Quantum Decision Theory

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Plan

1. Classical Decision Theory
   1.1. Notations and definitions
   1.2. Typical paradoxes
   1.3. Absence of solution

2. Quantum Decision Theory
   2.1. Definitions and axioms
   2.2. General properties
   2.3. Solution of paradoxes

Classical Decision Theory

Set of outcomes, set of payoffs, consumer set, field of events

$$X = \{x_n : n = 1, 2, \ldots N\}, \qquad x_n \in \mathbb{R}$$

Utility function

elementary utility function, satisfaction function

$$u(x) : X \to \mathbb{R}$$

(i) nondecreasing

$$u(x_1) \geq u(x_2) \qquad (x_1 \geq x_2)$$

(ii) concave

$$u(\alpha x_1 + (1 - \alpha) x_2) > \alpha u(x_1) + (1 - \alpha) u(x_2)$$

$$0 \leq \alpha \leq 1, \qquad \ddot{u}(x) < 0$$

Risk aversion

Coefficient of absolute risk aversion
Pratt (1964)

$$r(x) \equiv -\frac{\ddot{u}(x)}{\dot{u}(x)}$$

Coefficient of relative risk aversion
Pratt (1964), Arrow (1965)

$$R(x) \equiv x\, r(x) = -x\,\frac{\ddot{u}(x)}{\dot{u}(x)}$$

Portfolio problem

Exponential utility function

$$u(x) = 1 - \mathrm{e}^{-kx}$$

Risk aversion

$$r(x) = k\,, \qquad R(x) = kx$$

Expected Utility

Bernoulli (1738)
Von Neumann – Morgenstern (1944)

Probability measure over X

$$\{ p(x_n) :\ n = 1, 2, \ldots, N \}, \qquad \sum_{n=1}^{N} p(x_n) = 1$$

Lottery

$$L = \{ x_n,\ p(x_n) :\ n = 1, 2, \ldots, N \}$$

Linear combination

$$L_1 = \{ x_n,\ p_1(x_n) \}, \qquad L_2 = \{ x_n,\ p_2(x_n) \}$$

$$\alpha L_1 + (1 - \alpha) L_2 = \{ x_n,\ \alpha\, p_1(X_n) + (1 - \alpha)\, p_2(x_n) \}$$

$$0 \le \alpha \le 1$$

Lottery mean

$$\bar{x} = \frac{1}{N} \sum_{n=1}^{N} p(x_n) x_n$$

Lottery dispersion

$$\Delta^2(L) = \frac{1}{N} \sum_{n=1}^{N} p(x_n) x_n^2 - \bar{x}^2(L)$$

dispersion, measure of uncertainty

Expected utility of lottery

$$U(L) = \sum_{n=1}^{N} p(x_n) u(x_n)$$

Comparison of lotteries

Indifference: $L_1 = L_2 \quad \rightarrow \quad U(L_1) = U(L_2)$

Preference: $L_1 > L_2 \quad \rightarrow \quad U(L_1) > U(L_2)$

$$L_1 \geq L_2 \quad \rightarrow \quad U(L_1) \geq U(L_2)$$

Properties of expected utility

(1) completeness

for L_1 and L_2, one of relations

$$L_1 = L_2, \quad L_1 < L_2, \quad L_1 > L_2, \quad L_1 \leq L_2, \quad L_1 \geq L_2$$

(2) transitivity

$$\text{if} \quad L_1 \leq L_2 \quad \text{and} \quad L_2 \leq L_3, \quad \text{then} \quad L_1 \leq L_3$$

(3) continuity

for $L_1 \leq L_2 \leq L_3$, there exist $\alpha \in [0, 1]$

$$\alpha L_1 + (1 - \alpha) L_3 = L_2$$

(4) independence

for $L_1 \geq L_2$ and any L_3, $0 \leq \alpha \leq 1$,

$$\alpha L_1 + (1 - \alpha) L_3 \geq \alpha L_2 + (1 - \alpha) L_3$$

Classical decision-making scheme

Set of lotteries $\{L_j : j = 1, 2, ..., \}$

$$L_j = \{x_n, \; p_j(x_n)\}$$

Expected utility $U(L_j)$

compare $U(L_j)$

Optimal lottery L^*

$$U(L^*) \equiv \sup_j U(L_j)$$

Allais Paradox

Allais (1953)

Compatibility violation: Several choices are not compatible with utility theory

Payoff set $X = \{x_1, x_2, x_3\}$

units of x_n millions of dollars $\$10^6$

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 5$$

Set of 4 lotteries $\{L_j : j = 1, 2, 3, 4\}$

Probability measures $\{p_j(x_n)\}$

Balance conditions

$$p_1(x_n) + p_3(x_n) = p_2(x_n) + p_4(x_n)$$

for all $n = 1, 2, 3.$

$$\{p_1(x_n)\} = \{0, 1, 0\}, \qquad \{p_2(x_n)\} = \{0.01, 0.89, 0.10\}$$

$$\{p_3(x_n)\} = \{0.9, 0, 0.1\}, \qquad \{p_4(x_n)\} = \{0.89, 0.11, 0\}$$

$$p_1(x_1) + p_3(x_1) = p_2(x_1) + p_4(x_1) = 0.9$$
$$p_1(x_2) + p_3(x_2) = p_2(x_2) + p_4(x_2) = 1$$
$$p_1(x_3) + p_3(x_3) = p_2(x_3) + p_4(x_3) = 0.1$$

Lotteries

$$L_1 = \left\{ \begin{matrix} \$0, & 0 \\ \$1, & 1 \\ \$5, & 0 \end{matrix} \right\} \qquad \Delta^2(L_1) = 0$$

$$L_2 = \left\{ \begin{matrix} \$0, & 0.01 \\ \$1, & 0.89 \\ \$5, & 0.10 \end{matrix} \right\} \qquad \Delta^2(L_2) = 0.916$$

$L_1 > L_2$

L_2 is more uncertain

$$L_3 = \begin{Bmatrix} \$0, & 0.9 \\ \$1, & 0 \\ \$5, & 0.1 \end{Bmatrix} \qquad \Delta^2(L_3) = 0.805$$

$$L_4 = \begin{Bmatrix} \$0, & 0.89 \\ \$1, & 0.11 \\ \$5, & 0 \end{Bmatrix} \qquad \Delta^2(L_4) = 0$$

$L_3 > L_4$

L_3 is more uncertain, but stake is larger

$$U(L_1) = u(1)$$

$$U(L_2) = 0.01\,u(0) + 0.89\,u(1) + 0.1\,u(5)$$

$$U(L_3) = 0.9\,u(0) + 0.1\,u(5)$$

$$U(L_4) = 0.89\,u(0) + 0.11\,u(1)$$

$$L_1 > L_2 \rightarrow U(L_1) > U(L_2)$$

$$0.11\,u(1) > 0.01\,u(0) + 0.1\,u(5)$$

$$L_3 > L_4 \rightarrow U(L_3) > U(L_4)$$

$$0.11\,u(1) < 0.01\,u(0) + 0.1\,u(5)$$

Contradiction!

For any definition of $u(x)$!

Independence Paradox

Allais (1953)

Independence axiom:

$$\text{if } L_1 > L_2 \text{ and } L_3 \geq L_4, \quad \text{then for any } \alpha \in [0, 1]$$

$$\alpha L_1 + (1 - \alpha) L_3 > \alpha L_2 + (1 - \alpha) L_4$$

Lotteries as in the Allais paradox

$$\{L_j : j = 1, 2, 3, 4\}$$

take $\alpha = \frac{1}{2}$

$$\frac{1}{2}(L_1 + L_3) = \left\{\begin{array}{ll} \$0, & 0.45 \\ \$1, & 0.50 \\ \$5, & 0.05 \end{array}\right\}$$

$$\frac{1}{2}(L_2 + L_4) = \left\{\begin{array}{ll} \$0, & 0.45 \\ \$1, & 0.50 \\ \$5, & 0.05 \end{array}\right\}$$

$$U\left(\frac{L_1 + L_3}{2}\right) \equiv U\left(\frac{L_2 + L_4}{2}\right)$$

but by the independence axiom it should be

$$U\left(\frac{L_1 + L_3}{2}\right) > U\left(\frac{L_2 + L_4}{2}\right)$$

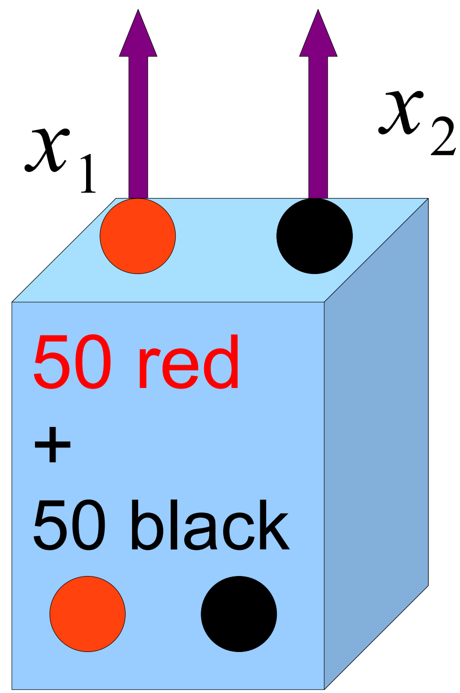
Contradiction!

For any definition of $u(x)$!

Ellsberg Paradox

Ellsberg (1961)

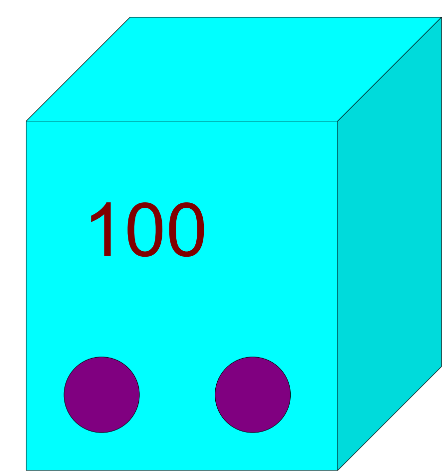
1 urn: 50 red balls + 50 black balls



2 urn: 100 balls, red or black in an unknown proportion

Payoffs

x_1 prize for getting a red ball



x_2 prize for getting a black ball

Units of x_n, say, $1000

Get a red ball from 1-st urn

$$L_1 = \begin{pmatrix} \$0, & \frac{1}{2} \\ \$1, & \frac{1}{2} \end{pmatrix}$$

Get a red ball from 2-nd urn

$$L_2 = \begin{pmatrix} \$0, & \alpha \\ \$1, & 1-\alpha \end{pmatrix} \qquad 0 \le \alpha \le 1$$

preference: $L_1 > L_2$

Get a black ball from 1-st urn

$$L_3 = \begin{Bmatrix} \$0, & \frac{1}{2} \\ \$1, & \frac{1}{2} \end{Bmatrix}$$

Get a black ball from 2-nd urn

$$L_4 = \begin{Bmatrix} \$0, & 1-\alpha \\ \$1, & \alpha \end{Bmatrix} \qquad 0 \le \alpha \le 1$$

Preference: $L_3 > L_4$

Indifference: $L_2 = L_4$

$$L_1 > L_2 \;\rightarrow\; U(L_1) > U(L_2)$$

$$\frac{1}{2}u(0) + \frac{1}{2}u(1) > \alpha u(0) + (1-\alpha)u(1)$$

$$L_3 > L_4 \;\rightarrow\; U(L_3) > U(L_4)$$

$$\frac{1}{2}u(0) + \frac{1}{2}u(1) > (1-\alpha)u(0) + \alpha u(1)$$

No such $\alpha \in [0,1]$

Contradiction!

Also:

$$L_2 = L_4 \;\rightarrow\; U(L_2) = U(L_4)$$

hence $\alpha = \dfrac{1}{2}$

$$p_j(x_n) = \frac{1}{2} = const$$

then

$$L_1 = L_2 = L_3 = L_4$$

$$U(L_j) = const$$

for any definition of $U(L)$

Kahneman-Tversky Paradox

Kahneman-Tversky (1979)

Invariance violation: Preference instead of indifference

Set of payoffs $\{x_n\} = \{1, 1.5, 2\}$

Units of x_n, thousands of dollars \$1000

$$u(1.5) = \frac{1}{2}\left[\, u(1) + u(2) \,\right]$$

After winning, one gets

$$L_1 = \begin{Bmatrix} \$1, & 0.5 \\ \$1.5, & 0 \\ \$2, & 0.5 \end{Bmatrix}, \qquad L_2 = \begin{Bmatrix} \$1, & 0 \\ \$1.5, & 1 \\ \$2, & 0 \end{Bmatrix}$$

$$\Delta^2(L_1) = 0.583 \qquad \Delta^2(L_2) = 0$$

L_2 is more certain

$L_2 > L_1$

After loosing, one gets

$$L_3 = \begin{Bmatrix} \$1, & 0.5 \\ \$1.5, & 0 \\ \$2, & 0.5 \end{Bmatrix}, \qquad L_4 = \begin{Bmatrix} \$1, & 0 \\ \$1.5, & 1 \\ \$2, & 0 \end{Bmatrix}$$

$$\Delta^2(L_3) = 0.583 \qquad \Delta^2(L_4) = 0$$

L_4 is more certain, but

$L_3 > L_4$

$$L_2 > L_1 \;\rightarrow\; U(L_2) > U(L_1)$$

$$L_3 > L_4 \;\rightarrow\; U(L_3) > U(L_4)$$

However

$$U(L_j) = \frac{1}{2}u(1) + \frac{1}{2}u(2) = u(1.5)$$

for all $j = 1,2,3,4$

$$U(L_j) = const$$

Contradiction!

For any definition of $u(x)$!

Rabin Paradox

Rabin (2000)

payoffs: $X_1 = \{x - l_1,\ x,\ x + g_1\}$

l_1 loss, g_1 gain

$$x \geq l_1, \quad l_1 > 0, \quad g_1 > 0$$

Small difference between gain and loss

$$g_1 \approx l_1$$

$$L_1 = \begin{Bmatrix} \$\,(x - l_1), & 0.5 \\ \$\,x, & 0 \\ \$\,(x + g_1), & 0.5 \end{Bmatrix}, \qquad L_2 = \begin{Bmatrix} \$\,(x - l_1), & 0 \\ \$\,x, & 1 \\ \$\,(x + g_1), & 0 \end{Bmatrix}$$

$$\Delta^2(L_1) > 0 \qquad \Delta^2(L_2) = 0$$

L_1 is more uncertain

$L_2 > L_1$

Payoffs: $X_2 = \{x - l_2,\ x,\ x + g_2\}$

$$x \geq l_2, \quad l_2 > 0, \quad g_2 > 0$$

Large difference between gain and loss: $g_2 \gg l_2$

$$L_3 = \begin{vmatrix} \$\,(x - l_2), & 0.5 \\ \$\,x, & 0 \\ \$\,(x + g_2), & 0.5 \end{vmatrix}, \qquad L_4 = \begin{vmatrix} \$\,(x - l_2), & 0 \\ \$\,x, & 1 \\ \$\,(x + g_2), & 0 \end{vmatrix}$$

$$\Delta^2(L_3) > 0 \qquad \Delta^2(L_4) = 0$$

$$L_3 > L_4$$

Although L_3 is more uncertain

But the stake is much larger

$$L_2 > L_1 \;\rightarrow\; U(L_2) > U(L_1)$$

$$u(x) > \frac{1}{2}u(x - l_1) + \frac{1}{2}u(x + g_1).$$

$$L_3 > L_4 \;\rightarrow\; U(L_3) > U(L_4)$$

$$u(x) < \frac{1}{2}u(x - l_2) + \frac{1}{2}u(x + g_2)$$

Rabin theorem (2000)

If for some $l > 0$, $g > 0$

$$u(x) > \frac{1}{2}u(x - l) + \frac{1}{2}u(x + g),$$

then it is so for all l, g, because of the concavity of $u(x)$.

Contradiction with above!

For any concave $u(x)$!

Disjunction Effect

Tversky-Shafir (1992)

Two-step gambles

1-st step: $\begin{cases} 1\text{-st gamble won } (B_1) \\ 1\text{-st gamble lost } (B_2) \end{cases}$

2-nd step: $\begin{cases} 2\text{-nd gamble accepted } (A_1) \\ 2\text{-nd gamble refused } (A_2) \end{cases}$

People accept the 2 -nd gamble independently whether they won the first,

$$p(A_1 B_1) > p(A_2 B_1),$$

or they lost the first gamble, $p(A_1 B_2) > p(A_2 B_2)$.

But, when the results of the 1-st gamble are not known,

$$B = B_1 + B_2 \qquad (B_1 B_2 = 0),$$

people restrain from the 2-nd gamble,

$$p(A_2 B) > p(A_1 B).$$

By probability theory,

$$p(A_1 B) = p(A_1 B_1) + p(A_1 B_2),$$

$$p(A_2 B) = p(A_2 B_1) + p(A_2 B_2).$$

If $p(A_1 B_j) > p(A_2 B_j)$ for $j = 1,2,$ then

$$p(A_1 B) > p(A_2 B).$$

Contradiction!

Sure-thing principle

Savage (1954)

Humans respect probability theory:

$$p(A_1 B_j) > p(A_2 B_j) \;\to\; p(A_1 B) > p(A_2 B)$$

However, disjunction effect:

Humans do not abide to probability theory!

Another example of Disjunction Effect

1-st step:
$$\begin{cases} \text{exam passed} & (B_1) \\ \text{exam failed} & (B_2) \end{cases}$$

2-nd step:
$$\begin{cases} \text{vacation accepted} & (A_1) \\ \text{vacation refused} & (A_2) \end{cases}$$

Students go to vacation in any case of known results:

$$p(A_1 B_1) > p(A_2 B_1), \qquad p(A_1 B_2) > p(A_2 B_2).$$

When results are not known, students forgo vacations:

$$\boxed{p(A_1 B) < p(A_2 B)} \qquad (B = B_1 + B_2)$$

Contradiction with sure-thing principle!

Conjunction Fallacy

Tversky-Kahneman (1983)

One event (A).

Another event $(B = B_1 + B_2)$, which

may happen (B_1),

or does not happen (B_2).

People often judge: $p(AB) < p(AB_1)$.

But, by probability theory,

$$p(AB) = p(AB_1) + p(AB_2),$$

hence, conjunction rule:

$$p(AB) \geq p(AB_j) \qquad (j = 1, 2).$$

Contradiction!

Examples: description of a person, of a subject, of an event,...

Decide on the existence of one feature (A).

Decide on the existence of another feature (B_1) or absence of it (B_2).

$$p(AB) < p(AB_1) \qquad (B = B_1 + B_2).$$

Save utility theory ?

Non-expected utility functionals.

For a lottery $L = \{x_n,\ p(x_n)\}$

Instead of expected utility $U(L)$, utility functionals

$$F(L) = F[x_n,\ p(x_n),\ u(x_n)]$$

Minimal requirements: Risk aversion

Between two lotteries L_1 and L_2, with the same mean

$$\bar{x}(L_1) = \bar{x}(L_2)$$

the lottery L_1 is preferred to L_2 ($L_1 > L_2$) if $\Delta^2(L_1) < \Delta^2(L_2)$.

Then $F(L_1) > F(L_2)$.

Safra and Segal (2008): Non-expected utility functionals do not remove paradoxes!

What to do?

1. Realistic problems are complicated, consisting of many parts.
2. Different parts of a problem interact and interfere with each other.
3. Several thoughts of mind can be intricately interconnected (entangled).

Life is complex!

Quantum Decision Theory

Main definitions

1. Action ring

$$\mathcal{A} = \{A_n : n = 1, 2, \ldots, N\}$$

Intended actions A_n

addition $A_m + A_n \in \mathcal{A}$

associative: $A_1 + (A_2 + A_3) = (A_1 + A_2) + A_3$

reversible: $A_1 + A_2 = A_3 \rightarrow A_1 = A_3 - A_2$

multiplication: $A_m A_n \in \mathcal{A}$

distributive: $A_1 (A_2 + A_3) = A_1 A_2 + A_1 A_3$

idempotent: $A_n A_n \equiv A_n^2 = A_n$

noncommutative: $A_m A_n \neq A_n A_m$ (generally)

empty action: $A_n 0 = 0 A_n = 0$

disjoint actions: $A_m A_n = A_n A_m = 0$

2. Action Modes

Composite actions

$$A_n = \bigcup_{\mu=1}^{M_n} A_{n\mu} \qquad (M_n > 1)$$

$A_{n\mu}$ action modes, representations

$$A_{n\mu} A_{n\nu} = \delta_{\mu\nu} A_{n\mu}$$

3. Action prospects

$$\pi_j = \bigcap_{n=1}^{N} A_{j_n} \qquad (A_{j_n} \in \mathcal{A})$$

conjunction, A_{j_n} composite or simple,

composite and simple prospects

4. Elementary prospects

binary multi-index

$$\alpha = \{ i_n , \mu_n : n = 1, 2, \ldots, N \}_\alpha$$

number of α, cardinality

$$\mathrm{card}\{\alpha\} = \prod_{n=1}^{N} \mathrm{M}_n$$

$$e_\alpha = \bigcap_{n=1}^{N} A_{i_n \mu_n}$$

conjunction of modes

$$e_\alpha e_\beta = \delta_{\alpha\beta} \, e_\alpha$$

5. Prospect lattice

$$L = \{\pi_j : \; j = 1, 2, \ldots, N_L\}$$

ordering: $\pi_i \leq \pi_j$ or $\pi_i \geq \pi_j$

6. Mode states

$A_{n\mu} \rightarrow$ complex function

$$|A_{n\mu}\rangle : \; \mathcal{A} \rightarrow \mathbb{C}$$

scalar product

$$\langle A_{n\mu} | A_{n\nu} \rangle = \delta_{\mu\nu}$$

7. Mode space

closed linear envelope

$$\mathcal{M}_n = \text{Span}\,\{|A_{n\mu}\rangle :\ \mu = 1, 2, \ldots, \mathrm{M}_n\}$$

$$\dim \mathcal{M}_n = \mathrm{M}_n$$

Hilbert space

8. Basic states

elementary prospect $e_\alpha \rightarrow$

$$|e_\alpha\rangle :\ \mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A} \rightarrow \mathbb{C}$$

$$|e_\alpha\rangle = |A_{i_1\mu_1} A_{i_2\mu_2} \ldots A_{i_N\mu_N}\rangle = \bigotimes_{n=1}^{N} |A_{i_n\mu_n}\rangle$$

$$\langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta}$$

9. Mind space

$$\mathcal{M} = \text{Span}\{\, |e_\alpha\rangle : \alpha \in \{\alpha\} \,\} = \bigotimes_{n=1}^{N} \mathcal{M}_n$$

$$\dim \mathcal{M} = \prod_{n=1}^{N} \text{M}_n$$

10. Prospect states

$$\pi_j \in L \rightarrow |\pi_j\rangle \in \mathcal{M}$$

11. Strategic states

reference states $|\psi_s\rangle \in \mathcal{M}$

$$\langle\, \psi_s | \psi_{s'} \,\rangle = \delta_{ss'}$$

12. Mind strategy

$$\Sigma = \{ |\psi_s\rangle, \; w_s : \; s = 1, 2, \ldots, S\}$$

$$\sum_{s=1}^{S} w_s = 1 \, , \qquad 0 \leq w_s \leq 1$$

Person character, basic beliefs and principles

13. Prospect operators

$$\hat{P}(\pi_j) = | \, \pi_j \rangle \langle \pi_j \, |$$

Involutive bijective algebra $\left\{ \hat{P}(\pi_j) : \; \pi_j \in L \right\}$

14. Operator averages

$$\langle \hat{P}(\pi_j) \rangle = \sum_{s=1}^{S} w_s \, \langle \psi_s \, | \hat{P}(\pi_j) | \, \psi_s \rangle$$

15. Prospect probability

$$p(\pi_j) = \langle \hat{P}(\pi_j) \rangle \, , \qquad \sum_{j=1}^{N_L} p(\pi_j) = 1$$

16. Prospect ordering

π_1 indifferent to π_2:

$$p(\pi_1) = p(\pi_2) \qquad (\pi_1 = \pi_2)$$

π_1 preferred to π_2:

$$p(\pi_1) > p(\pi_2) \qquad (\pi_1 > \pi_2)$$

Decisions are probabilistic

17. Partial probabilities

$\pi_j e_\alpha$ conjunction prospects

$$p(\pi_j e_\alpha) = \langle \hat{P}(e_\alpha) \hat{P}(\pi_j) \hat{P}(e_\alpha) \rangle , \quad \sum_{j,\alpha} p(\pi_j e_\alpha) = 1$$

18. Attraction factor

$$q(\pi_j) = \sum_{\alpha \neq \beta} \langle \hat{P}(e_\alpha) \hat{P}(\pi_j) \hat{P}(e_\beta) \rangle$$

Quantifies the attractiveness of the project with respect to risk, uncertainty, biases.

Caused by action interference.

19. Attraction ordering

π_1 is more attractive than π_2: $q(\pi_1) > q(\pi_2)$
(less risky, less uncertain)

π_1 and π_2 are equally attractive: $q(\pi_1) = q(\pi_2)$

(equally risky, equally uncertain)

20. Attraction conditions

π_1 is more attractive than π_2 if it is connected with:

(a) more certain gain,
(b) less certain loss,
(c) higher activity under certainty,
(d) lower activity under uncertainty.

Aversion to risk, uncertainty, and loss.

General properties

$$L = \{\pi_j : j = 1, 2, \ldots, N_L\}$$

Proposition 1.

$$p(\pi_j) = \sum_{\alpha} p(\pi_j e_\alpha) + q(\pi_j)$$

Proposition 2.

$$\sum_{j=1}^{N_L} q(\pi_j) = 0$$

Attraction alternation

Proposition 3.

π_1 preferred to π_2 if and only if

$$\sum_{\alpha} \left[\, p(\pi_1 e_\alpha) \; - \; p(\pi_2 e_\alpha) \,\right] \; > \; q(\pi_2) \; - \; q(\pi_1)$$

Return to classical decision theory:

$$q(\pi_j) \;\; \longrightarrow \;\; 0$$

Binary mind

Two actions

$$A = \bigcup_{j=1}^{\mathrm{M}_1} A_j \; , \qquad B = \bigcup_{\mu=1}^{\mathrm{M}_2} B_\mu \; ,$$

Two mode spaces

$$\mathcal{M}_1 = \mathrm{Span}\,\{\, |A_j\rangle : \; j = 1, 2, \ldots, \mathrm{M}_1\}$$

$$\mathcal{M}_2 = \mathrm{Span}\,\{\, |B_\mu\rangle : \; \mu = 1, 2, \ldots, \mathrm{M}_2\}$$

Mind space

$$\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$$

Elementary prospects $e_{j\mu} = A_j B_\mu$

Basic states

$$| \, e_{j\mu} \rangle \; = \; | \, A_j B_\mu \rangle \; = \; | \, A_j \rangle \; \otimes \; | \, B_\mu \rangle$$

Action prospects: $\pi_j \; = \; A_j B$

Prospect probabilities:

$$p(\pi_j) \; = \; \sum_{\mu=1}^{\mathrm{M}_2} p(A_j B_\mu) \; + \; q(\pi_j)$$

Conditional probability

$$p(A_j B_\mu) \; = \; p(A_j | B_\mu) \; p(B_\mu)$$

Correspondence

$A_j \longrightarrow$ lottery L_j

$B_\mu \longrightarrow$ payoffs

$p(B_\mu) \longrightarrow$ normalized measure of B_μ

$p(A_j|B_\mu) \longrightarrow p_j(B_\mu)$

probability of the payoffs B_μ in the lottery L_j

$\sum_\mu p(A_j B_\mu) \longrightarrow$ normalized utility of L_j

$q(A_j B) \longrightarrow$? No equivalent

Allais paradox

$$A = \bigcup_{j=1}^{4} A_j \,, \qquad B = \bigcup_{\mu=1}^{3} B_\mu$$

Balance condition for all $\mu = 1, 2, 3$

$$p(A_1 B_\mu) + p(A_3 B_\mu) = p(A_2 B_\mu) + p(A_4 B_\mu)$$

$$\pi_1 > \pi_2 : \quad p(\pi_1) > p(\pi_2)$$

π_1 is more attractive: $q(\pi_1) > q(\pi_2)$

$$\sum_\mu \left[p(A_2 B_\mu) - p(A_1 B_\mu) \right] < q(\pi_1) - q(\pi_2)$$

$$\pi_3 > \pi_4 : \quad p(\pi_3) > p(\pi_4)$$

π_3 more attractive: $q(\pi_3) > q(\pi_4)$

$$\sum_\mu \left[p(A_3 B_\mu) - p(A_4 B_\mu) \right] > q(\pi_4) - q(\pi_3)$$

Balance condition $\rightarrow$

$$-|q(\pi_3) - q(\pi_4)| < \sum_\mu \left[p(A_2 B_\mu) - p(A_1 B_\mu) \right] <$$

$$< |q(\pi_1) - q(\pi_2)|$$

in classical utility theory $q(\pi_j) \longrightarrow 0$, contradiction

In QDT no contradiction! $-\frac{1}{2} < 0.065 < \frac{1}{2}$

Disjunction Effect

Tversky – Shafir (1992)

A_1: second gamble accepted

A_2: second gamble refused

B_1: first gamble won

B_2: first gamble lost

Experiment

1-st gamble won + 2-nd accepted: $p(A_1 B_1) = 0.345$

1-st gamble won + 2-nd refused: $p(A_2 B_1) = 0.155$

$$p(A_1 B_1) = 0.345 > 0.155 = p(A_2 B_1)$$

1-st gamble lost + 2-nd accepted: $p(A_1 B_2) = 0.295$

1-st gamble lost + 2-nd refused: $p(A_2 B_2) = 0.205$

$$p(A_1 B_2) = 0.295 > 0.205 = p(A_2 B_2)$$

$$B = B_1 + B_2$$

1-st gamble not known + 2-nd accepted: $p(A_1 B) = 0.36$

1-st gamble not known + 2-nd refused: $p(A_2 B) = 0.64$

$$p(A_1 B) = 0.36 < 0.64 = p(A_2 B)$$

Theory

Active under uncertainty: $A_1B \rightarrow$

attraction factor $q(A_1B)$

Passive under uncertainty: $A_2B \rightarrow$

attraction factor $q(A_2B)$

$$q(A_2B) > q(A_1B)$$

Alternation theorem $\rightarrow$

$$q(A_2B) = -q(A_1B) > 0$$

$$q(A_2B) \rightarrow 0.25, \qquad q(A_1B) \rightarrow -0.25$$

Prediction

$$p(A_1 B) = p(A_1 B_1) + p(A_1 B_2) + q(A_1 B)$$

$$p(A_2 B) = p(A_2 B_1) + p(A_2 B_2) + q(A_2 B)$$

$$p(A_1 B) = 0.39, \qquad p(A_2 B) = 0.61$$

Agreement with experiment!

Theory:

$$P(A_1 B) = 0.39 < 0.61 = p(A_2 B)$$

Experiment

$$P(A_1 B) = 0.36 < 0.64 = p(A_2 B)$$

Conclusions

- Novel approach to decision making is developed based on a complex Hilbert space over a lattice of composite prospects.
- Risk and uncertainty are taken into account.
- Paradoxes of classical decision theory are explained.
- Good quantitative agreement with empirical data.
- Conjunction fallacy is a sufficient condition for disjunction effect.

References: V.I. Yukalov and D. Sornette, *Quantum Decision Theory*, arXiv.org.0802.3597 (2008); *Mathematical Basis of Quantum Decision Theory*, ssrn.com/abstract=1263853 (2008).