A Two-Factor Asset Pricing Model and the Fat Tail Distribution of Firm Sizes

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Asset pricing

✓Asset pricing is a major component of economic theory and practice.

✓The International Financing Reporting Standards (IFRS) formerly the International Accounting Standard (IAS) requires that firms' liabilities be valued at market value.

✓Asset pricing is involved in

- investment analysis,
- capital budgeting,
- merger and acquisition transactions,
- financial reporting,
- tax liability and litigation, ….

✓Price is set by supply-demand, consumption preference

✓Present value of future dividends (time-preference and discount factor)

✓Equilibrium (supply-demand) + No-arbitrage

✓Behavior and "convention", …

Asset pricing models

General prediction:

- Only non-diversifiable risks are remunerated
- Excess returns ~ load on factors;

- The CAPM $$\mathrm{E}\left[r_i - r_0\right] = \beta_i \cdot \mathrm{E}\left[r_m - r_0\right]$$
  - Assumption: equilibrium
- The APT $$r_i = \alpha + \beta_1 \cdot f_1 + \cdots + \beta_p \cdot f_p + \varepsilon_i$$ $$\mathrm{E}\left[r_i - r_0\right] = \beta_1 \cdot \pi_1 + \cdots + \beta_p \cdot \pi_p$$
  - Assumption: no arbitrage opportunity
- Compatibility:
  - each asset as an infinitesimal weight the economy
  - mean-variance efficiency of the replicating portfolios

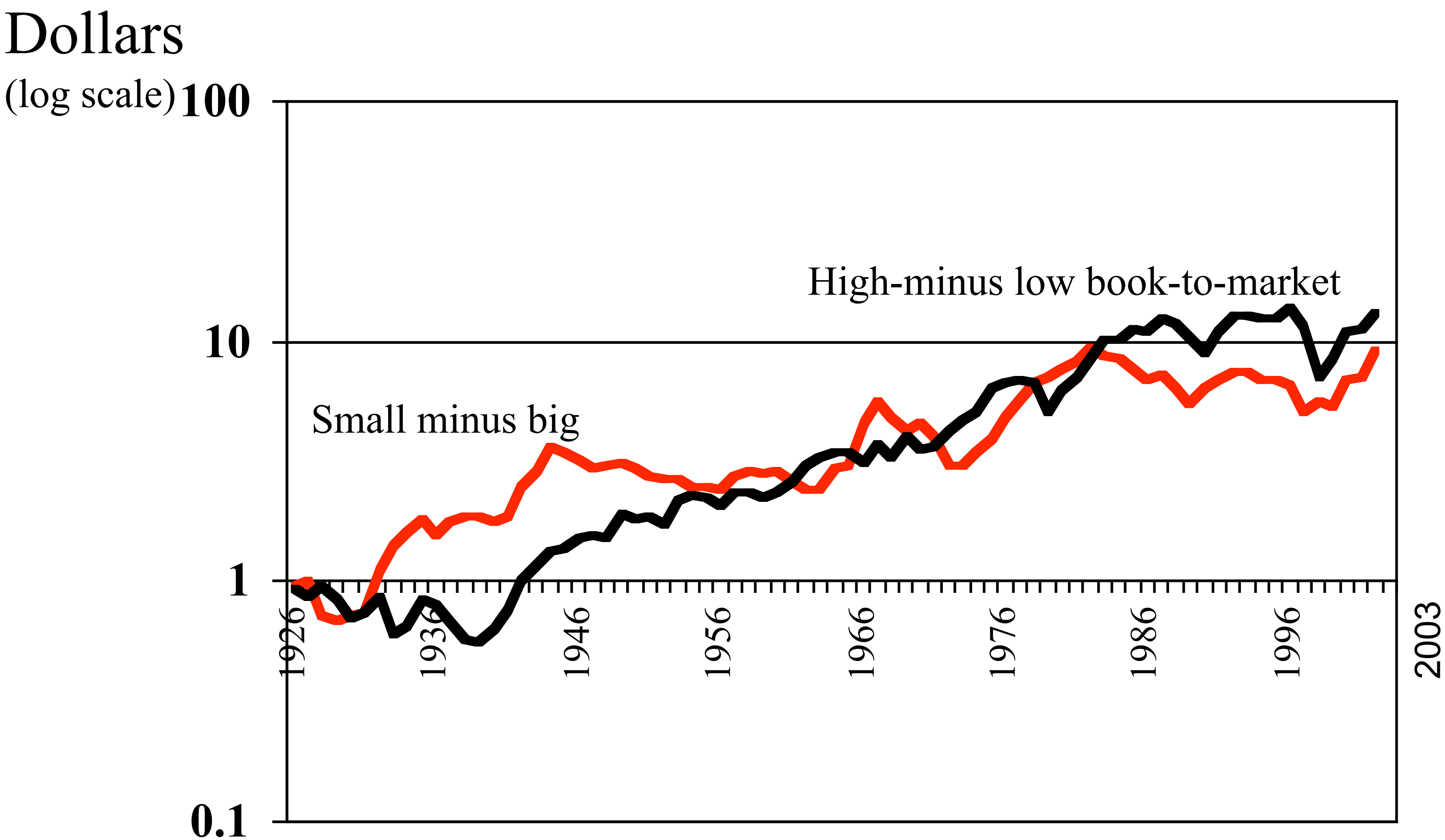
The pricing anomalies

- Small firm effect (Banz 1981)
- Book-to-market (Stattman 1980, Roseberg, Reid and Lanstein 1985, Daniel & Tittman 1997)

} Fama and French three factor model (1993, 1995)

- Reversal of long term returns (DeBondt and Thaler 1985, 1987)
- Continuation of short-term trends (Jegadeesh and Titman 1993)
- Preference for skewness (Rubinstein 1973, Harvey and Siddique 2000)
- …

Small-minus-Big vs. Book-to-Market



http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Our main results

- Our claims:
  - the lack of diversification of the market portfolio is responsible, to a large extent, for the failure of the CAPM to explain the cross-section of stock returns,
  - In addition to the market premium, investors require a *concentration premium.*
- Departure from the “traditional” explanations in terms of macro-economic factors, firm-specific factors, or behavioral factors.

Our two-factor model

•Our result is based on (i) the "internal consistency" condition that, in a complete market, the market portfolio is constituted of the assets whose returns it is supposed to explain and (ii) the distribution of the capitalization of firms is sufficiently heavy-tailed.

•Ingredient (i) leads mechanically to correlations between return residuals which are equivalent to the existence of a new "internal consistency" factor.

•By the generalized central limit theorem, ingredient (ii) ensures that the internal consistency factor does not disappear even for infinite economies and may produce significant undiversified non-priced risks for arbitrary well-diversified portfolios.

•The new self-consistency factor provides a rationalization of the SMB (Small Minus Big) factor and of the HML (High-minus-Low Book-to-Market) factor introduced by Fama and French (1993).

The distribution of firm sizes

A power law with unit tail index *μ=1*

$$\Pr\left[S \geq s\right] = \frac{1}{s^{\mu}} \cdot 1_{s \geq 1}$$

- A long history: Gibrat (1931), Zipf (1949), Simon & Bonini (1958), Axtell (2001), Marsili (2005), Gabaix *et al.* (2006)…

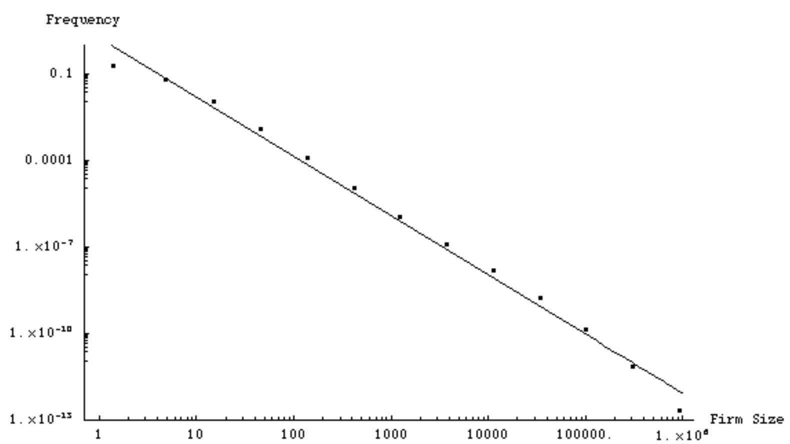


Figure 1: PDF of U.S. firm sizes, 1997 Economic Census data

(Axtell 2006)

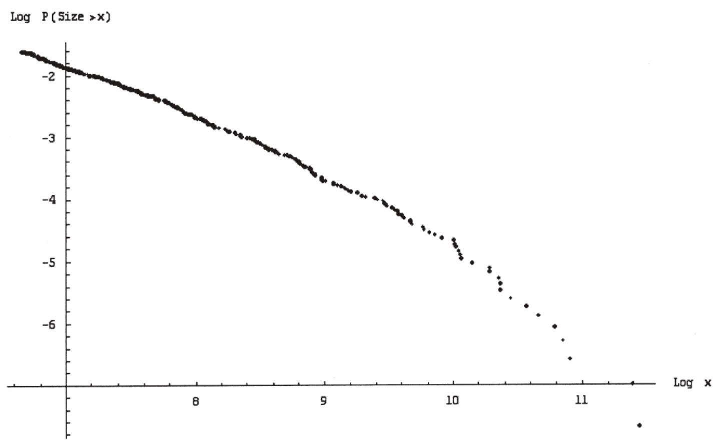
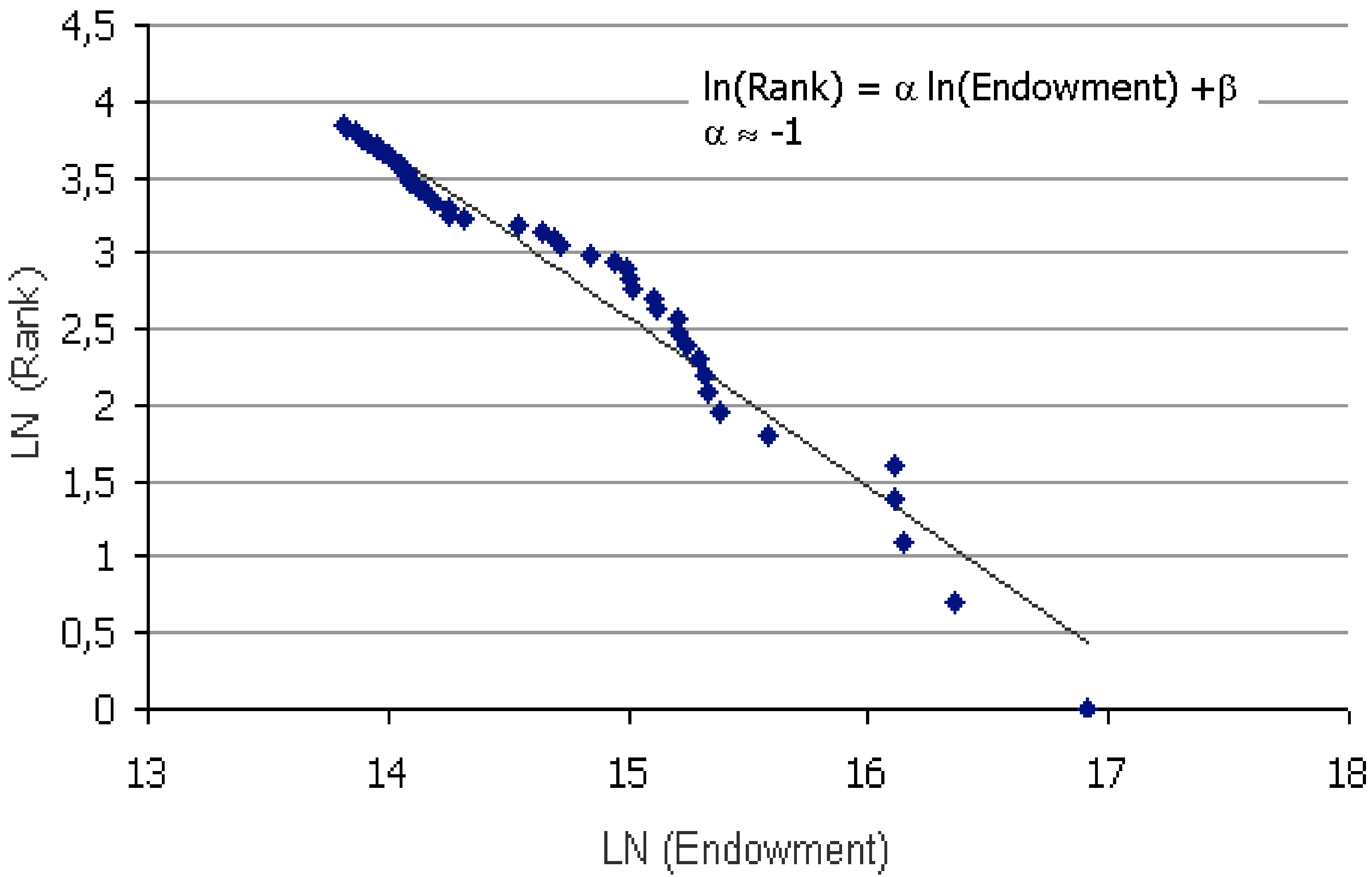


FIGURE VII

Cumulative distribution of the size (assets under management) of the top mutual funds in 1999. Source: Center for Research on Security Prices.

(Gabaix *et al.* 2006)

Distribution of US University endowments



Source: US Colleges and Universities with endowments greater than $1 billion in 2004

A power law with unit tail index

- A long history: Gibrat (1931), Zipf (1949), Simon & Bonini (1958), Axtell (2001), Marsili (2005), Gabaix *et al.* (2006)…
- Robustness *vis-à-vis* the proxy of the firm size: assets, market capitalizations, number of employees, profits, revenues, sales, value added…
- Several models: the law of proportional effect, economies of scale and costs reduction, the distribution of managerial talents and efficient allocation of productivity factors across managers, the partition of the set of workers…

Consequence on the concentration of the market portfolio

- The market portfolio: value-weighted portfolio of all the assets traded on the market
- Vector of composition: $\vec{w}_m = \left(w_{m,1}, \ldots, w_{m,N}\right)'$
- Definition: A portfolio is *well-diversified* if

$$H_N = \left\|\vec{w}_m\right\|^2 = \sum_{i=1}^{N} w_{m,i}{}^2 \xrightarrow{N \to \infty} 0$$

Consequence on the concentration of the market portfolio

- Consider an economy of *N* firms, whose sizes S_i, $i = 1, \ldots, N$, are drawn from a Pareto law with tail index μ

$$\Pr[S \geq s] = \frac{1}{s^{\mu}} \cdot 1_{s \geq 1}$$

- Let $w_{1,N} = \frac{\max S_i}{\sum_{i=1}^{N} S_i}$,

we have:

$$\mathrm{E}\left[w_{1,N}\right] \xrightarrow{N\to\infty} 0, \quad \text{as } \mu \geq 1$$

$$\mathrm{E}\left[1/w_{1,N}\right] \xrightarrow{N\to\infty} \frac{1}{1-\mu}, \quad \text{as } \mu < 1$$

Consequence on the concentration of the market portfolio

- Example 1: let the sizes, sorted in descending order, of the *N* firms be given by $S_{i,N} = \left(\frac{i}{N}\right)^{-1/\mu}$

  Then:

$$w_{m,1} \longrightarrow 0, \quad \text{if } \mu \geq 1,$$
$$w_{m,1} \longrightarrow \frac{1}{\zeta(1/\mu)}, \quad \text{if } \mu < 1,$$

where ζ denotes the Riemann zeta function $\varsigma(z) = \sum_{i=1}^{\infty} i^{-z}$

Consequence on the concentration of the market portfolio

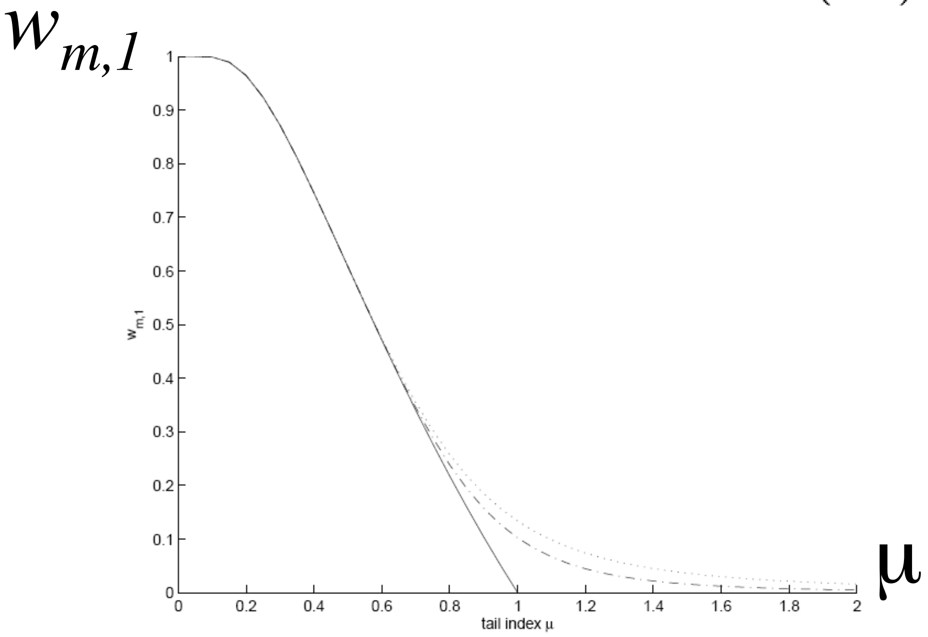
- Example 1: let the sizes, sorted in descending order, of the *N* firms be given by $S_{i,N} = \left(\frac{i}{N}\right)^{-1/\mu}$

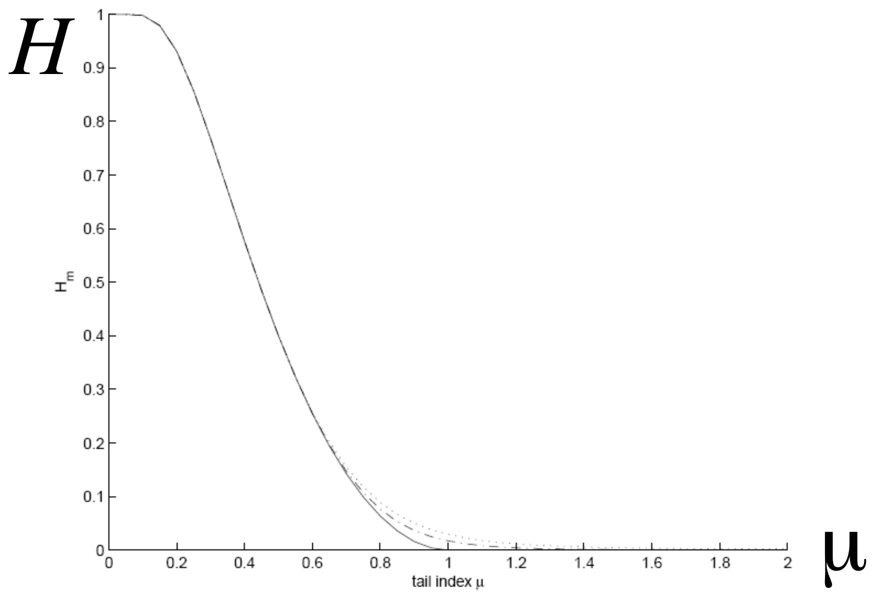
  Then:

$$H_N = \begin{cases} \frac{1}{1-\frac{1}{(1-\mu)^2}} \cdot \frac{1}{N} + O\left(N^{2/\mu-2}\right), & \mu > 2, \\ \frac{\ln N + \gamma}{4N} + O\left(N^{-3/2}\ln N\right), & \mu = 2, \\ \left(\frac{1-\mu}{\mu}\right)^2 \zeta(2/\mu) \cdot \frac{1}{N^{2-2/\mu}} + O\left(N^{3(1/\mu-1)}\right), & 1 < \mu < 2, \\ \frac{\pi^2}{6}\frac{1}{(\gamma+\ln N)^2} + O\left(N^{-1}(\gamma+\ln N)^{-2}\right), & \mu = 1, \\ \frac{\zeta(2/\mu)}{\zeta(1/\mu)^2} + O\left(N^{1-1/\mu}\right), & \mu < 1. \end{cases}$$

Consequence on the concentration of the market portfolio

- Example 1: let the sizes, sorted in descending order, of the *N* firms be given by $S_{i,N} = \left(\frac{i}{N}\right)^{-1/\mu}$





Plain line: *N*=infinity; Dotted line: *N*=1,000; Dash-dotted line: *N*=10,000.

Consequence on the concentration of the market portfolio

- Example 2: let the firm sizes be randomly drawn from a power law distribution of size with tail index μ, i.e. $s^{\mu} \cdot \Pr[S > s] \longrightarrow c \;\; as \;\; s \longrightarrow \infty,$

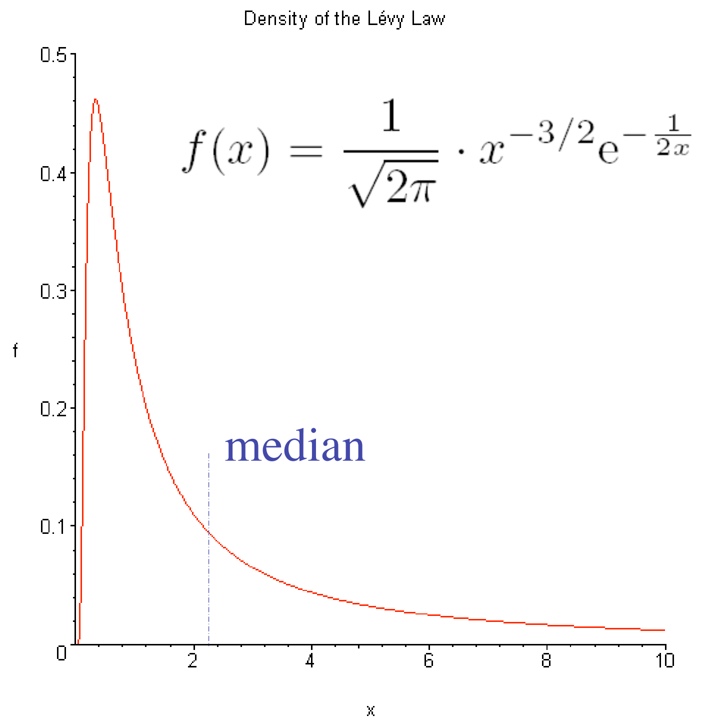
$$H_N = \begin{cases} \frac{1}{N}\frac{\mathrm{E}[S^2]}{\mathrm{E}[S]^2} + o_p(1/N), & \textit{provided that } \mathrm{E}[S^2] < \infty, \\ \frac{c}{\mathrm{E}[S]^2}\frac{\ln N}{N} + o_p\left(\frac{1}{N \ln N}\right), & \mu = 2 \\ \left[\frac{\pi c}{2\Gamma\left(\frac{\mu}{2}\right)\sin\frac{\mu\pi}{4}}\right]^{2/\mu} \frac{1}{\mathrm{E}[S]^2} \cdot \frac{1}{N^{2-2/\mu}} \cdot \xi_N + o_p\left(\frac{1}{N^{2-2/\mu}}\right), & \mu \in (1,2) \\ \frac{\pi}{2 \cdot \ln^2 N} \cdot \xi_N + O_p\left(\frac{1}{\ln^3 N}\right), & \mu = 1 \\ \frac{4}{\pi^{1/\mu}}\left[\Gamma\left(\frac{1+\mu}{2}\right)\cos\frac{\pi\mu}{4}\right]^{2/\mu} \cdot \frac{\xi_N}{\zeta_N^2}, & \mu \in (0,1) \end{cases}$$

Consequence on the concentration of the market portfolio

- Example 2: case $\mu=1$,

$$H_N \simeq \frac{\pi}{2 \cdot (\ln N)^2} \cdot \xi_N ,$$

ξ_N is a sequence of positive random variables with stable limit law *S(1/2,1)*, *i.e.*, the Levy law



Consequence on the concentration of the market portfolio

- Example 2: case $\mu=1$,

$$H_N \simeq \frac{\pi}{2 \cdot (\ln N)^2} \cdot \xi_N \,,$$

with ξ_N = 2.198, a typical value of H_N is *4-5%* for a market where 7000 to 8000 assets are traded.

H_N = 4-5% means that there are only about 20-25 independent lines in a typical portfolio supposedly well-diversified on 7000 - 8000 assets.

Consequence on the concentration of the market portfolio

- Two questions:
  - how can the market portfolio alone explain the expected return on any asset, irrespective of its size, as predicted by the CAPM?
  - is it actually optimal for a rational investor to put her money in this risky portfolio alone, as suggested by the theorem of separation in two funds?

Consequence on the concentration of the market portfolio

- Our claims:
  - the lack of diversification of the market portfolio is responsible, to a large extent, for the failure of the CAPM to explain the cross-section of stock returns,
  - In addition to the market premium, investors require a *concentration premium.*
- Departure from the “traditional” explanations in terms of macro-economic factors, firm-specific factors, or behavioral factors.

Consequence on the concentration of the market portfolio

- A justification:
  - Most of these factors provide a significant improvement in explaining the cross-section of asset returns.
  - BUT, they do not provide a clear identification of the most prominent ones.
  - Our approach focuses on the undisputable fact that the market portfolio is highly concentrated on a small number of very large companies and therefore cannot account for the behavior of the smallest ones.

Internal consistency of factor models

- Consider an economy with *N* firms whose returns on stock prices are determined according to the following equation

$$\vec{r} = \vec{\alpha} + \vec{\beta}_m \cdot [r_m - \mathrm{E}[r_m]] + B\vec{\phi} + \vec{\varepsilon},$$

- $\vec{r}$ is the random $N \times 1$ vector of asset returns;
- $\vec{\alpha} = \mathrm{E}[\vec{r}]$ is the $N \times 1$ vector of asset return mean values. We do not make any assumption neither on the *ex-ante* mean-variance efficiency of the market portfolio, nor on the absence of arbitrage opportunity, so that $\vec{\alpha}$ is not, *a priori*, specified;
- r_m is the random return on the market portfolio;
- $\vec{\beta}_m$ is the $N \times 1$ vector of the factor loadings of the market factor;
- $\vec{\phi}$ is the random $N \times 1$ vector of risk factors ϕ_i which are assumed to have zero mean $(\mathrm{E}[\phi_i] = 0)$, unit variance, are uncorrelated with each other and with r_m;
- B is the $N \times q$ matrix of factor loadings;
- $\vec{\varepsilon}$ is the random $N \times 1$ vector of disturbance terms with zero average $\mathrm{E}[\vec{\varepsilon}] = \vec{0}$ and covariance matrix $\Omega = \mathrm{E}[\vec{\varepsilon} \cdot \vec{\varepsilon}]$. The disturbance terms are assumed to be uncorrelated with the market return r_m and the factors ϕ_i.

Internal consistency of factor models

- Accounting for the fact that

$$r_m = \vec{w}'_m \cdot \vec{r}.$$

we get

$$\left[\vec{w}'_t \cdot \vec{\beta} - 1\right] \cdot (r_m - \mathrm{E}\left[r_m\right]) + w'_m B\vec{\phi} + \vec{w}'_m \cdot \vec{\varepsilon}_t = 0$$

which allows concluding that

$$\vec{w}'_m \cdot \vec{\varepsilon}_t = 0 \qquad \text{almost surely,}$$

The disturbance terms are correlated

$$\vec{w}'_m \cdot \vec{\beta} = 1 \quad \text{and} \qquad \vec{w}'_m B = 0 \ .$$

Correlation structure of the disturbance terms

- The fact that the disturbance terms are correlated means that there exists at least one common "factor" f to the ε's:

$$\vec{\varepsilon} = \vec{\gamma} \cdot f + \vec{\eta} \ ,$$

where $\vec{\gamma}$ is the vector of factor loadings

- Actually, f is not a factor in so far as it cannot be uncorrelated with $\vec{\eta}$ due to the internal consistency relation $\vec{w}'_m \cdot \vec{\varepsilon}_t = 0$, which yields

$$f = -\frac{\vec{w}'_m \vec{\eta}}{\vec{w}'_m \vec{\gamma}}$$

provided that $\vec{w}'_m \vec{\gamma} \neq 0$. (otherwise, we should have $\vec{w}_m ' \vec{\eta} = 0$)

Correlation structure of the disturbance terms

- The market model becomes

$$\vec{r}_t = \vec{\alpha} + \vec{\beta} \cdot r_m(t) + \vec{\gamma} \cdot f + \vec{\eta},$$

with:

- $\text{Cov }(r_m, f) = 0,$
- $\text{Cov }(r_m, \vec{\eta}) = 0,$
- $\text{Var } \vec{\eta} = \Delta,$ (Δ can be chosen as a diagonal matrix)
- $\text{Var } f = \frac{\vec{w}'_m \Delta \vec{w}_m}{(\vec{w}'_m \vec{\gamma})^2}$
- $\text{Cov }(f, \vec{\eta}) = -\frac{1}{\vec{w}'_m \vec{\gamma}} \cdot$

Correlation structure of the disturbance terms $\vec{\varepsilon} = \vec{\gamma} \cdot f + \vec{\eta}$,

- Remark: one can always choose $\vec{\gamma}$ such that $\vec{w}'\vec{\gamma} = 1$.

  Thus: $\Omega = (\vec{w}_m' \Delta \vec{w}_m)\, \vec{\gamma}\vec{\gamma}' - \vec{\gamma}\vec{w}_m' \Delta - \Delta \vec{w}_m \vec{\gamma}' + \Delta,$

  and

$$\rho_{ij} = \frac{(\vec{w}_m' \Delta \vec{w}_m)\, \gamma_i \gamma_j - \gamma_i w_{m,j} \Delta_{jj} - \gamma_j w_{m,i} \Delta_{ii}}{\sqrt{[(\vec{w}_m' \Delta \vec{w}_m)\, \gamma_i^2 - 2\gamma_i w_{m,i} \Delta_{ii} + \Delta_{ii}] \cdot [(\vec{w}_m' \Delta \vec{w}_m)\, \gamma_j^2 - 2\gamma_j w_{m,j} \Delta_{jj} + \Delta_{jj}]}}$$

  which simplifies to

$$\rho_{ij} = \frac{H_N - w_{m,i} - w_{m,j}}{\sqrt{(1 + H_N - 2w_{m,i})\,(1 + H_N - 2w_{m,j})}} = \frac{H_N}{1 + H_N} \cdot \left(1 + O(w_{m,i(j)}/H_N)\right)$$

  if $\gamma_i = 1$ and $\Delta_{ii} = \Delta$, for all $i = 1, \ldots, N$

- Since $\rho_{ij} \simeq \frac{H_N}{1 + H_N}$, the largest eigenvalue of the correlation matrix is $\lambda_{\max,N} \simeq N \cdot \frac{H_N}{1+H_N}$.

Correlation structure of the disturbance terms $\vec{\varepsilon} = \vec{\gamma} \cdot f + \vec{\eta}$,

- Consequences for the residual variance of a well-diversified portfolio w_p:

$$w_p' \Omega w_p = (\vec{w}_m' \Delta \vec{w}_m) \left(\vec{\gamma} \vec{w}_p'\right)^2 - 2 (\vec{w}_m' \Delta \vec{w}_p) (\vec{\gamma}' \vec{w}_p) + \vec{w}_p' \Delta \vec{w}_p'$$

If $\Delta_{ii} \leq c < \infty$ and $0 < c' \leq |\vec{\gamma} \vec{w}_p'| \leq c'' < \infty$, we get:

$$\begin{aligned} \vec{w}_p' \Delta \vec{w}_p' &\leq c \cdot ||\vec{w}_p||^2 \to 0, \\ |(\vec{w}_m' \Delta \vec{w}_p) (\vec{\gamma}' \vec{w}_p)| &\leq c \cdot c'' \cdot ||w_m|| \cdot ||w_p|| \to 0 \end{aligned}$$

so that

$$w_p' \Omega w_p \sim K \cdot H_N, \quad K > 0, \quad \text{as} \quad N \to \infty.$$

Asymptotic variance of the equally-weigthed portfolio

$$\mathrm{Var}\ r_e = \begin{cases} {\beta_e}^2 \cdot \mathrm{Var}\ r_m + O_p(1/N), & \textit{provided that } \mathrm{E}[S^2] < \infty, \\ {\beta_e}^2 \cdot \mathrm{Var}\ r_m + \frac{c \cdot \bar{\Delta}}{\mathrm{E}[S]^2} \frac{\ln N}{N} + o_p(\ln N/N), & \mu = 2 \\ {\beta_e}^2 \cdot \mathrm{Var}\ r_m + \left[\frac{\pi c \mathrm{E}\left[\Delta^{\mu/2}\right]}{2\Gamma\left(\frac{\mu}{2}\right) \sin \frac{\mu\pi}{4}} \right]^{2/\mu} \frac{1}{\mathrm{E}[S]^2} \cdot \frac{1}{N^{2-2/\mu}} \cdot \xi_N + o_p\left(\frac{1}{N^{2-2/\mu}}\right) & \mu \in (1,2) \\ {\beta_e}^2 \cdot \mathrm{Var}\ r_m + \frac{\pi\ \mathrm{E}\left[\Delta^{1/2}\right]^2}{2} \frac{\mathrm{E}[\gamma]^2}{\mathrm{E}[|\gamma|]^2} \frac{1}{\ln^2 N} \cdot \xi_N + o_p\left(1/\ln^2 N\right), & \mu = 1 \\ {\beta_e}^2 \cdot \mathrm{Var}\ r_m + \mathrm{E}\left[\Delta^{\mu/2}\right]^{2/\mu} \frac{\mathrm{E}[\gamma]^2}{\mathrm{E}[|\gamma|^{\mu}]^{2/\mu}} \frac{4}{\pi^{1/\mu}} \left[\Gamma\left(\frac{1+\mu}{2}\right) \cos \frac{\pi\mu}{4}\right]^{2/\mu} \cdot \frac{\xi_N}{\zeta_N^2} + o_p(1), & \mu \in (0,1) \end{cases}$$

Specific market risk

Non-diversified risk

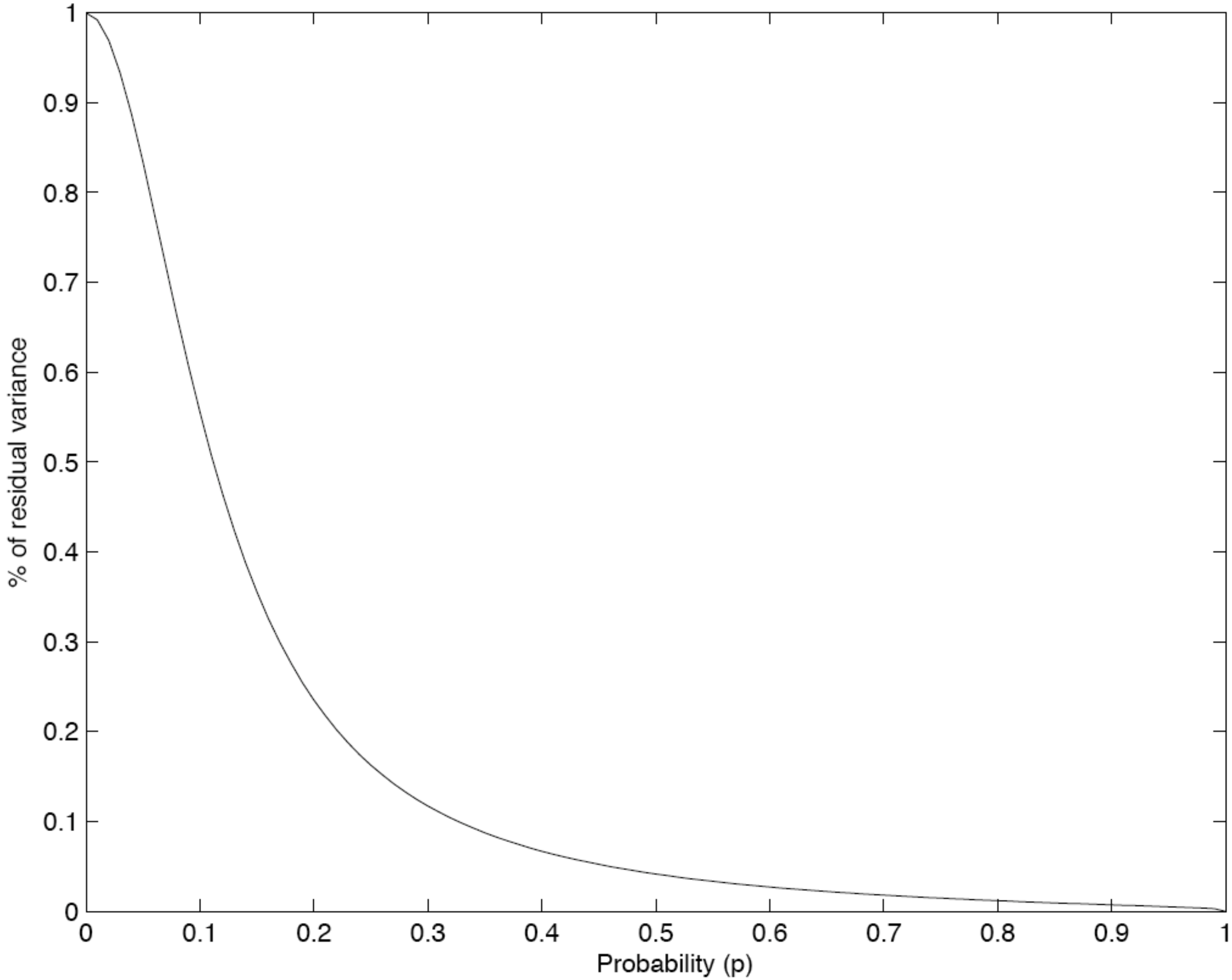
$$\mathrm{Var}\ r_e \overset{law}{\simeq} \beta_e^2 \cdot \mathrm{Var}\ r_m + K_\mu \cdot H_N$$

Additional contribution:

$$K_\mu / N_{eff}$$

Contribution of the residual variance

Contribution of the residual variance to the total variance. The figure shows the probability p to reach or exceed a given level of contribution, in percentage, of the residual variance to the total variance of the return on the equally weighted portfolio in a market with 7000-8000 traded asset and a distribution of firm sizes given by Zipf's law ($\mu = 1$).



Contribution of the residual variance

Assume distribution of firm sizes: $f(x) = \frac{1}{\sqrt{2\pi}} \cdot x^{-3/2} e^{-\frac{1}{2x}}, \quad x \geq 0.$

density of the marginal law of w_i $\quad g_N(w) = \frac{N-1}{\pi} \cdot \frac{w^{-1/2}(1-w)^{1/2}}{1+[(N-1)^2-1]\,w}$

then $\quad \mathrm{E}[H_N] = \frac{1}{2} \cdot \frac{N+1}{N}$

$$H_N = H + o_p(1), \quad \text{with } H = \lim_{N\to\infty} \frac{S_1^2 + \cdots + S_N^2}{(S_1 + \cdots + S_N)^2},$$

$$\mathrm{Var}\ f = \sigma_f^2 + o_p(1), \quad \text{with } \sigma_f^2 = \lim_{N\to\infty} \frac{\Delta_{11} \cdot S_1^2 + \cdots + \Delta_{NN} \cdot S_N^2}{(\gamma_1 \cdot S_1 + \cdots + \gamma_N \cdot S_N)^2},$$

$$\mathrm{Var}\ r_e = \underbrace{\beta_e^2 \cdot \mathrm{Var}\ r_m}_{\text{specific market risk}} + \underbrace{\mathrm{E}[\gamma]^2 \cdot \sigma_f^2}_{\text{non-diversified risk}} + o_p(1).$$

Numerical Simulations of the market model

$$\Pr\left[S \geq s\right] = \frac{1}{s^{\mu}} \cdot 1_{s \geq 1}$$

$$\vec{r} = \vec{\alpha} + \vec{\beta} \cdot \left[r_m - E\left[r_m\right]\right] + \vec{\gamma} \cdot f + \vec{\eta}$$

- Average, minimum and maximum value of the R^2 of the regression of the return of 20 equally weighted portfolios (randomly drawn from a market of 1000 and 10000 assets) on the market portfolio (r_m), on the market portfolio and the internal consistency factor (r_m, f), on the market portfolio and the (overall) equally weighted portfolio (r_m, r_e), on the market portfolio and an under-diversified portfolio (r_m, r_u) and on the market portfolio and a well-diversified arbitrage portfolios (r_m, r_a). Different market situations are considered with distributions of firm sizes with tail index μ which varies from 0.5 to 2.

		N=1000					N=10000				
		r_m	r_m, f	r_m, r_e	r_m, r_u	r_m, r_a	r_m	r_m, f	r_m, r_e	r_m, r_u	r_m, r_a
	Mean	94%	94%	95%	94%	94%	99%	99%	99%	99%	99%
$\mu = 2$	Min	90%	93%	93%	90%	90%	99%	99%	99%	99%	99%
	Max	96%	96%	96%	96%	96%	100%	100%	100%	100%	100%
	Mean	80%	95%	95%	86%	82%	88%	99%	99%	93%	89%
$\mu = 1$	Min	1%	91%	91%	42%	17%	20%	99%	99%	66%	20%
	Max	95%	100%	100%	95%	95%	99%	100%	100%	99%	99%
	Mean	56%	97%	97%	79%	64%	56%	100%	100%	83%	63%
$\mu = 1/2$	Min	2%	89%	89%	34%	15%	1%	96%	97%	15%	3%
	Max	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Numerical Simulations of the market model

$P(S) \sim 1/S^{1+\mu}$ $\mu=1$

$$\vec{r}_t = \vec{\alpha} + \vec{\beta} \cdot r_m(t) + \vec{\gamma} \cdot f + \vec{\eta}$$

$$\text{Var } r_e = {\beta_e}^2 \cdot \text{Var } r_m + \bar{\gamma}_N^2 \cdot \frac{\sum_{i=1}^N S_i^2 \Delta_{ii}}{\left(\sum_{i=1}^N S_i \gamma_i\right)^2} + O_p(1/N).$$

	N=1000				N=10000			
	Market factor	Market Factor + f	Market Factor + EW	Market + Under Diversified	Market factor	Market Factor + f	Market Factor + EW	Market + Under Diversified
$\mu=2$	94%	**94%**	**95%**	94%	99%	**99%**	**99%**	99%
$\mu=1$	80%	**95%**	**95%**	86%	88%	**99%**	**99%**	93%
$\mu=0.5$	56%	**97%**	**97%**	79%	56%	**99%**	**99%**	83%

Asset pricing equation

$$r_i = \alpha + \beta_1 \cdot f_1 + \cdots + \beta_p \cdot f_p + \varepsilon_i$$

$$\mathrm{E}\left[r_i - r_0\right] = \beta_1 \cdot \pi_1 + \cdots + \beta_p \cdot \pi_p$$

- The market model is:

$$\vec{r}_t = \vec{\alpha} + \vec{\beta} \cdot r_m(t) + \vec{\gamma} \cdot f + \vec{\eta},$$

- Therefore, the APT applies and tell us that

$$\mathrm{E}\left[r_i - r_0\right] = \beta_i \cdot \mathrm{E}\left[r_m - r_0\right] + (\gamma_i - \gamma_m \cdot \beta_i) \cdot \mathrm{E}\left[r_{icc} - r_0\right]$$

where r_{ICC} is the return on the equally-weighed portfolio r_e minus the return on the market portfolio r_m, which is used as a proxy for f.

Empirical consequences

- Multi-factor time series regression:

$$r_{i,t} - r_0 = \alpha_i + \beta_i \cdot [r_m(t) - r_0] + \beta_i^{ICC} \cdot r_{icc}(t) + \beta_i^{SMB} \cdot r_{smb}(t) + \beta_i^{HML} \cdot r_{hml}(t) + \varepsilon_i(t)$$

with r_{smb} and r_{hml}, the two Fama & French factors

- If our specification is correct:

$$\alpha_i = \beta^{SMB} = \beta^{HLM} = 0$$

25 portfolios sorted by size and book-to-market

- Parameter estimates of the linear regression of the excess returns on 25 equally-weighed portfolios (sorted by quintiles of the distribution of size – Small, 2, 3, 4 and Big – and by quintiles of the distribution of Book equity to Market equity ratio – Low, 2, 3, 4 and High) regressed on the excess return on the market portfolio, on the two Fama-French factors SMB and HML and on the proxy for the additional risk factor due to the internal consistency constraint given by the difference between the return on the equally-weighted portfolio and the return on the market portfolio.

  Time span: Jan. 1927 – Dec 2005;

  948 months

		α	β	β^{SMB}	β^{HML}	β^{ICC}	R^2
	Low	-0.0076	1.24**	-0.49**	-0.24**	2.33**	75%
	2	-0.0032	1.05**	0.78**	0.16*	1.17**	81%
Small	3	0.0007	1.01**	0.37**	0.21**	1.06**	89%
	4	0.0017	0.94**	0.47**	0.36**	1.05**	94%
	High	0.0037	0.93**	0.45**	0.65**	1.32**	92%
	Low	-0.0032	1.11**	0.70**	-0.38**	0.56**	90%
	2	-0.0009	1.11**	0.69**	0.14**	0.34**	94%
2	3	0.0011	0.98**	0.75**	0.33**	0.20**	93%
	4	0.0008	1.00**	0.74**	0.56**	0.11**	95%
	High	-0.0004	1.07**	0.79**	0.83**	0.19**	96%
	Low	-0.0021	1.16**	0.29**	-0.38**	0.61**	92%
	2	0.0010	1.03**	0.44**	0.03	0.11*	92%
3	3	0.0005	1.04**	0.38**	0.32**	0.08**	93%
	4	0.0011	0.97**	0.51**	0.52**	-0.01**	93%
	High	-0.0007	1.18**	0.31**	0.87**	0.28**	94%
	Low	0.0004	1.08**	0.07	-0.44**	0.26**	93%
	2	-0.0004	1.04**	0.14**	0.10**	0.11*	91%
4	3	0.0010	1.02**	0.17**	0.29**	0.09	92%
	4	0.0002	1.08**	0.08	0.57**	0.16**	93%
	High	-0.0024	1.27**	0.17**	0.98**	0.28**	93%
	Low	0.0002	1.06**	-0.24**	-0.35**	0.21**	96%
	2	0.0003	1.04**	-0.19**	0.07**	0.13**	94%
Big	3	-0.0001	1.04**	-0.20**	0.32**	0.11**	93%
	4	-0.0015	1.10**	-0.30**	0.66**	0.26**	92%
	High	-0.0012	1.10**	-0.26**	0.82**	0.27**	86%

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25 portfolios sorted by size and book-to-market (I)

		Rm	ICC	SMB	HML	HML + SMB	ICC + SMB	ICC + HML	All four factors
	Low	52.0% (43.1%,60.6%)	**74.8%** (68.5%,80.3%)	66.7% (60.1%,74.5%)	54.3% (44.3%,64.3%)	68.6% (62.1%,75.8%)	**74.9%** (68.7%,80.9%)	74.8% (69.3%,80.6%)	75.2% (69.6%,81.0%)
	2	51.8% (43.4%,61.5%)	**79.9%** (73.0%,86.0%)	76.4% (71.9%,81.3%)	54.9% (45.7%,66.6%)	78.9% (73.3%,84.3%)	**80.7%** (75.1%,86.1%)	79.9% (74.0%,86.2%)	80.9% (75.5%,86.5%)
Small	3	63.8% (57.2%,70.3%)	**89.0%** (85.8%,91.8%)	82.9% (80.0%,85.6%)	68.5% (60.7%,76.5%)	87.0% (83.9%,90.2%)	89.1% (86.4%,91.9%)	**89.1%** (85.8%,92.5%)	89.4% (86.6%,92.6%)
	4	61.7% (53.8%,69.8%)	**92.5%** (90.9%,94.2%)	84.4% (81.6%,87.6%)	69.4% (62.1%,77.2%)	91.3% (89.3%,93.2%)	92.6% (91.0%,94.3%)	**93.2%** (91.7%,95.0%)	93.7% (92.5%,95.3%)
	High	53.9% (46.3%,62.6%)	**89.5%** (86.0%,92.5%)	77.2% (71.2%,82.5%)	67.5% (60.9%,74.3%)	89.6% (85.9%,92.4%)	89.7% (86.1%,92.7%)	**92.1%** (89.5%,94.4%)	92.5% (89.9%,94.8%)
	Low	70.3% (66.1%,75.4%)	84.2% (81.0%,87.7%)	**88.9%** (86.1%,91.5%)	70.8% (66.5%,76.3%)	**89.6%** (87.1%,92.1%)	88.9% (86.4%,91.5%)	89.0% (86.7%,91.5%)	90.4% (88.5%,92.6%)
	2	78.0% (71.3%,84.1%)	92.2% (90.3%,94.1%)	**92.3%** (90.8%,94.0%)	79.3% (73.2%,85.0%)	93.4% (92.1%,94.9%)	**93.5%** (92.3%,95.0%)	92.3% (90.5%,94.2%)	93.7% (92.5%,95.2%)
2	3	74.6% (65.9%,83.0%)	**90.8%** (88.3%,93.8%)	89.6% (86.9%,92.8%)	78.4% (71.0%,85.9%)	**92.9%** (91.2%,95.3%)	91.6% (89.5%,94.2%)	91.1% (88.8%,94.1%)	93.0% (91.4%,95.4%)
	4	75.8% (69.2%,81.8%)	**91.0%** (88.4%,93.1%)	87.7% (84.7%,90.7%)	83.6% (78.5%,88.5%)	**94.9%** (93.7%,96.2%)	91.1% (88.6%,93.2%)	93.2% (91.4%,94.8%)	95.0% (93.8%,96.2%)
	High	71.3% (65.5%,76.6%)	**89.3%** (85.7%,92.0%)	83.4% (79.3%,87.5%)	84.4% (80.4%,88.0%)	**95.8%** (94.0%,97.0%)	89.4% (85.8%,92.0%)	94.3% (92.2%,95.8%)	95.9% (94.2%,97.1%)

25 portfolios sorted by size and book-to-market (II)

		Rm	ICC	SMB	HML	HML + SMB	ICC + SMB	ICC + HML	All four factors
	Low	80.3% (75.7%,84.8%)	88.6% (86.1%,90.8%)	**90.7%** (87.8%,93.0%)	80.8% (76.5%,85.5%)	91.4% (88.9%,93.5%)	90.8% (88.4%,93.1%)	**92.2%** (90.6%,93.7%)	92.5% (90.9%,94.0%)
	2	85.6% (82.7%,88.3%)	90.9% (89.1%,92.9%)	**91.8%** (89.7%,93.8%)	85.7% (82.9%,88.6%)	92.0% (90.1%,93.9%)	**92.0%** (90.1%,93.9%)	91.1% (89.2%,93.0%)	92.0% (90.2%,93.9%)
3	3	85.4% (81.9%,88.4%)	**91.4%** (89.2%,93.2%)	89.9% (87.3%,92.1%)	88.8% (86.3%,90.9%)	**93.0%** (91.5%,94.3%)	91.4% (89.3%,93.3%)	92.4% (90.8%,93.8%)	93.1% (91.5%,94.3%)
	4	80.4% (75.2%,84.9%)	**88.7%** (85.0%,91.6%)	86.0% (82.2%,89.4%)	87.8% (84.3%,91.1%)	**93.0%** (91.1%,94.7%)	88.7% (85.2%,91.7%)	91.9% (89.6%,93.9%)	93.0% (91.1%,94.7%)
	High	75.6% (70.8%,79.7%)	85.9% (82.5%,88.9%)	79.9% (75.8%,83.9%)	**90.5%** (87.1%,93.1%)	**94.3%** (92.4%,95.8%)	87.2% (83.7%,90.3%)	94.2% (92.0%,95.8%)	94.4% (92.6%,96.0%)
	Low	86.4% (84.0%,88.7%)	87.0% (84.8%,89.3%)	88.4% (86.2%,90.4%)	**90.2%** (88.3%,91.8%)	92.3% (90.8%,93.7%)	89.0% (86.9%,91.2%)	**92.6%** (91.4%,93.9%)	92.6% (91.4%,94.0%)
	2	89.4% (87.1%,91.5%)	**91.3%** (89.0%,93.3%)	90.8% (88.2%,93.1%)	90.0% (88.0%,91.9%)	**91.4%** (89.3%,93.5%)	91.3% (89.1%,93.4%)	91.4% (89.3%,93.4%)	91.5% (89.4%,93.5%)
4	3	87.3% (84.5%,89.8%)	90.3% (87.5%,92.5%)	88.9% (86.2%,91.5%)	**90.5%** (88.5%,92.6%)	**92.0%** (89.8%,94.0%)	90.5% (87.8%,92.7%)	91.9% (89.7%,93.8%)	92.0% (89.8%,94.0%)
	4	82.5% (78.6%,85.7%)	86.6% (82.6%,89.8%)	83.5% (79.9%,87.1%)	**91.8%** (89.2%,93.9%)	92.7% (90.3%,94.6%)	88.1% (84.1%,91.1%)	**92.8%** (90.2%,94.6%)	92.8% (90.3%,94.7%)
	High	74.4% (69.6%,78.9%)	82.1% (77.6%,85.9%)	76.6% (72.0%,81.1%)	**90.7%** (87.6%,93.2%)	92.5% (89.9%,94.5%)	84.5% (79.7%,88.7%)	**92.6%** (90.0%,94.6%)	92.7% (90.1%,94.6%)
	Low	92.0% (90.5%,93.3%)	92.5% (91.0%,93.8%)	92.2% (90.7%,93.5%)	**95.1%** (94.0%,96.1%)	**95.2%** (94.2%,96.2%)	92.7% (91.1%,94.1%)	95.1% (94.0%,96.1%)	95.5% (94.6%,96.4%)
	2	93.3% (91.0%,94.9%)	93.3% (91.0%,95.0%)	93.5% (91.5%,95.0%)	**93.7%** (91.6%,95.3%)	**93.9%** (92.0%,95.4%)	93.9% (92.1%,95.4%)	93.7% (91.8%,95.3%)	94.0% (92.2%,95.5%)
Big	3	88.2% (85.0%,90.6%)	88.3% (85.1%,90.9%)	88.4% (85.6%,90.9%)	**92.3%** (90.0%,94.2%)	**92.7%** (90.5%,94.5%)	90.6% (87.9%,93.0%)	92.5% (90.3%,94.3%)	92.7% (90.5%,94.6%)
	4	79.0% (74.3%,82.9%)	80.5% (75.8%,84.5%)	79.1% (74.7%,83.0%)	**91.9%** (89.2%,94.0%)	**92.0%** (89.3%,94.1%)	86.0% (81.5%,89.6%)	91.9% (89.2%,94.0%)	92.2% (89.6%,94.3%)
	High	70.1% (64.1%,75.2%)	72.6% (66.9%,77.2%)	70.1% (64.4%,75.3%)	**86.2%** (82.4%,89.9%)	86.2% (82.5%,89.9%)	78.5% (72.7%,83.3%)	**86.3%** (82.5%,89.9%)	86.5% (82.8%,90.1%)
Average		76.1% (72.6%,79.7%)	**87.3%** (85.3%,89.3%)	84.7% (82.4%,87.2%)	82.2% (79.4%,85.3%)	90.6% (89.1%,92.2%)	88.6% (86.7%,90.6%)	**90.8%** (89.5%,92.2%)	91.4% (90.2%,92.8%)
GRS		4.37	4.11	4.41	4.02	4.07	4.19	3.92	4.06
p-value		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

25 portfolios sorted by size and book-to-market

R^2 of the linear regression of the excess returns of 25 equally-weighed portfolios (sorted by quintiles of the distribution of size – Small, 2, 3, 4 and Big – and by quintiles of the distribution of Book equity to Market equity ratio – Low, 2, 3, 4 and High) on the market portfolio (Rm), on the market portfolio and the factor ICC (ICC), on the market portfolio and the size factor (SMB), on the market portfolio and the book to market factor (HML), on the market portfolio and the two Fama & French factors (HML + SMB), on the market portfolio, the factor ICC and the size factor (ICC + SMB), on the market portfolio, the factor ICC and the book to market factor (ICC + HML) and, finally on all these four factors (Market, ICC, SMB and HML). Figures in boldface represent the maximum value of the R^2 within the group of regression with two factors (columns ICC, SMB and HML) and with three factors (columns HML + SMB, ICC + SMB and ICC + HML). The two last rows reports Gibbons *et al*. (1989) test statistics and *p*-values.

	Rm	ICC	SMB	HML	HML + SMB	ICC + SMB	ICC + HML	All four factors
Average	76.1% (72.6%,79.7%)	**87.3%** (85.3%,89.3%)	84.7% (82.4%,87.2%)	82.2% (79.4%,85.3%)	90.6% (89.1%,92.2%)	88.6% (86.7%,90.6%)	**90.8%** (89.5%,92.2%)	91.4% (90.2%,92.8%)
GRS	4.37	4.11	4.41	4.02	4.07	4.19	3.92	4.06
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

10 equally-weighted industry portfolios

- Parameter estimates of the linear regression of the excess returns on 10 equally-weighed industry portfolios regressed on the excess return on the market portfolio, on the two Fama-French factors SMB and HML and on the proxy for the additional risk factor due to the internal consistency constraint given by the difference between the return on the equally-weighted portfolio and the return on the market portfolio.

  Time span: Jan. 1927 – Dec 2005; 948 months

Industry	α	β	β^{SMB}	β^{HML}	β^{ICC}	R^2
Consumer Non Durables	-0.0003	0.84**	0.08*	0.10**	0.77**	94%
Consumer Durables	-0.0024	1.12**	0.21**	0.07*	0.97**	92%
Manufacturing	-0.0004	1.07**	0.12**	0.17**	0.76**	97%
Energy	0.0019	0.95**	0.13	0.34**	0.55**	69%
Business Equipment	0.0016	1.22**	-0.29**	-0.65**	1.52**	92%
Telecom	0.0030	0.92**	-0.30**	-0.54**	0.98**	73%
Shops	0.0000	0.91**	0.11*	-0.11**	0.93**	90%
Health	0.0037	0.91**	-0.04	-0.54**	0.92**	80%
Utilities	0.0006	0.85**	0.21*	0.55**	-0.06	66%
Others	-0.0008	0.95**	0.07	0.39**	0.93**	95%

10 equally-weighted industry portfolios

R^2 of the linear regression of the excess returns of 10 equally-weighed industry portfolios on the market portfolio (Rm), on the market portfolio and the factor ICC (ICC), on the market portfolio and the size factor (SMB), on the market portfolio and the book to market factor (HML), on the market portfolio and the two Fama & French factors (HML + SMB), on the market portfolio, the factor ICC and the size factor (ICC + SMB), on the market portfolio, the factor ICC and the book to market factor (ICC + HML) and, finally on all these four factors (Market, ICC, SMB and HML). Figures in boldface represent the maximum value of the R^2 within the group of regression with two factors (columns ICC, SMB and HML) and with three factors (columns HML + SMB, ICC + SMB and ICC + HML). The two last rows reports Gibbons *et al*. (1989) test statistics and *p*-values.

10 equally-weighted industry portfolios

	Group 1	Group 2			Group 3			Group 4
	CAPM	ICC	Cap	Value Growth	Value + Cap	ICC + Cap	ICC + Value	All Four Factors
Consumer Non Durables	75.9%	**94.1%**	88.4%	79.7%	91.8%	94.1%	**94.3%**	94.3%
Consumer Durables	74.4%	**92.3%**	87.9%	76.9%	90.2%	**92.4%**	92.3%	92.4%
Manufacturing	82.2%	**96.7%**	92.0%	85.9%	95.4%	96.8%	**97.0%**	97.1%
Energy	58.3%	**67.8%**	63.7%	63.4%	68.5%	68.1%	**69.3%**	69.3%
Business Equipment	74.5%	**87.4%**	86.2%	74.8%	86.6%	88.0%	**91.6%**	91.8%
Telecom	62.7%	**68.2%**	68.1%	63.9%	69.4%	68.6%	**72.6%**	73.0%
Shops	71.8%	**90.1%**	86.7%	72.8%	87.6%	90.3%	**90.4%**	90.5%
Health	65.1%	74.5%	**75.9%**	66.4%	77.4%	76.2%	**80.5%**	80.5%
Utilities	58.3%	**60.8%**	58.9%	65.9%	**66.5%**	61.7%	66.3%	66.5%
Others	71.9%	**92.8%**	83.6%	81.6%	92.7%	93.4%	**95.2%**	95.2%
Average	69.5%	**82.4%**	79.1%	73.1%	82.6%	82.9%	**84.9%**	85.0%

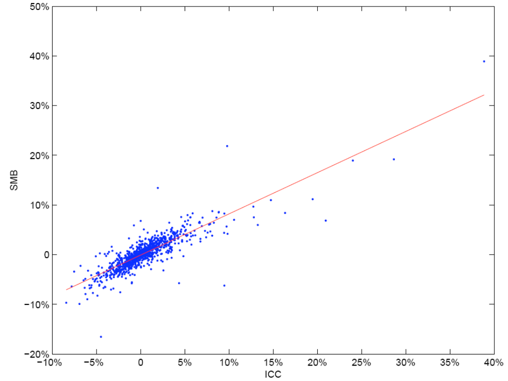
Best of the group

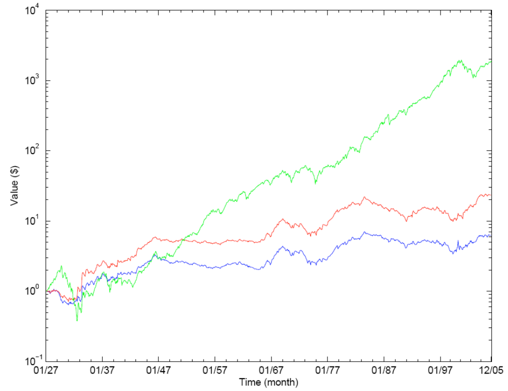
Worst of the group

Relation between ICC and the Fama & French two Factor

- In the presence of r_{ICC}, the relevance of the two Fama & French factors does not disappear but is weakened.

- The size effect: by construction ICC and SMB are close; indeed the ICC factor is long in the equally-weighted portfolio and short in the market portfolio, it is therefore long on the small caps and short on the large caps.





Relation between ICC and the Fama & French two Factor

- The book-to-market effect:
  - Empirical evidence: high book-to-market stocks have significantly lower beta's with respect to the market portfolio compared with low book-to-market stocks.
  - According to our model, the market premium related to the lack of diversification of the market portfolio is $(\gamma_i - \gamma_m \cdot \beta_i) \cdot \mathrm{E}\left[r_{icc} - r_0\right]$

=> *Ceteris paribus*, the internal consistency constraint leads to a higher expected rate of return for stock with a low beta if the term γ_m is positive.

Long term Prospective: Annual Returns

Owner or lender ?

- **Real US large Cap equity returns since 1802: 7%**
- Real US long Bond returns since 1802: 3.4%

Value or Growth ?

Real US Equity return over the last 75 years:

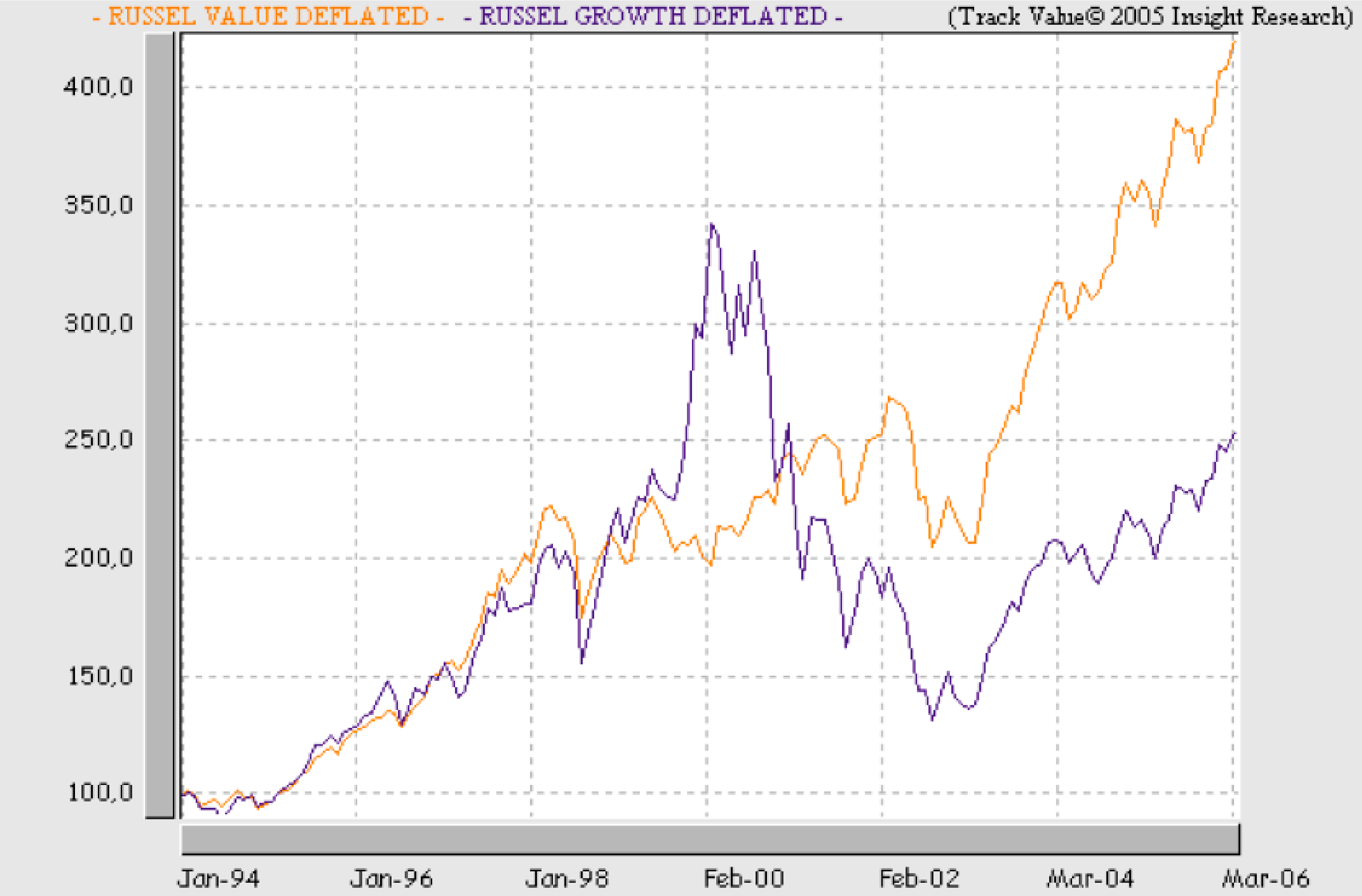
- Large cap growth 6,3%
- Small cap growth 6,7%
- **Large cap value 8,9%**
- **Small cap value 11,9%**

Total Return Strategies

- **Real return since 1991*: 11,7%**

**source: HFRI*

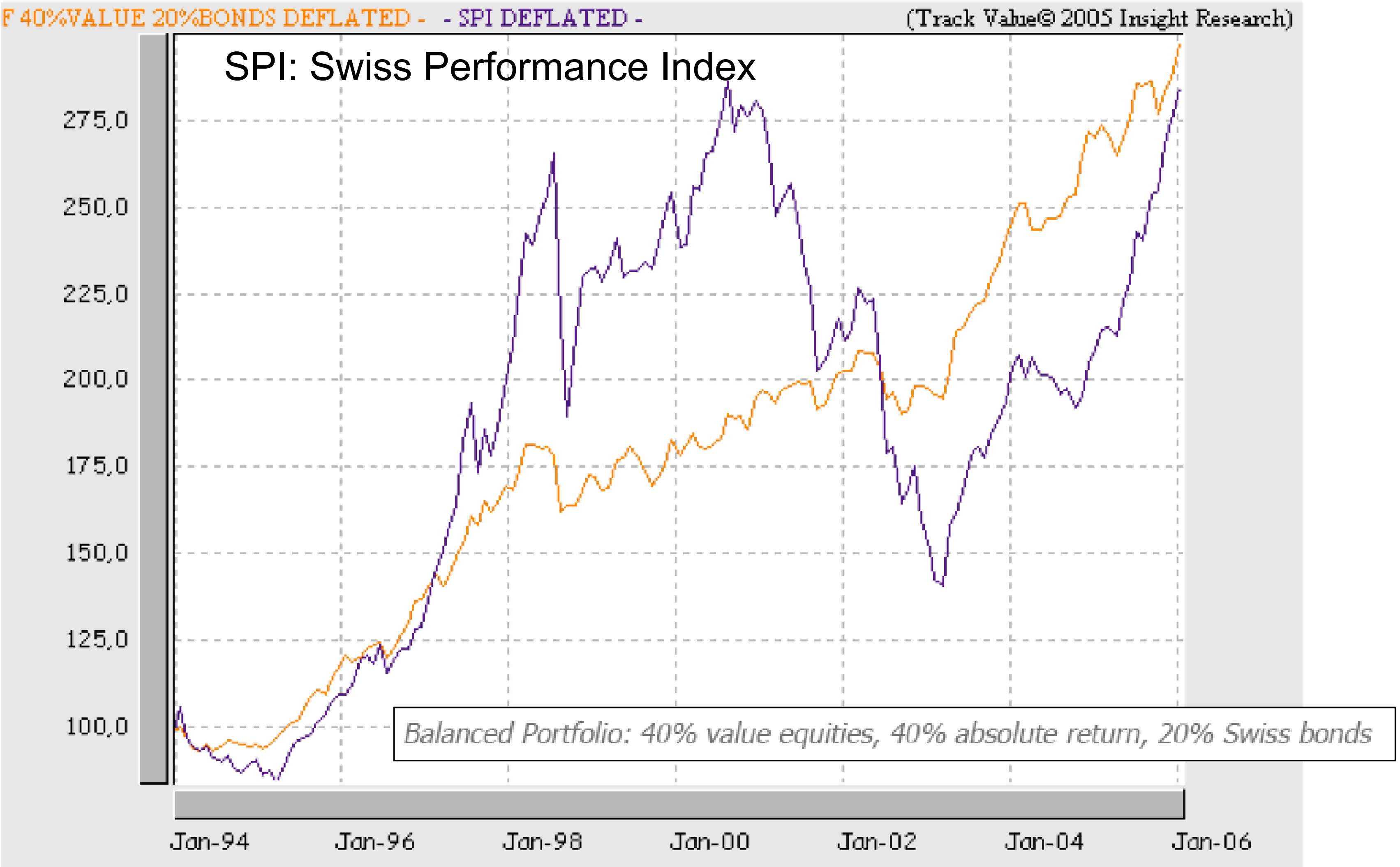
Value vs. Growth



Source: Insight Research / Russel Indexes *- Portfolio composition: 1/3 Large Cap 1/3 Mid-Cap 1/3 Small Cap*

Large Diversification across all Asset Classes

Balanced Portfolio vs. Total Return Swiss Equity Market



Concluding remarks

- Due to the fait tail nature of the distribution of firm size, the market portfolio is not well-diversified:

$$H_N = \|\vec{w}_m\|^2 = \sum_{i=1}^{N} w_{m,i}^{\ \ 2} \not\longrightarrow 0$$

- There exist a diversification premium related to the non-diversified nature of the market portfolio,
- The internal consistency of linear factor models allows accounting very naturally for the existence of a diversification factor,

Concluding remarks

- The diversification factor (ICC factor) can be closely related to the Size factor (SMB) introduced by Factor and French,
- To some extent, the diversification factor is also related to the book-to-market (HML) effect,
- The Fama-French three factor model does not provide a significant improvement, neither in terms of R^2 nor in terms of α, with respect to our two factor model (based on the undisputable fact that the market portfolio is highly concentrated on a small number of very large companies).