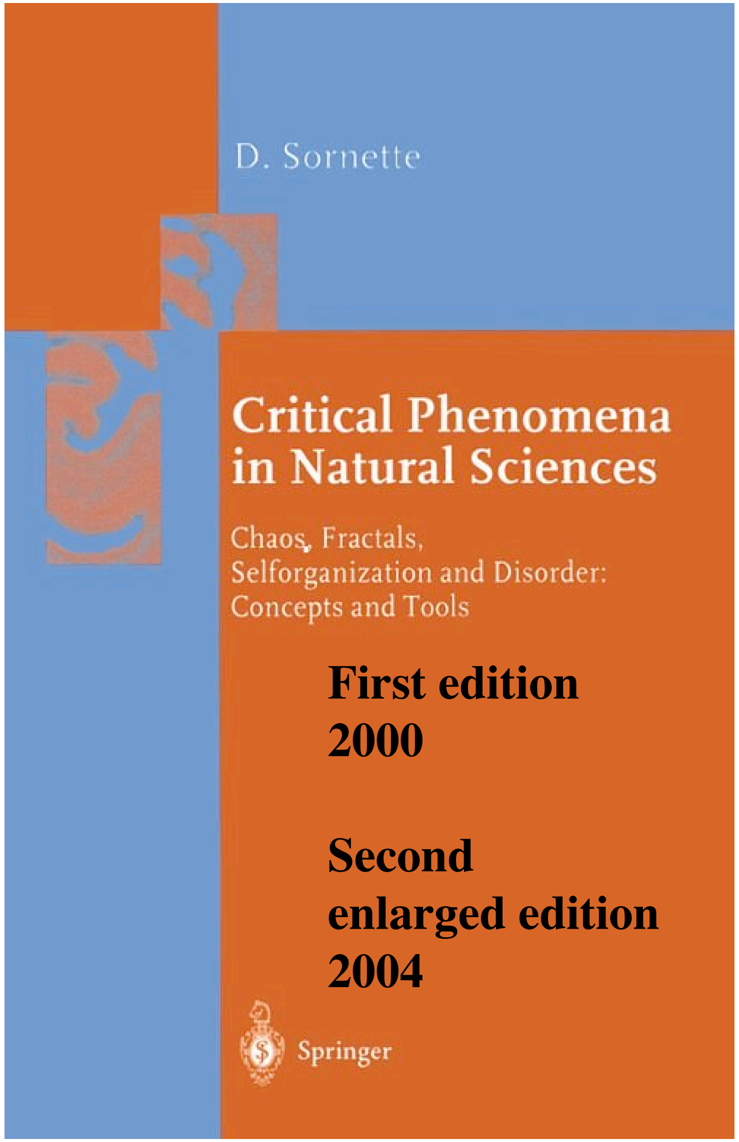
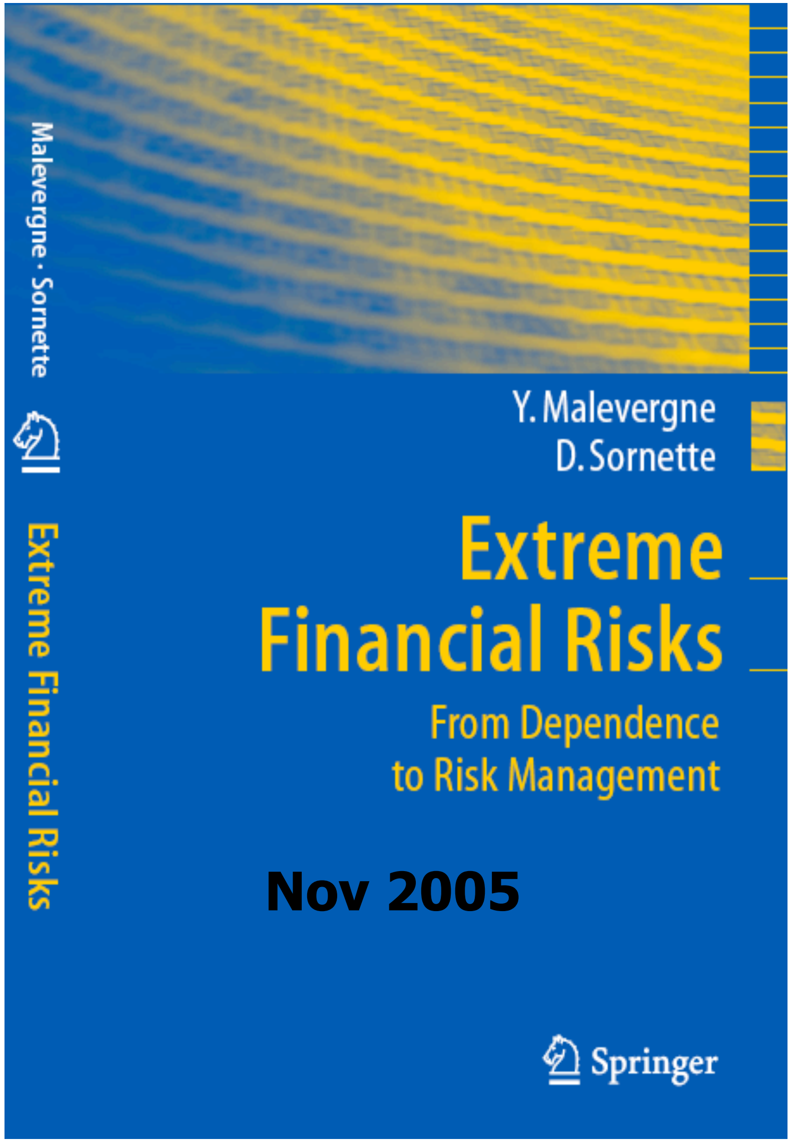
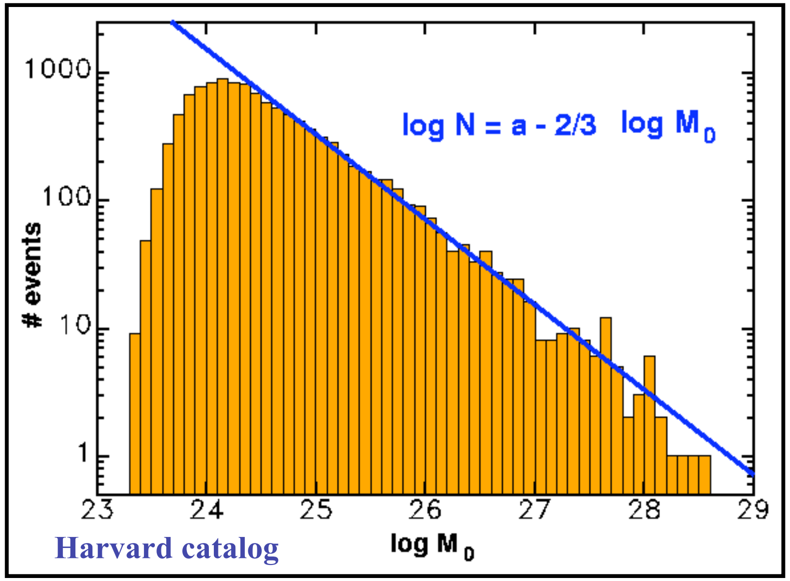
MECHANISMS FOR POWER LAWS

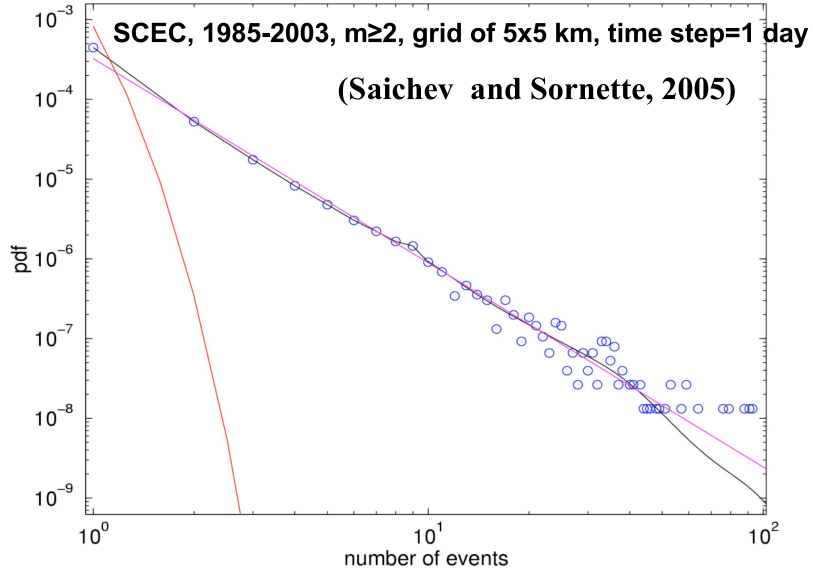




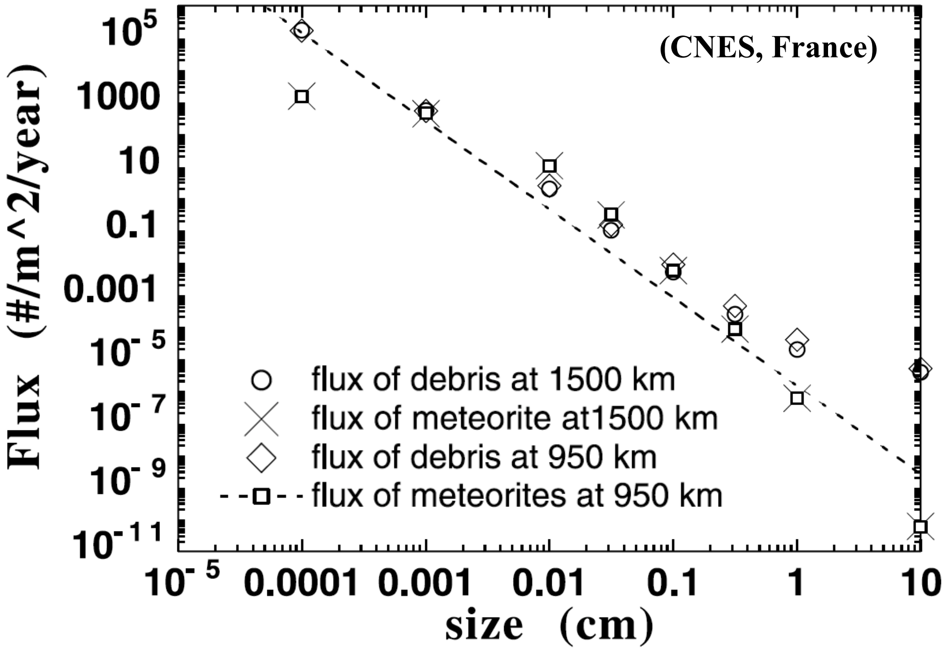
Heavy tails in pdf of earthquakes



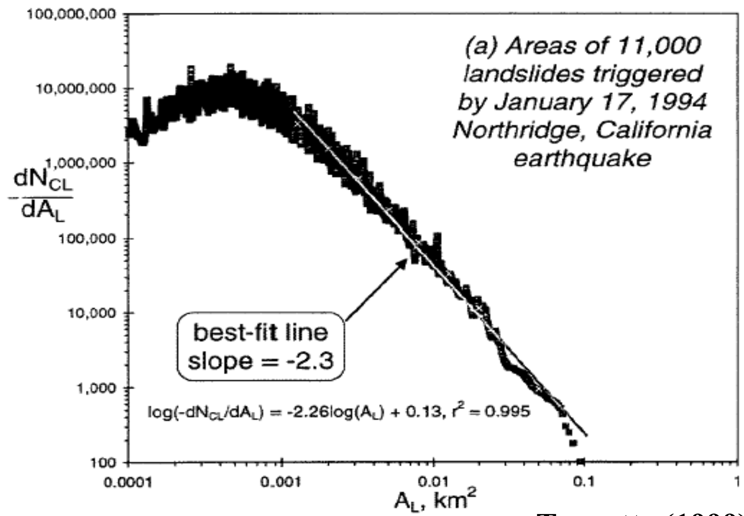
Heavy tails in pdf of seismic rates



Heavy tails in ruptures



Heavy tails in pdf of rock falls, Landslides, mountain collapses



Turcotte (1999)

Heavy tails in pdf of forest fires

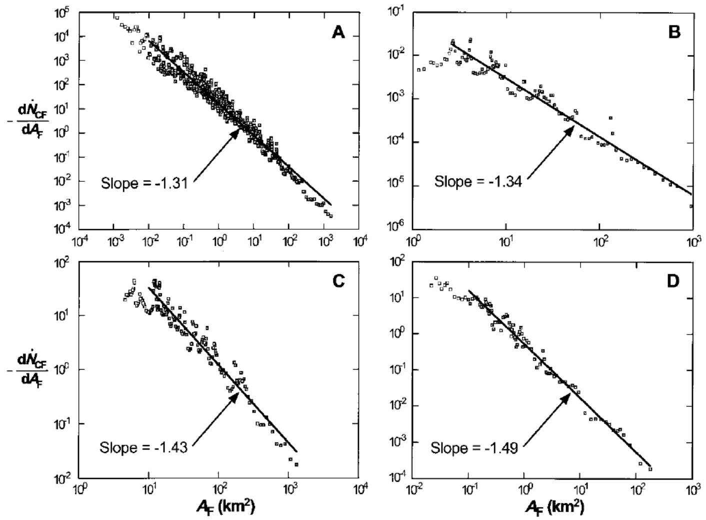
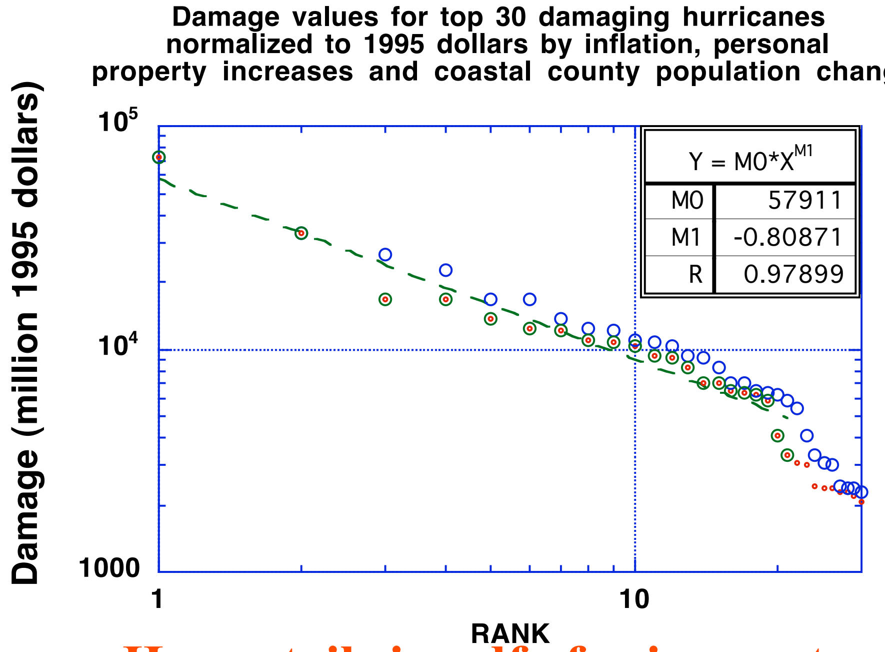


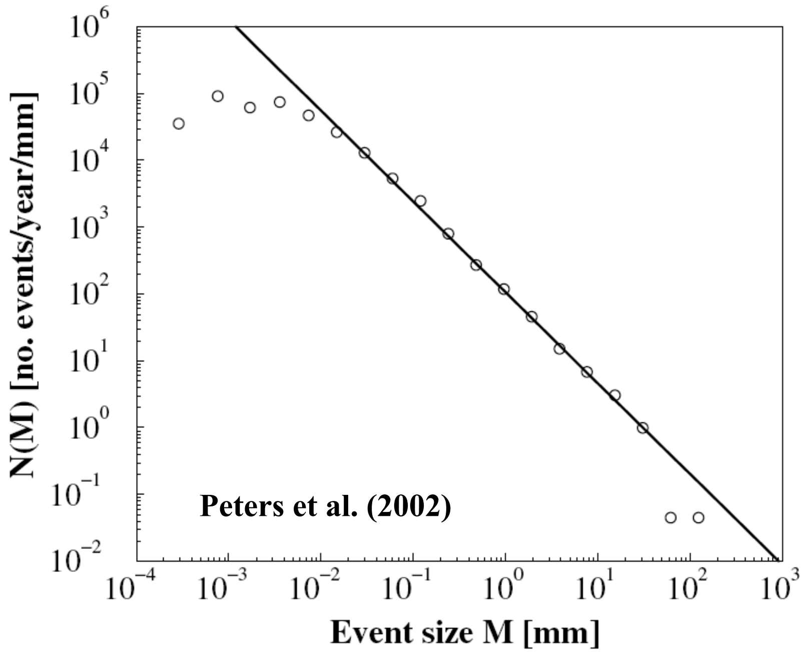
Fig. 2. Noncumulative frequency-area distributions for actual forest fires and wildfires in the United States and Australia: (**A**) 4284 fires on U.S. Fish and Wildlife Service lands (1986–1995) (*9*), (**B**) 120 fires in the western United States (1150–1960) (*10*), (**C**) 164 fires in Alaskan boreal forests (1990–1991) (*11*), and (**D**) 298 fires in the ACT (1926–1991) (*12*). For each data set, the noncumulative number of fires per year ($-d\dot{N}_{CF}/dA_F$) with area (A_F) is given as a function of A_F (*13*). In each case, a reasonably good correlation over many decades of A_F is obtained by using the power-law relation (Eq. 1) with α = 1.31 to 1.49; $-\alpha$ is the slope of the best-fit line in log-log space and is shown for each data set.

Malamud et al., Science 281 (1998)

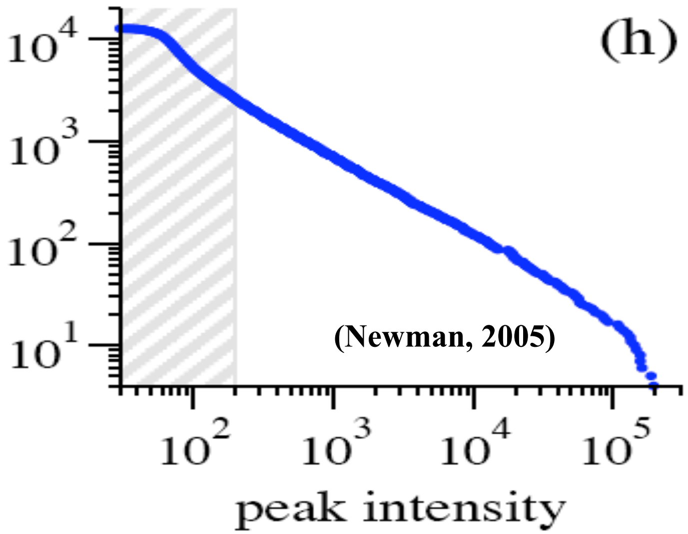
Heavy tails in pdf of Hurricane losses

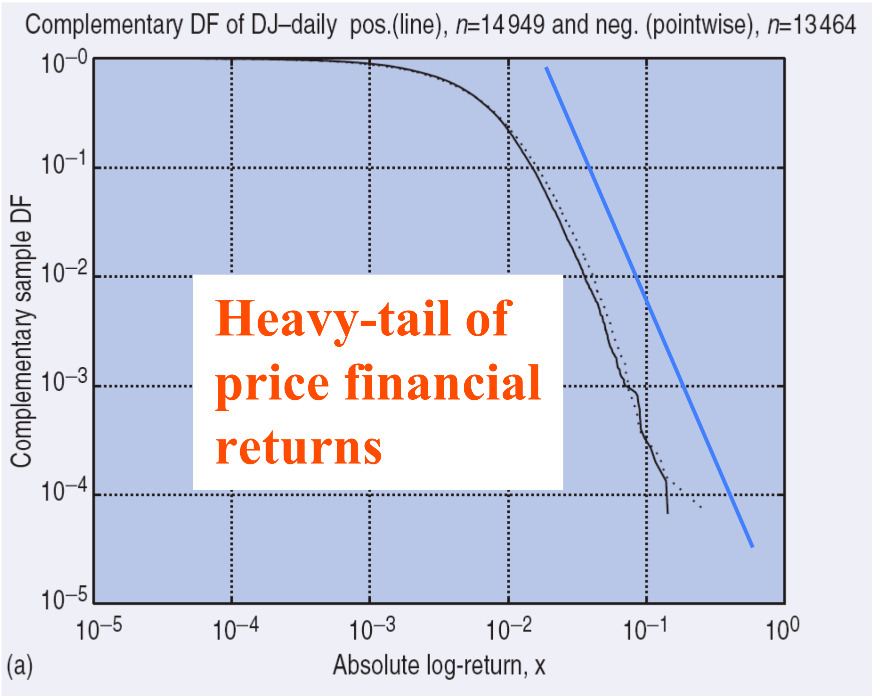


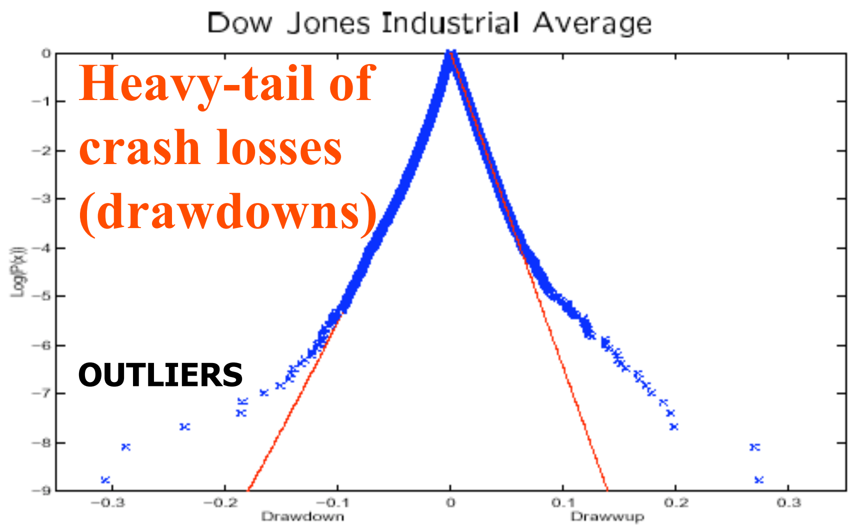
Heavy tails in pdf of rain events

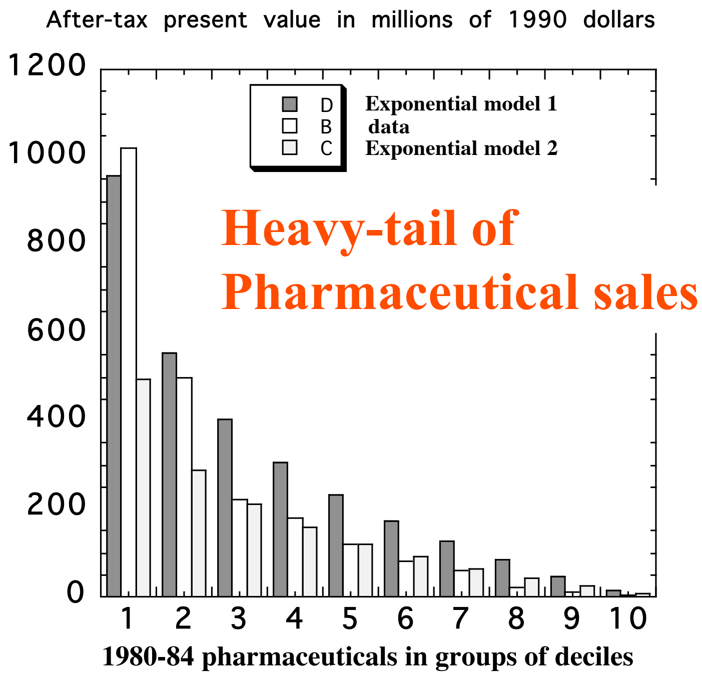


Heavy tails in pdf of Solar flares

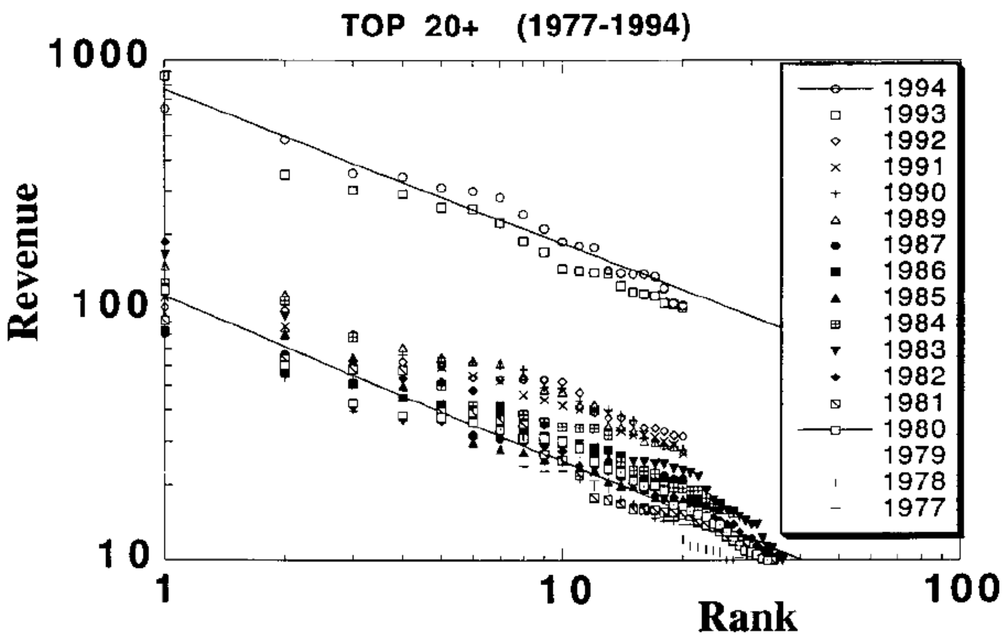




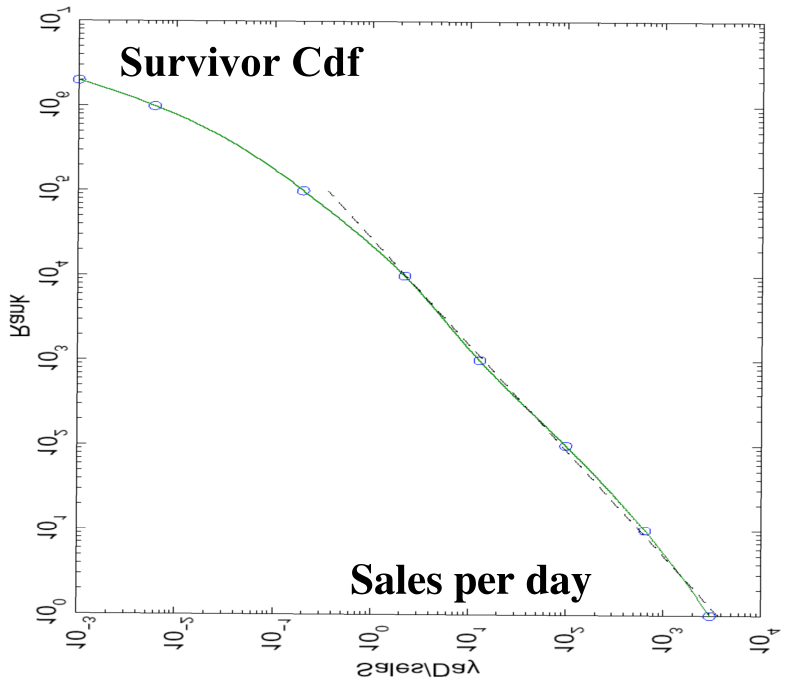




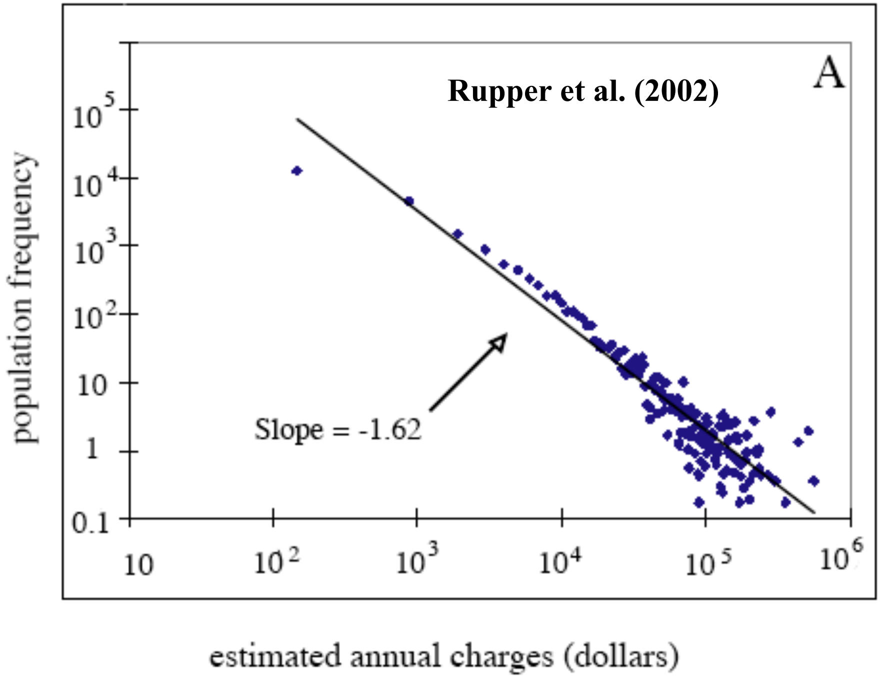
Heavy-tail of movie sales



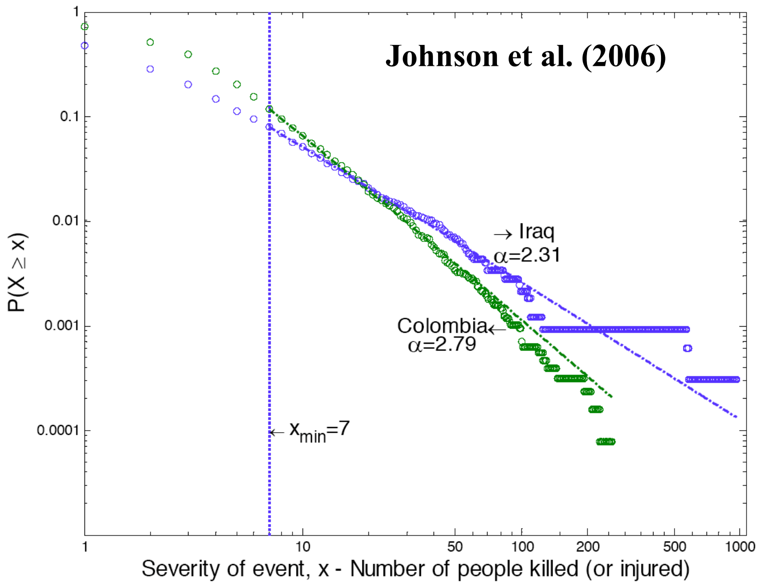
Heavy-tail of pdf of book sales



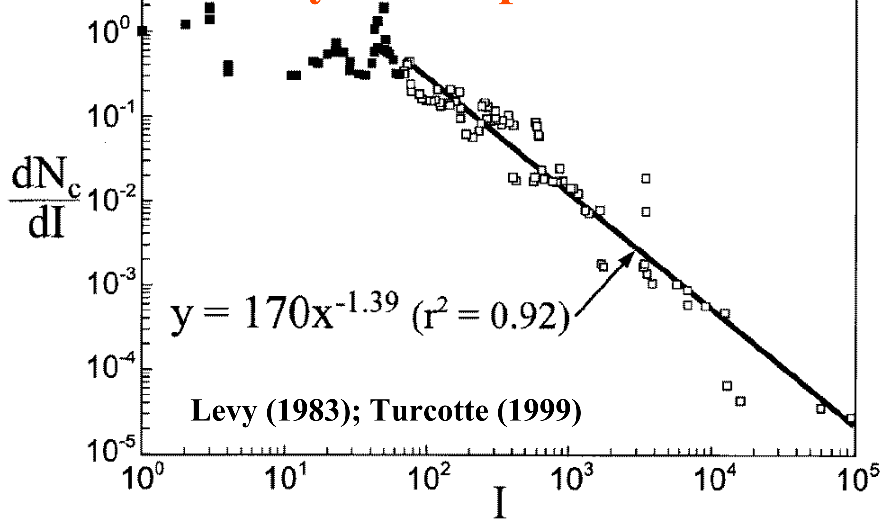
Heavy-tail of pdf of health care costs

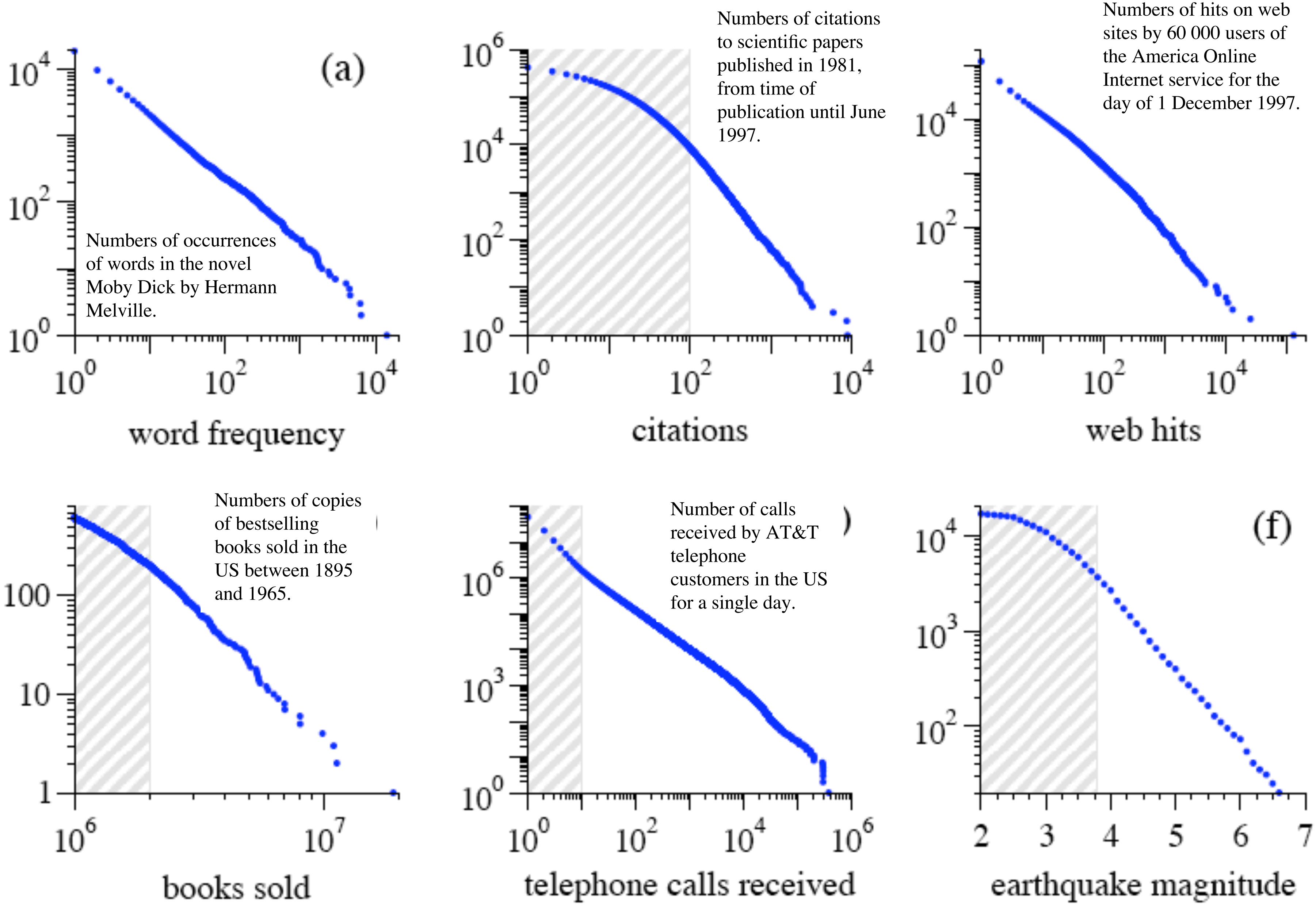


Heavy-tail of pdf of terrorist intensity

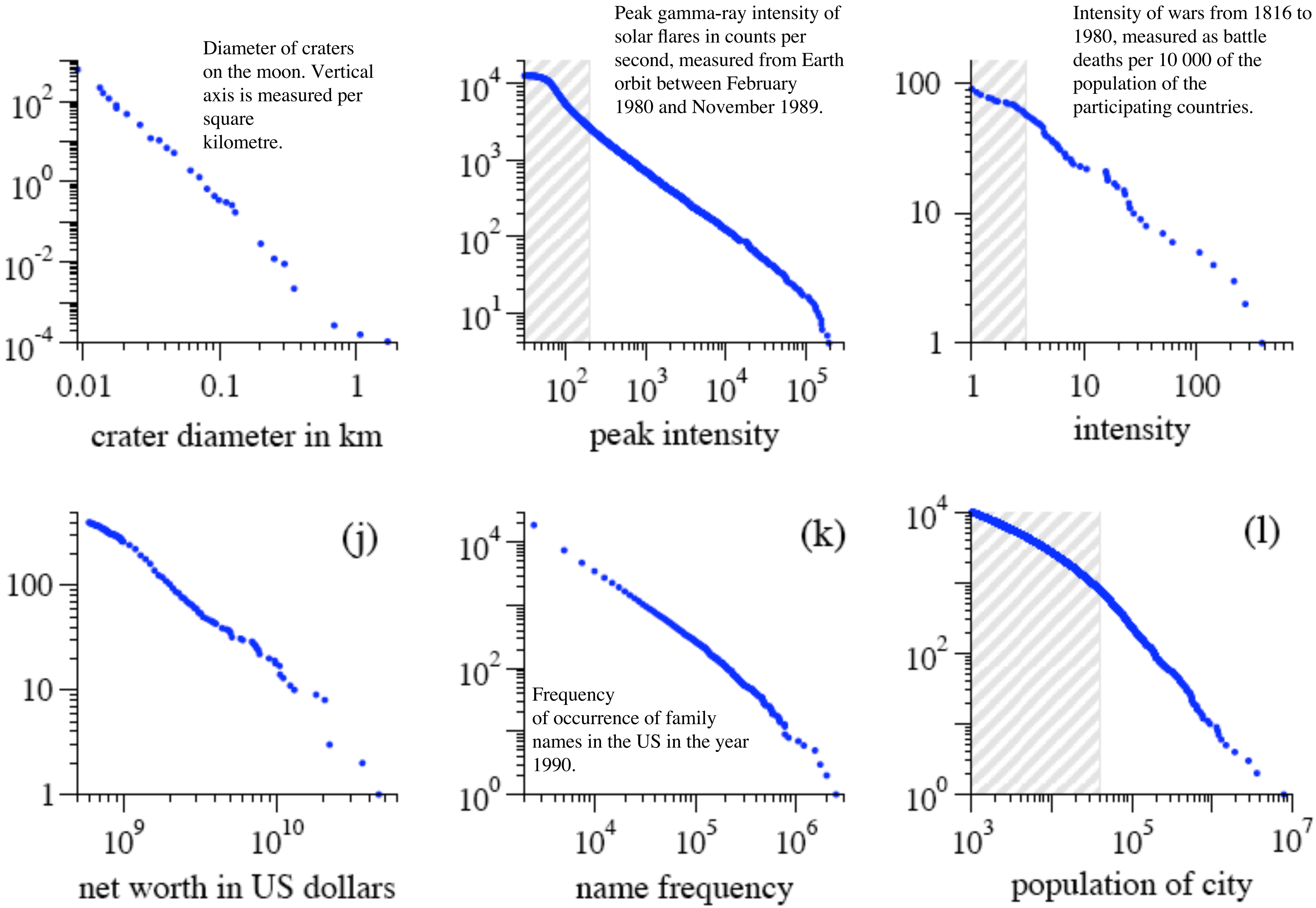


Heavy-tail of pdf of war sizes





M. E. J. Newman, Power laws, Pareto distributions and Zipf's law (2005)



M. E. J. Newman, Power laws, Pareto distributions and Zipf's law (2005)

Heavy Tails

- Probability density function *f*(*x*)

$$\Pr\{a \le x \le b\} = \int_a^b f(x)dx$$

- The distribution is called heavy tailed if it has infinite second moment:

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \infty$$

- For power-law distributions (Pareto distributions):

$$F(x) := \Pr\{\xi \le x\} = 1 - x^{-\beta}, x \ge 1$$

$$f(x) := dF(x)/dx = \beta x^{-\beta-1}, x \ge 1$$

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \int_1^{\infty} x^2 \beta x^{-\beta-1}dx = \frac{\beta}{2-\beta} x^{2-\beta}\Big|_1^{\infty} = \infty \Rightarrow \beta < 2$$

- Note also that if $\beta \le 1 \Rightarrow \int_1^{\infty} x\beta x^{-\beta-1}dx = \frac{\beta}{1-\beta} x^{1-\beta}\Big|_1^{\infty} = \infty$ (Infinite expectation)

MECHANISMS FOR POWER LAWS

1. percolation, fragmentation and other related processes,

2. directed percolation and its universality class of so-called “contact processes”,

3. cracking noise and avalanches resulting from the competition between frozen disorder and local interactions, as exemplified in the random field Ising model, where avalanches result from hysteretic loops [34],

4. random walks and their properties associated with their first passage statistics [35] in homogenous as well as in random landscapes,
5. flashing annihilation in Verhulst kinetics [36],

6. sweeping of a control parameter towards an instability [25, 37],

7. proportional growth by multiplicative noise with constraints (the Kesten process [38] and its generalization for instance in terms of generalized Lotka-Volterra processes [39], whose ancestry can be traced to Simon and Yule,

8. competition between multiplicative noise and birth-death processes [40],

D. Sornette, Probability Distributions in Complex Systems Encyclopedia of Complexity and System Science (Springer Science), 2007

9. growth by preferential attachment [32],

10. exponential deterministic growth with random times of observations (which gives the Zipf law) [41],

11. constrained optimization with power law constraints (HOT for highly optimized tolerant),

12. control algorithms, which employ optimal parameter estimation based on past observations, have been shown to generate broad power law distributions of fluctuations and of their corresponding corrections in the control process [42, 43],

13. on-off intermittency as a mechanism for power law pdf of laminar phases [44, 45],

14. self-organized criticality which comes in many flavors:

- cellular automata sandpiles with and without conservation laws,
- systems made of coupled elements with threshold dynamics,
- critical de-synchronization of coupled oscillators of relaxation,
- nonlinear feedback of the order parameter onto the control parameter
- generic scale invariance,
- mapping onto a critical point,
- extremal dynamics.

Mitzenmacher M (2004) A brief history of generative models for power law and lognormal distributions, Internet Mathematics 1, 226-251.
Newman MEJ (2005) Power laws, Pareto distributions and Zipf's law, Contemporary Physics 46, 323-351.

Change of Variable

Fundamental identity:

$$P(x)\,dx = P(y)\,dy\ , \tag{1}$$

• Power Law Change of Variable Close to the Origin

$$y = x^{-1/\alpha}\ , \tag{2}$$

then

$$P(y) = \frac{P(x)/\alpha}{y^{1+\frac{1}{\alpha}}}\ . \tag{3}$$

Suppose that $P(x)$ goes to a constant for $x \to 0$, then the distribution of y for large y is a power law with

$$\mu = \frac{1}{\alpha} \,. \qquad (4)$$

The uniform fluctuation of x close to zero lead to scale-free and arbitrarily large fluctuations of its inverse power y. The power law form is kept obviously (with a modification of the value of μ) if $P(x)$ itself goes to zero or diverges close to the origin as a power law.

Physical examples

• pdf of transport coefficients such as conductance, permeability and rupture thresholds and of necks between random holes or random cracks,

• pdf $P(u)$ of velocities u due to vortices:

$$P(u) \sim \frac{1}{u^3}$$

• The Holtsmark's distribution of gravitional forces created by a random distribution of stars in an infinite universe is a stable Lévy law with exponent $3/2$ and its power law tail results directly from the inversion mechanism (2), where y is the force

- Generalization to any other field with a suitable modification of the exponent, such as electric, elastic or hydrodynamics, with a singular power law dependence of the force as a function of the distance to the source.

- $-7/2$ power law pdf of density in the Burgers/adhesion model and singularities

- Berry’s “battles of catastrophes”

Distribution of $\delta M/M$ Ising model

This exemplifies the widespread and misled belief that power laws are equivalent to critical behavior.

At $T = T_c$, pdf $P(M)$ of magnetization is

$$P(M) \propto M^{(\delta-1)/2} \exp\{-\text{const}\ M^{\delta+1}\} \quad (5)$$

with $\delta + 1 \approx 5.8$ for the 3D Ising universality class. The central part of the distribution $P(M)$ is extremely well represented by the following ansatz

$$P(M) \propto \exp\{-(\frac{M^2}{M_0^2} - 1)^2(a\frac{M^2}{M_0^2} + c)\}\,. \quad (6)$$

Pdf of ΔM under a fixed number of Monte Carlo steps per site is a Gaussian at $T = T_c$.

Pdf of the relative changes $X \equiv \Delta M/M$ is a power law with exponent -2.

Explaination The distribution of $\Delta M = M_i - M_f$ is Gaussian, where M_i and M_f are the initial and final magnetisations over a time interval from 2 to 500 Monte Carlo steps per site.

We are interested in $P(X \to \infty)$, and large values of X come from the limit $M \to 0$ rather than $\Delta M \to \infty$.

The probability $P(M)$ is a constant δ for $M \to 0$, while ΔM can be approximated by the width Δ of the Gaussian; thus:

$$P(X)dX = P(\Delta M/M)d(\Delta M/M)$$

$$= P(M)dM \approx \text{const}\ dM$$

$$P(X) = P(M)/(dX/dM)$$

$$= \text{const}\ /(d(\Delta M/M)/dM) \propto 1/X^2$$

in agreement with the simulations (Metropolis or Heat Bath) of Jan et al.. The $1/X^2$ power law is in fact not restricted to the critical point and is very general since it results simply from the inversion mechanism.

Student's distribution with μ degrees of freedom

$$P_\mu(w) = \frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\sqrt{\mu \Pi}\,\Gamma\left(\frac{\mu}{2}\right)} \frac{1/s}{\left[1+\left(\frac{w}{s\sqrt{\mu}}\right)^2\right]^{(1+\mu)/2}}, \tag{7}$$

and is defined for $-\infty < w < +\infty$.

STUDENT DISTRIBUTION

The Student's distribution $P_\mu(w)$ has a bell-like shape like the Gaussian (and actually tends to the Gaussian in the limit $\mu \to \infty$) but is a power law $C/w^{1+\mu}$ for large $|w|$ with a tail exponent equal to the number μ of degrees of freedom and with a scale factor

$$C_\mu = \frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\sqrt{\mu \Pi}\,\Gamma\left(\frac{\mu}{2}\right)} \mu^{\frac{1+\mu}{2}} s^\mu . \tag{8}$$

s is the typical width of the Student's distribution.

This statistics is often used in the construction of tests and confidence intervals relating to the value $\langle x \rangle$ if σ is known. If σ is not known, it is reasonable to replace it by the estimator

$$S = [(n-1)^{-1} \sum_{j=1}^{n} (x_j - \bar{x})^2]^{1/2}$$

and study the statistics of

$$T = \frac{\sqrt{n}(\bar{x} - \langle x \rangle)}{S} = \frac{\sqrt{n}(\bar{x} - \langle x \rangle)}{[(n-1)^{-1} \sum_{j=1}^{n} (x_j - \bar{x})^2]^{1/2}} \tag{9}$$

In 1908, Student has derived the distribution of T as given by (7) with $w = T$, $\mu = n - 1$ is the number of degrees of freedom and $s = 1$.

Simple argument giving the power law shape $C/w^{1+\mu}$ with $\mu = n - 1$

A value of T larger or equal to X typically arises when all terms $|x_j - \bar{x}|$ in the denominator of (9) are smaller than a value proportional to $1/X$.

The probability that this occurs is proportional to the integral of their probability density from 0 to $1/X$ for each of the variable.

As the Gaussian part of the distribution goes to a constant for $1/T \to 0$, the probability that $|x_j - \bar{x}|$ be smaller than $1/X$ is only controlled by the width of the interval and thus proportional to $1/X$.

Since there are only $n-1$ independent variables in the sum defining the denominator S, the probability that T is larger than X is proportional to $1/X^{n-1}$, hence the power law tail with exponent $\mu = n - 1$.

Combination of Exponentials

Pdf $P(x) = e^{-x/x_0}/x_0$, for $0 \leq x < +\infty$

Let us assume that y is exponentially large in x:

$$y = e^{x/X} \,, \tag{10}$$

where X is a constant. Then by (1),

$$P(y) = \frac{\mu}{y^{1+\mu}} \,, \quad \text{with } \mu = \frac{X}{x_0} \text{ for } 1 \leq y < +\infty \tag{11}$$

The exponential amplification (10) of the fluctuations of x compensates the exponentially small probability for large excursions of x.

Example of activated escape of a system from a well

Kramers' problem $\rightarrow$ Arrhenius activation law

$$\tau \propto \tau_0 \, e^{\beta \Delta E} \,, \qquad (12)$$

where β is the inverse temperature and ΔE is the height of the barrier.

Many different wells, with Poisson distribution $P(E) \propto e^{-\Delta E / E_0}$. The result (11) implies a power law distribution of residence times

$$P(\tau) \propto \tau^{-1-\mu}$$

with

$$\mu = 1/\beta E_0$$

The cooler the system, the larger the inverse temperature β, the smaller the exponent μ and the "fatter" the tail of the power distribution of residence times. A good example of this situation occurs in magnetic systems with quenched random impurities exhibiting the phenomenon of "aging".

$\mu = 1$ is a glass transition since lower temperatures correspond to smaller exponents such that the average residence time becomes infinite. This infinite average residence time implies non-stationarity and "aging": any expectation of a transition that is estimated at any given time depends on the past in a manner which does not decay. This is a hallmark of aging.

Yule's process

Nb of species of a given genus;
Size of a given firm

$$S_i(t) = s_0 e^{c(t-t_i)}$$

Growth of the number of genus;
growth of the number of firms

$$\nu(t) = \frac{dH(t)}{dt} = H_0 \, D \, e^{Dt}$$

$$\mathrm{Pdf}(S) \sim 1/S^{1+\gamma}$$

$$\gamma = \frac{D}{c}$$

Zipf's law $\gamma=1$ for $D=c$

The rv y, conditioned on a characteristic width σ, is distributed according to the "bare" distribution

$$P_\sigma(y) = C_1 \, e^{-f(y/\sigma)} \; , \tag{13}$$

where $f(y/\sigma) \to 0$ sufficiently fast for large y/σ to ensure normalization.

SUPERPOSITION OF DISTRIBUTIONS

Mixture of different values of σ with pdf

$$\Sigma(\sigma) \equiv C_2 e^{-g(\sigma)} \tag{14}$$

Then, the observed distribution of y is the following mixture of distributions

$$P_{\text{ren}}(y) = \int_0^\infty d\sigma \, \Sigma(\sigma) \; P_\sigma(y)$$

$$= C_1 C_2 \int_0^\infty d\sigma \; e^{-f(y/\sigma) - g(\sigma)} \; . \tag{15}$$

Saddle-node method:

$$P_{\text{ren}}(y) \sim \frac{1}{\sqrt{F''(\sigma^*(y))}}\, e^{-F(\sigma^*(y))} , \qquad (16)$$

$$F(\sigma) \equiv f(y/\sigma) + g(\sigma) \qquad (17)$$

and $\sigma^*(y)$ is the solution of the saddle-node condition

$$\frac{y}{\sigma}\, f'(y/\sigma) = \sigma\, g'(\sigma) . \qquad (18)$$

The only solution of (18), such that $P_{\text{ren}}(y)$ is asymptotically proportional to $y^{-1-\mu}$ can be shown to be $\sigma^*(y) \propto y$, which implies from (16) and by solving (18) that $g(\sigma)$ must be of the form

$$g(\sigma) = (2+\mu) \ln y + \text{constant} . \qquad (19)$$

Therefore, the distribution of the widths must be an asymptotic power law distribution with an exponent larger than the required one by one unit, due to the effect of the $1/\sqrt{F''\sigma^*(y)}$ factor in (16):

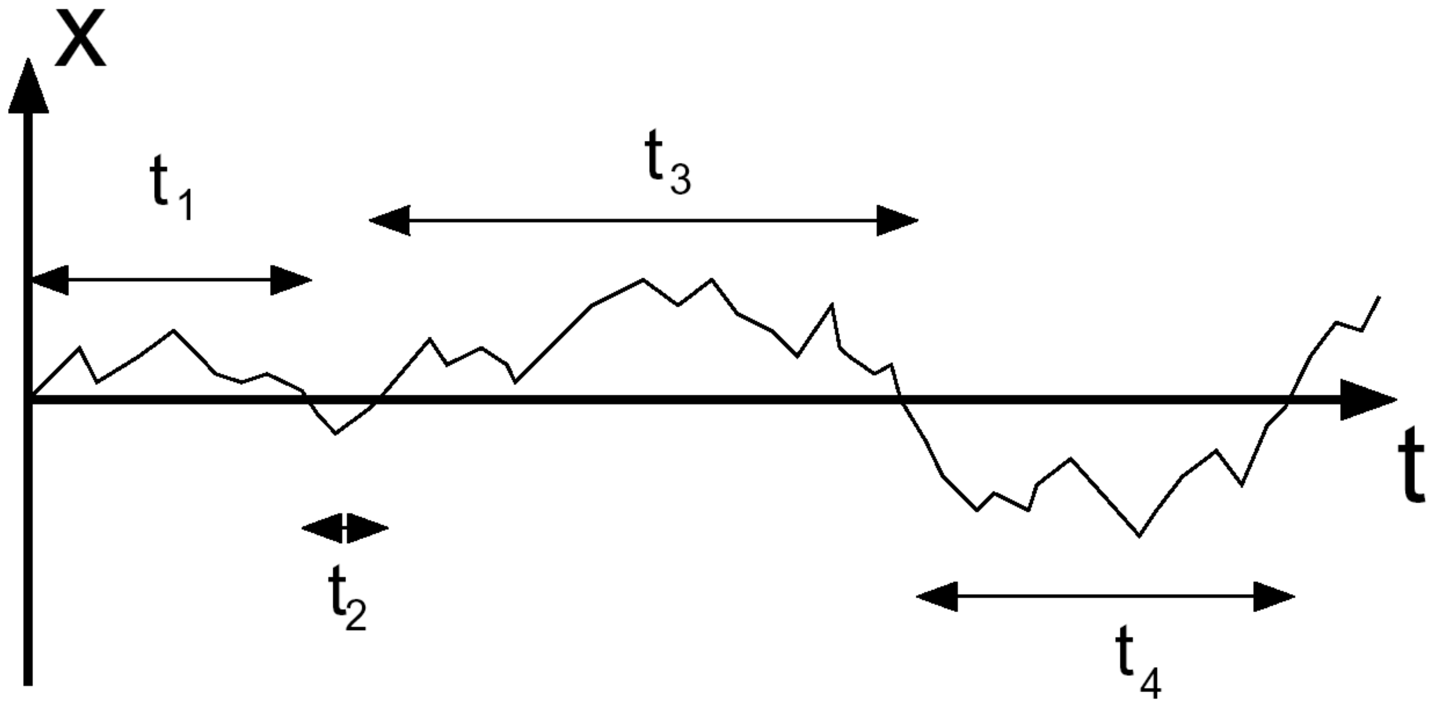
$$g(\sigma) \sim \frac{1}{\sigma^{2+\mu}} \,. \tag{20}$$

This mechanism holds as long as $P_\sigma(y)$ falls off faster than a power law at large y. This ensures that large realizations of y correspond to large width $\sigma \propto y$ occurrences.

Pdf to return to the origin

$$P_G(t) = \frac{1}{\sqrt{2\Pi}} \frac{1}{\sqrt{2Dt}} \sim t^{-1/2}\,, \qquad (21)$$

where D is the diffusion coefficient.



The probability $F(t)$ to return to the origin for the first time at time t after starting from the origin at time 0 :

$$P_G(t) = \delta(t) + \int_0^t dt' \; P_G(t') \; F(t - t') \; . \quad (22)$$

Laplace transform:

$$\widehat{P}_G(\beta) = 1 + \widehat{P}_G(\beta) \; \widehat{F}(\beta) \quad (23)$$

$$\widehat{F}(\beta) = \frac{\widehat{P}_G(\beta) - 1}{\widehat{P}_G(\beta)} \quad (24)$$

$$\hat{P}_G(\beta) \approx_{\beta \to 0} \int_{t_{min}}^{\infty} dt \, \frac{e^{-\beta t}}{\sqrt{4\Pi D t}} \approx \frac{1}{C \beta^{1/2}} \,, \tag{25}$$

where $1/C = \int_0^\infty dx \frac{e^{-x}}{\sqrt{4\Pi D x}}$ is a constant.

$$\hat{F}(\beta) \approx_{\beta \to 0} 1 - C\beta^{1/2} \approx e^{-C\beta^{1/2}} \,. \tag{26}$$

Its inverse Laplace transform has a tail

$$F(t) \sim_{t \to \infty} t^{-3/2} \,. \tag{27}$$

An exact calculation gives

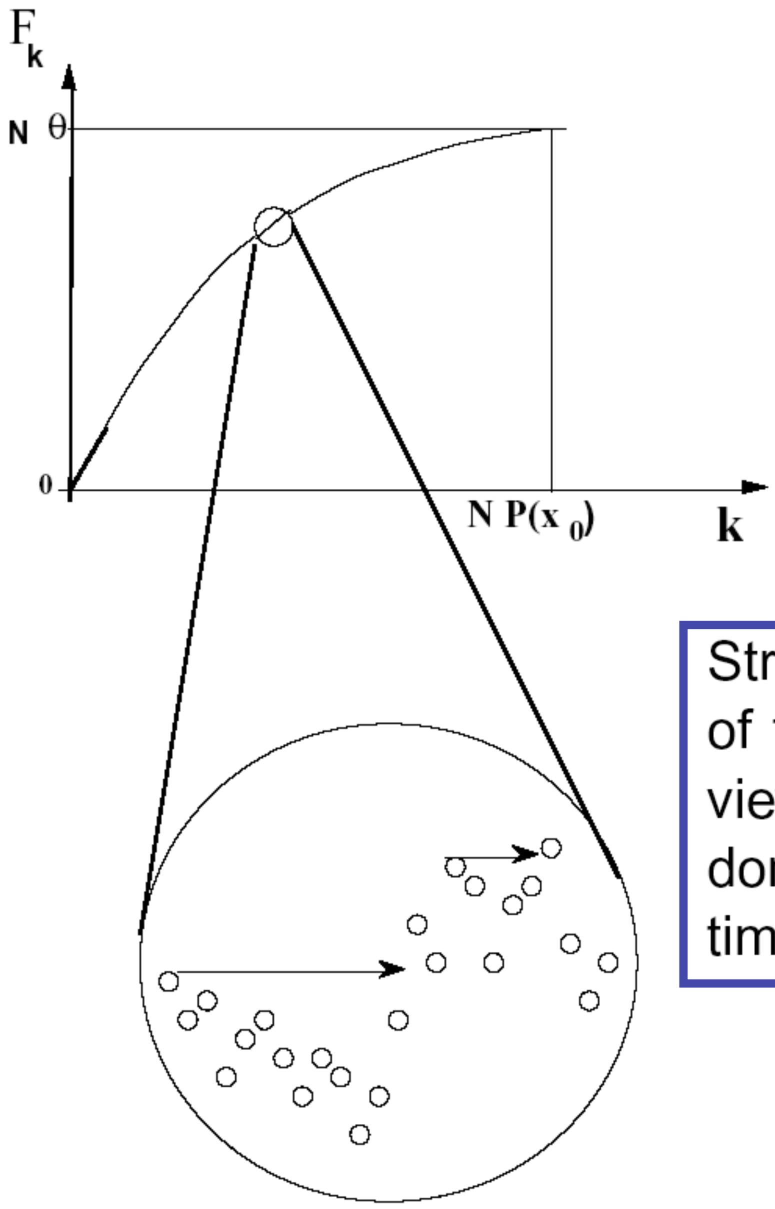
$$F(t) = \frac{C}{\sqrt{2\Pi}\, t^{3/2}} \, e^{\frac{-C^2}{2t}} \,. \tag{28}$$

Stable Lévy distribution with exponent $\mu = 1/2$

An application: pdf of fiber rupture bursts

Rupture thresholds $X_1 \leq X_2 \leq ... \leq X_N$

$$F_k = X_k(N - k + 1)$$



Strength F_k of a bundle of N fibers as a function of the number k of broken fibers; the magnified view of the dependence of F_k illustrates the random walk in the space of forces F_k, the role of time being played by k

Hemmer and Hansen (1992)

Sweeping of Control Parameter Towards Instability

Example on the Ising and Percolation models

Fluctuations: spatial cooperative domains of all sizes between the microscopic scale up to the correlation length ξ in which the order parameter takes a non-zero value over a finite duration .

$$P_p(s)ds \sim s^{-a} f(\frac{s}{s_0(p)})ds \ , \qquad (29)$$

with $a = 2 + \frac{1}{\delta}$ ($= 2.05$ with $\delta = 91/5$ in 2D)

Typical cluster size $s_0 \sim \mid p_c - p \mid^{-\frac{1}{\sigma}}$ with Fisher's notation $\frac{1}{\sigma} = \gamma + \beta$.

Mean cluster size $\langle s\rangle(p) \sim\mid p_c - p \mid^{-\gamma}$. γ is the susceptibility exponent defined by the number of spins which are affected by the flip of a single spin $=$ mean cluster size.

$f(s/s_0(p))$ decays rapidly (exponentially or as an exponential of a power law) for $s > s_0$. $P_p(s)ds \sim s^{-a}$ for $s < s_0(p)$ and $P_p(s)$ is negligibly small for $s > s_0(p)$.

Let us monitor the fluctuation amplitudes (*i.e.* cluster sizes) as the control parameter p is swept across its critical value p_c, say from the value $p = 0$ to $p = 1$. The total number of clusters of size s which are measured is then proportional to

$$N(s) = \int_0^1 P_p(s)dp \; , \qquad (30)$$

$$N(s) = \int_0^{p_c} P_p(s)dp + \int_{p_c}^1 P_p(s)dp \; . \qquad (31)$$

Change of variable $p \to s_0(p)$:

$$N(s) = s^{-a} \int_1^{+\infty} s_0^{-\sigma(1+\frac{1}{\sigma})} f(\frac{s}{s_0(p)}) ds_0$$

$$\simeq s^{-a} \int_s^{+\infty} s_0^{-(1+\sigma)} ds_0 \; ,$$

using the fact that $f(s/s_0(p))$ is negligible for $s_0(p) < s$.

$$N(s) \simeq s^{-(a+\sigma)} \; . \qquad (32)$$

Avalanches in Hysteretic Loops

An hysteresis loop is the graph of the response which lags behind the force.

In many materials, the hysteresis loop is composed of small bursts, or avalanches, which cause acoustic emission (crackling noises); in magnets, they are called Barkhausen noise.

Disorder and pinning $\rightarrow$ wide range of avalanche scales

Random field f_i at each site of the Ising model

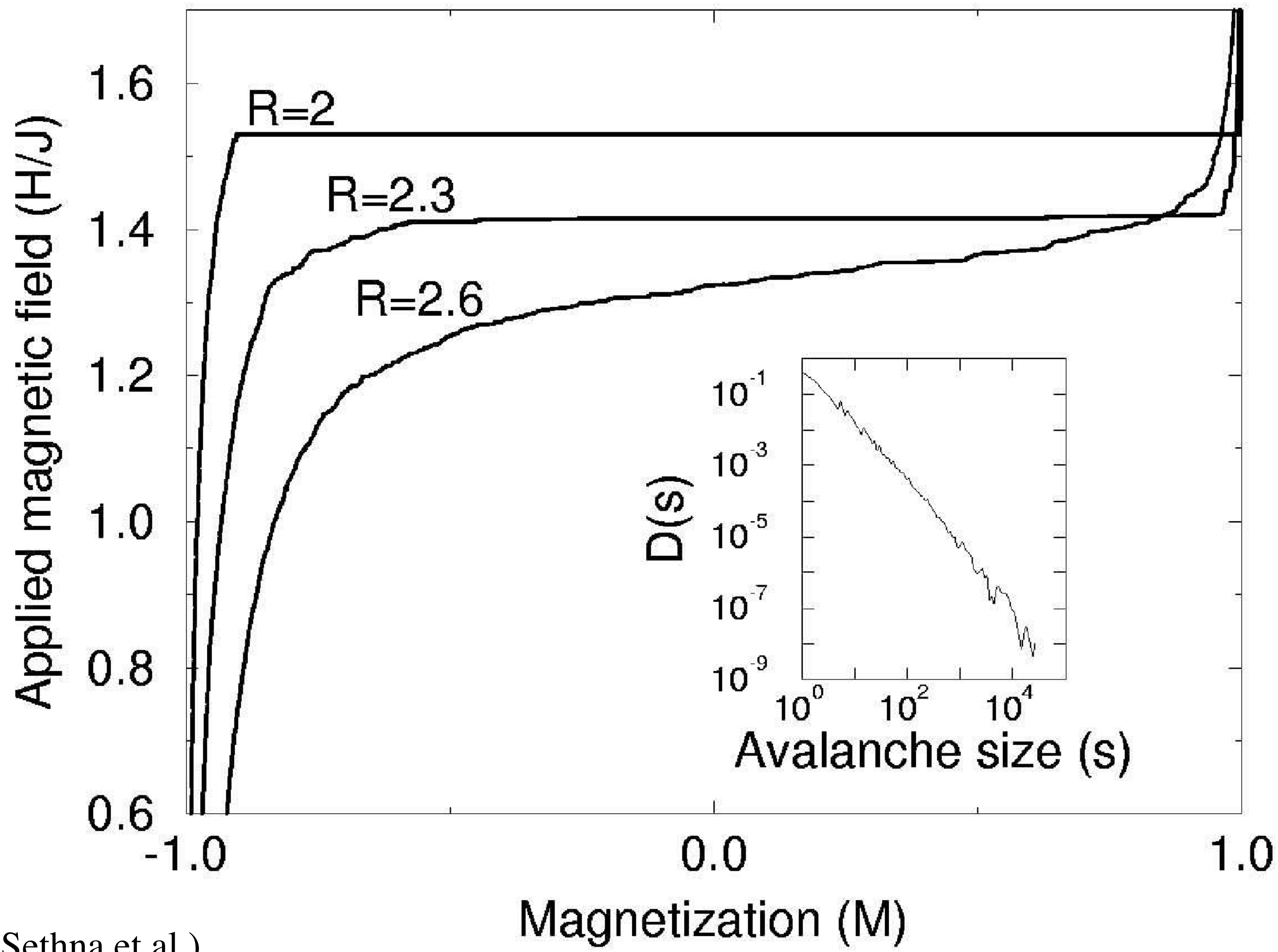
$$\mathcal{H} = -\sum_{ij} J_{ij} s_i s_j - \sum_i (f_i s_i + H s_i) \quad . \tag{33}$$

Rule: each spin flips when the direction of its total local field changes

$$F_i \equiv \sum_j \Big(J_{ij} s_j + f_i + H \Big) \tag{34}$$

Mean field theory:

$$M(H) = 1 - 2 \int_{-\infty}^{-JM(H)-H} P(f)\, \mathrm{d}f$$



R=2
R=2.3
R=2.6
Applied magnetic field (H/J)
1.6
1.4
1.2
1.0
0.8
0.6
-1.0
0.0
1.0
Magnetization (M)
D(s)
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{0}
10^{2}
10^{4}
Avalanche size (s)

(Sethna et al.)

Three $H(M)$ curves for different levels of disorder for a 60^3 system. The estimate of the critical disorder is $R_c = 2.16J$ (J is set equal to 1 in the figure). At $R = 2 < R_c$, there is an infinite avalanche which seems to appear abruptly. For $R = 2.6 > R_c$, the dynamics is macroscopically smooth, although of course microscopically it is a sequence of sizable avalanches. At $R = 2.3$, near the critical level of disorder, extremely large events become common.

Inset: Log-Log Plot of the avalanche-size distribution $D(s)$ vs. avalanche size s for the 60^3 system at $R = 2.3$ for $1.3 < H < 1.4$, averaged over 20 systems. Here $D(s) \sim s^{-1.7}$, compared to the mean–field exponent τ of $3/2$. Reproduced from Sethna et al.

Growth with Preferential Attachment

This is nothing but Simon (1956)'s model for firms translated for the Internet

Let us start with a single page, with a link to itself. At each time step, a new page appears, with outdegree 1. With probability $p < 1$, the link for the new page points to a page chosen uniformly at random. With probability $1 - p$, the new page points to a page chosen proportionally to the indegree of

$X_j(t)$: number of pages with in-degree j at time t

Probability tha $X_j(t)$ increases is $\frac{pX_{j-1}}{t} + \frac{(1-p)(j-1)X_{j-1}}{t}$

probability that X_j decreases is $\frac{pX_j}{t} + \frac{(1-p)jX_{j-1}}{t}$

for $j > 1$, the growth of X_j is approximated by

$$\frac{\mathrm{d}X_0}{\mathrm{d}t} = 1 - \frac{pX_0}{t}$$

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = \frac{1}{t}\left[p(X_{j-1} - X_j) + (1-p)\left((j-1)X_{j-1} - jX_j\right)\right]$$

Steady state $X_j(t) = c_j \, t,$

c_j as the steady-state fraction of pages with indegree j.

Recurrence equation

$$c_j(1 + p + j(1 - p)) = c_{j-1}(p + (j - 1)(1 - p))$$

For large j: $$\frac{c_j}{c_{j-1}} = 1 - \frac{2 - p}{1 + p + j(1 - p)} \sim 1 - \frac{2 - p}{1 - p} \, \frac{1}{j}$$

$$c_j \sim \frac{C}{j^{1+\mu}}, \qquad \text{with} \quad \mu = \frac{1}{1 - p}$$

Yule (1925); Simon (1955); Lokta, Zipf, ….

Kesten process
stochastic recurrence equations

$$X_{t+1} = aX_t + b, \qquad (1)$$

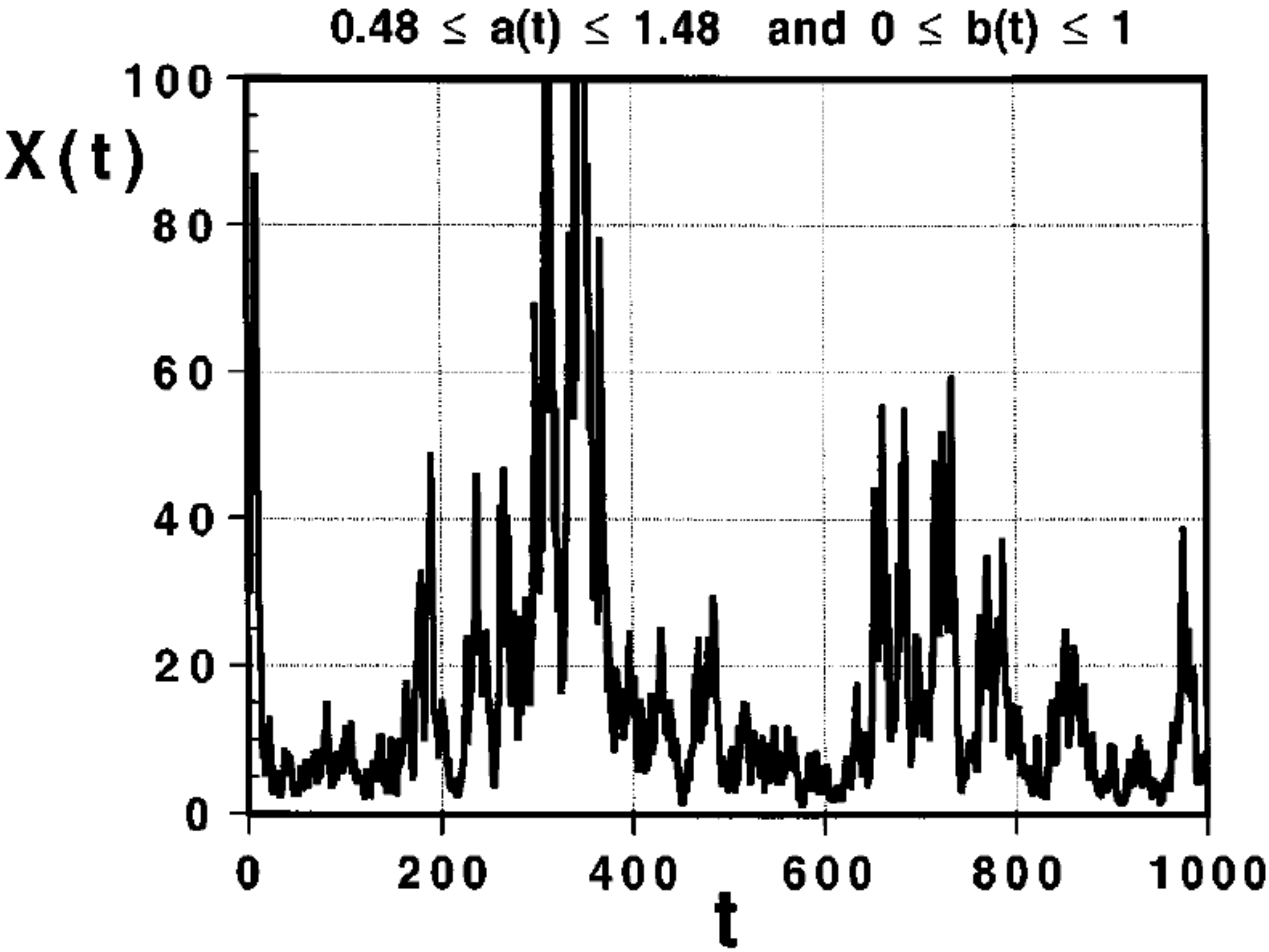
The distribution of X_t gives an histogram characterized by a power law tail

$$P(X) \sim X^{-(1+\mu)} \quad , \qquad (2)$$

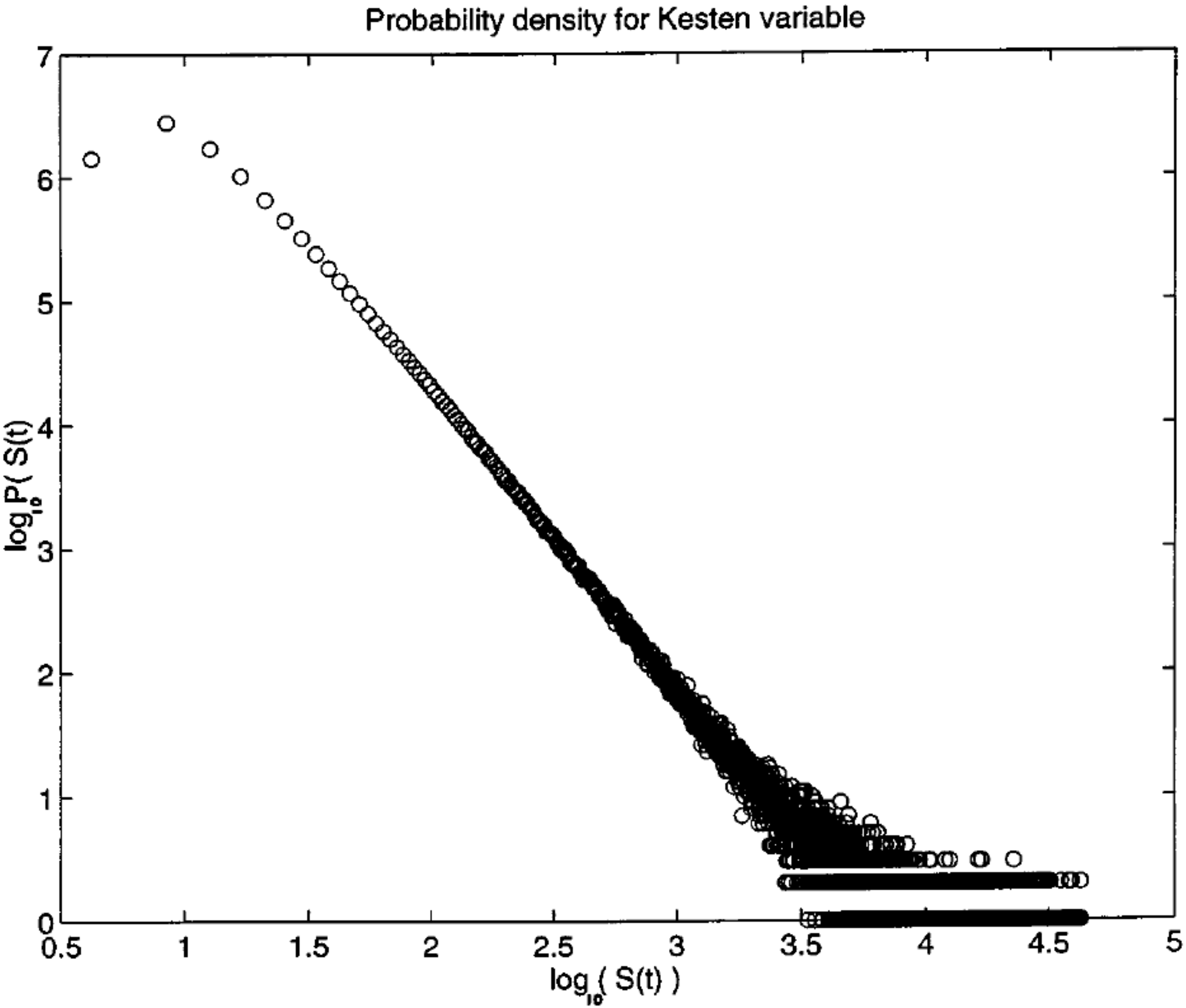
if there is a solution $\mu > 0$ of the equation

$$\langle a^{\mu} \rangle = 1 \quad . \qquad (3)$$

$$X_{t+N} = \left(\prod_{l=0}^{N-1} a_{t+l} \right) X_t + \sum_{l=0}^{N-1} b_{t+l} \prod_{m=l+1}^{N-1} a_{t+m},$$



0.48 ≤ a(t) ≤ 1.48 and 0 ≤ b(t) ≤ 1
X(t)
100
80
60
40
20
0
0
200
400
600
800
1000
t



Probability density for Kesten variable
log10 P(S(t))
log10 (S(t))
0
1
2
3
4
5
6
7
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

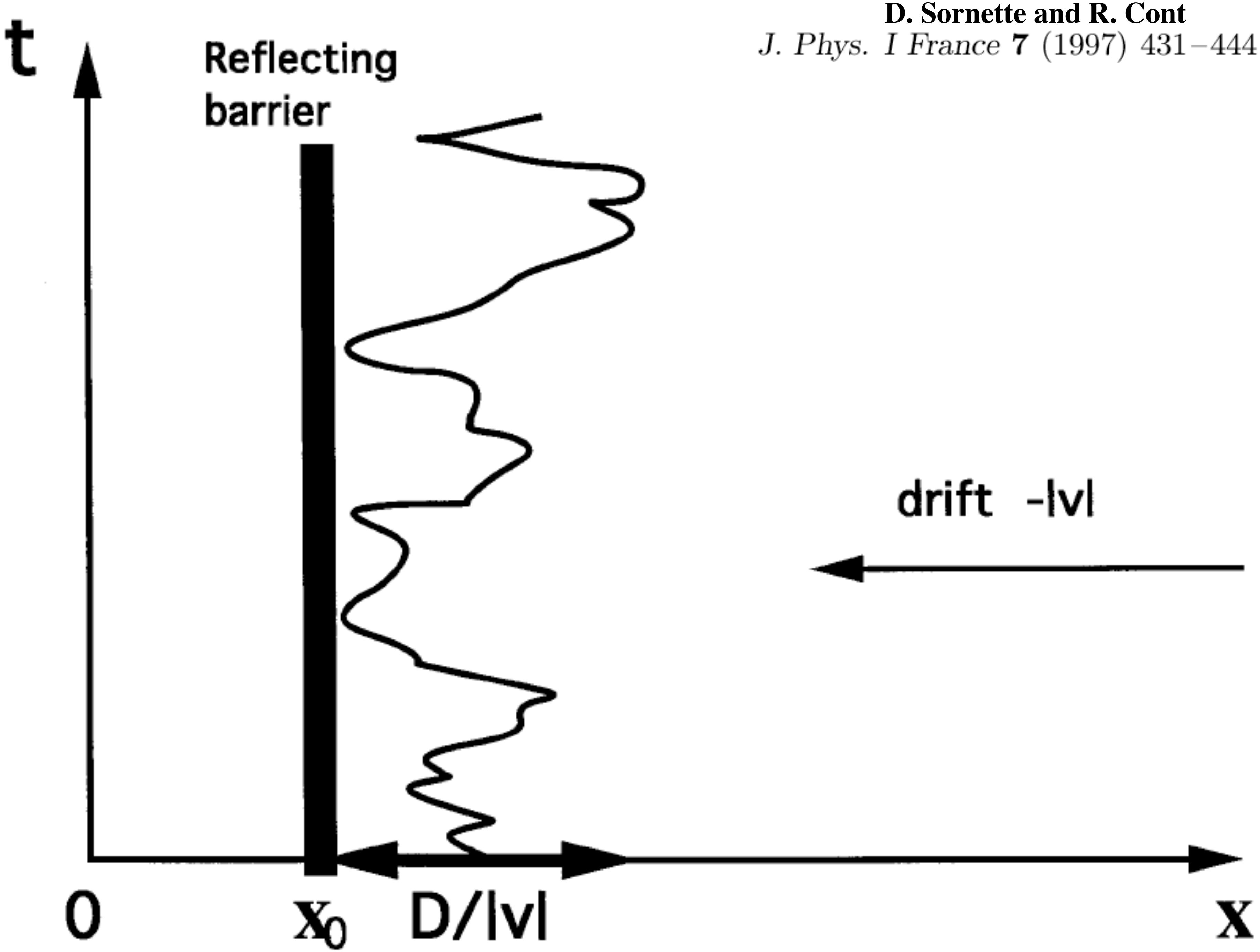
$$P_{X_{t+1}}(X) = \int_{-\infty}^{\infty} P_{a_t}(a)\,da \int_{-\infty}^{\infty} P_{b_t}(b)\,db \int_{-\infty}^{\infty} P_{X_t}(Y)$$

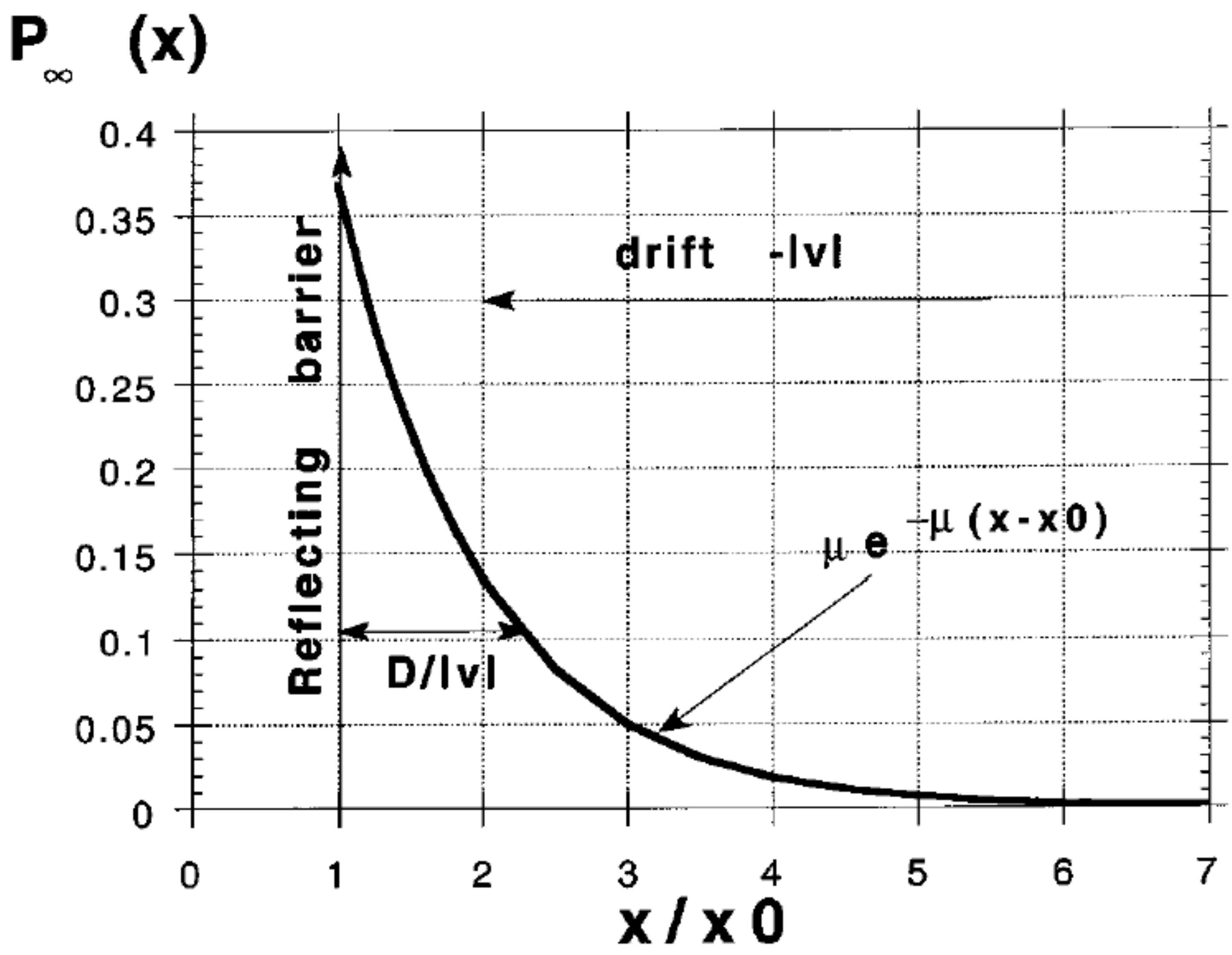
$$\times\, \delta(X - aY - b)\,dY$$

or

$$P_{X_{t+1}}(X) = \int_{-\infty}^{\infty} \frac{P_{a_t}(a)}{a}\,da \int_{-\infty}^{\infty} P_{b_t}(b)\, P_{X_t}\!\left(\frac{X-b}{a}\right) db.$$

$$P(X) = \int_{-\infty}^{\infty} \frac{P_{a_t}(a)}{a}\, P\!\left(\frac{X}{a}\right) da \qquad \text{for large } X,$$


t
Reflecting
barrier
drift -|v|
0
x0
D/|v|
x



P∞ (x)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
x / x 0
drift -|v|
Reflecting barrier
D/|v|
μ e^{-μ (x - x0)}

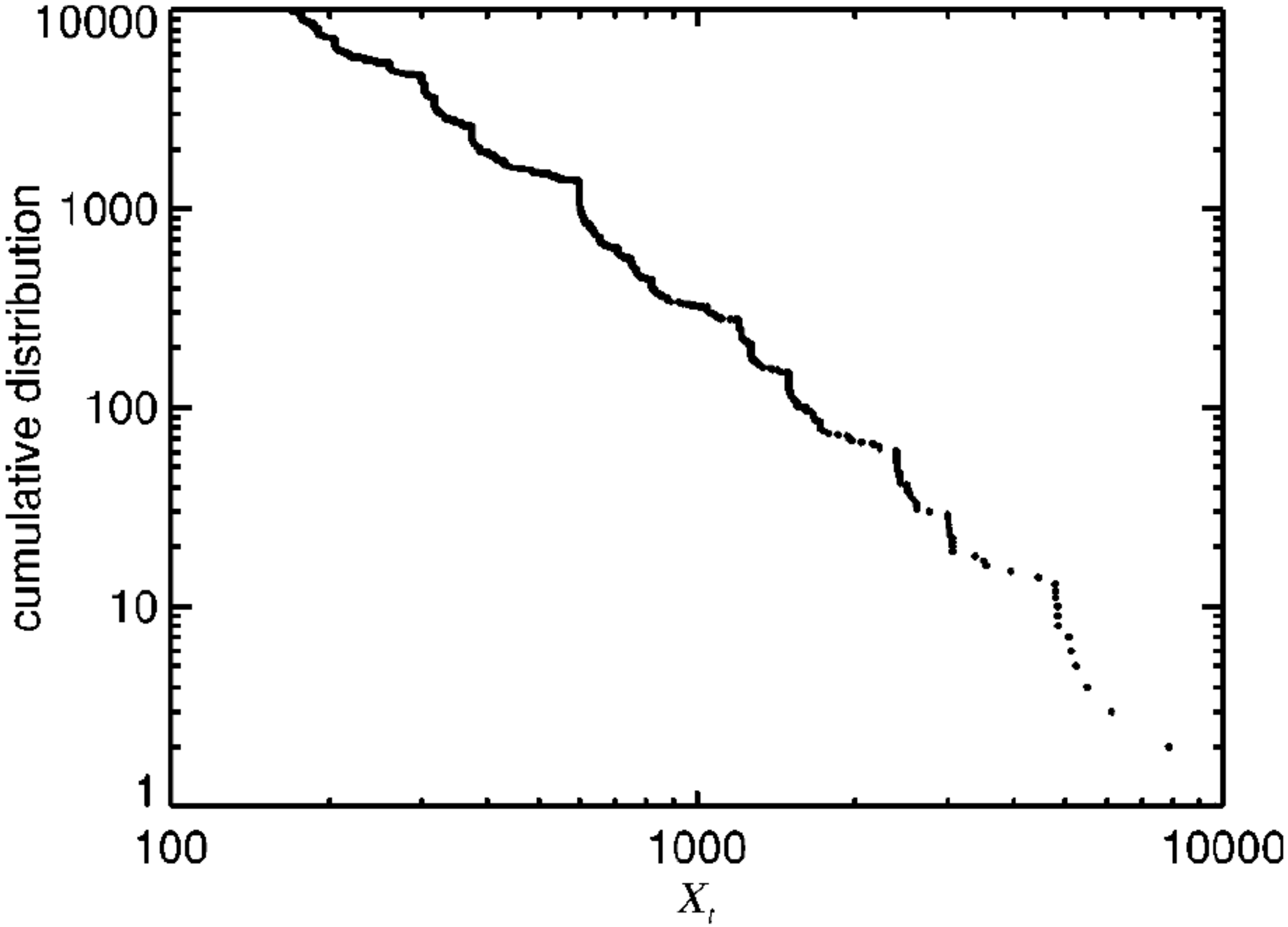


FIG. 2. Cumulative distribution of the 10^4 largest iterates among 10^8 realized for a_t with a two-point distribution at $a=2$, $p=0.95$, and $b_t=1$.

$$P(X)=paP(aX)+(1-p)a^{-\xi}P(a^{-\xi}X).$$

P. Jogi, D. Sornette and M. Blank, Fine structure and complex exponents in power law distributions from random maps, Phys. Rev. E 57 120-134 (1998)

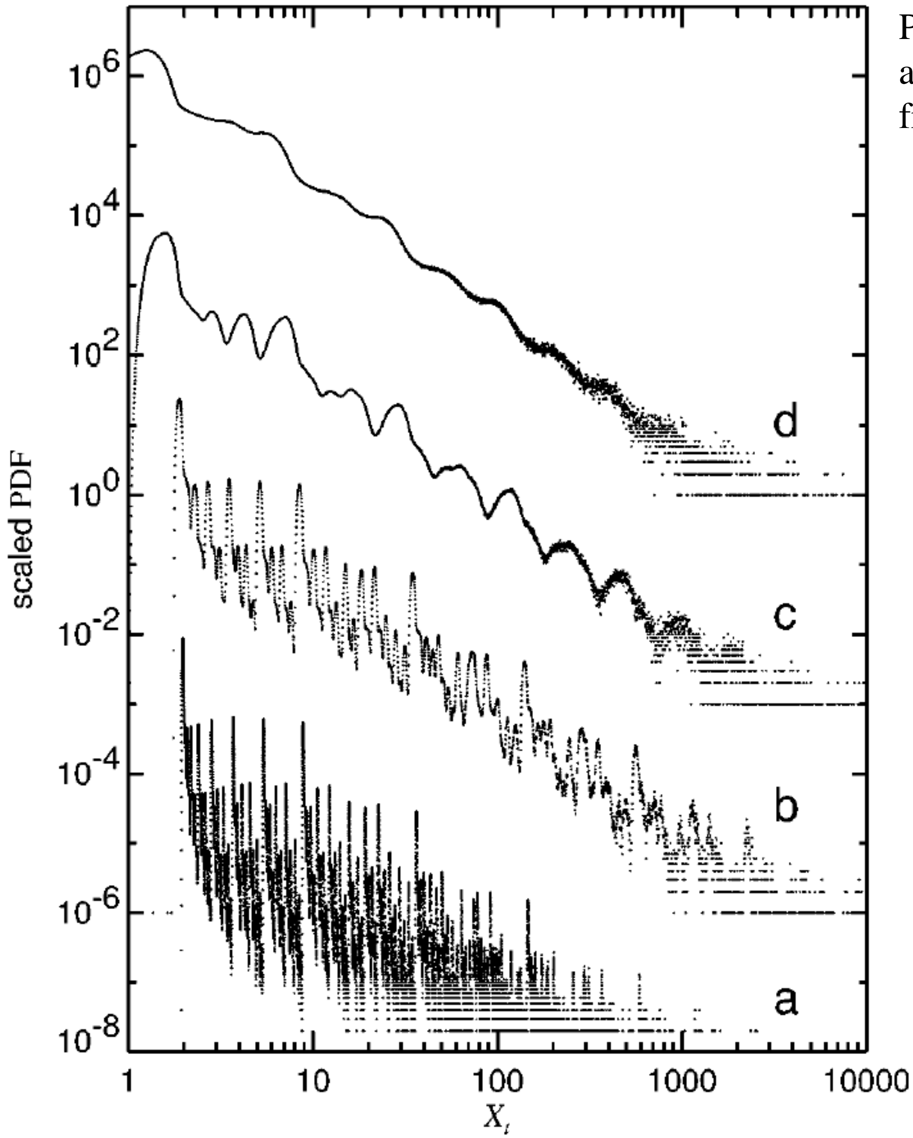


FIG. 4. Scaled PDF of X_t given by Eq. (1) for a_t with a two-point distribution at $a=2$, $\xi=2$, $p=0.95$, and b_t uniform with (a) $1\times$PDF and $\beta=\frac{31}{32}$(10^8 iterates, 10^4 equispaced bins per unit of $\log X_t$), (b) $10^3\times$PDF and $\beta=\frac{7}{8}$ [10^9 iterates, 10^3 equispaced bins per unit of $\log X_t$, the same for (c) and (d)], (c) $10^6\times$PDF and $\beta=\frac{1}{2}$, and (d) $10^9\times$PDF and $\beta=0$.

Immigration; Investment growth with capitalization

Consider the case $\langle \log a \rangle > 0$ and define

$$r \equiv lim_{t \to \infty} \frac{1}{t} \langle \log b \rangle \ . \tag{7}$$

Case $\langle \log a \rangle < r$: we define

$$b(t) = e^{r(t+1)} \hat{b}(t) \ , \tag{8}$$

where $\hat{b}(t)$ is a stochastic variable of order one. Let

$$a(t) = e^{r} \ \hat{a}(t) \ . \tag{9}$$

Since $r > \langle \log a \rangle$, then $\langle \log \hat{a} \rangle < 0$. Finally let

$$X_t = e^{rt} \hat{X}_t \ . \tag{10}$$

Equation (1) is then transformed into

$$\hat{X}_{t+1} = \hat{a} \hat{X}_t + \hat{b} \ , \tag{11}$$

where $\hat{a}$ and $\hat{b}$ obey the conditions of our previous analysis exactly. Thus $\hat{X}_t$ has a well-defined asymptotic non-singular pdf with a power law tail for large values.

Generalization

(1) is only one among many convergent ($\langle \ln b(t) \rangle < 0$) multiplicative processes with repulsion from the origin (due to the $b(t)$ term in (1)):

$$X_{t+1} = e^{f(X_t, \{a,b,...\})} \; a \; X_t \; . \qquad (12)$$

f has the following properties:

$$f(X_t, \{a, b, ...\}) \to 0 \quad \text{for} X_t \to \infty \; , \qquad (13)$$

i.e. X_t is a pure multiplicative process when it is large,

$$f(X_t, \{a, b, ...\})) \to \infty \quad \text{for} X_t \to 0 \; . \qquad (14)$$

6.1 Relation with auto-catalytic stochastic ODE

$$\frac{dX}{dt} = -rX + pX^{1-p} + \eta X \quad , \qquad (15)$$

where r is the decay rate in the absence of the last two terms and η is a multiplicative *Gaussian* noise with zero mean.

6.2 Generalization to multi-dimensional processes

6.3 Complex exponents and log-periodicity

6.4 Relation with intermittency in nonlinear dynamical systems

6.5 Self-affinity and multiself-affinity

6.6 Extremes and durations of the intermittent bursts

6.7 Non-linear extension: optimization of restocking strategy

Multi-dimensional Stochastic Recurrence Equations

$$X_{t+1} = a_t X_t + b_t Y_t + \eta_t$$

$$Y_{t+1} = c_t X_t + d_t Y_k + \epsilon_t$$

A generalization of the two-dimensional case to arbitrary dimensions leads to the following stochastic random equation (SRE)

$$\boldsymbol{X}_t = \boldsymbol{A}_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t \tag{16}$$

where $(\boldsymbol{X}_t, \boldsymbol{B}_t)$ are d-dimensional vectors.

Theorem 1. *Let* $(\boldsymbol{A}_n)$ *be an i.i.d. sequence of matrices in* $GL_d(\mathbb{R})$ *satisfying the following set of conditions:*

H1: for some $\epsilon > 0$, $E_{\mathbb{P}_A}[||\boldsymbol{A}||^{\epsilon}] < 1$, **(stationarity)**

H2: for every open $U \subset S_{d-1}$ *(the unit sphere in* $\mathbb{R}^d$*) and for all* $x \in S_{d-1}$ *there exists an n such that*

(ergodicity)

$$\Pr\left\{\frac{\boldsymbol{x}A_1\ldots\boldsymbol{A}_n}{||\boldsymbol{x}A_1\ldots\boldsymbol{A}_n||} \in U\right\} > 0. \tag{26}$$

(Aperiodicity) *H3: the group* $\{\ln|q(M)|, M \text{ is } \mathbb{P}_A\text{-feasible}\}$ *is dense in* $\mathbb{R}$.

H4: for all $\boldsymbol{r} \in \mathbb{R}^d$, $\Pr\{\boldsymbol{A}_1\boldsymbol{r} + \boldsymbol{B}_1 = \boldsymbol{r}\} < 1$. **(no traps)**

H5: there exists a $\kappa_0 > 0$ *such that* **(strong fluctuations)**

$$E_{\mathbb{P}_A}([\lambda(\boldsymbol{A}_1)]^{\kappa_0}) \geqslant 1. \tag{27}$$

H6: with the same $\kappa_0 > 0$ *as for the previous condition, there exists a real number* $u > 0$ *such that*

(tail not controlled by A or B)

$$\begin{cases} E_{\mathbb{P}_A}\left(\left[\sup\{||\boldsymbol{A}_1||, ||\boldsymbol{B}_1||\}\right]^{\kappa_0+u}\right) < \infty\,, \\ E_{\mathbb{P}_A}\left(||\boldsymbol{A}_1||^{-u}\right) < \infty. \end{cases} \tag{28}$$

Provided that these conditions hold,

- *there exists a unique solution* $\kappa_1 \in (0, \kappa_0]$ *to the equation*

$$\lim_{n\to\infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \left[||\boldsymbol{A}_1 \ldots \boldsymbol{A}_n||^{\kappa_1} \right] = 0, \qquad (29)$$

- *if* $(\boldsymbol{X}_n)$ *is the stationary solution to the SRE in (16) then* $\boldsymbol{X}$ *is regularly varying with index* κ_1*. In other words, the tail of the marginal distribution of each of the components of the vector* $\boldsymbol{X}$ *is asymptotically a power law with exponent* κ_1*.*

Kesten H 1973 Random difference equations and renewal theory for products of random matrices *Acta Mathematica* **131** 207–48

Landau-Ginzburg Theory of Self-Organized Criticality

Dynamics of an order parameter (OP) and of the corresponding *control* parameter (CP): within the sandpile picture, $\frac{\partial h}{\partial x}$ is the slope of the sandpile, h being the local height, and S is the state variable distinguishing between static grains ($S = 0$) and rolling grains ($S \neq 0$).

L. Gil and D. Sornette "Landau-Ginzburg theory of self-organized criticality", Phys. Rev.Lett. 76, 3991-3994 (1996)

$$\frac{\partial S}{\partial t} = \chi \{\mu S + 2\beta S^3 - S^5\} \quad (1)$$

where

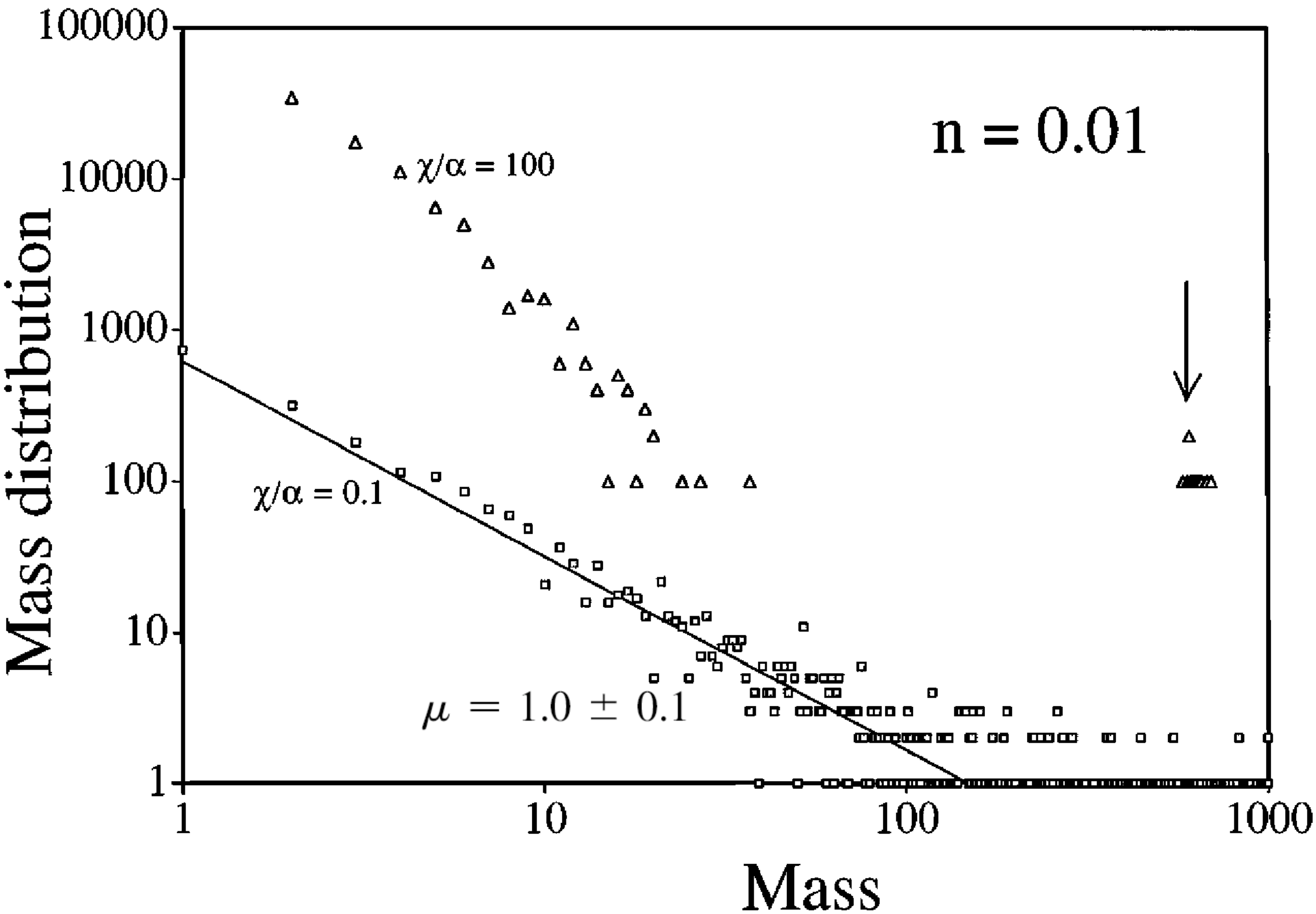
$$\mu = \left[\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial x}\Big|_c\right)^2 \right] \quad (2)$$

and $\beta > 0$ (subcritical condition).

Mechanism: Negative effective Diffusion coefficient

$$F\left(S, \frac{\partial h}{\partial x}\right) = -\alpha \frac{\partial h}{\partial x} S^2, \qquad \alpha > 0$$

$$\frac{\partial h}{\partial t} = -\frac{\partial F\left(S, \frac{\partial h}{\partial x}\right)}{\partial x} + \Phi \quad (3)$$



System sizes range from $L/a = 64$ to 2048.

$$P(M)dM \simeq M^{-(1+\mu)}dM,$$

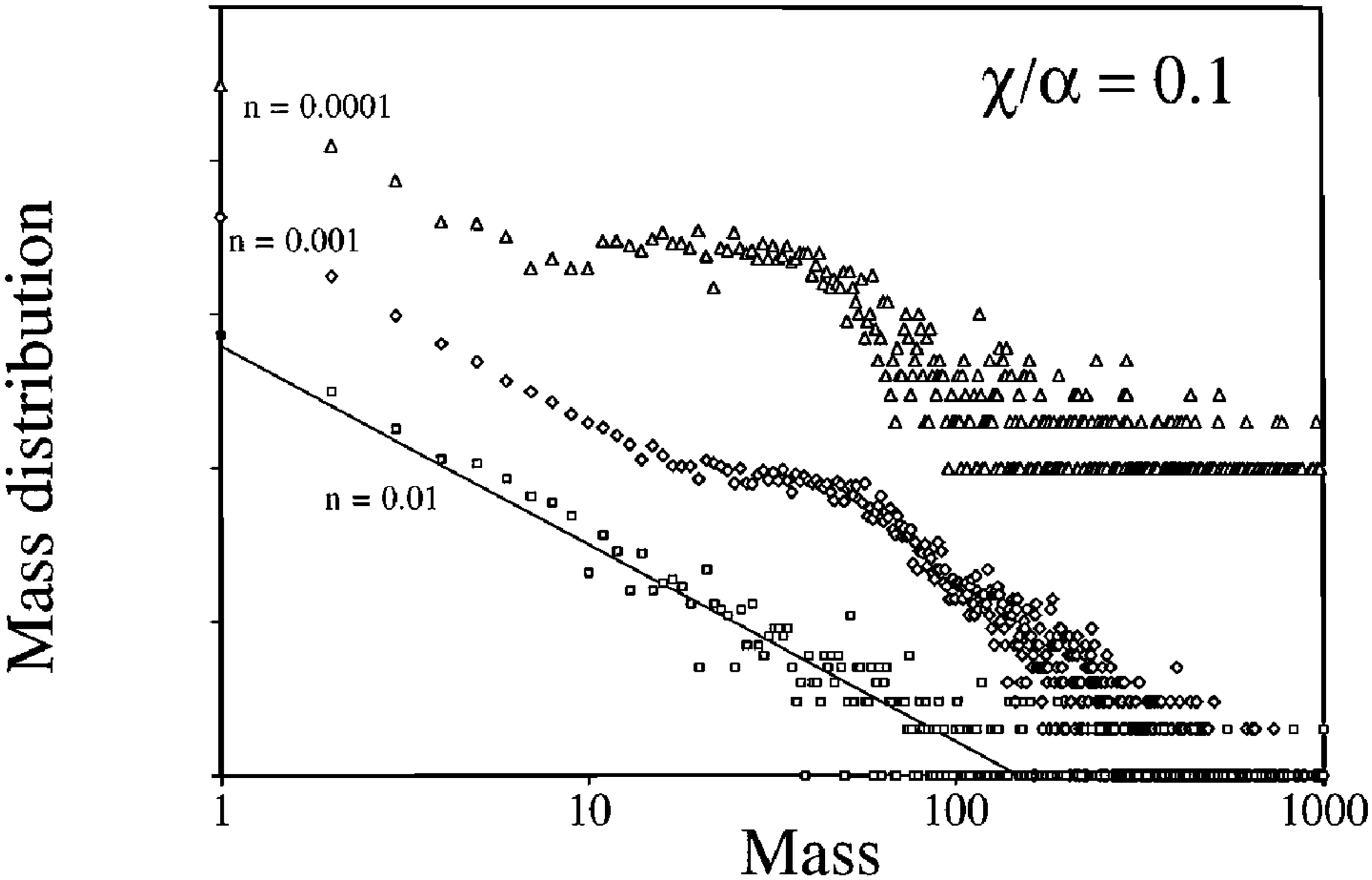


FIG. 2. Distributions $P(M)$ of avalanche sizes for the same $\chi/\alpha - 0.1$ but decreasing values, from bottom to top, of the noise. The curves have been moved with respect to each other for better clarity.

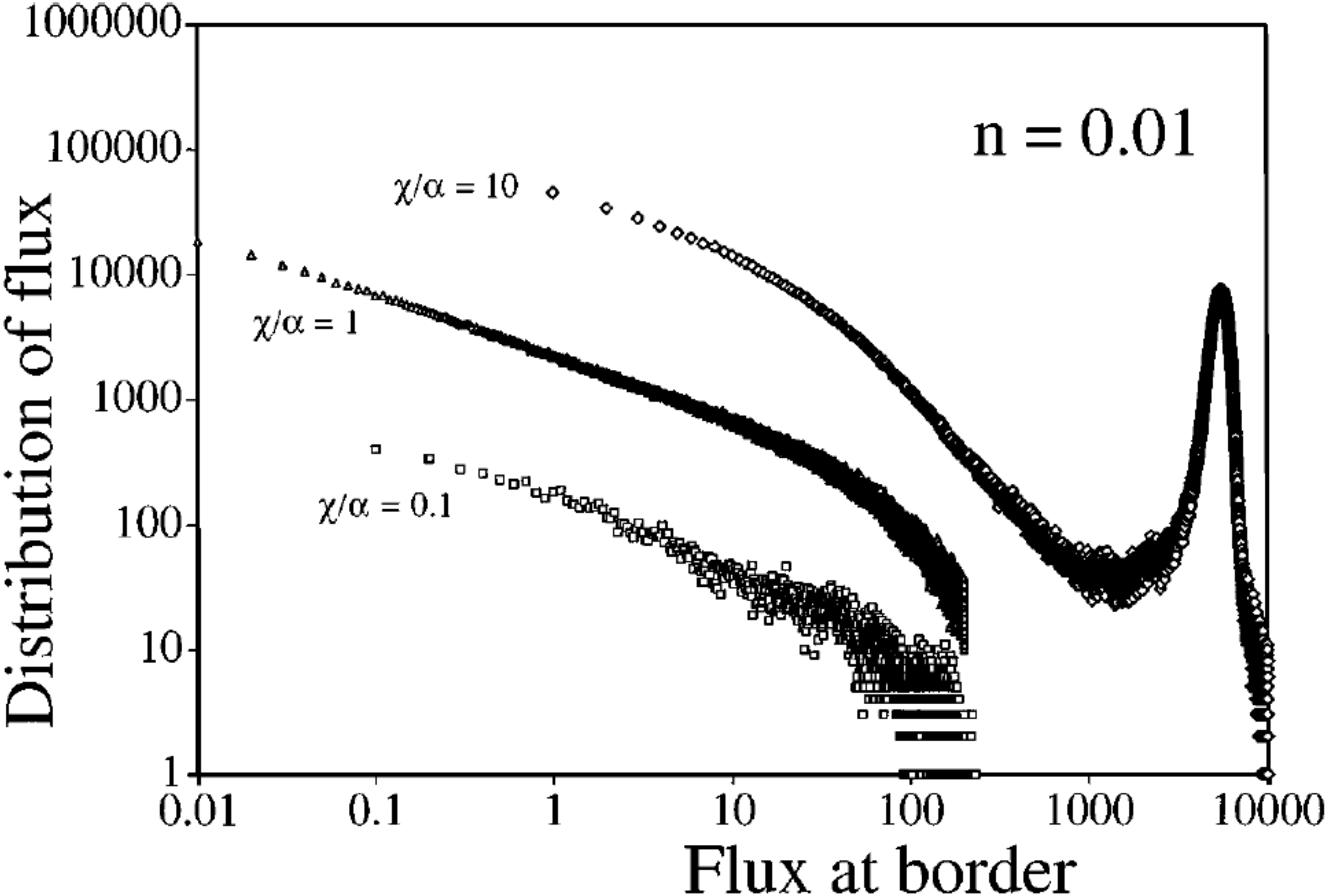


FIG. 3. Distribution $P(J)$ of flux amplitudes at the right border, in the same conditions as for Fig. 1.

List of Other Mechanisms

- Coherent-noise mechanism
- Highly Optimized Tolerant (HOT) Systems
- Sandpile models and Threshold dynamics (faults and earthquakes)
- Nonlinear Feedback of the “Order Parameter” onto the ‘Control Parameter”
- Generic Scale Invariance

- Mapping onto a Critical Point (contact processes)

- Critical Desynchronization

- Extremal Dynamics

- Dynamical system theory of self-organized criticality

Securitization of credit risks: is it the next "systemic collapse"?

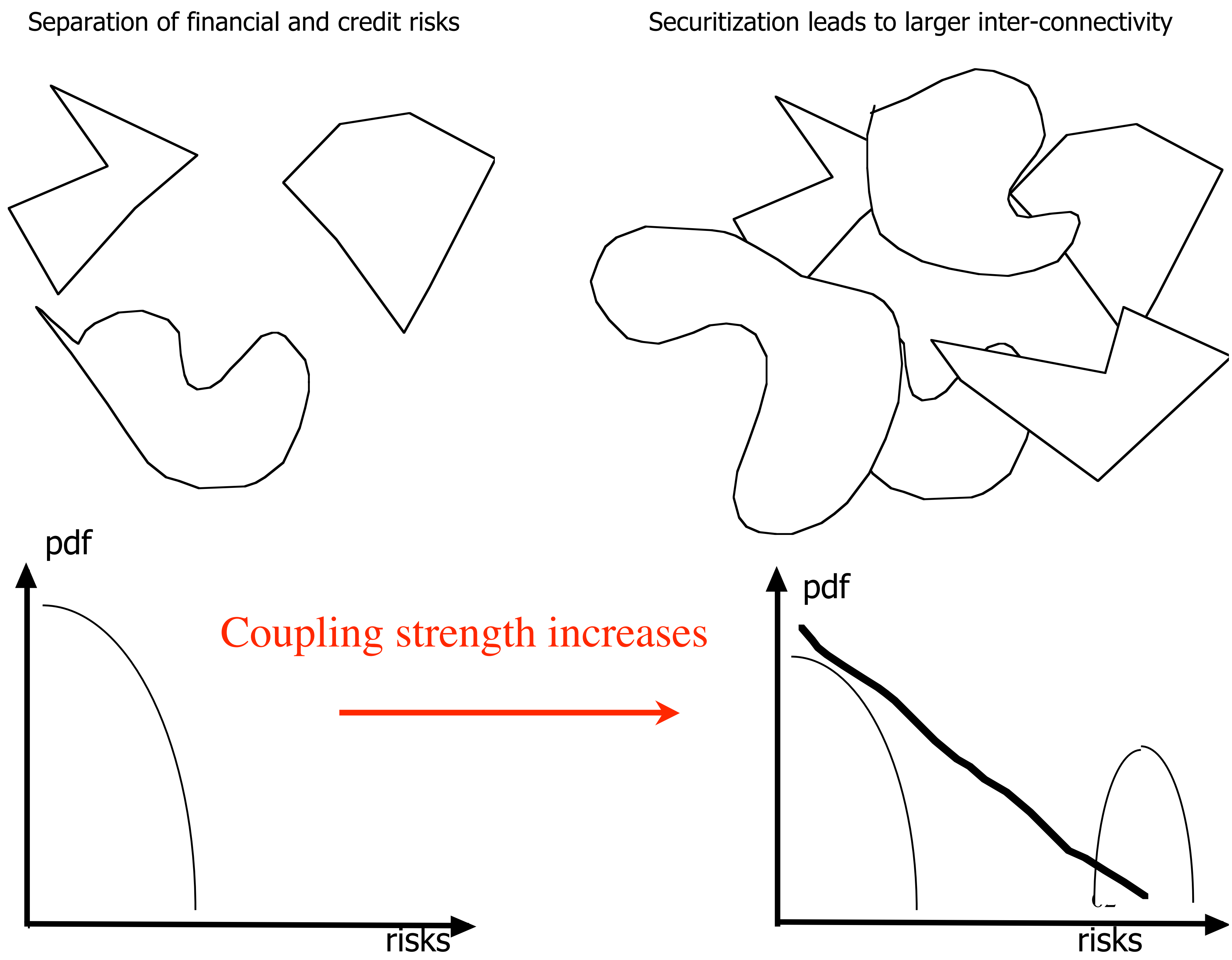
- Securitization of credit risks leads to smaller risks
- But more inter-connected

  $\Rightarrow$ global risk?

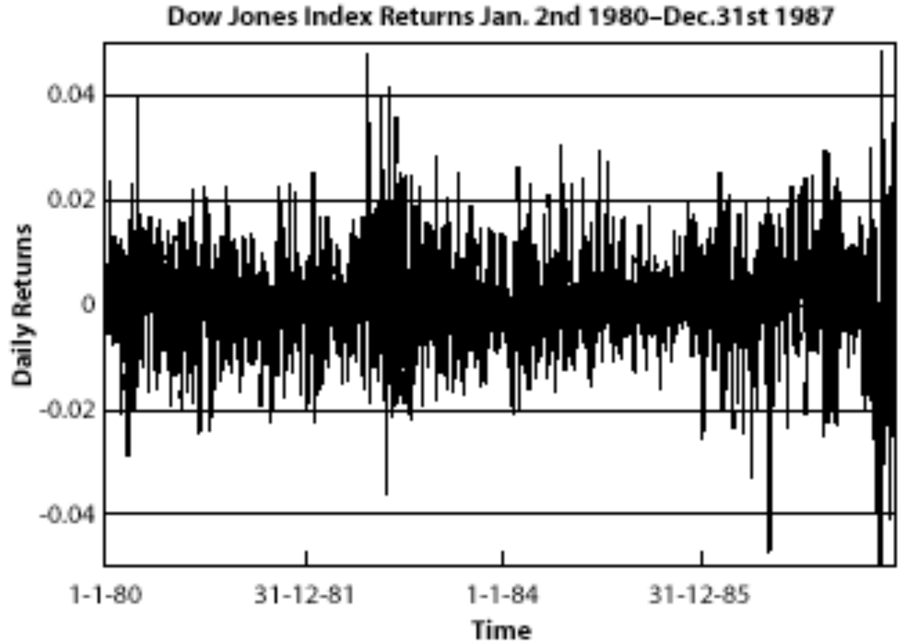
CDS and CDO: form of insurance contracts linked to underlying debt that protects the buyer in case of default.

The market has almost doubled in size every year for the past five years, reaching $20 trillion in notional amounts outstanding last June 2007, according to the Bank for International Settlements.

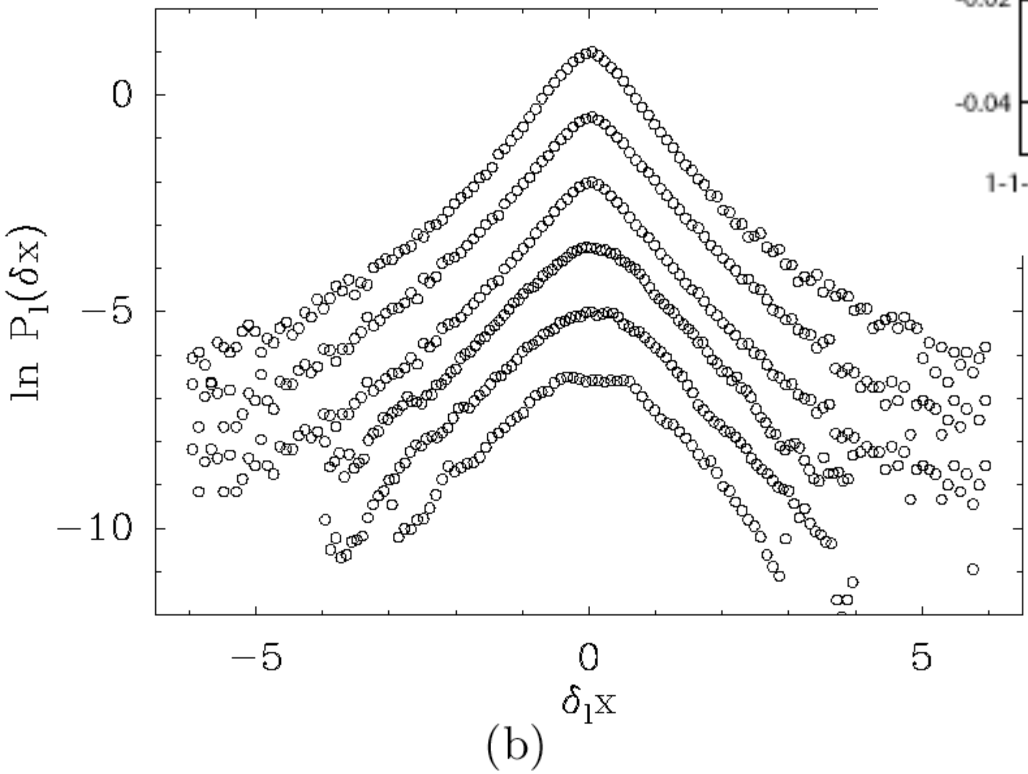
Bundling of indexes of CDSs together and slicing them into tranches, based on riskiness and return. The most toxic tranche at the bottom exposes the holder to the first 3% of losses but also gives him a large portion of the returns. At the top, the risks and returns are much smaller-unless there is a systemic failure.


Separation of financial and credit risks
Securitization leads to larger inter-connectivity
pdf
risks
Coupling strength increases
pdf
risks

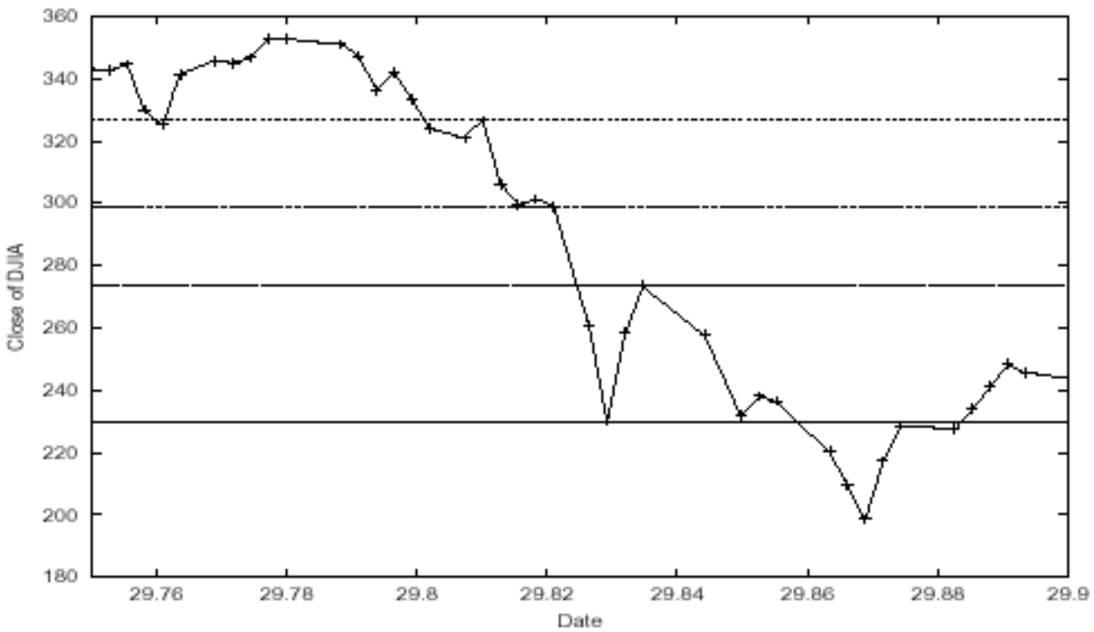
THE CONCEPT OF "Kings"



Traditional emphasis on Daily returns do not reveal any anomalous events



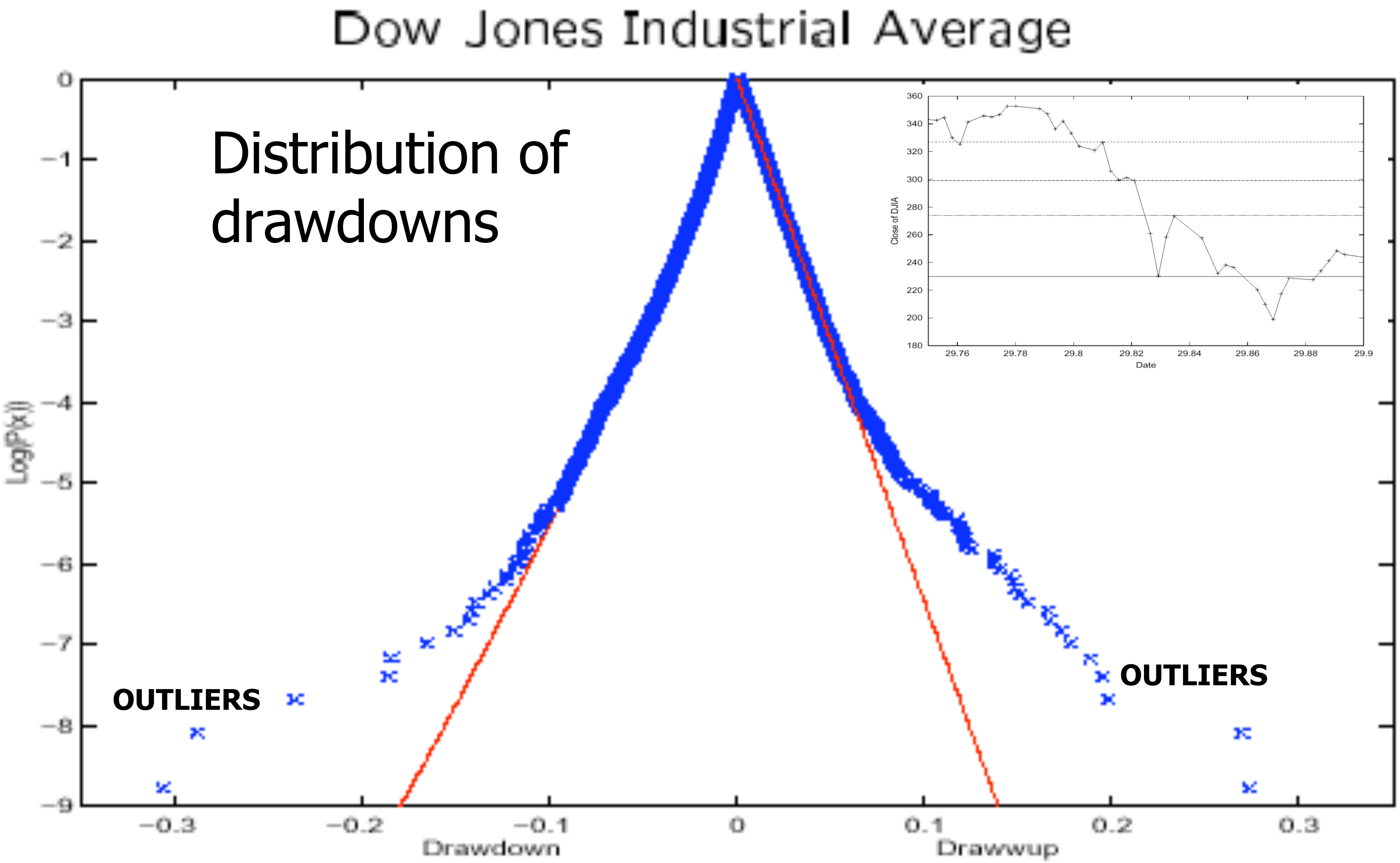
“Drawdowns/Drawups” nonlinear measure



- “Elastic” time horizon determined by market dynamics.

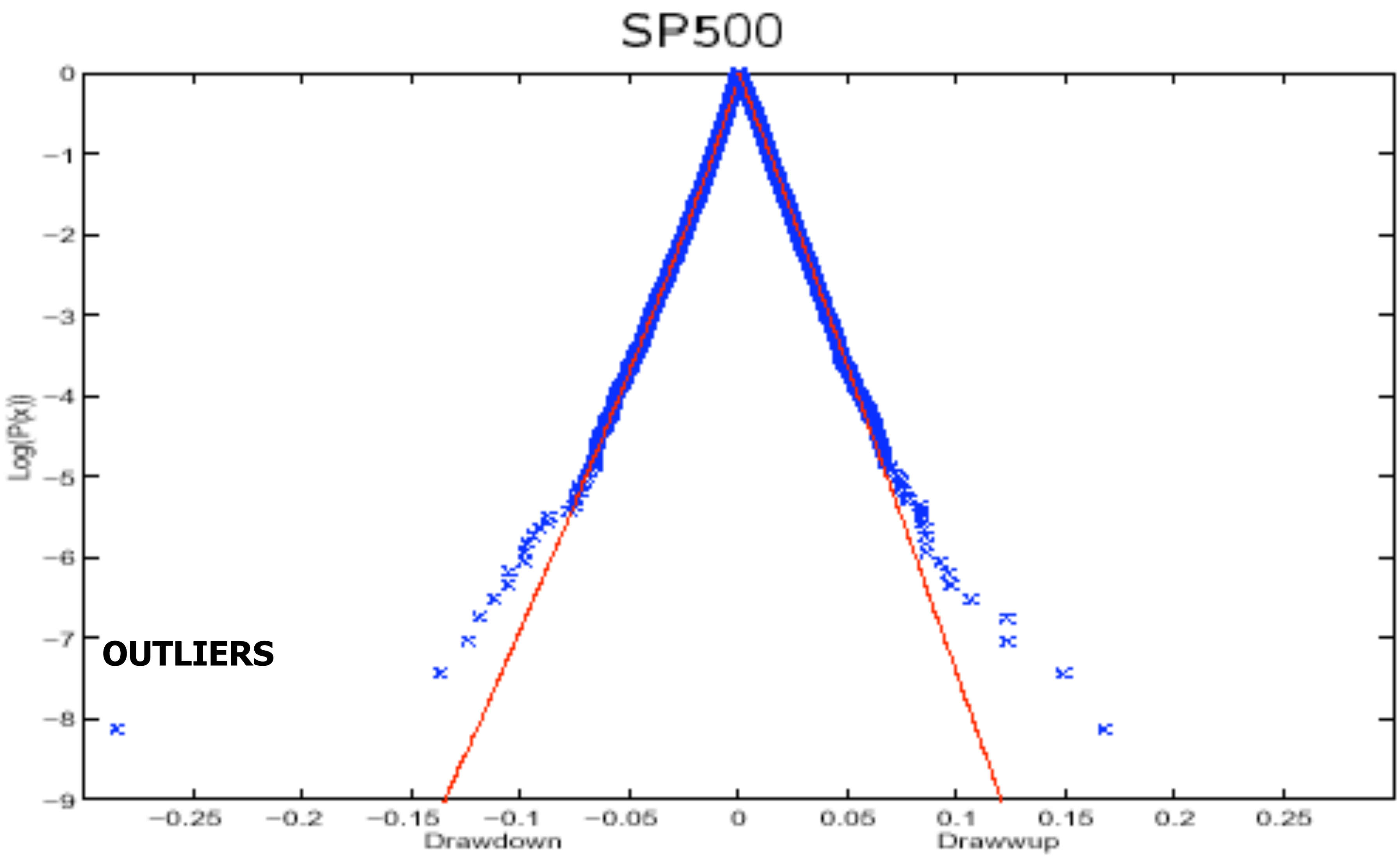
- Worst case scenario (risk management).

- Amplification of extreme market dynamics through “filtering”.



A. Johansen and D. Sornette, Stock market crashes are outliers,
European Physical Journal B 1, 141-143 (1998)

A. Johansen and D. Sornette, Large Stock Market Price Drawdowns Are Outliers,
Journal of Risk 4(2), 69-110, Winter 2001/02


SP500
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
Log(P(x))
OUTLIERS
−0.25
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
0.25
Drawdown
Drawwup

Thirty Major US companies

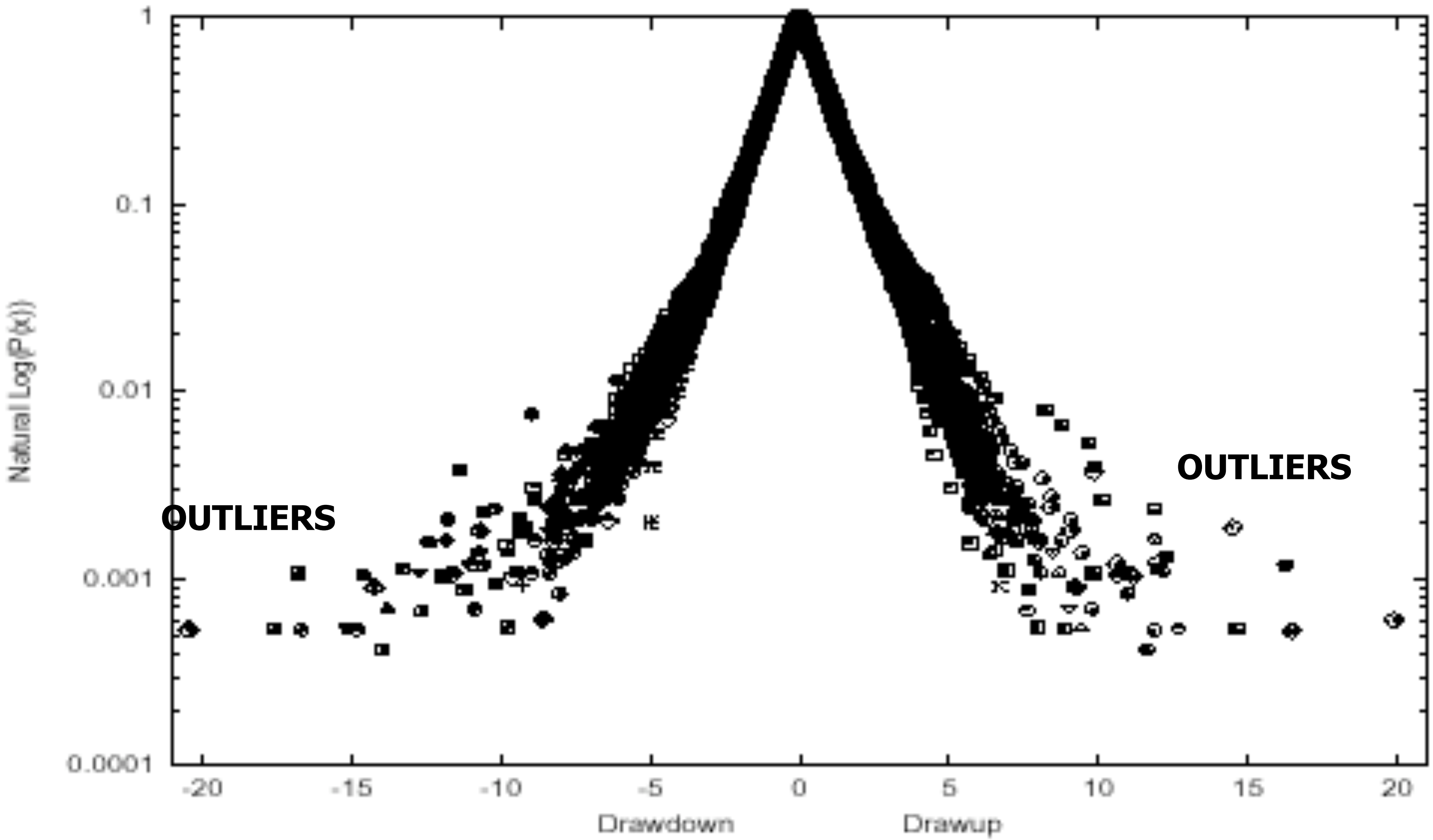
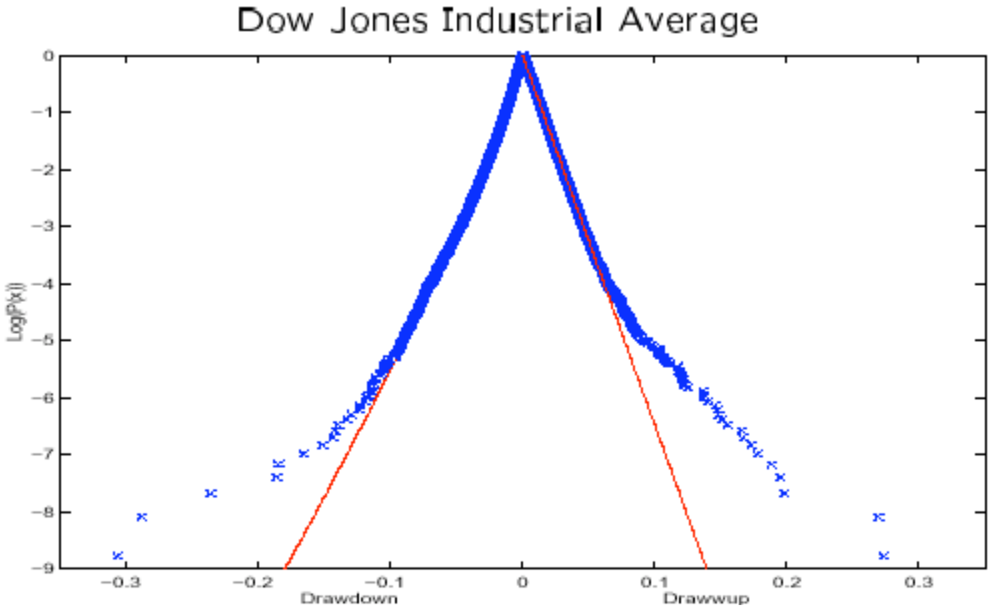


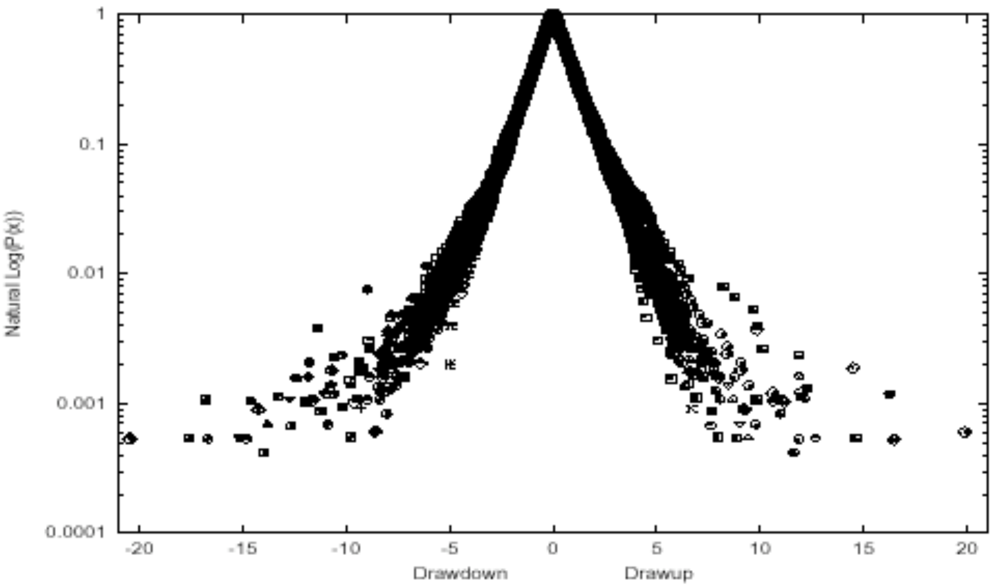
Table 1. NASDAQ composite index. The total number of drawdowns is 1495. The first column is the cut-off u such that the MLE of the two competing hypotheses (standard (SE) and modified (MSE) stretched exponentials) is performed over the interval $[0, u]$ of the absolute value of the drawdowns. The second column gives the fraction 'quantile' of the drawdowns belonging to $[0, u]$. The third column gives the exponents z found for the SE (first value) and MSE (second value) distributions. The fourth and fifth columns give the logarithm of the likelihoods (12) and (13) for the SE and MSE, respectively. The sixth column gives the variable T defined in (14). The last column 'proba' gives the corresponding probability of exceeding T by chance. For $u > 18\%$, we find that T saturates to 13.6 and 'proba' to 0.02%.

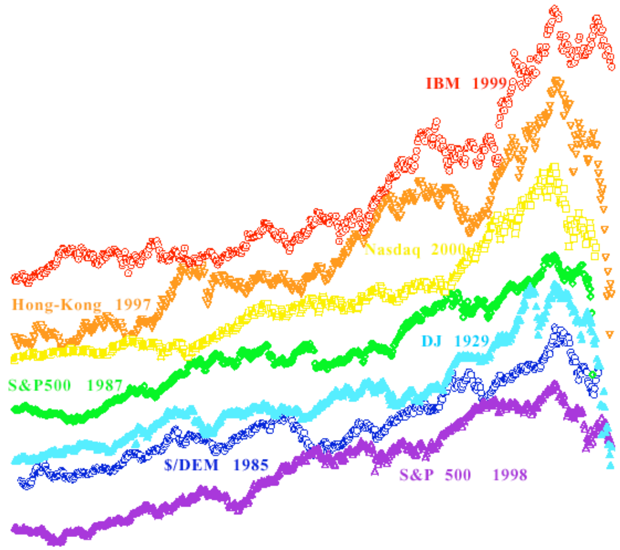
Cut-off u	Quantile	z	$\ln(L_0)$	$\ln(L_1)$	T	Proba
3%	87%	0.916, 0.940	4890.36	4891.16	1.6	20.5%
6%	97%	0.875, 0.915	4944.36	4947.06	5.4	2.0%
9%	99.0%	0.869, 0.918	4900.75	4903.66	5.8	1.6%
12%	99.7%	0.851, 0.904	4872.47	4877.46	10.0	0.16%
15%	99.7%	0.843, 0.898	4854.97	4860.77	11.6	0.07%
18%	99.9%	0.836, 0.890	4845.16	4851.94	13.6	0.02%

D. Sornette and A. Johansen
Significance of log-periodic precursors to financial crashes,
Quantitative Finance 1 (4), 452-471 (2001)

A. Johansen and D. Sornette,
Endogenous versus Exogenous Crashes in Financial Markets,
in press in ``Contemporary Issues in International Finance''
(Nova Science Publishers, 2004)
(http://arXiv.org/abs/cond-mat/0210509)







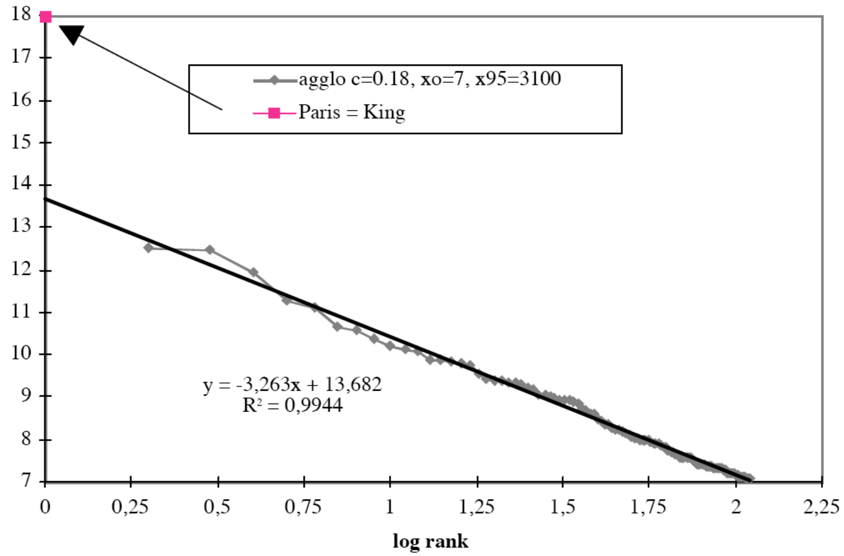


Fig. 7. French agglomerations: stretched exponential and "King effect".

Beyond power laws: five examples of “kings”

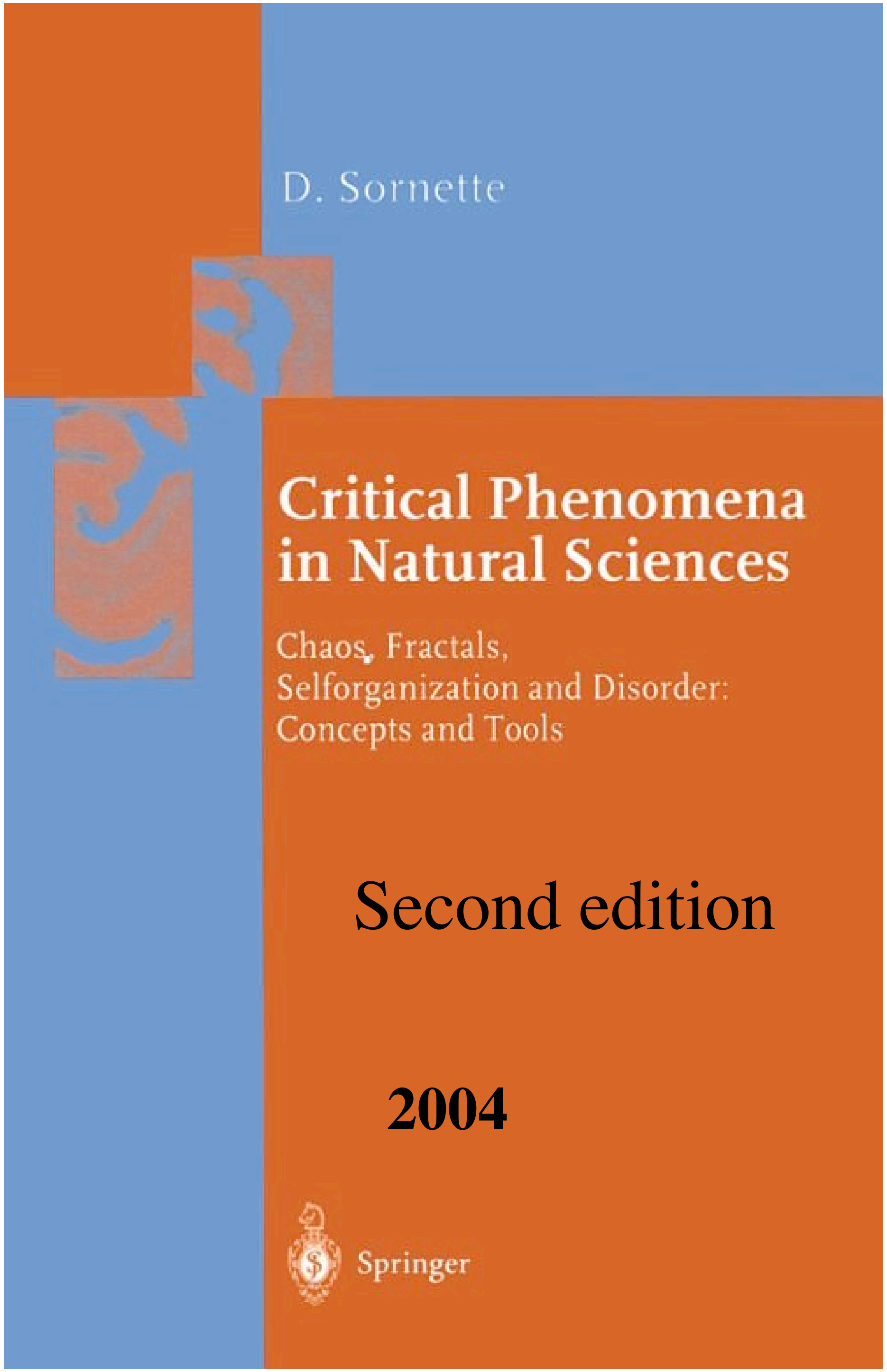
Material failure and rupture processes.

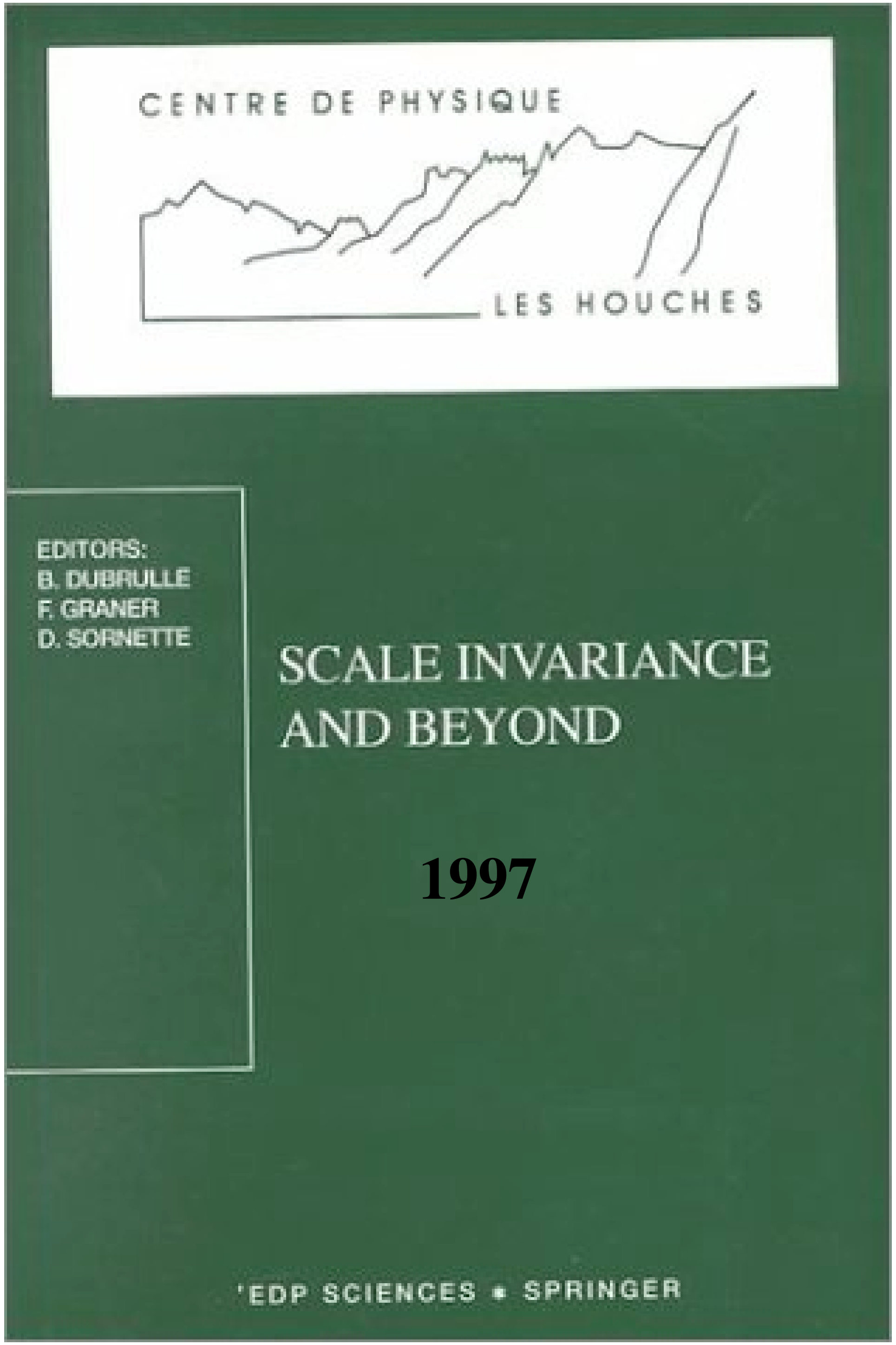
Gutenberg-Richter law and characteristic earthquakes.

Extreme king events in the pdf of turbulent velocity fluctuations

Outliers and kings in the distribution of financial drawdowns.

Paris as the king in the Zipf distribution of French city sizes.


D. Sornette
Critical Phenomena in Natural Sciences
Chaos, Fractals, Selforganization and Disorder: Concepts and Tools
Second edition
2004
Springer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