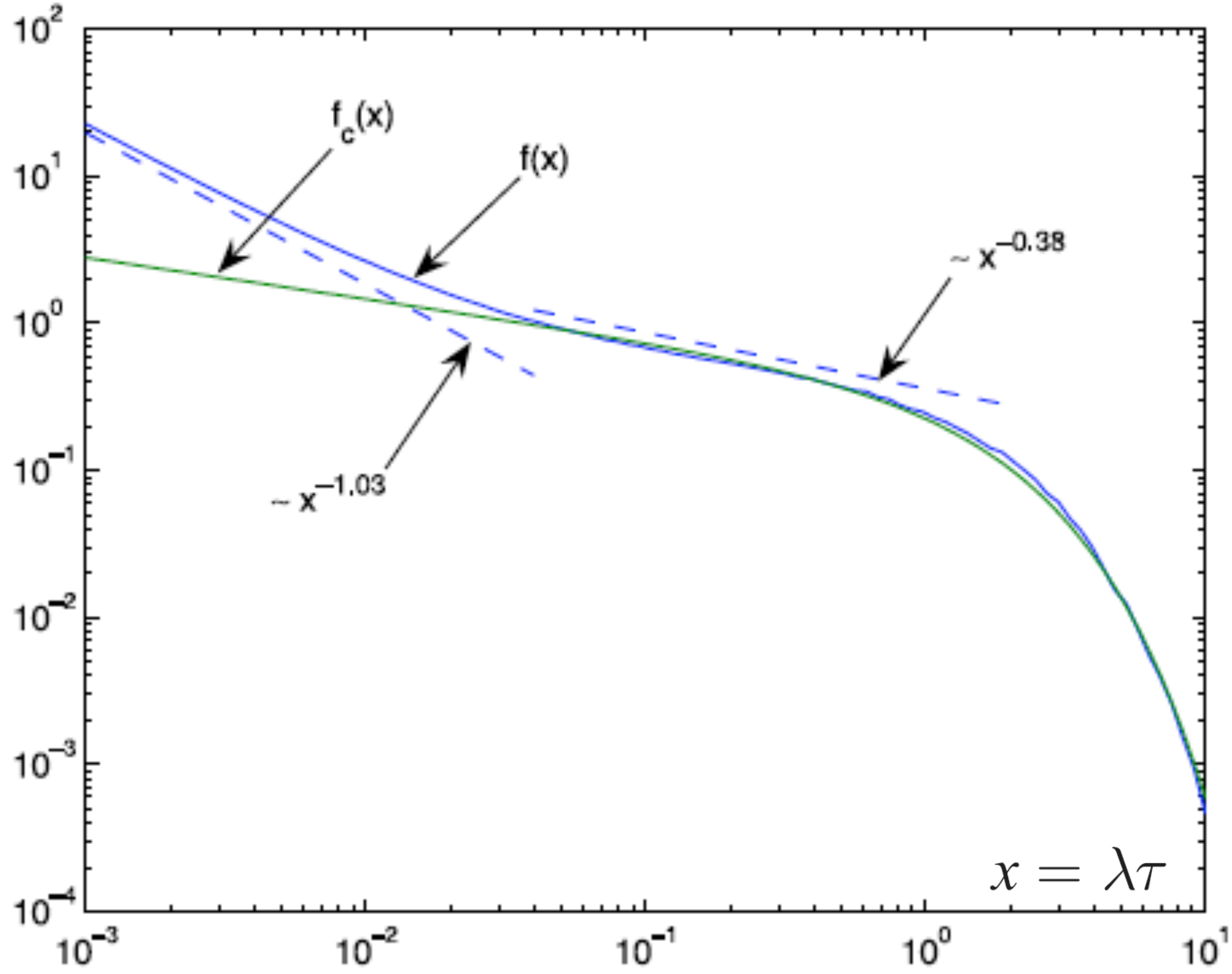
Result of our theory: $x = \lambda\tau, \qquad \epsilon = \lambda c$

$$P(\tau) \approx \varphi(x,\epsilon) = \exp\left(-(1-n)x - \frac{n\epsilon^\theta}{1-\theta}x^{1-\theta}\right)$$

$$f(x) = \left(n\epsilon^\theta \theta x^{-1-\theta} + \left[1 - n + n\epsilon^\theta x^{-\theta}\right]^2\right)\varphi(x,\epsilon) \quad = \langle\tau\rangle \frac{d^2P(\tau)}{d\tau^2}$$

Corral (2004): $$f_c(x) = \frac{C\bar{\delta}}{d\Gamma(\bar{\gamma}/\bar{\delta})}\left(\frac{x}{\bar{a}}\right)^{\bar{\gamma}-1} e^{-(x/\bar{a})^{\bar{\delta}}}$$

$$\bar{\gamma} = 0.67 \pm 0.05, \quad \bar{\delta} = 1.05 \pm 0.05, \quad \bar{a} = 1.64 \pm 0.15$$



Plot of the PDF $f(x)$ given by expression (36) for the parameters $n = 0.9$, $\theta = 0.03$, $\epsilon_1 = 0.76$, $m - m_1 = 2$, and Corral's fitting curve (38) indicated as $f_c(x) =$ for $\gamma = 0.38$, $\delta = 1$, $d = 1.7$ and $C = 0.75$. We also show an intermediate asymptotics $\sim 1/x^{0.38}$ discussed by Corral (see below) as well as the short-time asymptotics $\sim 1/x^{1.03}$ corresponding to the Omori law.

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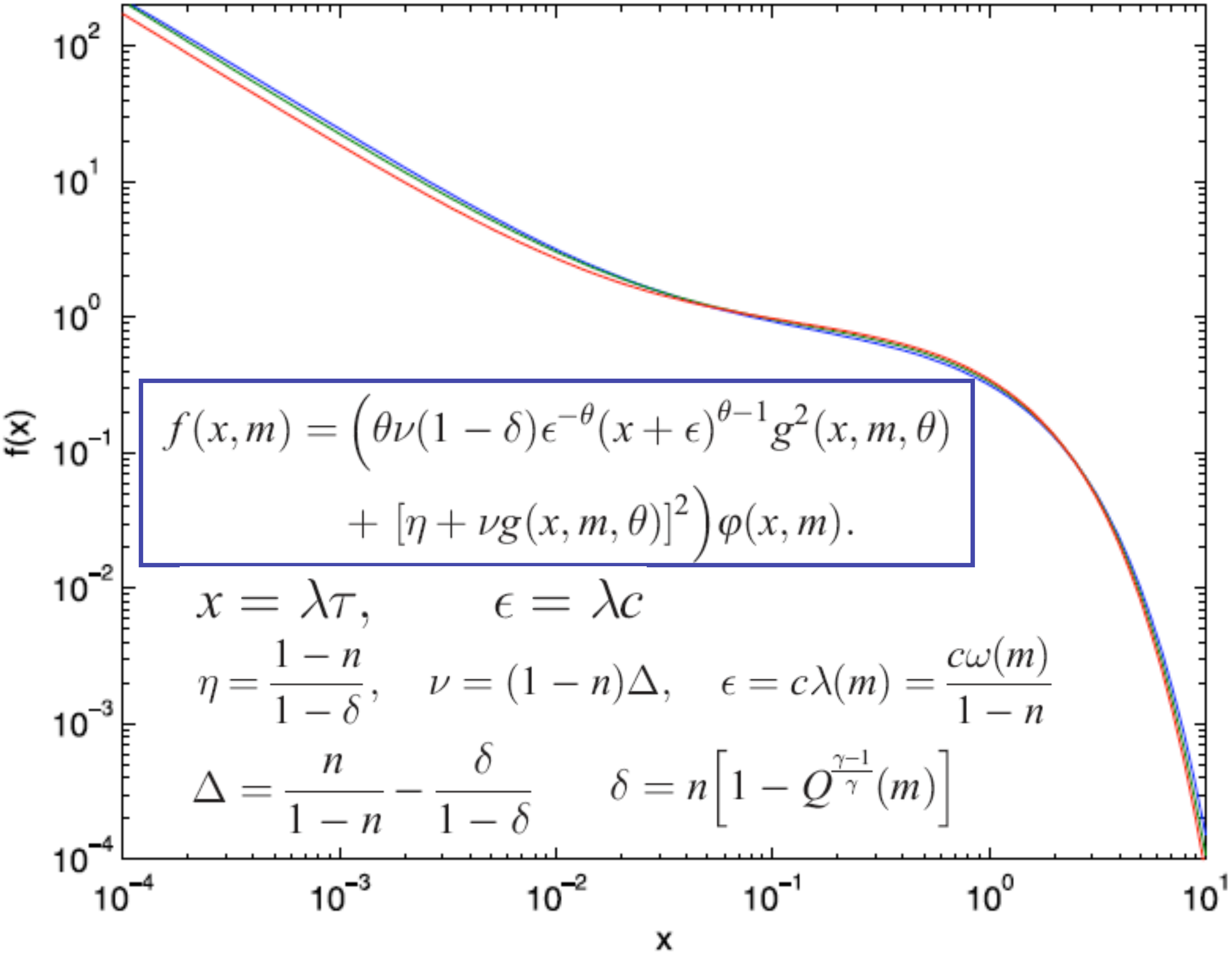


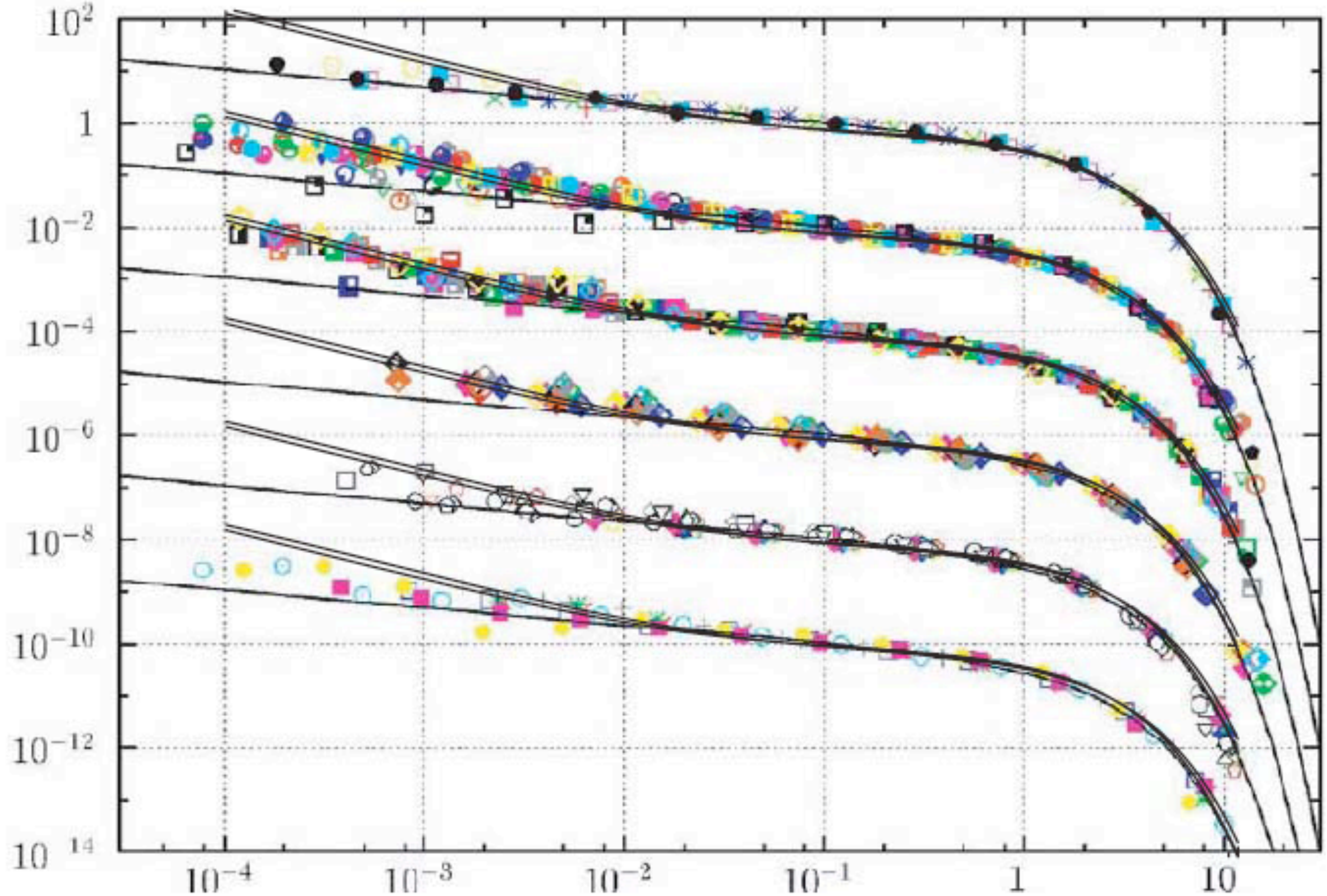
$$f(x,m) = \Big(\theta\nu(1-\delta)\epsilon^{-\theta}(x+\epsilon)^{\theta-1}g^2(x,m,\theta) + \left[\eta + \nu g(x,m,\theta)\right]^2\Big)\varphi(x,m).$$

$$x = \lambda\tau, \qquad \epsilon = \lambda c$$

$$\eta = \frac{1-n}{1-\delta}, \quad \nu = (1-n)\Delta, \quad \epsilon = c\lambda(m) = \frac{c\omega(m)}{1-n}$$

$$\Delta = \frac{n}{1-n} - \frac{\delta}{1-\delta} \qquad \delta = n\left[1 - Q^{\frac{\gamma-1}{\gamma}}(m)\right]$$

Plots of the PDF (55) of the scaled recurrence times x for $\theta = 0.03$, $\gamma = 1.2$, n = 0.9 and for different magnitude thresholds $m\text{-}m_0 = 2, 4, 6$.



Plots of the empirical PDFs of scaled interevent times taken from *Corral* [2004a], Corral's fitting curves using expression (38) (single solid curves), and our theoretical prediction (55) for $\theta = 0.03$, $\gamma = 1.2$ and $n = 0.9$ (double solid curves). Our ETAS prediction is shown as double solid curves because we show the theoretical prediction (55) for two different magnitude threshold levels $m - m_0 = 2$ and 6, showing the very weak dependence on the magnitude of completeness. Top to bottom curves are for the NEIC worldwide catalog for regions with $L \geq 180$ degrees, 1973–2002; NEIC with $L \leq 90$ degrees, (same period of time); southern California, 1984–2001, 1988–1991, and 1995–1998; northern California, 1998–2002; Japan, 1995–1998, and New Zealand, 1996–2001; (bottom), Spain, 1993–1997, New Madrid, 1975–2002, and Great Britain, 1991–2001. The curves are translated for clarity by the factors 1, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, and 10^{-10} from top to bottom.

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PDF of Recurrence Times for Multiple Regions: Accounting for Kossobokov and Mazhkenov (1988) and Bak et al.'s [2002] Empirical Analysis

Principle of statistical self-similarity: if the branching ratios n_i and the linear size L_i of different regions are identical, then the seismic rates can be written

$$\omega_i = \bar{\omega} u_i \quad \text{with iid PDF} \quad \mathcal{E}(u)$$

$$\mathcal{H}(\tau) = \frac{1}{\bar{\lambda}} \frac{d^2}{d\tau^2} \int_0^\infty P(\tau) \mathcal{E}(u) \frac{du}{u}, \qquad \bar{\lambda} = \frac{\bar{\omega}}{1-n}$$

1. Assumption: fractal spatial organization of the subset of spontaneous events

2. Fractal geometry translates into the PDF $\mathcal{E}$(u) of source activity.

3. Duality between two reciprocal random variables.
   - The first random variable is the area S(k) occupied by k spontaneous events which occurred during a fixed time interval of duration t.
   - The second random variable k(S) is the inverse function of S(k), which is nothing but the random number of spontaneous events within the given area S.

$$h(x) \approx \mu\Gamma(1+\chi)\left[(1+\chi)\left(1-n+n\bar{\epsilon}^{\theta}x^{-\theta}\right)^2 K^{-2-\chi}(x) + \theta n\bar{\epsilon}^{\theta}x^{-1-\theta}K^{-1-\chi}(x)\right]$$

$$K(x) \approx (1-n)x + \frac{n}{1-\theta}\epsilon^{\theta}x^{1-\theta}$$

$d_f = 2 - \chi$

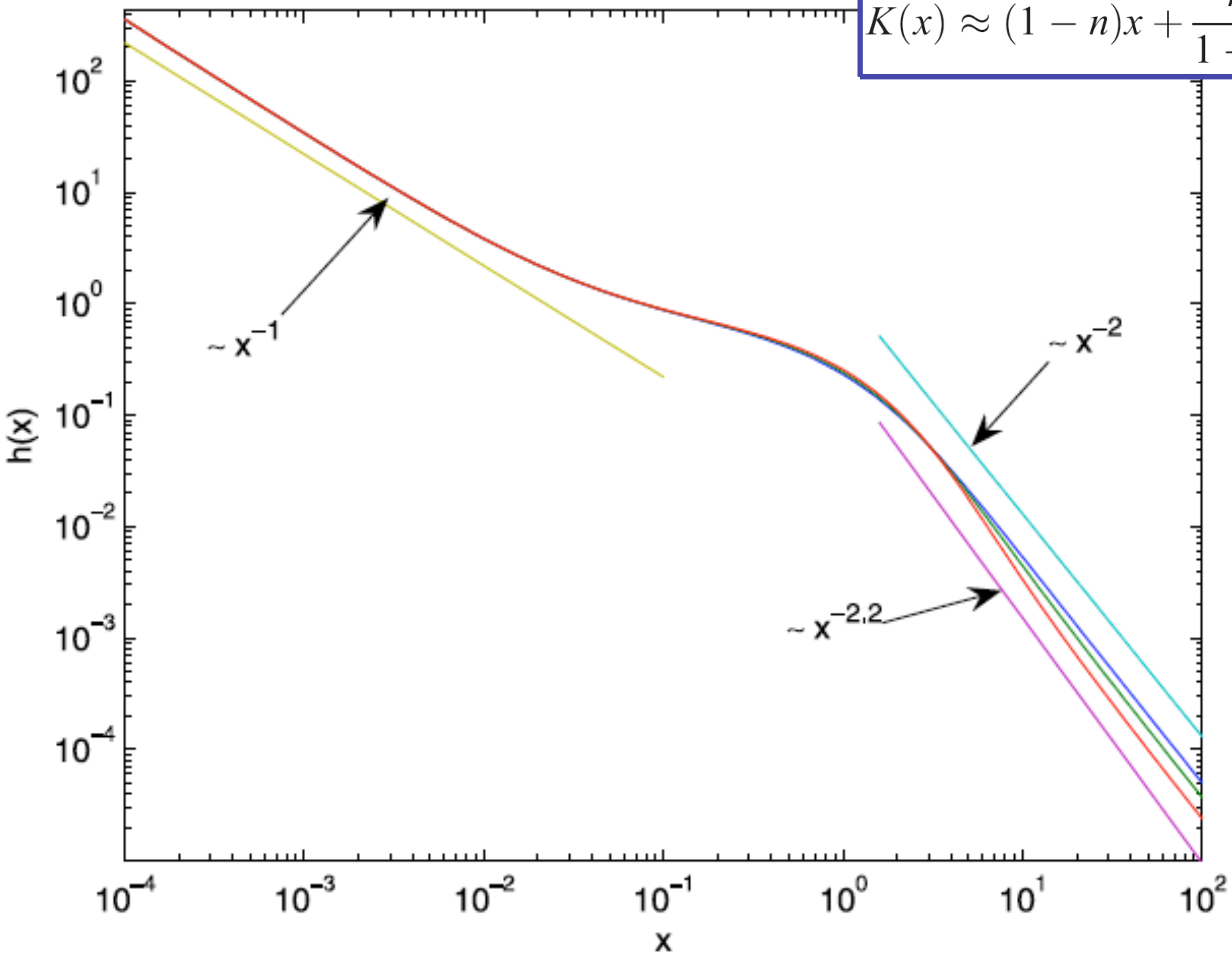
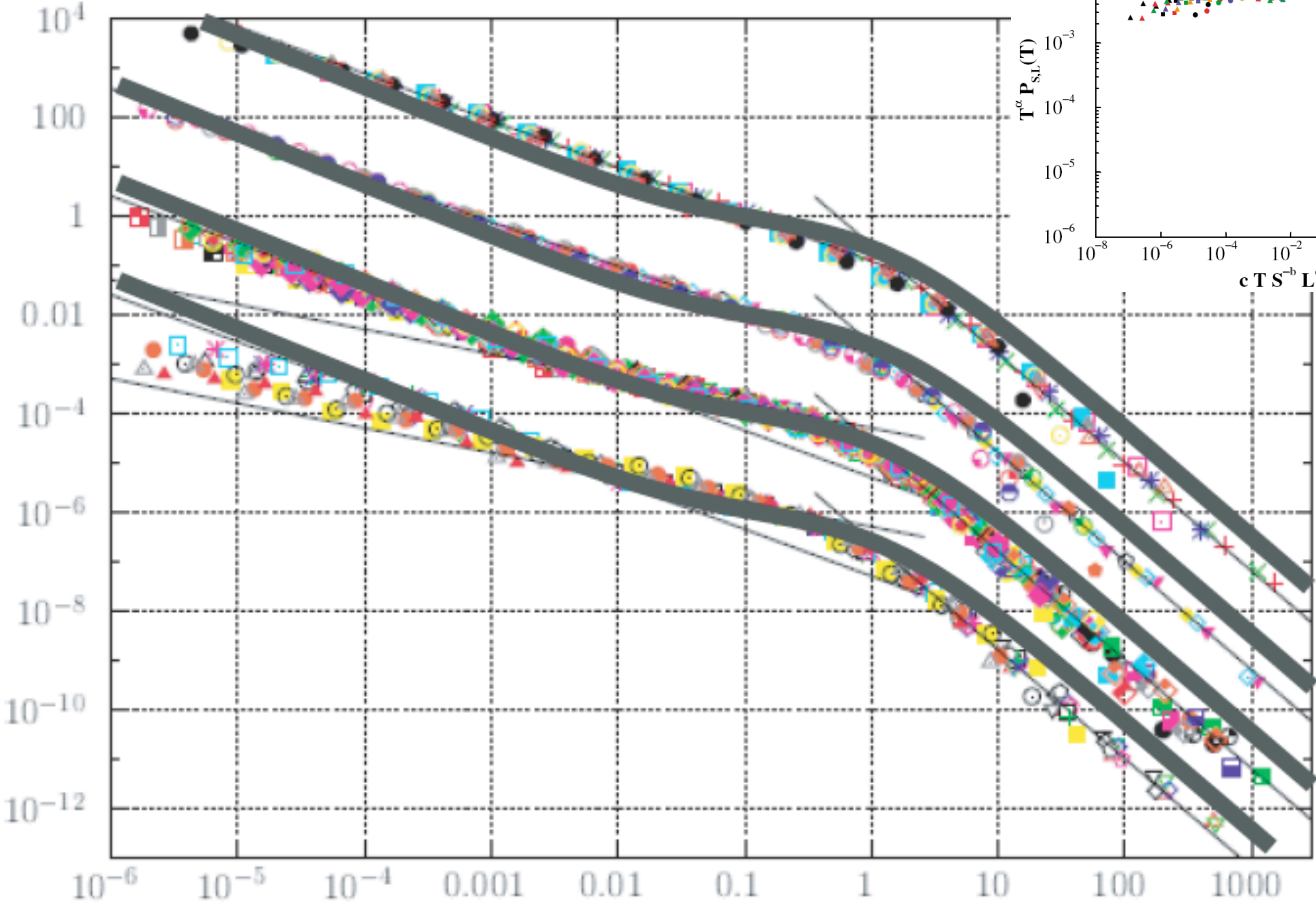
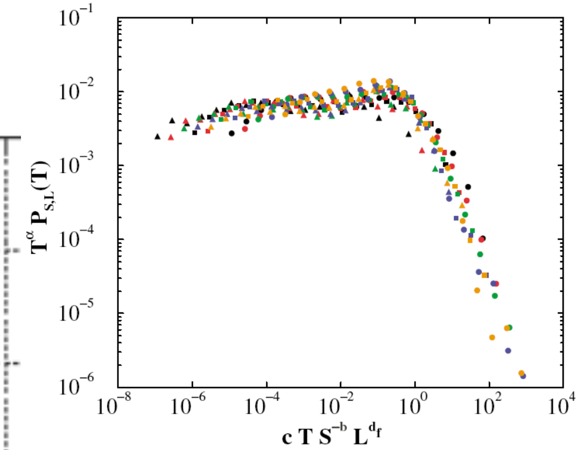


Figure 10. Plots of the PDF $h(x)$ of interevent times given by (64) corresponding to the PDF (94) of the average regional seismic rates. Top to bottom on the right side of the picture $d = 1.2, 1.4, 1.6$. The straight lines show asymptotic power laws, as proposed by *Bak et al.* [2002].





Empirical PDFs of the recurrence times between earthquakes over multiple regions following *Bak et al.*'s [2002] procedure obtained from Figure 2 of *Corral* [2004a], on which has been superimposed our prediction for $h(x)$ obtained with the Gamma distribution (77) for $\mathcal{E}(u)$ with $\theta = 0.03$, $n = 0.9$, $\epsilon = 0.76$, and $\chi = 0.2$. The curves have been translated from top to bottom by the factors 1, 10^{-2}, 10^{-4}, 10^{-6}.

(d_f=1.8)

Main results

1. The so-called universal scaling laws of interevent times do not reveal more information than what is already captured by the well-known laws of seismicity (Gutenberg-Richter and Omori-Utsu, essentially), together with the assumption that all earthquakes are similar (no distinction between foreshocks, main shocks and aftershocks).
2. This conclusion is reached by a combination of analyses,
   - generalization of Molchan's [2005] argument,
   - simple models of triggered seismicity taking into account the Omori-Utsu law,
   - detailed study of what the ETAS model with the GPF formalism.
3. Formalism of generating probability functions allowed us to derive analytically specific predictions for the PDF of recurrence times between earthquakes in a single homogeneous region as well as for multiple regions.
4. Our theory account quantitatively precisely for the empirical power laws found by Bak et al. [2002] and Corral [2003, 2004a].
5. The empirical statistics of inter-event times result from subtle crossovers rather than being genuine asymptotic scaling laws.
6. Universality does not strictly hold.
   - Generalization to conditional PDF and higher-order statistics
   - Non-linear theory

Applications to inter-events in geo-events in general and in other fields (financial shocks, epileptic seizures, and so on)