St. Petersburg Paradox - a VaR approach

A Master's Thesis by

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Abstract

In this thesis, we study the St. Petersburg paradox and (i) resolve the issue between median heuristic and expectation heuristic in previous research and (ii) develop a risk pricing framework called Value-at-Risk Probability (VaR-P) for the St. Petersburg paradox. Our findings show a strong support for the median heuristic theory. We also suggest that every player is willing to accept an individual probability of loss, p, that is constant when game returns are modified, holding the underlying distribution identical.

Keywords: St. Petersburg paradox, Median Heuristic, Expectation Heuristic, Value at Risk, VaR-P

Contents

1 Introduction

The purpose of this thesis has been to develop a risk-pricing framework using the St. Petersburg paradox as a starting point. Great emphasis and focus has been put on the background and properties of this historical game to derive our ideas in an intuitive path. St. Petersburg paradox has puzzled mathematicians, statisticians and economists for centuries, leading to many new theories and ideas in other fields. Despite many attempts, the paradox itself is considered unsolved yet today.

> **The St. Peterburg paradox**
> Peter offers to let Paul toss a fair coin an indefinite number of times, paying him 2 ducats if heads first come up at the first toss, 4 ducats if it first comes up at the second toss,..., 2^i ducats for heads first coming up at the i-th toss, ..., and so forth ad infinitum.

Our academic starting point are two, equally viable theories backed up by two, recently conducted, empirical studies. The first theory; median heuristic, expects players to bid for the median return of game [Hayden & Platt, 2009]. The second theory; expectation heuristic, uses an intuitive two-step approach calculating the return, given number of expected tosses until the first head come up [Treisman, 1983]. Both theories yield a very similar expectation and are therefore indistinguishable.

Using a theory, previously developed but never tested by David Holtgrave, we modify the St. Petersburg game so median heuristic and expected heuristic yield different expectations [Bottom et al., 1989]. After conducting an empirical study with the modified games, the results show a very strong support for the median heuristic.

Despite the fact that median heuristic is an explicit single answer solution, all empirical studies show very dispersed distributions of bids. This means that all players do not bid according to a 50% win/loss scenario (definition of the median heuristic). Instead, this suggests that players have individual risk-preferences where they are willing to accept a probability of loss p. We call this the Value at Risk Probability theory (VaR-P) and show, using our own and other empirical studies, that players tend to keep their risk-preference (p) constant when St. Petersburg game returns are modified, holding the underlying distribution identical.

The St. Petersburg paradox, being a well defined power law distribution absent of moments describing its properties, has been an ideal testing platform for the VaR-P theory. We suggest that it can find its real use and be conveniently applied in other fields. An example in finance is VaR-pricing of interest rate forwards developed by Bouchaud, described in the last section [Bouchaud et al., 1999].

2 Background

2.1 Bernoulli

In late 17th century, mathematicians joined forces with gamblers, calculating probabilities and developing strategies on how to maximize returns. The theory of expected value developed to be a central measure to price a game.

In a letter to Montmort, dated 9th September 1713, Nicholas Bernoulli proposed several gambling problems. Bernoulli noted that one of the problems had very interesting properties.

> "...Fourth Problem. A promise to give a coin to B, if with an ordinary die he achieves 6 points on the first throw, two coins if he achieves 6 on the second throw, 3 coins if he achieves this point on the third throw, 4 coins if he achieves it on the fourth and thus it follows; one asks what is the expectation of B?
>
> Fifth Problem. One asks the same thing if A promises to B to give him some coins in this progression 1, 2, 4, 8, 16 etc. or 1, 3, 9, 27 etc. or 1, 4, 9, 16, 25 etc. or 1, 8, 27, 64 in stead of 1, 2, 3, 4, 5 etc. as beforehand. Although for the most part these problems are not difficult, you will find however something most curious."

The game with the progression $1, 2, 4, 8, 16, \ldots$ in the fifth problem lead to an infinite expected value

$$\sum_{n=0}^{\infty} 2^n \left(\frac{5}{6}\right)^n \frac{1}{6} = \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n = \infty \tag{1}$$

Although the problem is phrased differently today, this was the birth of the St. Petersburg paradox. Nicholas Bernoulli described the game to his brother Daniel, who was at the time working in St. Petersburg. In 1738, Daniel Bernoulli wrote an article about the game and presented it in a simpler form; Peter offers to pay Paul \$2 if a coin toss turns up heads, or 2^n if head turns up on n-th toss [Bernoulli, 1738]. The expected value of the game is

$$\sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} = \sum_{n=1}^{\infty} 1 = \infty \tag{2}$$

2.2 Utility theory

The infinite expected value property challenged scientists and started a discussion that is still actively ongoing. Daniel Bernoulli proposed a solution based on Paul's decreasing marginal utility of money, U.

$$U = a \log(W + x) + c. \tag{3}$$

W is Paul's wealth, x is the prospective gain and a, c are constants. By replacing the monetary unit in the expected value formula with the change in utility, the expected value is limited. Wealth W is set to 0 for simplified calculations.

$$\sum_{n=1}^{\infty} (a \log(2^n) + c) \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} a \log(2^n) \left(\frac{1}{2}\right)^n + c = a \log 2 \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n + c = 2a \log 2 + c \tag{4}$$

In a letter to Nicholas Bernoulli in 1728, Cramer proposes a similar approach using a utility function $U = \sqrt{x}$, called the moral value [Cramer, 1728]. A finite moral expectation can then be calculated

$$\sum_{n=1}^{\infty} \sqrt{2^n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^n = 1 + \sqrt{2} \tag{5}$$

Utility theory has since then evolved and been applied in many economics problems. Menger challenged its explanation of the St. Petersburg by changing the return to e^{2^n} [Menger, 1934]. Then, it yields an infinite expected utility

$$\sum_{n=1}^{\infty} (a \log\left(e^{2^n}\right) + c) \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} a2^n \left(\frac{1}{2}\right)^n + 2c = \infty \tag{6}$$

It does therefore not matter what utility function is used, the return can always be altered to yield an infinite expected value.

2.3 Limited capital and Low Probabilites

Poisson stated that no human being can possibly expect to be paid an infinite amount of money since the physical amount in the world is limited [Poisson, 1837]. If the monetary base is M, the expected value of the game is k+1, k being the $\log_2$ base of M

$$\sum_{n=1}^{k} 2^n \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} + \sum_{m=k+1}^{\infty} 2^k \left(\frac{1}{2}\right)^{m-1} \frac{1}{2} = k+2^k \left(2 - \frac{1 - \frac{1}{2}^{k+1}}{1 - \frac{1}{2}}\right) = k+2^k \left(2 - 2 + \frac{1}{2^k}\right) = k+1 \tag{7}$$

This argument is based on the assumption that people are unable to imagine an individual with an infinite amount of wealth.

Cramer and D'Alembert argued that probabilities lower than $1/10,000$ are so improbable that they can be set equal to zero [Cramer, 1728] [DAlembert, 1767]. Since $\left(\frac{1}{2}\right)^{13} = \frac{1}{8192}$ the cutoff point will be 13 tosses and the expected value

$$\sum_{n=1}^{13} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{13} 1 = \$13$$

This approach has been heavily criticized for its arbitrariness by Condorcet [Dutka, 1988] [Arrow, 1951]. Also, recent studies have shown that small probabilities are in fact exaggerated [Blavatskyy, 2005] [Kahneman & Tversky, 1979] .

2.4 Properties of the St. Petersburg Distribution and Contemporary Views

Definition 1 : The St. Petersburg Paradox Distribution. *Properties of the St. Petersburg Paradox Distribution*

Mass Function : *The mass function of the St. Petersburg Paradox*

$$f_X(x) = \begin{cases} \frac{1}{2^{\log_2 x}} = \frac{1}{x}, & x \in \{2,4,8,...\} \\ 0, & x \notin \{2,4,8,...\} \end{cases} \tag{8}$$

Distribution Function : *The distribution function of the St. Petersburg paradox*

$$F_X(x) = \sum_{k=2,4,8,...}^{x} \frac{1}{2^{\log_2 k}} = \sum_{n=1}^{\log_2 x} \frac{1}{2^n} = \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{log_2 x}}{1-\frac{1}{2}} = 1 - \frac{1}{x}, x \in \{2,4,8,...\} \tag{9}$$

$$P(X > x) = 1 - F_x(x) = \frac{1}{x}, x \in \{2,4,8,...\} \tag{10}$$

$$P(X \geq x) = 1 - F_x(x) + f_X(x) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}, x \in \{2,4,8,...\} \tag{11}$$

Median : *The median of the St. Petersburg paradox*

$$P(X \leq 2) = 1 - \frac{1}{2} = \frac{1}{2} \geq \frac{1}{2} and\ P(X \geq 2) = \frac{2}{2} = 1 \geq \frac{1}{2} \tag{12}$$

$$P(X \leq 4) = 1 - \frac{1}{4} = \frac{3}{4} \geq \frac{1}{2} and\ P(X \geq 4) = \frac{2}{4} = \frac{1}{2} \geq \frac{1}{2} \tag{13}$$

$$m = \frac{2+4}{2} = 3 \tag{14}$$

Liebovitch and Scheurle challenge the use of expected value as valid tool for power law distributions. If the complimentary cdf is of the form Ax^{-b}, it is considered to be a power law distribution. Its moments m are only well-defined for $m < b$. The St. Petersburg paradox is a power law distribution with $b = 1$ as shown in (10) [Liebovitch & Scheurle, 2000].

This means that the expected value for the St. Petersburg is in fact undefined as the mean never converges and continues to grow as the sample size increases. For an infinite sample, this demands for other measures that can be used instead of the traditional expected value, variance and other moments. Apart from the St. Petersburg paradox, power laws have found a strong explanation power in other fields. Mandelbrot and others have shown that the tail behavior of stock asset returns can be modeled using power laws with b between 2 and 4 [Mandelbrot, 1963] [Sornette, 2004].

2.5 Median Heuristic, Expectation Heuristic and Decision Theory

Paul Samuelson recalls that his friend declined an offer to win $200 and loose $100 with 50 percent probability. The friend was, however, willing to play the game if it was offered 100 times. Paul was disturbed by the, in his view, irrational behavior [Samuelson, 1963]. Inspired by this thinking, Lopa Lopes investigated whether people price gambles differently based on the number of times they play the game. She argued that, in the short-run, people adjust their strategy to the probability of coming out ahead [Lopes, 1981]. Lopes proposed that a player sets a probability value y from 0 to 1, indicating the probability of losing a game they are willing to take. Two empirical studies propose $y = 0.5$, the median, as a solution to the St. Petersburg Paradox [Bottom et al., 1989] [Hayden & Platt, 2009]. Given by (14), the median for the St. Petersburg paradox is 3.

Definition 2 : Expectation Median. *The expectation median predicts that players are willing to win in 50% of the cases and lose in 50%.*

Median *: A median m of a discrete distribution satisfies the following equations*

$$P(X \leq m) \geq \frac{1}{2} \tag{15}$$

and

$$P(X \geq m) \geq \frac{1}{2} \tag{16}$$

St. Petersburg Game Median *: The median for the basic St. Petersburg game is given by (14) and is 3.*

A similiar proposition was formulated by Treisman. He came up with a two-step approach called the expectation heuristic. The first step is to calculate the expected toss, k, when head comes up. The second step is to find the return of 2^k, the expectation heuristic [Treisman, 1983]. In the St. Petersburg paradox, head is expected to turn up on second turn.

Definition 3 : Expectation Heuristic. *How to calculate the expectation heuristic for the St. Petersburg Paradox*

Step 1 *: Calculate the expected trial k, when head comes up*

$$k = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2 \tag{17}$$

Step 2 *: Calculate the expected return given that head comes up on the trial given in Step 1*

$$2^k = 2^2 = 4 \tag{18}$$

The expectation heuristic is then \$4. Treisman suggests that when people are faced with a complicated problem it is common that they attempt to simplify it by breaking it down to smaller pieces, which are then individually examined. This approach may be used as a preliminary hunch, by the sophisticated, Treisman notes, but the laymen may mistake it for a valid procedure.

3 Empirical Studies

3.1 Overview

The first empirical study was done by Buffon. He hired a child to toss a coin until it came up heads and repeat the procedure 2048 times [Buffon, 1777]. De Morgan later added 2048 trials

Theory	Explanation
EV (Expected Value)	Standard EV approach
MH (Median Heuristic)	Players are willing to win/lose with a probability of 0.5
EH (Expectation Heuristic)	Two-step calculation proposed by Treisman
FW (Finite Wealth)	Limited capital M, expected value is $\log_2 M + 1$
CT14 (Cutoff 14 after tosses)	Probabilities lower than 2^{-14} are set to zero
CT20 (Cutoff 20 after tosses)	Probabilities lower than 2^{-20} are set to zero
EUSQ (Expected Utility Squared)	Return utility value to the player is $\sqrt{2^n}$
EUNL (Expected Utility LOG)	Return utility value to the player is $a \log 2^n + c$

Table 1: Summary of hypotheses for the St. Petersburg game

to the study [De Morgan, 1838]. Both experiments showed that the sample frequency for low probabilities are much lower than theoretical expected value. Buffon and De Morgan concluded that this was the reason behind the low bids for the game. Recent studies done with computer simulation using Monte Carlo show an average return that is increasing with number of trials. Although the amount of trials have been limited, the real expected return is higher than people have been willing to pay [Ceasar, 1984] [Liebovitch & Scheurle, 2000]. Because of the inherent risk and obvious limitations, there has not been any real St. Petersburg game trials up to date. A modified version of the game was offered to 30 students [Cox et al., 2009]. There were 9 different games G_i ($i = 1-9$), each with the maximum return 2^i. If heads come up on the n-th turn the return is 2^n and if $n > i$ the return is 0. The expected return for game G_i is i. Each subject had to make a bid for each of the 9 games after which a random generator selected a game G_i that was commenced. Two particular facts were evident,

> (a) 26 out of 30 (or 87%) of the individual subjects refused at least one opportunity to play a St. Petersburg lottery when paying less than its expected value; and (b) over all subjects, 127 out of 270 (or 47%) of their choices were inconsistent with expected value, bidding lower than i for game G_i.

Other empirical studies, with and without real money returns, have shown that people are in fact willing to pay much less for the game [Bottom et al., 1989].

3.2 Bottom, Bontempo and Holtgrave study

An extensive study examines the empirical evidence and explanation power of the solutions presented earlier in this text, all summarized in Table 1 [Bottom et al., 1989].

To be able to distinguish the hypotheses the authors developed several versions of the game with different returns presented in Table 2.

Expected returns
The expected return of the Expected Value approach for all games is given by (2) and is ∞. Median Heuristic for the basic game is given by (14). The median is similarly calculated to be \$6, \$8 and \$13 for the T2, P5 and P10 games. The Expectation Heuristic approach for the

Versions of the game	
Game	**Return**
B (Basic Game)	2^n
P5 (Plus 5)	$2^n + 5$
P10 (Plus 10)	$2^n + 10$
T2 (Times 2)	$2 * 2^n$

Table 2: Games in Bottom, Bontempo and Holtgrave's survey

basic game is given by (18) and is then easily calculated to be \$8 for T2, \$9 for P5 and \$14 for P10. The Finite Wealth theory assumes a maximum capital of \$1048576 (\$$2^{20}$) [Lopes, 1981] and according to (7) the expected value for the basic game is $\log_2 2^{20} + 1$ or \$21. The expected returns for the other games are calculated similarly to be \$41 for T2, \$26 for P5 and \$31 for P10.

The cutoff theory assumes that probabilities lower than 2^{-14} and 2^{-20} are ignored. The returns for the cutoff and expected utility games are calculated below.

Definition 4 : Expected returns. *The expected returns for the rest of the games.*

CT14 : *Expected return when probabilities lower than 2^{-14} are 0.*

$$\sum_{n=1}^{14} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{14} 1 = \$14 \tag{19}$$

The other games expectations are similarly calculated to be \$28 for T2, \$19 for P5 and \$24 for P10.

CT20 : *Expected return when probabilities lower than 2^{-20} are 0.*

$$\sum_{n=1}^{20} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{20} 1 = \$20 \tag{20}$$

The other games expectations are similarly calculated to be \$40 for T2, \$25 for P5 and \$30 for P10.

EUNL for B : *Expected return for basic game with logarithmic utility function.*

$$\sum_{n=1}^{\infty} (a \log 2^n + c) \left(\frac{1}{2}\right)^n = a \log 2 \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n + c = 2a \log 2 + c \tag{21}$$

EUNL for T2 : *Expected return for T2 game with logarithmic utility function.*

$$\sum_{n=1}^{\infty} (a \log 2^{n+1} + c) \left(\frac{1}{2}\right)^n = a \log 2 \left(\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \right) + c = 3a \log 2 + c \tag{22}$$

Expected returns				
Theory	**B**	**T2**	**P5**	**P10**
EV (Expected Value)	∞	∞	∞	∞
MH (Median Heuristic)	\$3	\$6	\$8	\$13
EH (Expectation Heuristic)	\$4	\$8	\$9	\$14
FW (Finite Wealth)	\$21	\$41	\$26	\$31
CT14 (Cutoff 14 Tosses)	\$14	\$28	\$19	\$24
CT20 (Cutoff 20 Tosses)	\$20	\$40	\$25	\$30
EUSQ (Expected Utility Squared)	\$2.41	\$3.41	\$3.43	\$4.16
EUNL (Expected Utility LOG)	$a\$1.38 + c$	$a\$2.08 + c$	$a\$2.30 + c$	$a\$2.74 + c$

Table 3: Expected returns for the hypotheses in Bottom, Bontempo and Holtgrave survey (Source: [Bottom et al., 1989])

The expected return for games P5 and P10 is numerically calculated to be $a\$2.30 + c$ *and* $a\$2.74 + c$.

The W variable in the Expected Utility function (3) is omitted because it yields overly optimistic bid results [Gorovitz, 1979].

From (5) we get that the expected return for the Expected Utility Squared approach is $\$1+\sqrt{2}$. Similarly, T2 is calculated to have the expected return is $\$\sqrt{2}(1+\sqrt{2}) = \$2+\sqrt{2}$. For P5 and P10, the expected return is calculated numerically to be \$3.43 and \$4.16.

Summary of the expected returns
The expected return according to the different hypotheses is presented in Table 3.

For games with more than two tosses, Times 2 has larger returns than Plus 5. For games with more than three tosses, Times 2 has larger returns than Plus 10.

The Survey
Original description of the survey:

> **Subjects** One hundred thirty-nine undergraduates from the University of Illinois participated for credit in an introductory psychology course. In addition, a survey questionnaire was sent to two hundred fifty members of a professional society consisting of specialists in statistics, economics, and management science. Forty-seven of these subjects returned completed forms.
>
> **Procedure** Both expert and student subjects were presented with survey forms which described the four games in detail. Subjects were initially told to imagine that they were in a sealed bid auction for the right to play each game. (For students, the Basic game was also explained orally by an experimenter, and a practice round of bidding was conducted.) Subjects then wrote down their bids for each of the games. After bidding, subjects were told to imagine that they had won the right to play the Basic game but that they would actually be given the choice of which game

Results				
Sample	**B**	**T2**	**P5**	**P10**
Experts	4.0	8.0	11.0	14.0
	$(2.0, 8.0)$	$(5.0, 13.5)$	$(7.0, 11.0)$	$(12.0, 15.0)$
Novices	4.0	8.0	10.0	14.0
	$(2.0, 7.0)$	$(5.0, 10.0)$	$(8.0, 14.0)$	$(10.5, 17.0)$

Table 4: Median bids and interquartile range (in parenthesis) of the Bottom, Bontempo and Holtgrave survey (Source: [Bottom et al., 1989])

Expected returns for fitted EUNL				
Sample	**B**	**T2**	**P5**	**P10**
Experts	5.55	8.32	9.2	10.95
Novices	6.55	9.32	10.2	11.95

Table 5: EUNL fit to the bid data (Source: [Bottom et al., 1989])

> they wanted to play for that price. Subjects were asked to rank order the games according to their preferences in this situation. This procedure made the cost of winning [playing (auth. note)] a game irrelevant to their preference.

Results

Table 4 shows the median game bids and interquartile range (in parenthesis) for the expert and novice groups. The values a and c of the EUNL utility function are fitted in a least-squares sense using a grid search procedure. The novice group estimate for a is 4 and c is 0. For the expert group, a was estimated to be 4 and c to be 1. The bids predicted for the novice and expert groups are shown in Table 5.

The bid medians are closer to the MH and EH predicted values and the fitted EUNL is an unrealistic utility function for money [Bottom et al., 1989]. Therefore, this empirical study shows a strong support for the EH and MH theory.

3.3 Hayden and Platt study

Hayden and Platt released an empirical study proposing the median, described in the previous chapter (given by (14)), calling it the median heuristic [Hayden & Platt, 2009].

Conducting the survey, Hayden and Platt used two groups. In the first, there were 20 undergraduate students playing the game for real money and the second study was conducted over the Internet with 200 subjects giving hypothetical answers. The game was the original St. Petersburg paradox with the return scheme $(1, 2, 4, 8, ...)$ and subjects were to make one bid to play the game. Because of the different return scheme the median of the distribution is 1.5 and the expectation heuristic is 2.

The median bid for the real version gamble was \$1.75 and for the hypothetical one \$1.50. These bids do not differ significantly (Wilcoxon rank-sum test, $p > 0.4$). The results are presented in

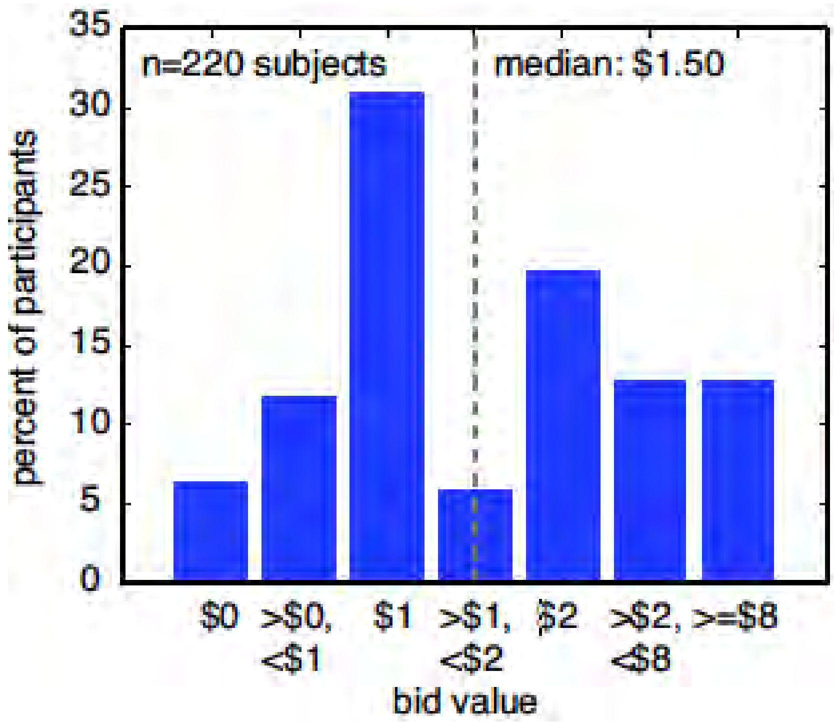


Figure 1: Distribution of the bids in Hayden and Platt's study of the St. Petersburg game with return (1,2,4,8,...) (Source: [Hayden & Platt, 2009])

Figure 1.

The empirical results support the median heuristic but does not differ significantly from the expectation heuristic. It is very hard to distinguish the theories and both studies lacks a satisfactory explanation. To correct this, the distribution characteristics of the game played needs to be altered in order for the expectation heuristic to differ significantly from the median. This can be done using the generalized St. Petersburg game created by David Holtgrave [Bottom et al., 1989].

4 Expectation Heuristic vs. Median Heuristic

4.1 The Generalized St. Petersburg paradox

David Holtgrave developed a generalized St. Petersburg game to distinguish between the expectation heuristic and the median heuristic. It goes as follows, you have a deck of z cards where $z-1$ of them are blank and one says STOP. You keep pulling cards from the deck until you hit the STOP card, with returning each card and reshuffling the deck every time. The probability that the STOP card will show up on the n-th draw is

$$\left(\frac{z-1}{z}\right)^{n-1}\frac{1}{z} = \frac{(z-1)^{n-1}}{z^n}$$

The game will have an infinite expected value if the return P is equal or larger than the reciprocal of the probability that the STOP card will show up on n draws.

$$P \geq \frac{z^n}{(z-1)^{n-1}} \iff EX = \infty$$

When z is 2 the game is equal to the St. Petersburg paradox. When z is 6 you can use a regular die to play the game. To distinguish between the expectation heuristic and the median heuristic, one can offer the die game with the same return of 2^n.

Definition 5 : Expectation Heuristic. *How to calculate the expectation heuristic for the die game*

Step 1 : *Calculate the expected trial when head comes up*

$$k = \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} = \frac{1}{5} \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n} = \frac{1}{5} \frac{\frac{5}{6}}{(1-\frac{5}{6})^2} = 6 \tag{23}$$

Step 2 : *Calculate the expected return given that head comes up on the trial given in Step 1*

$$2^k = 2^6 = 64 \tag{24}$$

Definition 6 : The Die Game Distribution. *Properties of the die game distribution*

Mass Function : *The mass function of the die game*

$$f_X(x) = \begin{cases} \left(\frac{5}{6}\right)^{\log_2 x - 1} \frac{1}{6} = \left(\frac{5}{6}\right)^{\log_2 x} \frac{1}{5}, & x \in \{2,4,8,...\} \\ 0, & x \notin \{2,4,8,...\} \end{cases} \tag{25}$$

Distribution Function : *The distribution function of the die game*

$$F_x(x) = \sum_{k=2,4,8,...}^{x} \left(\frac{5}{6}\right)^{\log_2 k} \frac{1}{5} = \sum_{n=1}^{\log_2 x} \left(\frac{5}{6}\right)^{n} \frac{1}{5} = \frac{1}{6} \frac{1-\left(\frac{5}{6}\right)^{log_2 x}}{1-\frac{5}{6}} = 1 - \left(\frac{5}{6}\right)^{log_2 x}, x \in \{2,4,8,...\} \tag{26}$$

$$P(X > x) = 1 - F_x(x) = \left(\frac{5}{6}\right)^{log_2 x} = \frac{1}{x^{\log_2 \frac{6}{5}}}, x \in \{2,4,8,...\} \tag{27}$$

$$P(X \geq x) = 1 - F_x(x) + f_X(x) = \left(\frac{5}{6}\right)^{\log_2 x} + \left(\frac{5}{6}\right)^{\log_2 x} \frac{1}{5} = \left(\frac{5}{6}\right)^{\log_2 x - 1}, x \in \{2,4,8,...\} \tag{28}$$

Median : *The median of the die game*

$$P(X \leq 16) = 1 - \left(\frac{5}{6}\right)^{\log_2 16} = \frac{671}{1296} \geq \frac{1}{2} \textit{and } P(X \geq 16) = \left(\frac{5}{6}\right)^{\log_2 16 - 1} = \frac{125}{216} \geq \frac{1}{2} \tag{29}$$

$$m = 16 \tag{30}$$

The expectation heuristic predicts that the game will stop on the 6th toss; therefore a person will be willing to bet 64. The median heuristic is 16. Here, the two cases are clearly distinguishable. The die game has a power law distribution with $b = \log_2 \frac{6}{5} \approx .26$. For a 10-sided dice, the Expectation Heuristic is similarly calculated to be 1024 and the Expectation Median to be 128.

Versions of the game	
Game	**Description**
C	Coin game (Original St. Petersburg Paradox)
6D	6-sided dice (Generalized St. Petersburg Paradox)
10D	10-sided dice (Generalized St. Petersburg Paradox)

Table 6: Versions of the game in the survey

Versions of the game	
Version	**Return**
Basic (B)	CHF 2^n
Cutoff at 1024 (CT10)	CHF 2^n (cutoff at 1024)
Cutoff at 32 (CT5)	CHF 2^n (cutoff at 32)
Power of return rate of 4 (T4)	CHF $2 * 4^{n-1}$

Table 7: return structures of the games in the survey. n is the number of tosses until first head for the coin game and first 1 for the dice games.

4.2 The Survey

Background

A group of 55 students, taking a graduate class about financial risk at ETH Zurich, were asked to give bids for 3 games with 4 different return schemes, 12 in total. It was the basic St. Petersburg paradox with a coin and the generalized St. Petersburg paradox with 6-sided and 10-sided dices, summarized in Table 6. For the dice games, the player tosses the dice until 1 comes up.

The returns used was the original basic return of CHF 2^n and two identical return schemes limited to CHF 32 and CHF 1024. In the last version, the power of return was increased to 4 which gave the return CHF $2 * 4^{n-1}$. Table 7 summarizes the return structures.

Predictions

Median heuristic and expectation heuristic for the original basic St. Petersburg game were calculated to be 3 and 4 in (14) and (18) respectively. For the 6-sided dice game, the median heuristic and expectation heuristic were calculated to be 16 and 64 in (30) and (24) respectively. The heuristics are calculated in a similar way for the rest of the games and presented in Table 8 together with the results.

Demographics

Game	**Coin**				**6-sided Dice**				**10-sided Dice**			
Variant	**B**	**CT10**	**CT5**	**T4**	**B**	**CT10**	**CT5**	**T4**	**B**	**CT10**	**CT5**	**T4**
MH	3	3	3	5	16	16	16	128	128	128	32	8192
EH	4	4	4	8	64	64	32	2048	1024	1024	32	524288
Survey Median	*5*	*4*	*3.35*	*12*	*18.5*	*16*	*8*	*128*	*62*	*32*	*15*	*200*

Table 8: Predictions and survey median for each game in our survey.

The majority of the group were men (87.5%) and the average age was 26 years. 59% had previously studied probability theory and 16% were familiar with the St. Petersburg paradox. 52% could answer as if real money were at stake.

Results
The results of the survey are presented together with the heuristics expectations in Table 8. It is now possible to distinguish a strong support for the median heuristic amongst the 6-sided dice games. Although the heuristics predict much higher bids for the 10-sided dice game, the bids are still closer to the median heuristic.

The low bids can possibly be explained by a wealth effect where poor students are not willing to bid large amounts of money. A solution for this could be to adjust the return scheme for the 6-sided dice and 10-sided dice so the median heuristic for all games is in the same range.

It is also evident that the subjects are willing to bet less money on the limited games and more money when the power of return rate increases. We will return to this issue in the following section.

5 VaR-P Theory

5.1 Introduction

A very interesting observation can be made from the bid distribution of Hayden and Platt's study in Figure 1. It is bimodal with large modes at \$1 and \$2 dollars. Although the median bid was \$1.5 and \$1.75, the majority of the bids are actually spread out around the median. Over 30 % of the bids are \$1 which indicates that these players are risk-averse while 12.5 % of the participants gave a bid larger than \$8. A considerate amount of participants are therefore risking a loss in 87.5 % of games played and are risk-takers. This offers a more exhaustive explanation of the bidders risk preferences. If every player has an individual risk preference, willing to lose in a certain percent of the games, an empirical distribution can be derived for the risk preferences of a game. This is equivalent to Lopa Lopes proposition where the players are willing to lose the game in y percent [Lopes, 1981]. If the median risk-preference of all players in a particular game is 0.5, this theory is also consistent with the median heuristic theory presented earlier.

A simple example
A group of 6 subjects are offered a simple gamble; a regular die is tossed one time and the return is given by the toss result. The expected value and median of the game is 3.5. If the subjects all bid 3.5 this shows that they are all risk-neutral and consistent with the median heuristic. But if the subjects bid (1, 2, 3, 4, 5, 6), the median and mean will also be 3.5 and therefore consistent with the median heuristic. The results, however, tell something completely different about the bidders risk preferences. Either, each subject has a unique risk tolerance, or they are just random which means the results are in fact inconclusive.

5.2 Theory

The St. Petersburg paradox will soon celebrate its 300 year anniversary but there is still no widely accepted solution. We suggest that there is no specific method used to get a single explicit answer for the St. Petersburg paradox. The wide dispersion of bids without a single mode in the Hayden and Platt study (Figure 1) suggests that players have individual risk-preferences. They can conveniently be quantified by the probability of loss that a player is willing to accept, a simple single dimension measure described for the first time by Lopa Lopes. A players individual risk-preference will be called his Value-at-Risk probability, or simply VaR-P.

This theory can be tested by changing the properties of the game whilst keeping the underlying distribution probabilities. The players risk-preferences should not be affected, they should still be willing to have the same probability of loss. The St. Petersburg Paradox has an explicit and well defined distribution that is easy to model and study. Examining empirical studies of the St. Petersburg paradox games might give more answers about behavioral aspects and risk-preferences of players in the financial markets where the underlying distributions are unknown.

A simple example (Why these risk-preferences are important)
The group of 6 subjects from our previous example are now divided into two groups. They will play the same gamble; a regular die is tossed one time and the return is given by the toss result. The expected value and median of the game is 3.5. This time, the first group will bid to play the game, the other group will be the casino and give their ask price, offering to play the game. If both the players and casino are completely risk-neutral, the bid and ask price will be 3.5 (the mean) and 3 games will be played. If 4 of the players and casino are risk-averse and bid B=(2,3,3.5) and ask A=(3.5,4,5), only one game will be played. This is equivalent to the gamble being more liquid in the first scenario than in the second, a very important aspect of the financial markets.

The first step is to define the VaR-P model for the St. Petersburg-paradox:

Given the players bid, each game is simplified to have only two events: ”loss” and ”win”, without quantifiying the amount of money lost or won.

Definition 7 : Events. *An arbitrary game can be simplified into these two events*

Loss : *The return (Q) is less than or equal the bid.*

$$Q \leq bid \tag{31}$$

Win : *The return (Q) is more than the bid.*

$$Q > bid \tag{32}$$

For any game, the probability of a ”loss” event is implied by the player’s bid and is equal to the probability of the return being lower than the bid.

$$P(Loss|Bid) = p \tag{33}$$

Probabilities of loss		
Bin	**Bid**	**P(Loss|Bid)=p**
b_0	$0-2^-$	0
$b_{0.5}$	$2-4^-$	0.5
$b_{0.75}$	$4-8^-$	0.75
$b_{0.875}$	$8-16^-$	0.875
$b_{0.9375}$	$16-32^-$	0.9375
$b_{0.96875}$	$32-64^-$	0.96875
$b_{0.984375}$	$64-128^-$	0.984375
$b_{0.9921875}$	$128-256^-$	0.9921875
...	...	...

Table 9: Bids of the standard St. Petersburg game are binned into groups of loss probability p

For the St. Petersburg paradox, the bids can be binned into the probabilities of loss presented in Table 9.

Each player is indirectly indicating his risk-preference by the probability of "loss" p, he is willing to tolerate. A player bidding $\$0-2^-$ is never willing to lose and a player bidding $\$2-4^-$ is willing to lose with a probability p of 0.5 and so on.

Aggregate bid data from a sample of players can map the risk-preferences for different games. After all the bids are binned into their respective probability of loss, a histogram can be made, showing the relative frequency (percentage) of players willing to accept a certain probability of loss p. We are not looking to find the true distribution of risk-preferences for the entire population. Such a distribution might not exist, and it is highly likely that it changes with time due to exogenous behavioral inputs. The interesting aspect is to see if investors will stick to their risk preferences when properties of the game are changed. By making a smaller sample of players bid for different versions of the St. Petersburg one can observe if, and how the distribution of risk preferences changes. It is appropriate to then assume that the population will act similarly, independent of their original risk preferences for the basic game.

The St. Petersburg paradox can be modified in the following ways:

Change the seed value

By changing the seed value of the game we can study whether players are willing to have the same probability of loss, p. The VaR-P theory predicts that players will adjust their bids according to their risk-preferences, the probability of loss p will not change. This theory is supported by the financial markets where investors can buy forward contracts at one rate independent of the notional amount as long as the markets are liquid. The liquidity is dependent on a wealth effect.

When the stock price of a company goes up significantly, it is common to do a stock split so more investors can afford to buy the stock. Berkshire Hathaway is known to never split their stock. In the end of 2010, one share was trading at \$120,000, out of reach for the average investor. This significantly decreases the liquidity of the stock but is done on purpose to only attract long-term investors and keep away speculators.

Increase the rate of return
The power of return for each additional toss in the St. Petersburg game is 2. If this power is increased, the player will earn more money for each additional toss at the same probability. The Value at Risk theory predicts that players will adjust their bids upwards to reflect the increased return but maintain their risk-preferences and probability of loss, p.

Limit the upside
If the game is cutoff after a certain amount of tosses, the upside will be limited. This changes the tail of the St. Petersburg paradox distribution and might change the players risk-preferences to be more risk-averse if the cutoff is set very low. This is equivalent to fixed income securities in finance, which are limited to a notional amount. Bond investors demand either a lower risk or a better price. If the cutoff is set very high and the probability of reaching the limit is low, the VaR-P theory predicts that players will not change their bids and keep their risk preference, p.

The VaR-P theory and Median Heuristic (MH)
The VaR-P theory can be seen as a generalization of the median heuristic theory. If, by ranking all players by their risk preferences p, the median player will have $p = 0.5$ for all games, the median heuristic theory will be valid. The Value at Risk theory predicts the median player to have the same risk preference p of any value (it does not have to be 0.5 specifically), for all games.

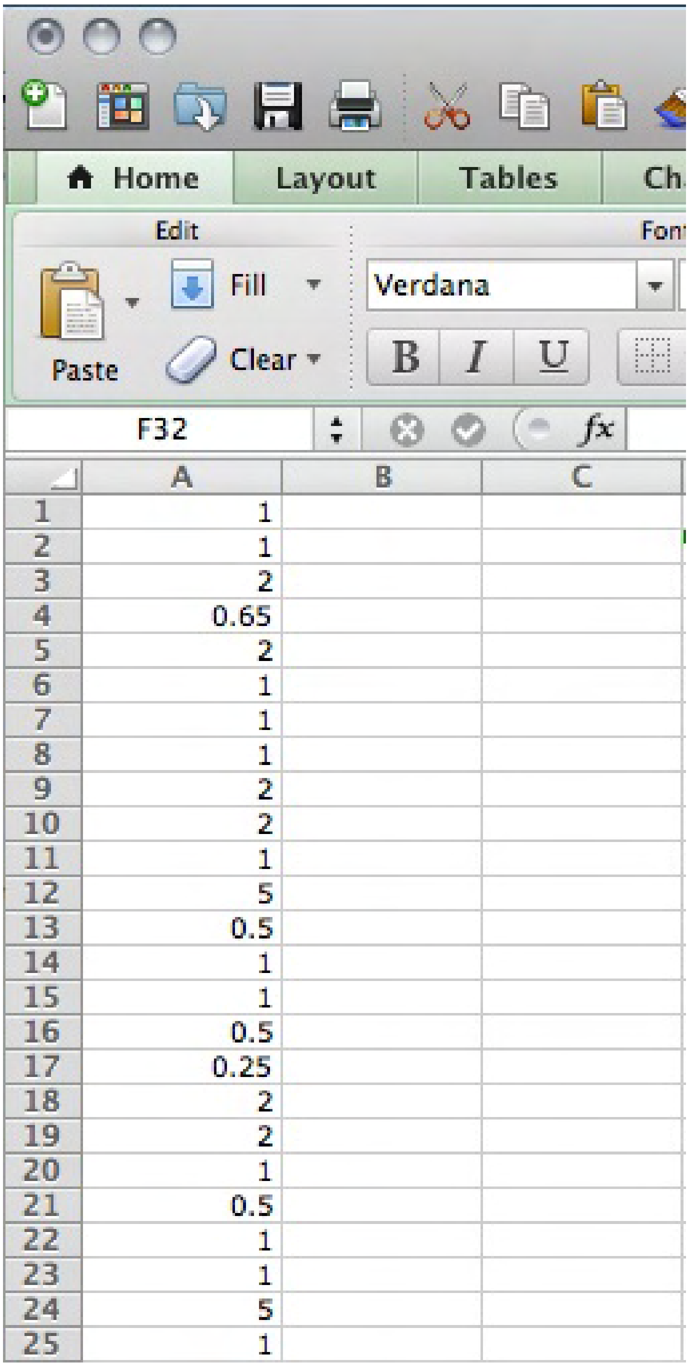
5.3 Testing the Hayden and Platt data

In an attempt to prove the median heuristic theory, Michael Platt and Benjamin Hayden conducted a survey consisting of different versions of the St. Petersburg paradox [Hayden & Platt, 2009]. They were willing to provide us with the unmodified raw bid data on the conditions that it would not be redistributed and that they would have the proprietary rights of its use. It was sent in two Excel files, one for the basic game and one for the other versions of the game. A sample of the files is presented in Figure 2 and 3.

Background
In 2009, Hayden and Platt, surveyed two groups, each consisting of 100 New York Times readers. The survey was online based and open to people who had agreed to participate in various surveys of the magazine. The first survey consisted of the basic St. Petersburg game and several versions of the game presented in Table 10. When 100 subjects had answered the survey it was closed and another survey was put up, consisting of the standard St. Petersburg game and other gambles not relevant to our work. Hayden and Platt provided us with the complete data from the first survey and only the basic game from the second survey.

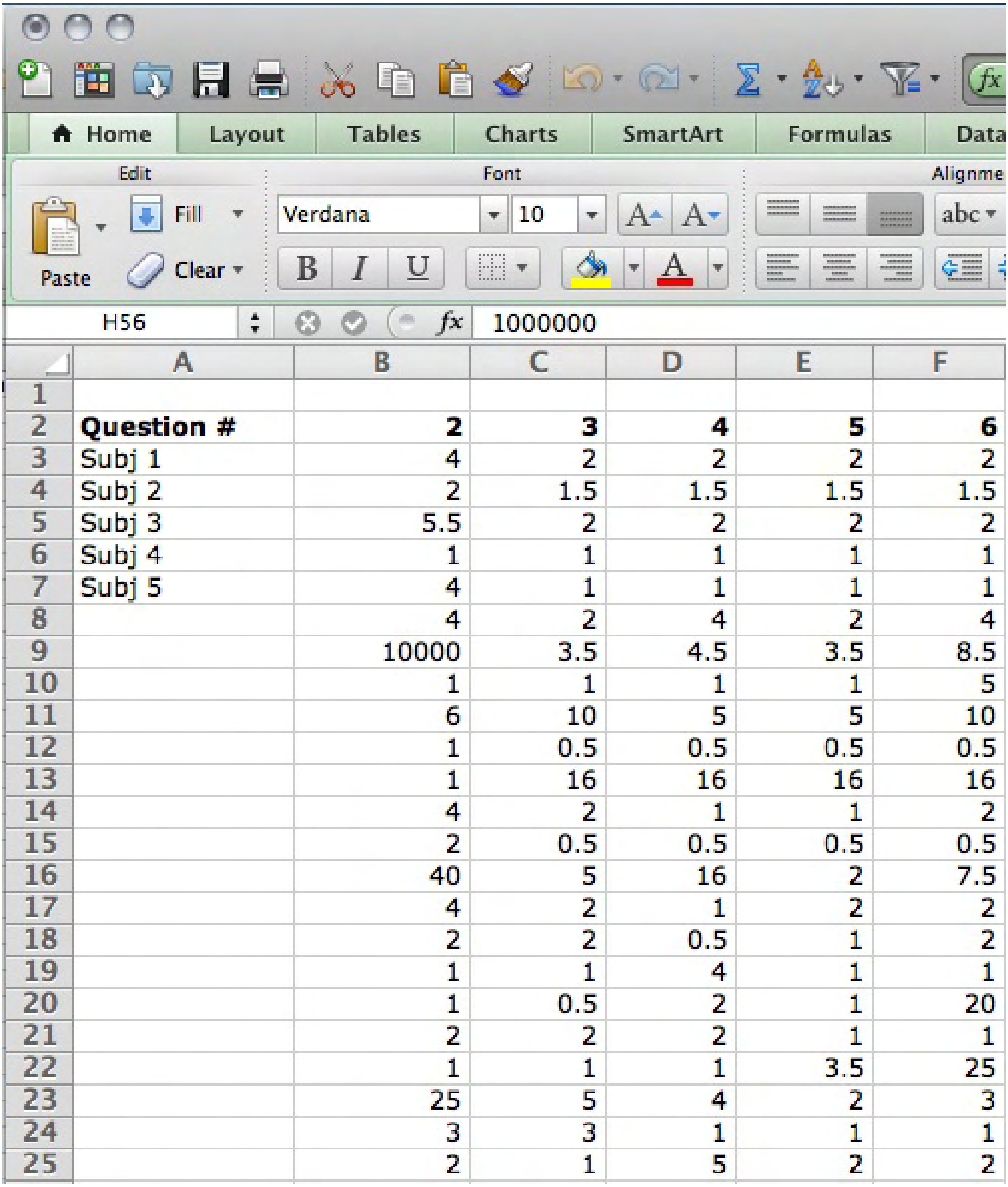
Note: The St. Petersburg game in this study has the return scheme $(1, 2, 4, 8, ...)$

Analysis
For each game, the bids are binned into their respective probability of loss p. For the basic game, these bins are illustrated in Table 9. The **frequency** for a bin b_p represents the amount of bids in that bin. **Relative frequency** is the number of bids in a bin as a percentage of total bids. Other relevant measures are the empirical cumulative distribution function (cdf)



	A	B	C
1	1		
2	1		
3	2		
4	0.65		
5	2		
6	1		
7	1		
8	1		
9	2		
10	2		
11	1		
12	5		
13	0.5		
14	1		
15	1		
16	0.5		
17	0.25		
18	2		
19	2		
20	1		
21	0.5		
22	1		
23	1		
24	5		
25	1		

Figure 2: Sample of raw bid data for the basic game of the St. Petersburg paradox. (Source: Courtesy of Benjamin Hayden and Michael Platt)



	A	B	C	D	E	F
1						
2	**Question #**	**2**	**3**	**4**	**5**	**6**
3	Subj 1	4	2	2	2	2
4	Subj 2	2	1.5	1.5	1.5	1.5
5	Subj 3	5.5	2	2	2	2
6	Subj 4	1	1	1	1	1
7	Subj 5	4	1	1	1	1
8		4	2	4	2	4
9		10000	3.5	4.5	3.5	8.5
10		1	1	1	1	5
11		6	10	5	5	10
12		1	0.5	0.5	0.5	0.5
13		1	16	16	16	16
14		4	2	1	1	2
15		2	0.5	0.5	0.5	0.5
16		40	5	16	2	7.5
17		4	2	1	2	2
18		2	2	0.5	1	2
19		1	1	4	1	1
20		1	0.5	2	1	20
21		2	2	2	1	1
22		1	1	1	3.5	25
23		25	5	4	2	3
24		3	3	1	1	1
25		2	1	5	2	2

Figure 3: Sample of raw bid data for the different game versions of the St. Petersburg paradox. (Source: Courtesy of Benjamin Hayden and Michael Platt)

Versions of the St. Petersburg game in Hayden and Platt study	
Game	**Description**
B	Basic game
B2	Basic game (from 2nd survey)
CT16	Game cutoff after 16 tosses (max return: 32768)
CT11	Game cutoff after 11 tosses (max return: 1024)
CT9	Game cutoff after 9 tosses (max return: 256)
CT6	Game cutoff after 6 flips (max return: 32)
S0.01	Start the game at \$0.01 (return $\$0.01 * 2^{n-1}$)
S0.5	Start the game at \$0.5 (return $\$0.5 * 2^{n-1}$)
S0.99	Start the game at \$0.99 (return $\$0.99 * 2^{n-1}$)
S4	Start the game at \$4 (return $\$4 * 2^{n-1}$)
T4	The return increases with a power of 4 in the order $(1, 4, 16, 64, ...)$

Table 10: Various games tested in the experiment conducted by Hayden and Platt. All games except B2 are from the first group of 100 subjects. (Source: [Hayden & Platt, 2009])

$P(X \leq p)$, the relative frequency of people willing to loose with probability *p or less*

$$P(X \leq p) = \left(\sum_{i \leq p} b_i\right) \Big/ \left(\sum_i b_i\right) \tag{34}$$

Even more important is the relative frequency of people willing to loose with probability *p or more*, $P(X \geq p)$

$$P(X \geq p) = \left(\sum_{i \geq p} b_i\right) \Big/ \left(\sum_i b_i\right) \tag{35}$$

Properties of the different games are presented in Table 18-28 and figures 15-47

Comparison

The probability of loss gives the implied risk-preference for all bids in a defined bin. VaR-P theory predicts that the empirical distribution of risk-preferences should be similar for all games, except games where the cut off limit is set very low. The games are dependent samples from nonparametric distributions which limits our testing to Wilcoxon sign-ranked test and sign test for the empirical pdf of each game. The equality of cdf $P(X \leq p)$ for each game can be tested with the Kolmogorov-Smirnov test.

More about the tests

Sign Test

The sign test is the most simple nonparametric test making few assumptions about the underlying distribution. If X and Y are two distributions from which a paired sample is drawn, the sign test tests if one of the distributions is larger than the other in absolute terms. Since our data consists of p-values from different games this is exactly what we want to test. If players of one game are more risk averse, the p-values will be consistently lower than for the other game.

Pairs where x_i and y_i are equal are omitted. w is the number of pairs where $x_i - y_i > 0$. If X and Y are completely random, the probability that X is larger than Y $P(X > Y)$ and also $P(Y > X)$ is 0.5. This implies that W, the random distribution of w follows a binomial distribution $B(m, 0.5)$ where m is the reduced number of pairs where $x_i \neq y_i$. We are testing if $X > Y$ and $X < Y$ and the hypotheses are $H_0 : p = 0.5$ (p as in probability of the Binomial distribution W) vs $H_1 : p \neq 0.5$.

Wilcoxon Signed-Rank Test
Wilcoxon signed-rank test is a further development of the sign test. If X and Y are two distributions from which a paired sample is drawn, then $Z = Y - X$ with a median θ. Each z_i is assigned a rank, R_i, by the size of its absolute value $|z_i|$, smallest first. Tied scores are assigned a mean rank. The Wilcoxon signed ranked statistics W_+ and W_- are defined as:

$$W_+ = \sum_{i=1}^{n} \delta_i R_i \tag{36}$$

$$W_- = \sum_{i=1}^{n} (1 - \delta_i) R_i \tag{37}$$

where δ_i is an indicator function, $\delta_i = 1$ if $z_i > 0$. The least of the sums W_+ and W_- is then compared to a distribution table. If the number is less than, or equal to a certain limit, the hypothesis $H_0 : \theta = 0$ cannot be rejected in favor of $H_1 : \theta \neq 0$.

Kolmogorov-Smirnov Test
Kolmogorov-Smirnov tests utilizes the empirical CDF of distribution X and Y to verify if there are any significant differences. For each pair x_i and y_i, the difference of their CDF is ranked according to size $|P(X \leq x_i) - P(X \leq y_i)|$. The largest deviation is then compared to a specific Kolmogorov distribution, preferably using a statistics software [Sprent & Smeeton, 2001].

Procedure
The raw bid data for all games was obtained in Excel-format. We used Excel and MATLAB to conduct the tests in the following steps.

1. The raw bids were binned by the probability of loss, implied by the distribution for a specific game. For the standard game, bids $0 - 1^-$ were assigned the 0 probability of loss bin, bids $1 - 2^-$ were assigned the 0.5 probability of loss bin etc.

2. When the raw bid data was translated into risk preferences, the vectors for each game was exported into MATLAB using a csv file. The bid order was maintained for a correct testing of a paired sample.

3. In MATLAB, the vectors were tested using *signtest* (Sign Test), *signrank* (Wilcoxon Signed Rank Test) and *kstest*2 (Kolmogorov-Smirnov Test) functions with the standard $\alpha = 0.05$.

Game Versions

Change the seed value
The VaR-P theory predicts no change in the distribution of risk-preferences amongst the different games. This is confirmed by Wilcoxon sign rank test and sign test that fail to reject (at 95%-level) the hypothesis that risk-preferences of games B, S0.01, S0.5, S0.99 and S4 comes

Statistical test results for Hayden and Platt data									
Game	**CT16**	**CT11**	**CT9**	**CT6**	**S0.01**	**S0.5**	**S0.99**	**S4**	**T4**
Sign test	R	NR	NR	R	NR	NR	NR	NR	NR
Signed-Rank	R	NR	NR	NR	NR	NR	NR	NR	NR
Kolmogorov-Smirnov	R	NR	NR	NR	NR	NR	NR	NR	NR

Table 11: Statistical tests if the basic game and altered versions come from the distribution. R=Rejected, NR=Not Rejected. At 95% confidence level.

from the same distribution. Kolmogorov-Smirnov test also fails to reject (at 95 %-level) the hypothesis that the games have the same cdf of risk preferences except S0.01 with S4.

Increase the rate of return
The VaR-P theory predicted no change in risk preferences between the basic game and if we set the power of the distribution to 4. This is confirmed by Wilcoxon sign rank test, the sign test and the Kolmogorov-Smirnov test. They all fail to reject that the T4 distribution pdf and cdf ($P(X \leq p)$) are different from the basic game at the 95 %-level.

Limit the upside
Although the underlying distribution is modified by limiting the upside, the VaR-theory might still be valid if the cutoff point is set very high. Wilcoxon sign rank test, sign test and Kolmogorov-Smornov test all fail to reject (95%-level) that the pdf and cdf ($P(X \leq p)$) for the games B, CT11, CT9 and CT6 differs. All tests rejected the hypothesis that CT16 have the same pdf and cdf as either B or CT6.

The basic game of the second survey
The B2 game is an independent sample from the second survey. Interestingly, the Wilcoxon rank sum (similar test for independent samples) failed to reject (95%-level) that the B2 distribution was different to any other game except CT16.

Table 11 shows a summary of all test statistics.

5.4 Our Empirical study

Using the empirical data from our survey described in section 4.2, an identical analysis was performed to test the VaR-P theory. The background information and details about the analysis is given in section 5.3. The different games, versions and returns in our survey are presented in Table 6 and 7 .

(Note: These games have the return schedule starting with 2 and not 1 as in the Hayden and Platt study)

Comparison
Each game has been modified in two ways; increase the rate of return and limit the upside. The data in Hayden and Platt studies showed some intuitive inconsistencies for the limited upside games and it is interesting to test this once more. It was decided to not do any further testing by changing in the seed value because the previous results were satisfactory. The size of the survey was a critical factor to keep up the motivation and concentration of participants.

Statistical Test results for our survey									
Game	**Coin**			**6-sided Dice**			**10-sided Dice**		
Variant	**CT10**	**CT5**	**T4**	**CT10**	**CT5**	**T4**	**CT10**	**CT5**	**T4**
Sign test	R	R	NR	R	R	R	R	R	R
Sign rank	R	R	NR	NR	R	R	R	R	R
KS	NR	NR	NR	NR	R	NR	NR	R	R

Table 12: Statistical tests if the basic game and altered versions come from the distribution. R=Rejected, NR=Not Rejected. 95% confidence level.

VaR-P theory predicts that the empirical distribution of risk-preferences should be similar for all games, except games where the cut off limit is set very low. We will use the same statistical tools described in section 5.3. Since the underlying distribution is different for the coin, 6-sided dice and 10-sided dice games, we can only compare the versions for each game separately.

Properties of the different games are presented in Table 29-40 and figures 48-83

Increase the rate of return
The VaR-P theory predicted no change in risk preferences between the basic game and if we set the power of the distribution to 4. For the coin game, this is confirmed by Wilcoxon sign rank test, the sign test and the Kolmogorov-Smirnov test. They all fail to reject that the T4 distribution pdf and cdf ($P(X \leq p)$) are different from the basic game at 95 %-level. For both dice games, all tests, except KS for the 6-sided dice game, reject the hypothesis that the risk-preferences come from the same distribution at the 95%-level. Table 12 shows a summary of the test statistics.

Limit the upside
Although the underlying distribution is modified by limiting the upside, the VaR-P theory might still be valid if the cutoff point is set very high. The results vary across the games, Table 12 shows a summary of all test statistics at 95%-level. It can be seen that CT10 gets some support that it is coming from the same distribution as the basic game.

5.5 Conclusion

The analysis of Hayden and Platt data show a strong support for the VaR-P theory. Players do not tend to change their risk-preferences when the seed value or rate of return changes. A further study with larger changes in seed value would be interesting. However, a wealth effect is anticipated from a student population that tend to have less money than the average individual.

For games with limited upside, Hayden and Platt data showed inconsistent results where players were more risk-averse when playing the basic game compared to the limited versions. That is why the decision was made to test these versions once more.

In our survey, the investors were more risk-averse playing the limited upside versions. The statistical tests showed minimal support for the hypothesis that the limited games come from the same distribution as the basic games. Our hypothesis was that when the limit is high

Forward Rate Variables	
Variables	**Description**
m	The underlying rate, e g 3-month LIBOR, 1-month EONIA
t	Time of contract agreement, e g today or any past date
θ	Contract maturity date

Table 13: The variables used to describe the foward rate.

enough, players should not make a difference between the basic and limited game. As the probability of winning increases in the dice games, 1024 becomes a low limit as 32 is for the basic game. In a limited St. Petersburg game, the players can asses their risk using the standard distribution moments; expectation value, variance, skew and kurtosis which are more powerful than VaR-P. However, Var-P offers a very good alternative when this is not possible in power law distributions lacking moments or other distributions with large skew and heavy tails.

For the coin game, statistical tests showed a strong support that the risk-preference distribution do not change when the rate of return increases. This did not occur for the dice games and a possible explanation could be the wealth effect in the dice games mentioned previously. It is unrealistic for students to play with these large amounts of money. The solution proposed in section 4.2 to do a scalar adjustment to the return scheme so the medians for the coin and dice games could solve this problem.

6 Forward Interest Rate Term Structure and Value-at-Risk Pricing

6.1 Background

In the financial markets, a forward contract allows you to agree to a predefined rate when borrowing or lending money at a future date. In January, you can agree to borrow $100,000, starting in July, until the end of September at the forward 3-month LIBOR rate r. Standardized forward contracts for a set amount of time with fixed maturity dates are exchange traded for the largest currencies in the world. The rate of a forward contract for an m-month underlying rate, agreed at time t, maturing within time θ, is $f_m(t, \theta)$. The variables are summarized in Table 13.

In normal market conditions, holding m constant, $f_m(t, \theta)$ is usually an increasing function of θ at any given time t. Figure 4 shows an example of a forward rate curve as a function of time to maturity, θ, averaged over a period of time from t_1 to t_2.

The rate increase with maturity is explained as lenders demanding a risk premium for the uncertainty of future spot rates. The classic models (Vasicek, Hull-White) fails to take this into consideration without introducing the market price of risk, which is not a directly measurable quantity [Bouchaud et al., 1999]

Bouchaud et al. studied a dataset of Eurodollar futures prices between 1987 and 1999. A Eu-

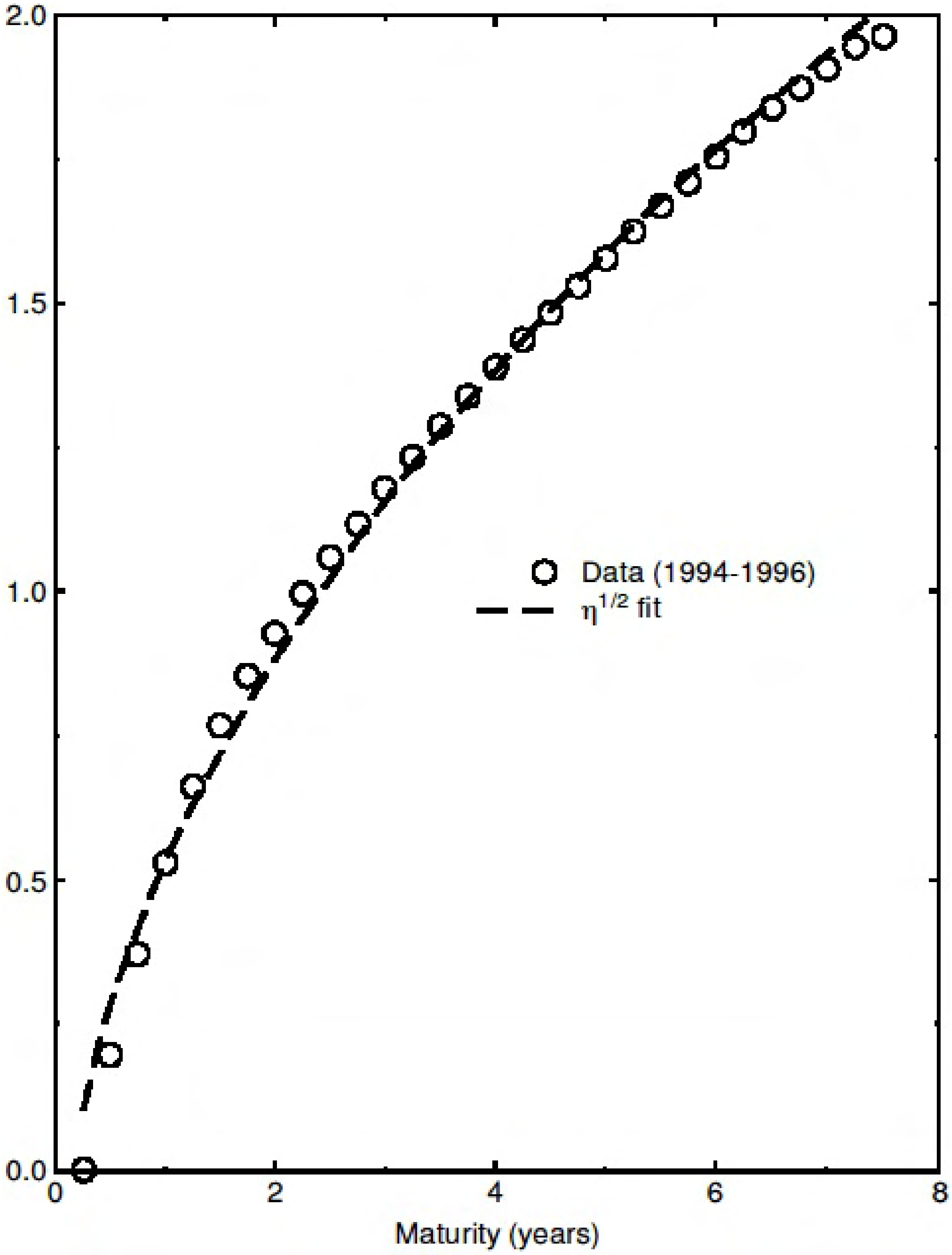


Figure 4: The average FRC in the period 94-96, as a function of maturity θ (Source: [Bouchaud et al., 1999])

rodollar future is an exchange traded contract very similar to a forward. The underlying rate is the 3-month USD LIBOR on expiry day. Eurodollar futures are traded on the Chicago Mercantile Exchange (CME) with fixed expiration dates, separated by three-month intervals (March, June, September, December). The Eurodollar futures are highly liquid and very responsive to economic changes. Although they have slighly other terms than forwards the differences are neglected in this study [Rendleman Jr & Carabini, 1979].

For simplification, the denomination $f(t, \theta) = f_{3M}(t, \theta)$ will be used since Eurodollar futures have a standardized term m of 3 months. Because of fixed expiration dates T_i, time series are available for $f(t, T_i - t)$. To extend the time series to dates in-between the fixed expirations a simple linear interpolation is done so that $T_i - t \leq \theta \leq T_{i+1} + t$.

The Eurodollar futures rate indicate the markets anticipation of the future spot rate. Because the spot rate is bounded, the long-term anticipation and therefore, the average difference between the future and the spot rate, $\langle s(t, \theta)\rangle_t$,

$$\langle s(t, \theta)\rangle_t = \langle f(t, \theta) - r(t)\rangle_t \tag{38}$$

should in theory be zero. As we can see in Figure 5, the spread is not 0 because of the demanded risk premium mentioned earlier.

In the study, the forward rate with the closest maturity θ_{min} is used as a proxy for the spot rate $r(t)$.

$$r(t) = f(t, \theta_{min}) \tag{39}$$

The spread is plotted in Figure 5 and as can be seen, increases with θ. The graph also shows the fitted function

$$\langle s(t, \theta)\rangle_t = a(\sqrt{\theta} - \sqrt{\theta_{min}}) \tag{40}$$

It is found that $a/\sqrt{250}$ is very close to the daily spot rate volatility σ (see Table 14). Bouchaud et al. suggests that money lenders want to protect themselves against rising spot rates in the future and add a risk premium that is $\sigma\sqrt{\theta}$. Compared to the previous models (Vasicek, Hull-White), this measure offers an intuitive explanation. Lenders take a bet that the future spot rate at time $t + \theta$, $r(t + \theta)$ will not exceed the forward rate $f(t, \theta)$. The pdf of the future spot rate given the current spot rate is $g_R(r(t + \theta)|r(t))$ Then, the probability p of the future spot rate $r(t + \theta)$ exceeding than the forward rate $f(t, \theta)$, given the current spot rate $r(t)$, is

$$p = \int_{f(t,\theta)}^{\infty} g_R(r(t + \theta)|r(t))dr \tag{41}$$

If $r(t + \theta)$ follows a random walk centered around $r(t)$, the forward rate is

$$f(t, \theta) = r(t) + \sqrt{(}2)erfc^{-1}(2p)\sigma\sqrt{(}\theta) \tag{42}$$

The equation only describes the forward rate on average and cannot be used from a no-arbitrage point of view. Bouchaud et al. found that this matched the empirical data with $p \approx 0.16$. Similar investigations were done with Short Sterling (GBP), Euromark (DEM), Bank Accepted Bills (AUD) and Euroyen (JPY) contracts between 94-99 [Bouchaud & Matacz, 2000]. It was found that GBP and AUD contracts also followed the $\sigma\sqrt{\theta}$ best-fit (Table 14), which further

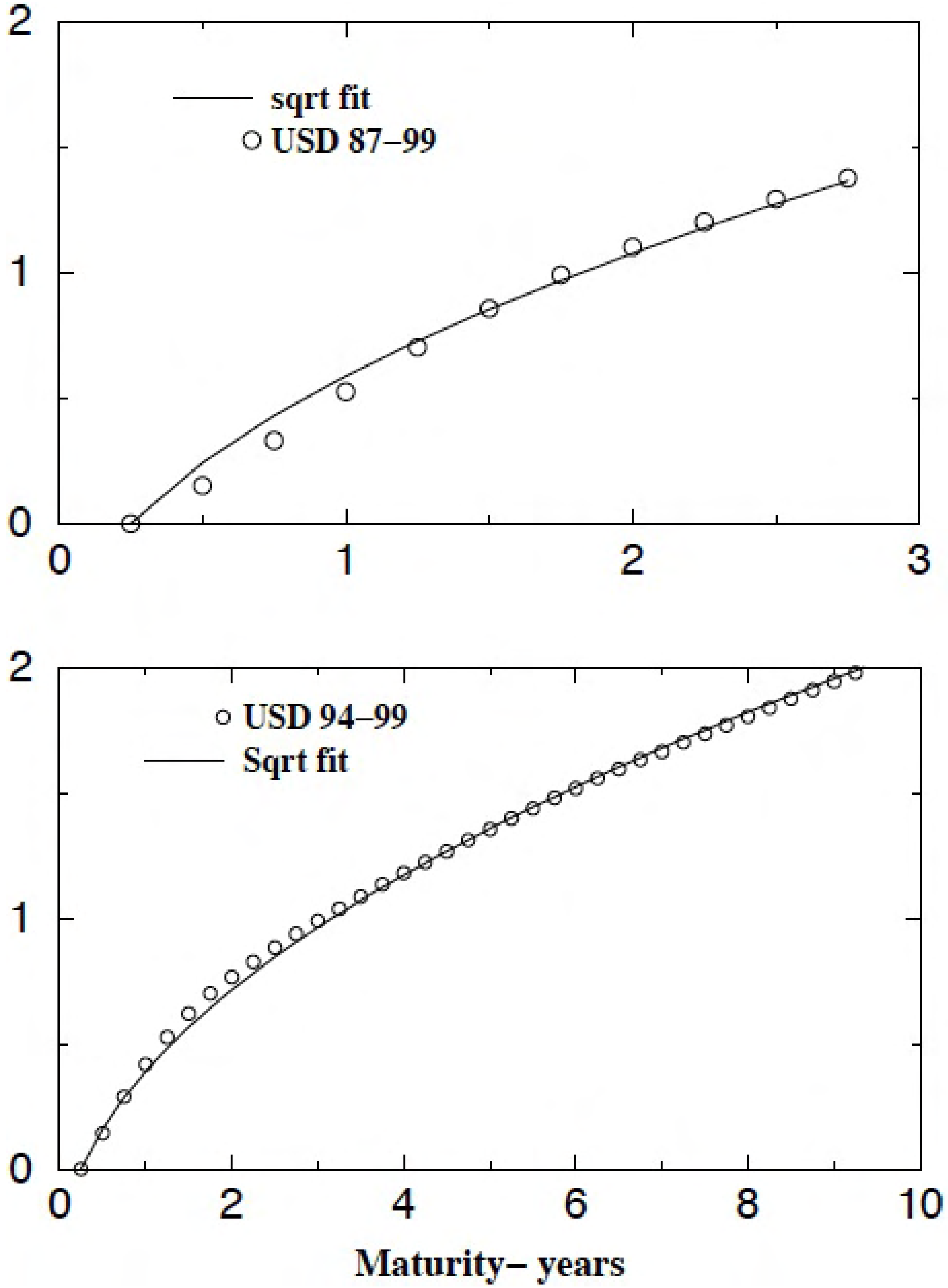


Figure 5: Top figure: the average FRC in % for USD 87-99, given empirically by (38), and best fit to (40). The fitting parameter a, is shown in Table 14. Bottom Figure: the same but now for USD 94-99. These figures, along with Table 14, demonstrate that the USD average FRC is well fitted by a square-root law with a prefactor given approximately by the spot volatility. (Source: [Bouchaud & Matacz, 2000])

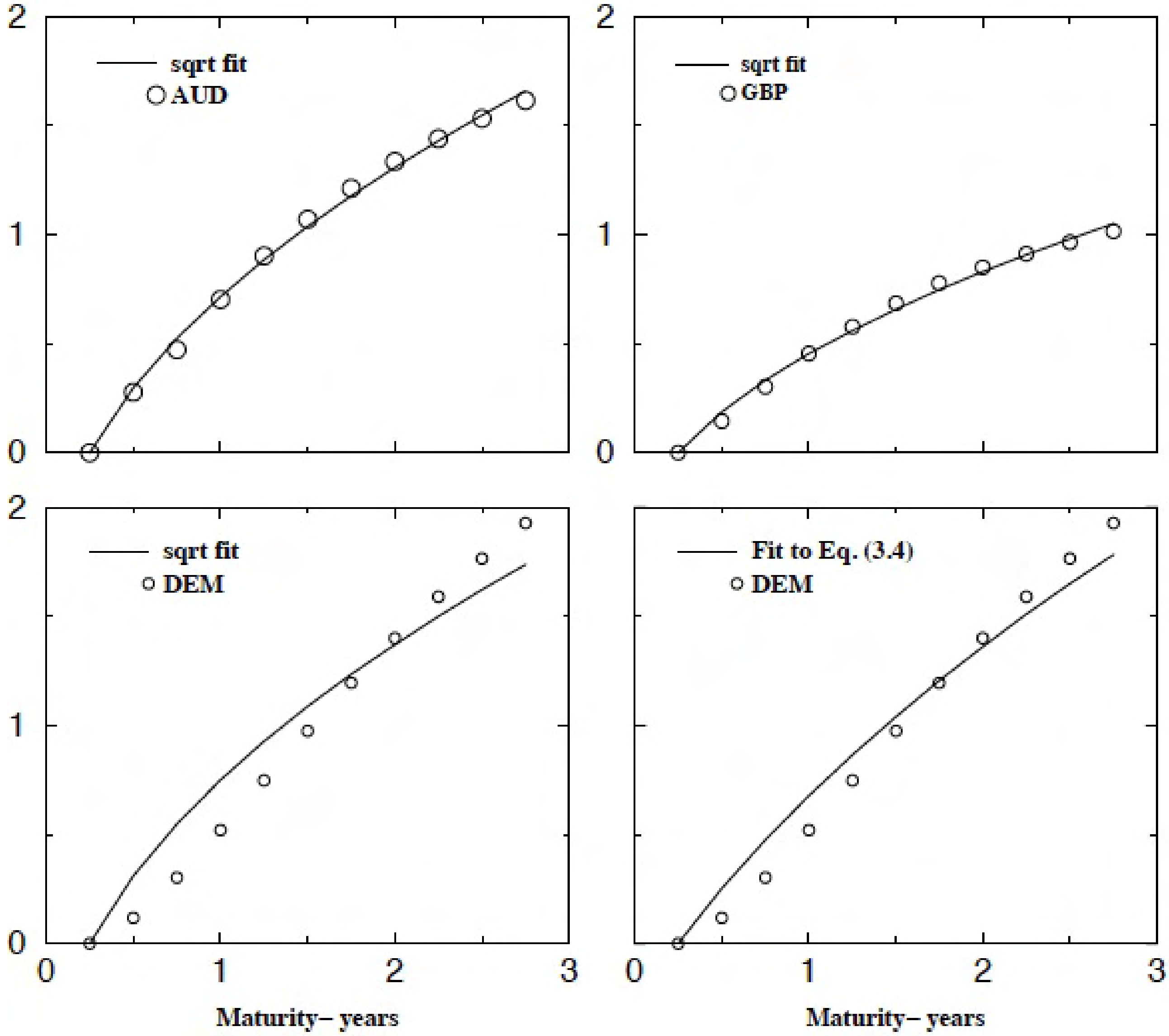


Figure 6: Top left: the same as figure 5 but now for AUD. Top right: the same but now for GBP. Bottom left: the same but now for DEM. Bottom right: here we show the same empirical DEM average FRC, but now with a best fit to (44) rathen than (40). These figures, along with Table 14, demonstrate that the AUD and GBP average FRC's are also well fitted by square root law with a prefactor given approximately by the spot volatility. (Source: [Bouchaud & Matacz, 2000])

Parameters			
Sample	a	$a/\sqrt{250}$	σ
USD 94-99	0.78	0.049	0.047
USD 87-99	1.18	0.075	0.067
GBP	0.91	0.058	0.053
AUD	1.43	0.090	0.078
DEM	1.50	0.095	0.033
JPY	No data provided		

Table 14: Parameters for the $\sigma\sqrt{\theta}$-law in (40). The units for σ are % per square-root day. (Source: [Bouchaud & Matacz, 2000])

strengthens the Value-at-Risk theory. The spreads and fits are plotted in Figure 6. JPY and DEM spot rates had a long-term drift that needs to be accounted for in the model. Bouchaud et al. introduces a spot trend in the spread function $s(t,\theta)$, inspired by the Vasicek model

$$s(t,\theta) = \sigma(\sqrt{\theta} - \sqrt{\theta}_{min}) + (r_0 - r(t))(1 - e^{-\lambda(\theta-\theta_{min})}) \tag{43}$$

The equilibrium spot rate is r_0 with reversion time $1/\lambda$. The average spread $\langle s(t,\theta)\rangle_t$ is

$$\langle s(t,\theta)\rangle_t = \sigma(\sqrt{\theta} - \sqrt{\theta}_{min}) + (r_0 - \langle r(t)\rangle_t)(1 - e^{-\lambda(\theta-\theta_{min})}) \tag{44}$$

Best fit for DEM data, using the empirical spot volatility $\theta = 0.033$, was $\lambda = 0.22$ and $r_0 = 6.88\%$. The trend is a mean reversion to approximately 7% over a time period of 5 years, a very realistic scenario. Similar results were found for the JPY forward rate.

Bouchaud et al. also showed that the forward rates showed a strong correlation with the past spot trend over a period of time. Their model was developed further to incorporate this anticipated bias [Bouchaud et al., 1999] but that part is not relevant for our work.

Comparison with the Heath-Jarrow-Morton (HJM) model
One of the most used interest rate term structured models is the HJM

$$f(t,\theta) = f(t_i, t - t_i + \theta) + \int_{t_i}^{t} \upsilon(t+\theta-s)ds + \int_{t_i}^{t} \sigma(t+\theta-s)dW(s) \tag{45}$$

where

$$\upsilon(\theta) = \sigma(\theta)\int_0^{\theta} \sigma\theta' d\theta' - \lambda\sigma(\theta) \tag{46}$$

where λ is the market price of risk and dW is a Brownian motion.

Bouchaud et al. compared the HJM model to the Value at Risk pricing. The average spread over a period $\tau = t - t_i$ is:

$$\langle s(t,\theta)\rangle_t = \langle f(t,\theta) - r(t)\rangle_t = f(t_i, \tau+\theta) - f(t_i,\tau) + \int_{\tau}^{\tau+\theta} \upsilon(u)du - \int_0^{\theta} \upsilon(u)du \tag{47}$$

Assuming a constant volatility for large maturities, the average spread will be independent of τ

$$\langle s(t,\theta)\rangle_t = \lambda\left(\int_0^{\theta} \sigma(u)du - \theta\sigma(\theta_{max})\right) \tag{48}$$

A comparison between the best fits of the VaR-pricing model and the HJM-model fit for the USD forward rate 94-99 are shown in Figure 7. It is clear that the HJM fit is very bad compared to the VaR-model [Matacz & Bouchaud, 1999] [Heath et al., 1992].

6.2 Testing with new data

Bouchaud et. al tested the VaR-pricing theory 10 years ago and it is interesting to see whether it still holds. We used Bloomberg to get access to historic futures prices on short term 90-day rate contracts for 7 currencies; USD, GBP, EUR, JPY, AUD, CAD and CHF. The previous tests were conducted in 1999 and we used historic data between the dates 01/01/2000-19/11/2010 for all rates except the JPY. Starting in April 2003, there were 20 contracts available for JPY so the period used is 4/29/2003-19/11/2010. Table 15 and 16 show a summary of contracts and maturities.

Currency	**Futures Name**	**Exchange**	**Underlying**
US Dollar (USD)	Eurodollar	CME	3-m USD LIBOR
British Pound (GBP)	Short Sterling	NYSE Liffe London	3-m Sterling LIBOR
Euro (EUR)	Euribor	NYSE Liffe London	3-m Euro Euribor
Japanese Yen (JPY)	Euroyen	TIFFE	3-m Yen TIBOR
Australian Dollar (AUD)	Bank Accepted Bill Future	ASX	Bank Accepted Bills
Canadian Dollar (CAD)	Canadian Bankers' Acceptance	Montreal Exchange	CDOR
Swiss Franc (CHF)	Euroswiss	NYSE Liffe London	3-m Swiss Francs LIBOR

Table 15: 90-day short term interest rate contracts used in our study. All contracts are in the top 10 of the daily volume list on Bloomberg.

Currency	**# Contracts**	**Longest maturity (days)**
US Dollar (USD)	40	2540
British Pound (GBP)	20	1233
Euro (EUR)	20	1236
Japanese Yen (JPY)	20	1236
Australian Dollar (AUD)	10	586
Canadian Dollar (CAD)	12	716
Swiss Franc (CHF)	8	456

Table 16: Number of generic contracts and longest maturity, θ counted in trading days.

The time series is expanded from the fixed maturity dates T_i by a linear interpolation so that $T_i - t \leq \theta \leq T_{i+1} + t$ for all trading days. The average spread is defined defined by (38). As in Bouchaud's study, the forward rate with the closest maturity is used as a proxy for the spot rate. The lowest maturity for all future contracts is 3 months (70 trading days). All calculations are done in MATLAB. The average curve $\langle s(t,\theta)\rangle_t$ for each currency is fitted with (40) using the Curve Fitting Toolbox.

Figures 8-14 show the average spread (38) and best fit of (40) for all currencies. Table 17 show parameter a, R^2 and the daily spot volatility σ of the underlying rate.

Although satisfying, the fit is not as good as in Bouchaud's study. Judging by the plots and R^2, the model works fairly for USD, EUR, AUD, CAD and CHF. A trend seen for all currencies is

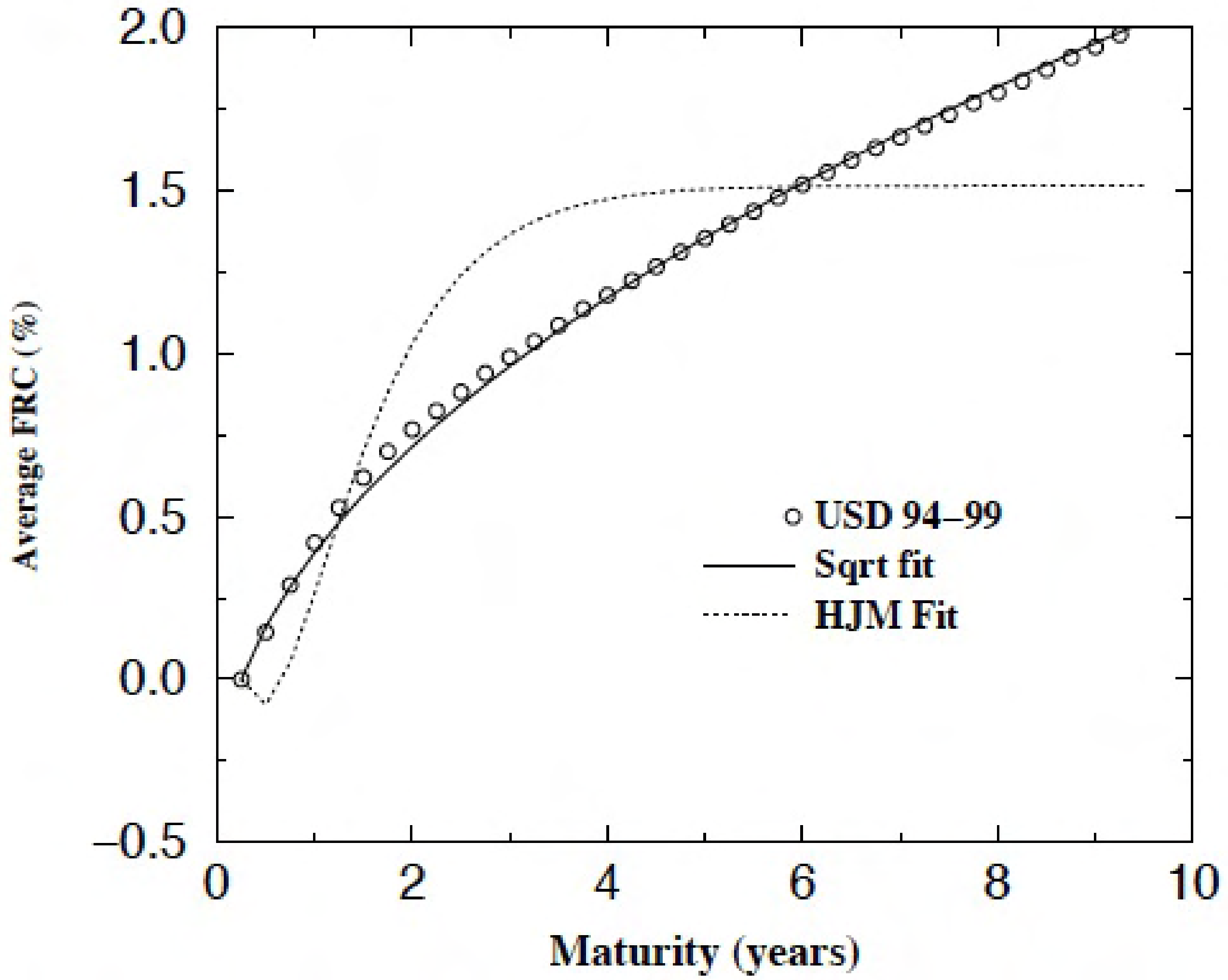


Figure 7: The average FRC spread for USD 94-99, given empirically by (38), and a best fit by (40). Also shown is the best fit of the HJM model which is the market price of risk contribution to the average FRC in the HJM framework, which fails to adequately describe this phenomenon. (Source: [Matacz & Bouchaud, 1999]

Currency	a	R^2	σ
Dollar (USD)	0.07943	0.9823	0.1270
Sterling (GBP)	0.03835	0.9341	0.0621
Euro (EUR)	0.05231	0.9669	0.0560
Yen (JPY)	0.03896	0.906	0.1329
Australian Dollar (AUD)	0.03844	0.9695	0.1089
Canadian Dollar (CAD)	0.07398	0.9627	0.1322
Swiss Francs (CHF)	0.05545	0.9719	0.2313

Table 17: Fit parameter a, R^2 and historical average daily spot % volatility over 30-days.

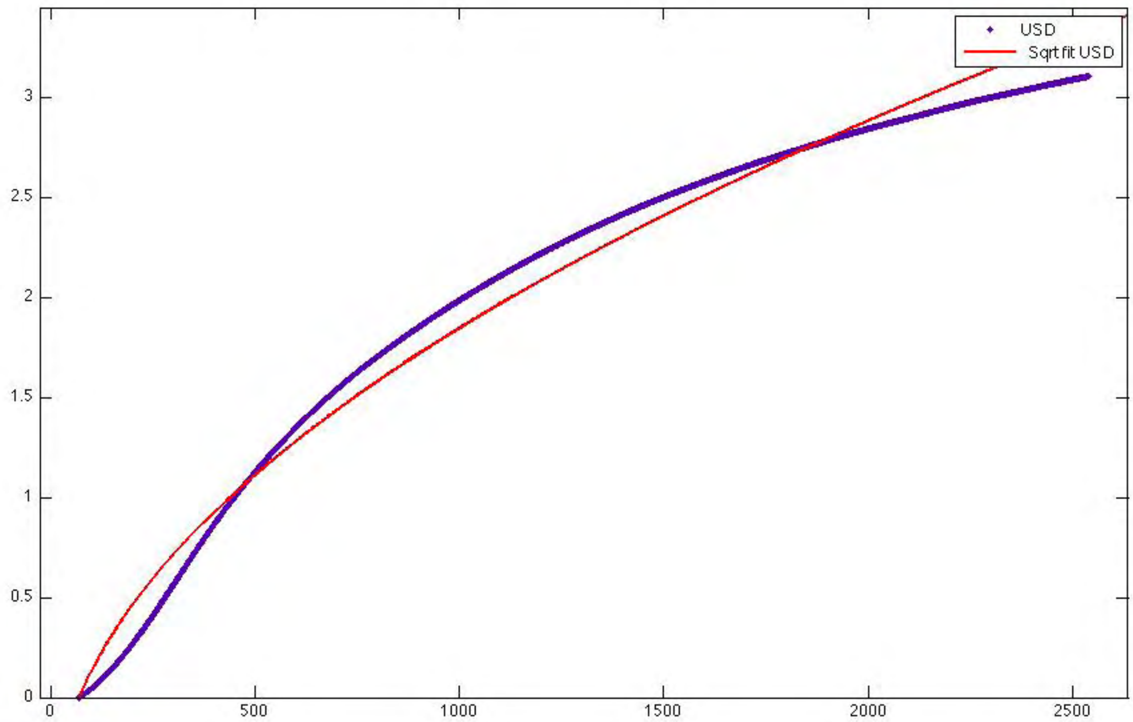


Figure 8: The average forward rate curve for USD and best fit to (40). Fitting parameter a is shown in Table 17

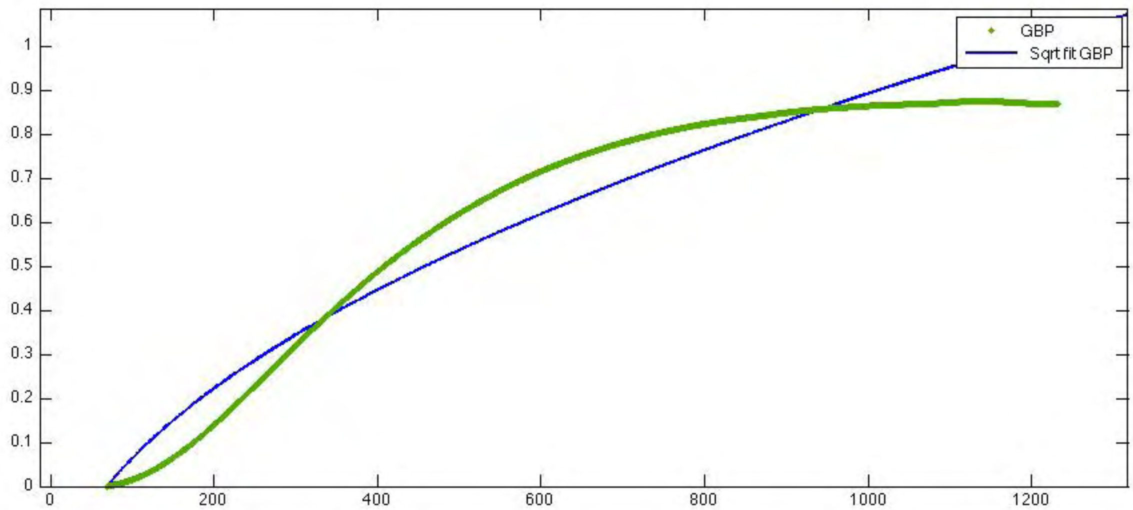


Figure 9: The average forward rate curve for GBP and best fit to (40). Fitting parameter a is shown in Table 17

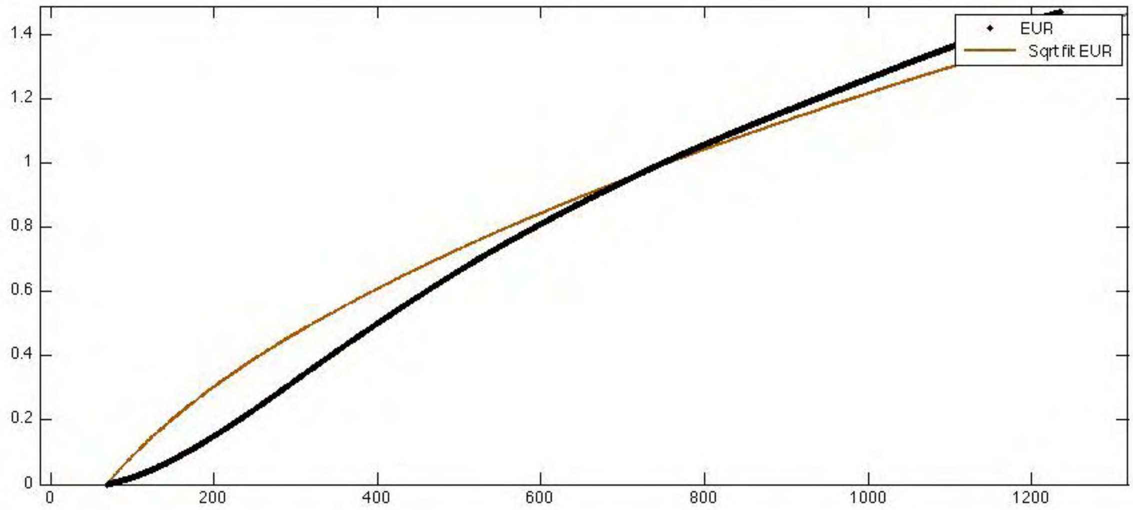


Figure 10: The average forward rate curve for EUR and best fit to (40). Fitting parameter a is shown in Table 17

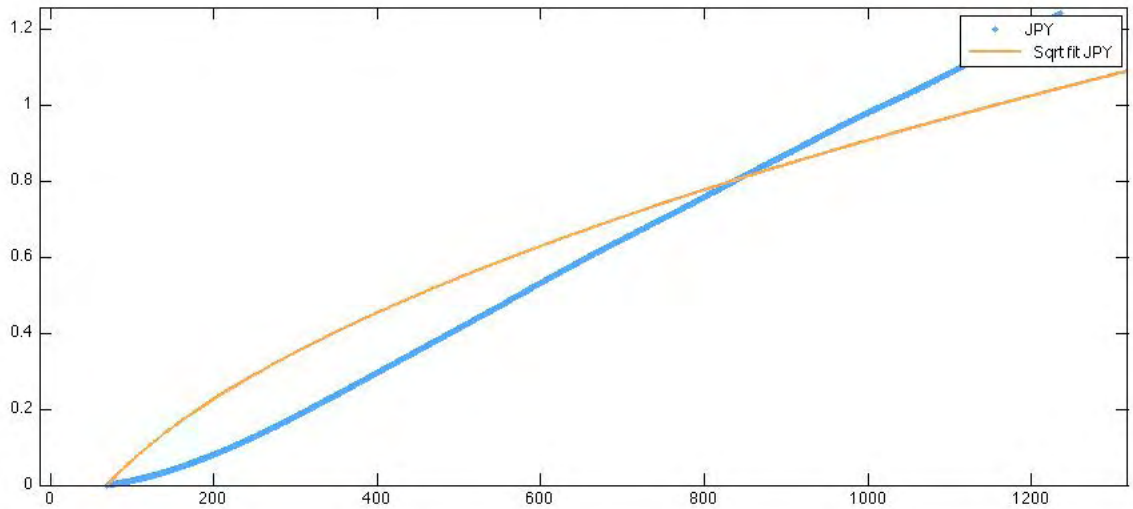


Figure 11: The average forward rate curve for JPY and best fit to (40). Fitting parameter a is shown in Table 17

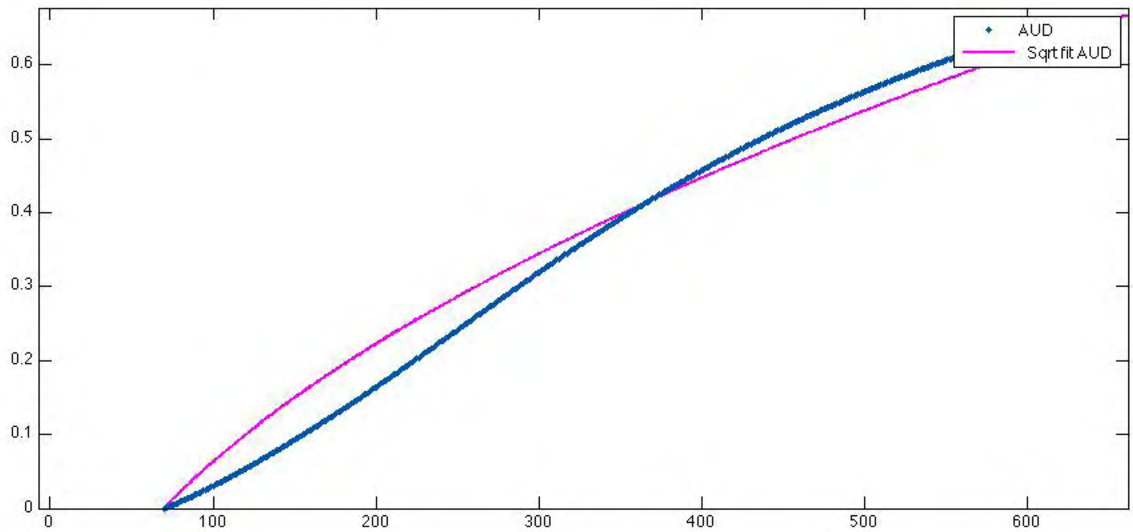


Figure 12: The average forward rate curve for AUD and best fit to (40). Fitting parameter a is shown in Table 17

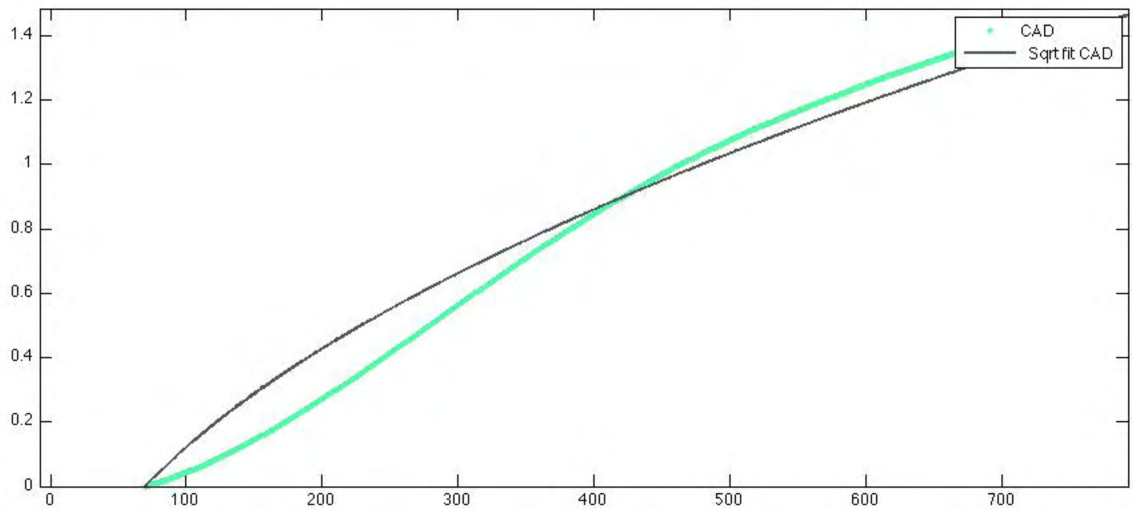


Figure 13: The average forward rate curve for CAD and best fit to (40). Fitting parameter a is shown in Table 17

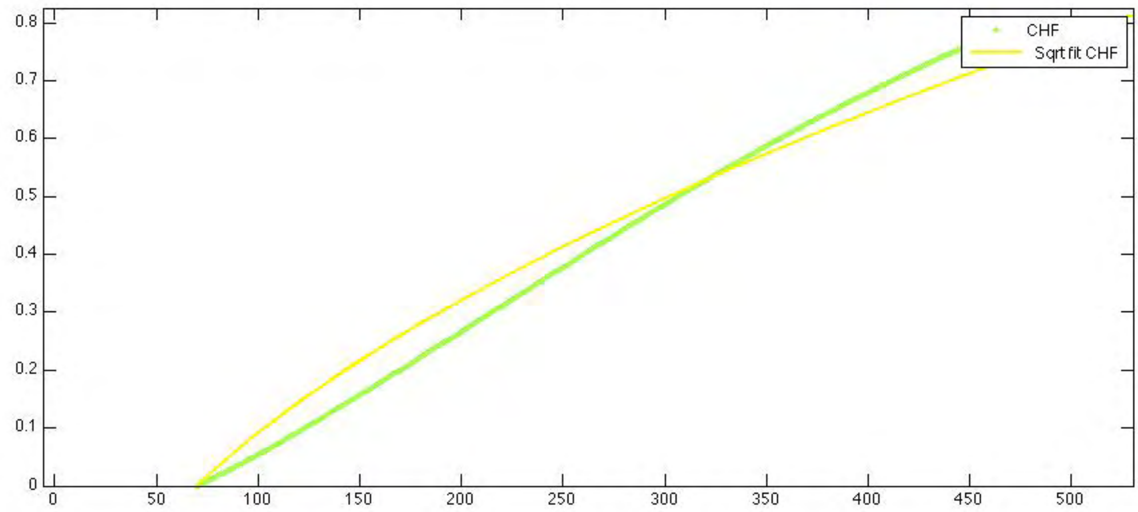


Figure 14: The average forward rate curve for CHF and best fit to (40). Fitting parameter a is shown in Table 17

that the steepest increase in the spread occurs when maturity is around 1 year. This is evident in the plots where the spread function starts out as convex and then transforms to concave at maturity 250 days. A reason could be the anticipated trend that investors include in their pricing for up to one year at which point futures prices are most volatile [Bouchaud et al., 1999]. The square root VaR-model is strictly concave and fails to include this. This is very evident for GBP and JPY. The data should be tested with the trend model. There is also a larger discrepancy between the fitted a values and daily spot volatilities compared to Bouchaud's tests. EUR is nearly identical but JPY, AUD and CHF differs by several times. The extraordinary conditions during the financial crisis in 2008 could be a reason that the spot volatilities are higher. Excluding this period lowers the average daily spot volatility. The total time period is very long and could contain multiple trends that distort the model. If the analysis is split up into multiple, shorter periods, the model might correspond better with the average volatilities of the rates.

7 Conclusion

With the help of the generalized St. Petersburg paradox, the first part of the thesis showed that the Median Heuristic theory is supported in favor of the Expectation Heuristic theory. The players of the 6-sided dice game gave bids close to the Median Heuristic expectations. The bids for the 10-sided dice game were lower than the Median Heuristic and this could be due to a wealth effect evident amongst students with limited capital. A possible solution is to change the seed value of the games so the medians are close or equal.

Using the results from our own and Hayden and Platt's survey, we show that the Median Heuristic can be generalized into the VaR-P theory where players have an individual risk-preference. Each investor assumes a probability of loss, p, that he is willing to take. When seed value and power of return of the St. Petersburg game changes, keeping the underlying distribution the same, players seem to adjust their bids according to their p value. For the dice games in the generalized St. Petersburg game, some of the VaR-P expectation bids are unrealistically high considering that the players are students with limited capital. To remove the wealth effect, the solution mentioned earlier can be used, changing the seed value so medians for all games are low and equal.

To show that the VaR-P theory has practical use outside the St. Petersburg games, we bring up a paper that proposes VaR-pricing of interest rate futures [Bouchaud et al., 1999]. Testing this theory once more on new historic data gives mixed results. The overall fit is adequate, with R^2 ranging from 0.9 to 0.98. The average spreads show the steepest increase at maturity of 1 year which suggests an anticipated trend in the price. Fitted values, a, are within a reasonable range for spot volatilities but do not correspond to the historical average volatilities for all currencies. Because of trends and other events, rate volatilities are rarely persistent over a 11 year period. The model might work better if multiple shorter time periods are used instead.

We have developed an intuitive and viable risk-pricing model but there is still much work to be done with the practical aspects. Some suggestions on future work are:

1. Conduct a survey similar to ours, where the median of the games are scaled to be equal or close to remove a possible wealth effect.
2. Test the VaR-pricing theory of interest rate futures in smaller periods including a trend model to see whether the fit is better and a-values are more consistent.
3. Test the VaR-pricing theory on equity and commodity futures using a lognormal model.

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A Hayden and Platt study

Tables and Figures from Hayden and Platt study

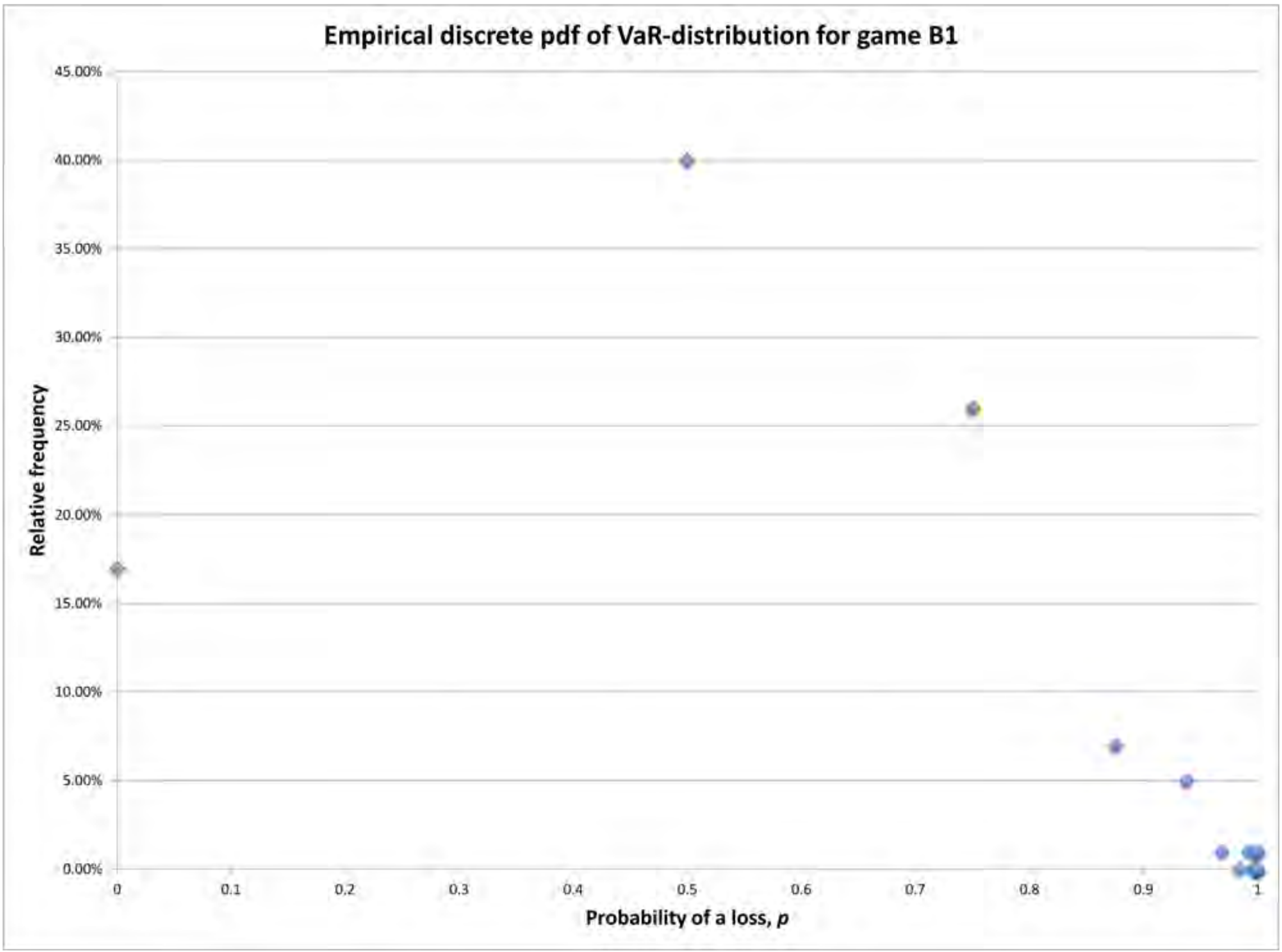

Figure 15: Relative frequency of each bin b_p with p probability of loss, for the basic game in first survey (B1)

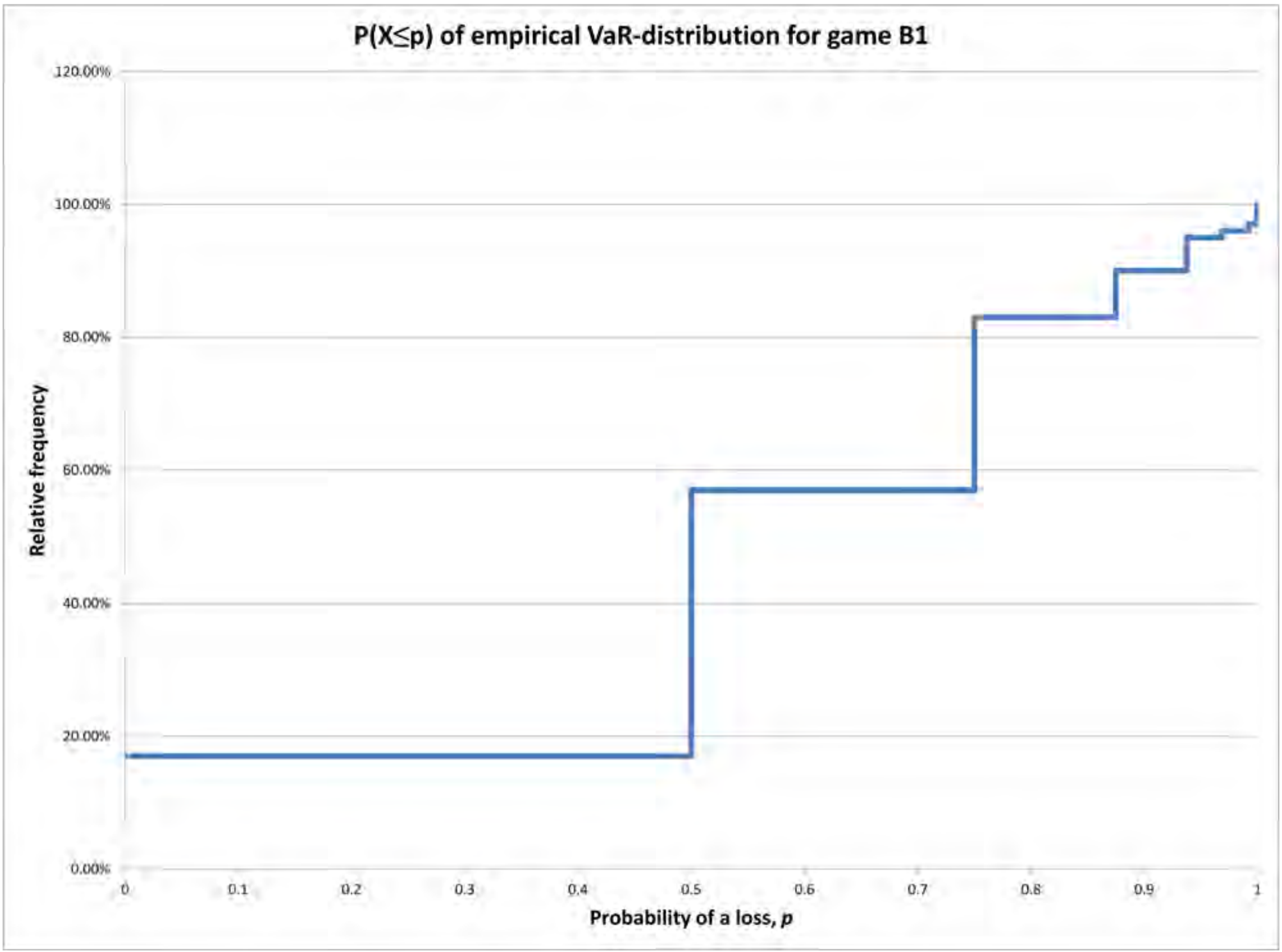

Figure 16: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the basic game in the first survey (B1).

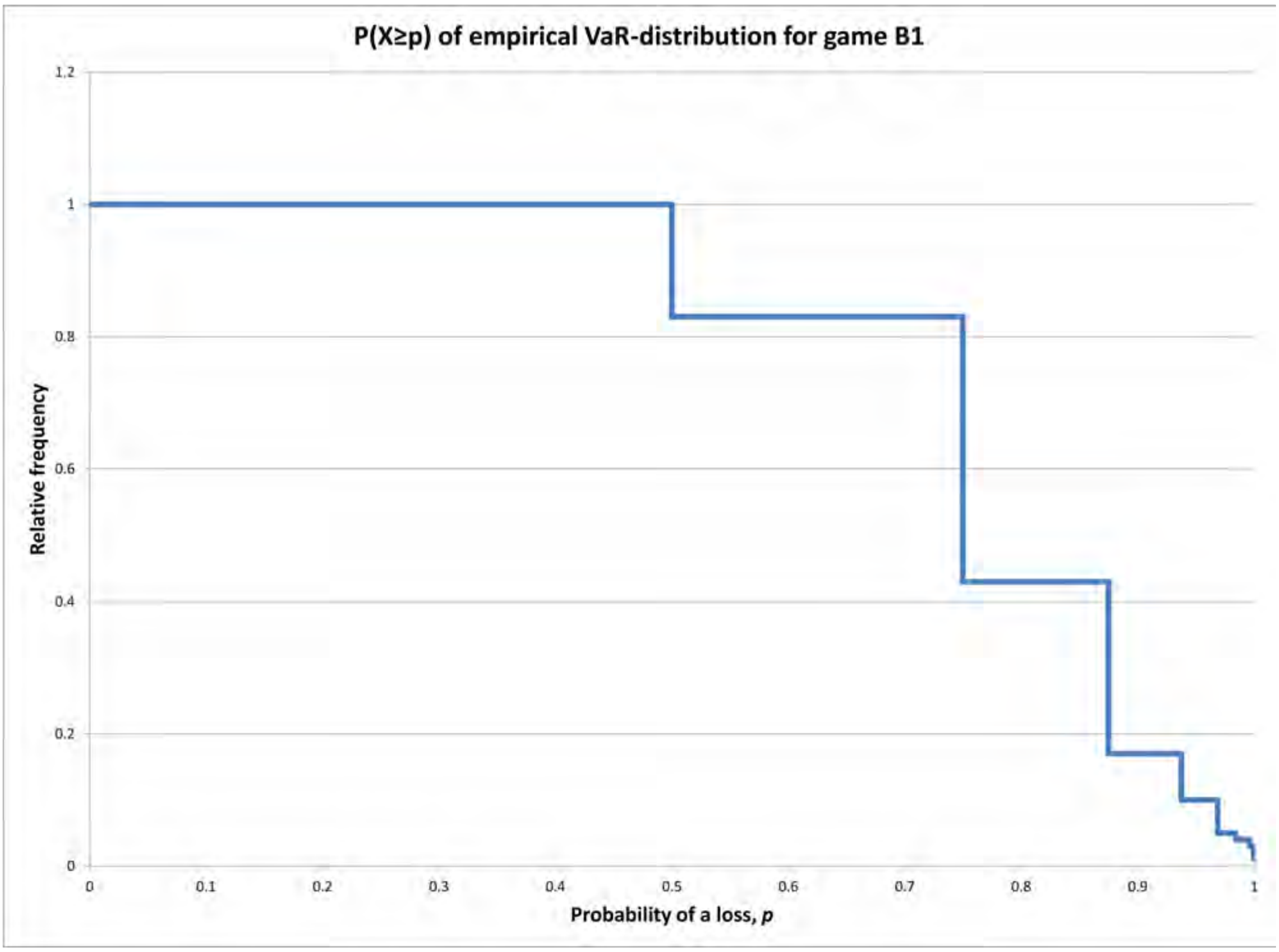

Figure 17: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the basic game in the first survey (B1).

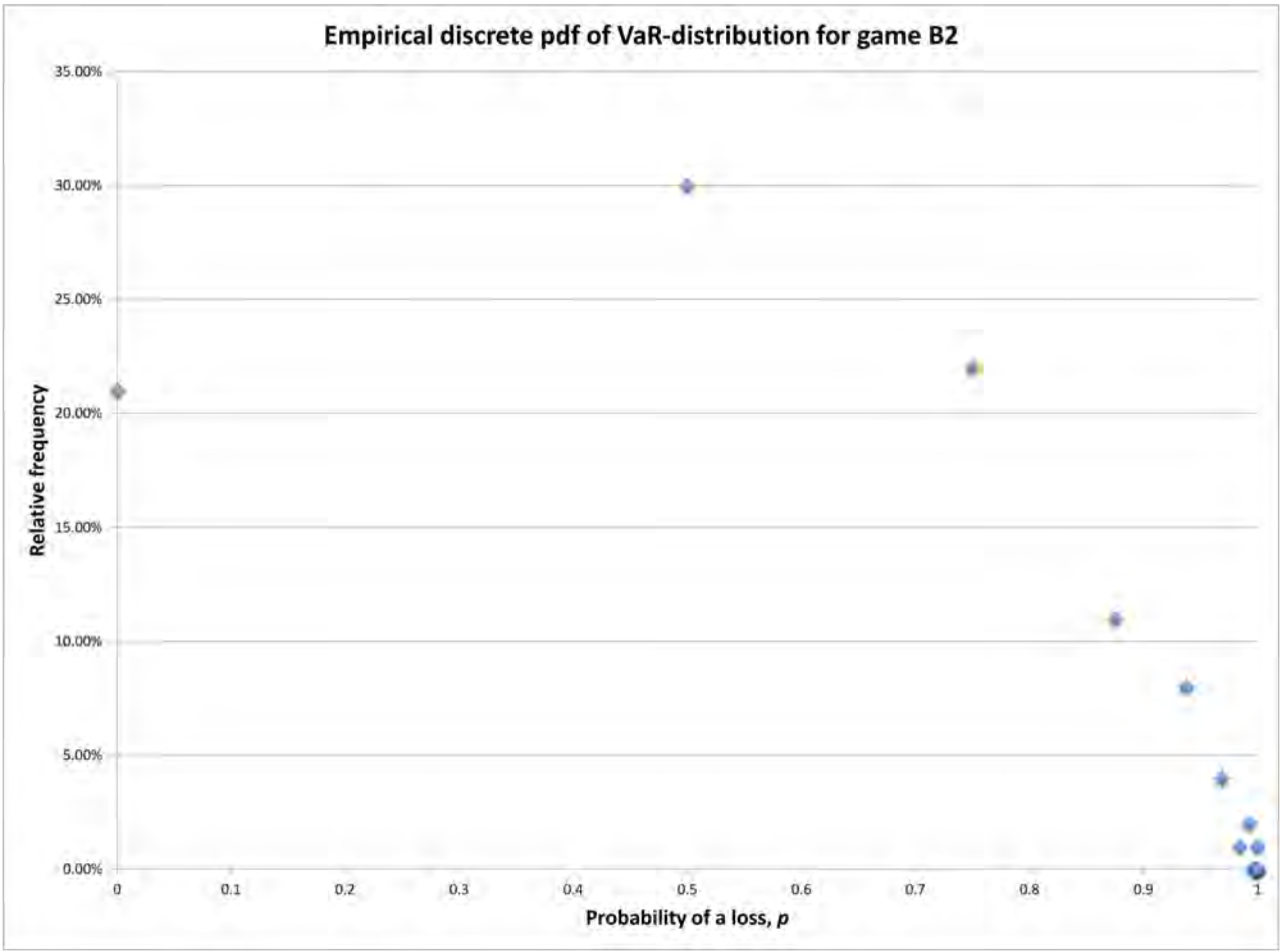

Figure 18: Relative frequency of each bin b_p with p probability of loss, for the basic game in the second survey (B2)

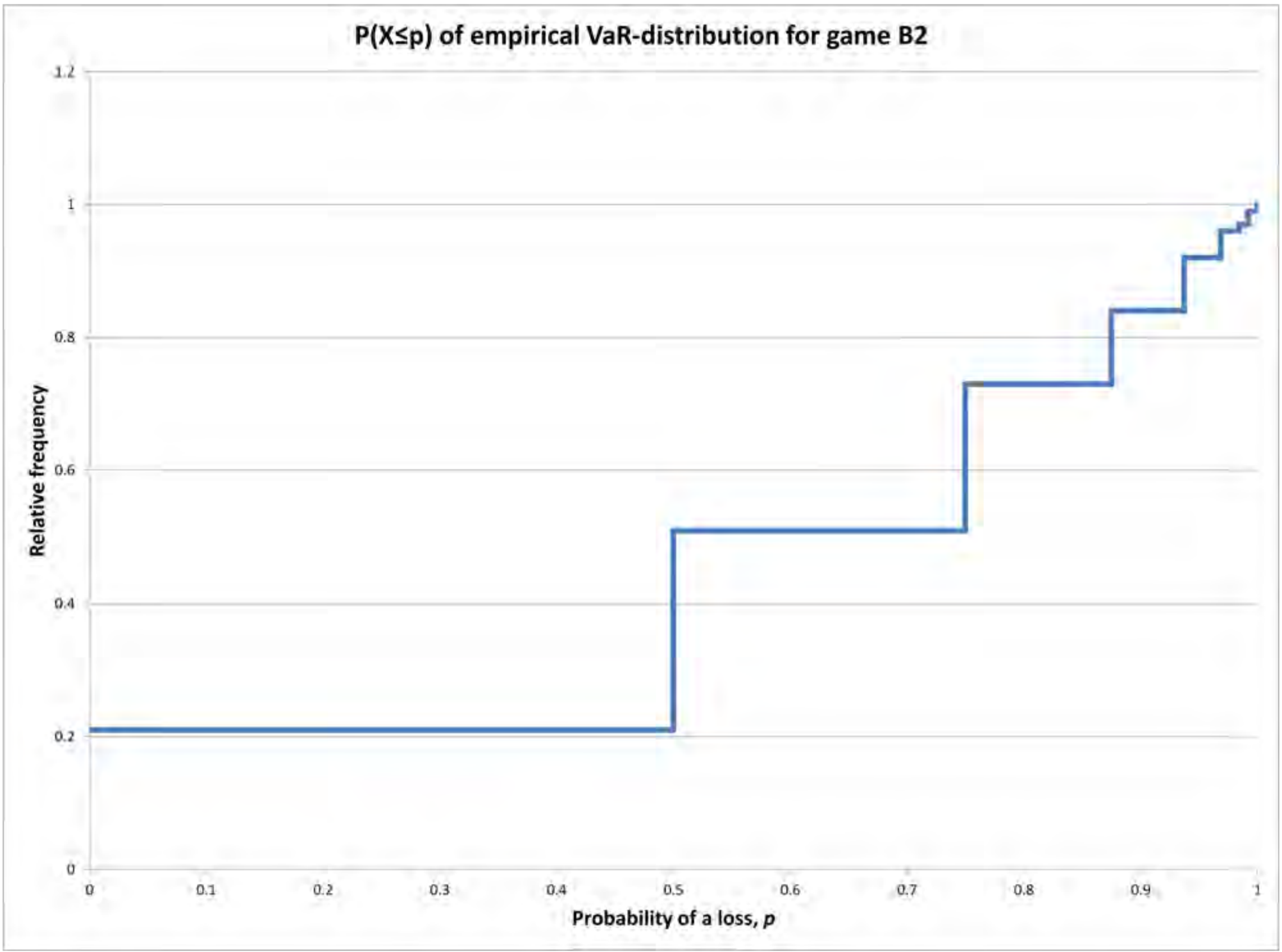

Figure 19: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the basic game in the second survey (B2).

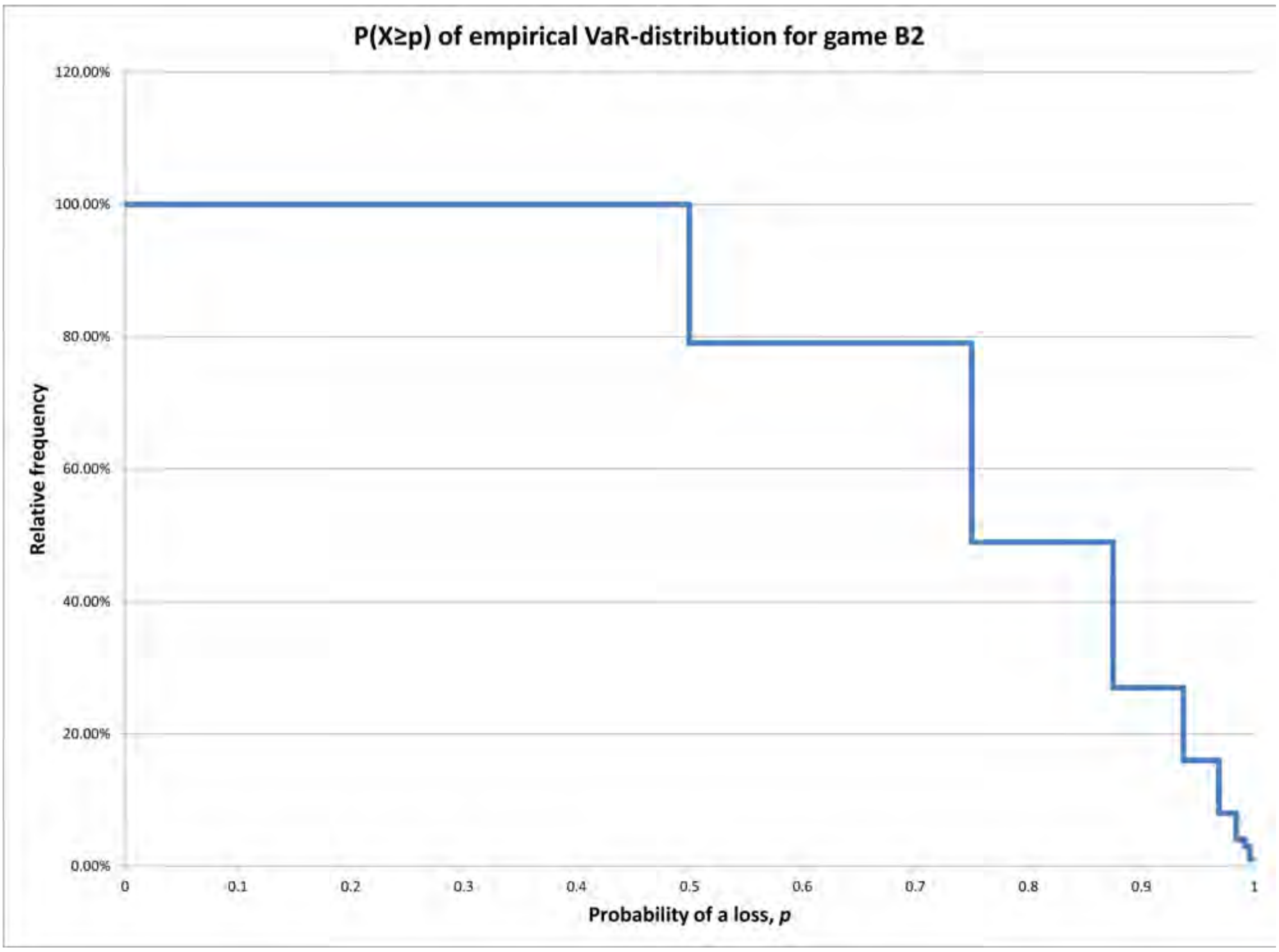

Figure 20: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the basic game in the second survey (B2).

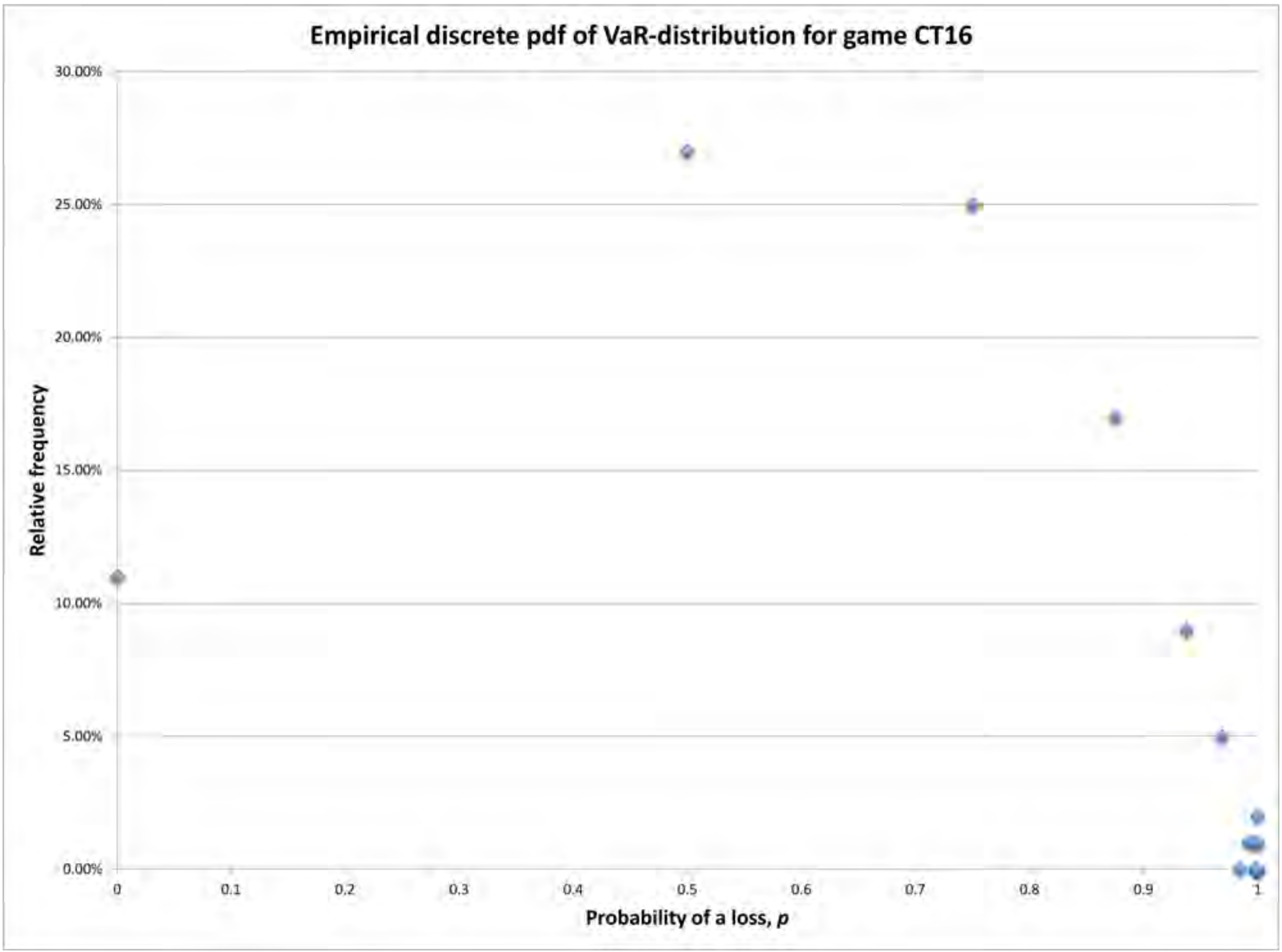

Figure 21: Relative frequency of each bin b_p with p probability of loss, for the game cut off at 32768 (CT16)

Empirical distribution of the basic game in the first survey (B1)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	17	17%	17%	100%
$b_{0.5}$	$1 - 2^-$	0.5	40	40%	57%	83%
$b_{0.75}$	$2 - 4^-$	0.75	26	26%	83%	43%
$b_{0.875}$	$4 - 8^-$	0.875	7	7%	90%	17%
$b_{0.9375}$	$8 - 16^-$	0.9375	5	5%	95%	10%
$b_{0.96875}$	$16 - 32^-$	0.96875	1	1%	96%	5%
$b_{0.984375}$	$32 - 64^-$	0.984375	0	0%	96%	4%
$b_{0.9921875}$	$64 - 128^-$	0.9921875	1	1%	97%	4%
...	...	...	...	...	...	...

Table 18: Value at Risk distribution for the basic game of the first group (B1), 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the basic game in the second survey (B2)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	21	21%	21%	100%
$b_{0.5}$	$1 - 2^-$	0.5	30	30%	51%	79%
$b_{0.75}$	$2 - 4^-$	0.75	22	22%	73%	49%
$b_{0.875}$	$4 - 8^-$	0.875	11	11%	84%	27%
$b_{0.9375}$	$8 - 16^-$	0.9375	8	8%	92%	16%
$b_{0.96875}$	$16 - 32^-$	0.96875	4	4%	96%	8%
$b_{0.984375}$	$32 - 64^-$	0.984375	1	1%	97%	4%
$b_{0.9921875}$	$64 - 128^-$	0.9921875	2	2%	99%	3%
...	...	...	...	...	...	...

Table 19: Value at Risk distribution for the basic game of the second group (B2), 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game cutoff after 16 flips, max return $32768 (CT16)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	11	11%	11%	100%
$b_{0.5}$	$1 - 2^-$	0.5	27	27%	38%	89%
$b_{0.75}$	$2 - 4^-$	0.75	25	25%	63%	62%
$b_{0.875}$	$4 - 8^-$	0.875	17	17%	80%	37%
$b_{0.9375}$	$8 - 16^-$	0.9375	9	9%	89%	20%
$b_{0.96875}$	$16 - 32^-$	0.96875	5	5%	94%	11%
$b_{0.984375}$	$32 - 64^-$	0.984375	0	0%	94%	6%
$b_{0.9921875}$	$64 - 128^-$	0.9921875	1	1%	95%	6%
...	...	...	...	...	...	...

Table 20: Value at Risk distribution for the game cut off after 16 flips, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game cutoff after 11 flips, max return $1024 (CT11)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	14	14%	14%	100%
$b_{0.5}$	$1 - 2^-$	0.5	31	31%	45%	86%
$b_{0.75}$	$2 - 4^-$	0.75	28	28%	73%	55%
$b_{0.875}$	$4 - 8^-$	0.875	12	12%	85%	27%
$b_{0.9375}$	$8 - 16^-$	0.9375	6	6%	91%	15%
$b_{0.96875}$	$16 - 32^-$	0.96875	2	2%	93%	9%
$b_{0.984375}$	$32 - 64^-$	0.984375	1	1%	94%	7%
$b_{0.9921875}$	$64 - 128^-$	0.9921875	1	1%	95%	6%
...	...	...	...	...	...	...

Table 21: Value at Risk distribution for the game cut off after 11 flips, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game cutoff after 9 flips, max return $256 (CT9)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	13	13%	13%	100%
$b_{0.5}$	$1 - 2^-$	0.5	33	33%	46%	87%
$b_{0.75}$	$2 - 4^-$	0.75	25	25%	71%	54%
$b_{0.875}$	$4 - 8^-$	0.875	16	16%	87%	29%
$b_{0.9375}$	$8 - 16^-$	0.9375	6	6%	93%	13%
$b_{0.96875}$	$16 - 32^-$	0.96875	3	3%	96%	7%
$b_{0.984375}$	$32 - 64^-$	0.984375	0	0%	96%	4%
$b_{0.9921875}$	$64 - 128^-$	0.9921875	1	1%	97%	4%
...	...	...	...	...	...	...

Table 22: Value at Risk distribution for the game cut off after 9 flips, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game cutoff after 6 flips, max return $32 (CT6)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	15	15%	15%	100%
$b_{0.5}$	$1 - 2^-$	0.5	36	36%	51%	85%
$b_{0.75}$	$2 - 4^-$	0.75	38	38%	89%	49%
$b_{0.875}$	$4 - 8^-$	0.875	4	4%	93%	11%
$b_{0.9375}$	$8 - 16^-$	0.9375	4	4%	97%	7%
$b_{0.96875}$	$16 - 32^-$	0.96875	1	1%	98%	3%
$b_{0.984375}$	$32 - \infty$	1	2	2%	100%	2%

Table 23: Value at Risk distribution for the game cut off after 6 flips, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game starting at $0.01 (S0.01)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 0.01^-$	0	32	32%	32%	100%
$b_{0.5}$	$0.01 - 0.02^-$	0.5	18	18%	50%	68%
$b_{0.75}$	$0.02 - 0.04^-$	0.75	16	16%	66%	50%
$b_{0.875}$	$0.04 - 0.08^-$	0.875	11	11%	77%	34%
$b_{0.9375}$	$0.08 - 0.16^-$	0.9375	3	3%	80%	23%
$b_{0.96875}$	$0.16 - 0.32^-$	0.96875	3	3%	83%	20%
$b_{0.984375}$	$0.32 - 0.64^-$	0.984375	1	1%	84%	17%
$b_{0.9921875}$	$0.64 - 1.28^-$	0.9921875	7	7%	91%	16%
...	...	...	...	...	...	...

Table 24: Value at Risk distribution for the game starting at $0.01, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game starting at $0.5 (S0.5)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 0.5^-$	0	22	22%	22%	100%
$b_{0.5}$	$0.5 - 1^-$	0.5	26	26%	48%	78%
$b_{0.75}$	$1 - 2^-$	0.75	30	30%	78%	52%
$b_{0.875}$	$2 - 4^-$	0.875	9	9%	87%	22%
$b_{0.9375}$	$4 - 8^-$	0.9375	5	5%	92%	13%
$b_{0.96875}$	$8 - 16^-$	0.96875	4	4%	96%	8%
$b_{0.984375}$	$16 - 32^-$	0.984375	0	0%	96%	4%
$b_{0.9921875}$	$32 - 64^-$	0.9921875	1	1%	97%	4%
...	...	...	...	...	...	...

Table 25: Value at Risk distribution for the game starting at $0.5, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game starting at $0.99 (S0.99)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 0.99^-$	0	24	24%	24%	100%
$b_{0.5}$	$0.99 - 1.98^-$	0.5	30	30%	54%	76%
$b_{0.75}$	$1.98 - 3.96^-$	0.75	29	29%	83%	46%
$b_{0.875}$	$3.96 - 7.92^-$	0.875	3	3%	86%	17%
$b_{0.9375}$	$7.92 - 15.84^-$	0.9375	7	7%	93%	14%
$b_{0.96875}$	$15.84 - 31.68^-$	0.96875	2	2%	95%	7%
$b_{0.984375}$	$31.68 - 63.36^-$	0.984375	1	1%	96%	5%
$b_{0.9921875}$	$63.36 - 126.72^-$	0.9921875	0	0%	96%	4%
...	...	...	...	...	...	...

Table 26: Value at Risk distribution for the game starting at $0.99, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game starting at \$4 (S4)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 4^-$	0	29	29%	29%	100%
$b_{0.5}$	$4 - 8^-$	0.5	33	33%	62%	71%
$b_{0.75}$	$8 - 16^-$	0.75	23	23%	85%	38%
$b_{0.875}$	$16 - 32^-$	0.875	7	7%	92%	15%
$b_{0.9375}$	$32 - 64^-$	0.9375	2	2%	94%	8%
$b_{0.96875}$	$64 - 128^-$	0.96875	1	1%	95%	6%
$b_{0.984375}$	$128 - 256^-$	0.984375	0	0%	95%	5%
$b_{0.9921875}$	$256 - 512^-$	0.9921875	0	0%	95%	5%
...	...	...	...	...	...	...

Table 27: Value at Risk distribution for the game starting at \$4, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of game increasing with the power of 4 (T4)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 1^-$	0	10	10%	10%	100%
$b_{0.5}$	$1 - 4^-$	0.5	35	35%	45%	90%
$b_{0.75}$	$4 - 16^-$	0.75	36	36%	81%	55%
$b_{0.875}$	$16 - 64^-$	0.875	11	11%	92%	19%
$b_{0.9375}$	$64 - 256^-$	0.9375	2	2%	94%	8%
$b_{0.96875}$	$256 - 1024^-$	0.96875	1	1%	95%	6%
$b_{0.984375}$	$1024 - 4096^-$	0.984375	2	2%	97%	5%
$b_{0.9921875}$	$4096 - 16384^-$	0.9921875	3	3%	100%	3%

Table 28: Value at Risk distribution for the game increasing with the power of 4, 100 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

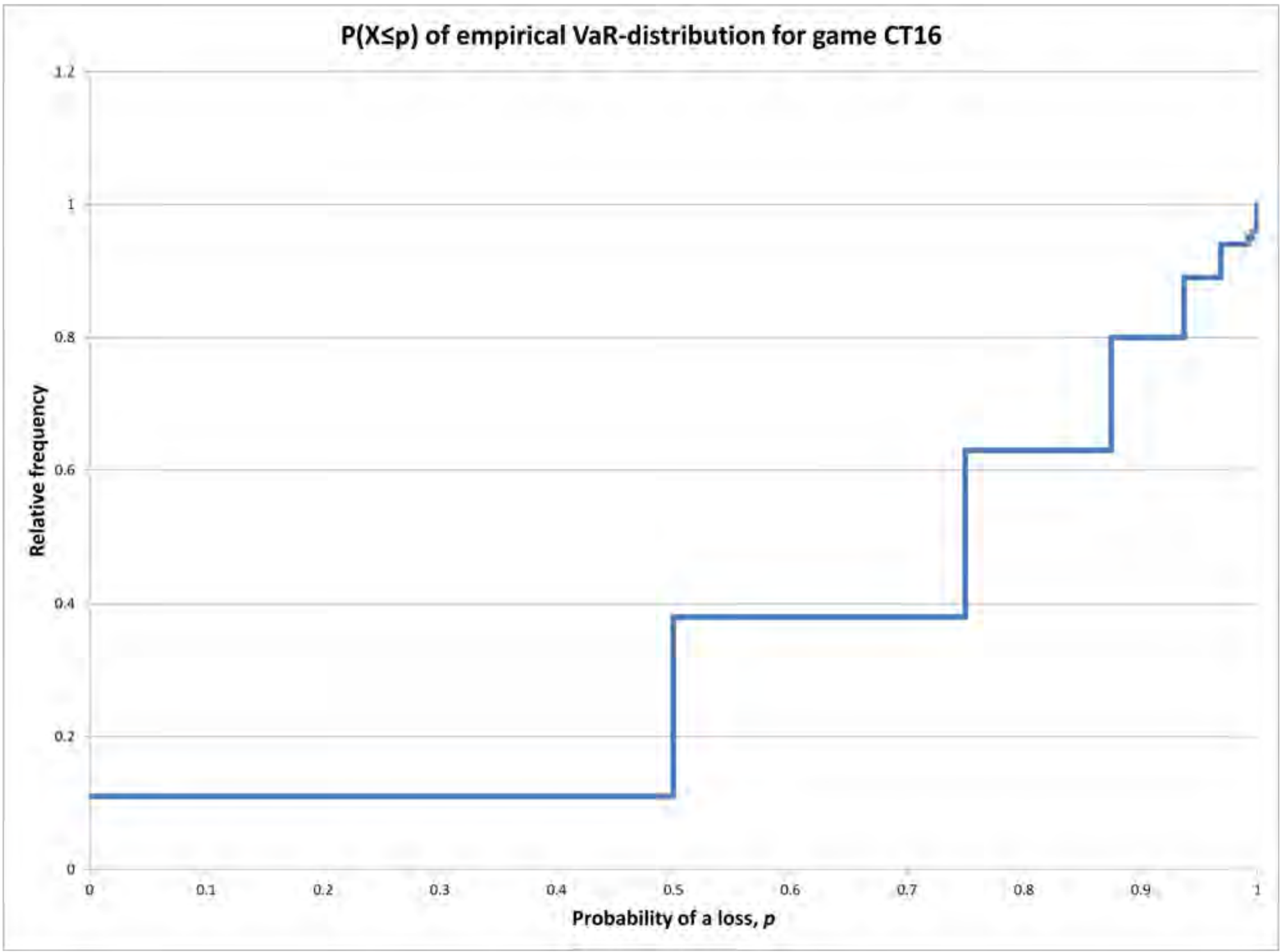

Figure 22: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game cut off at 32768 (CT16).

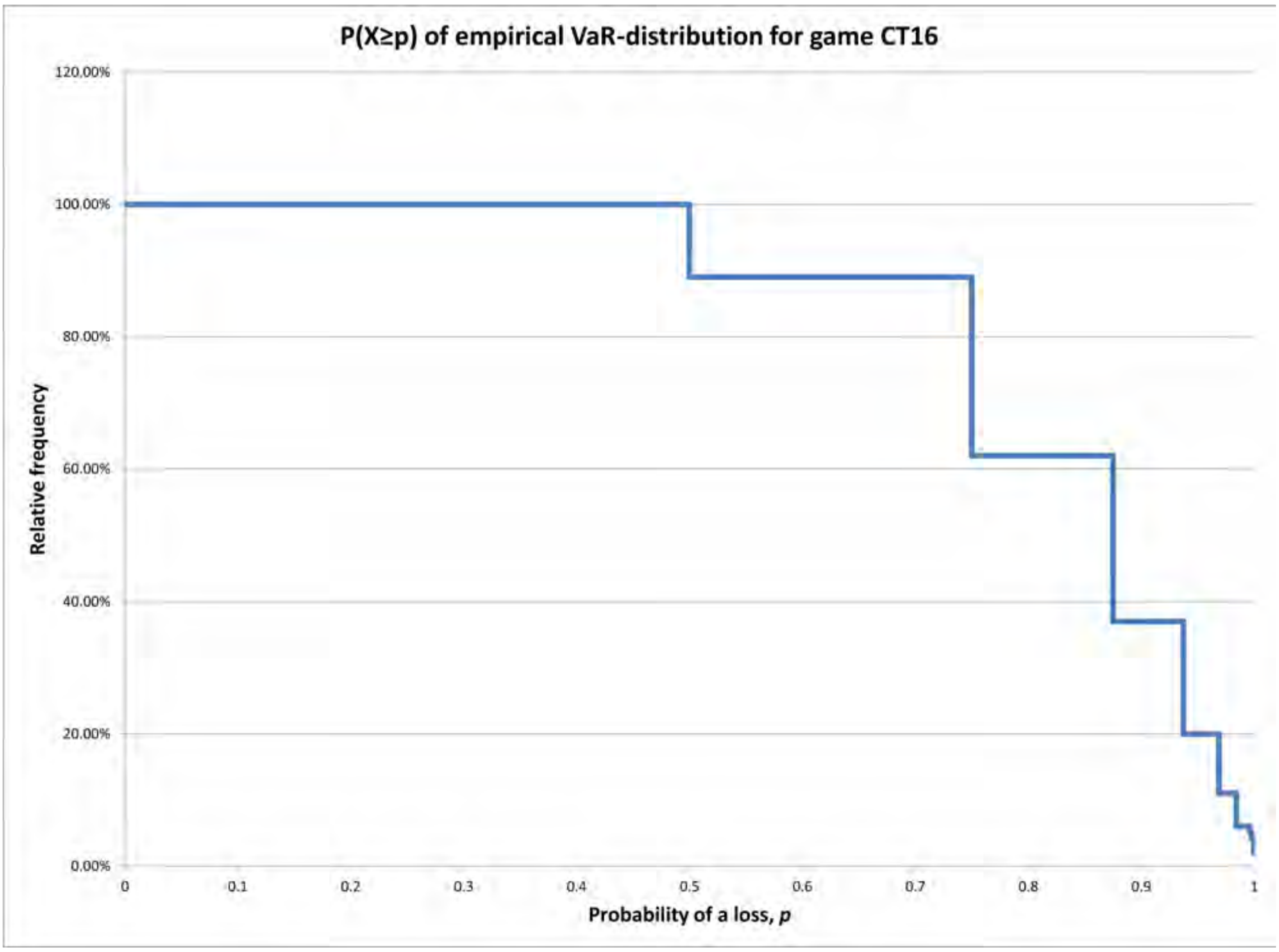

Figure 23: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game cut off at 32768 (CT16).

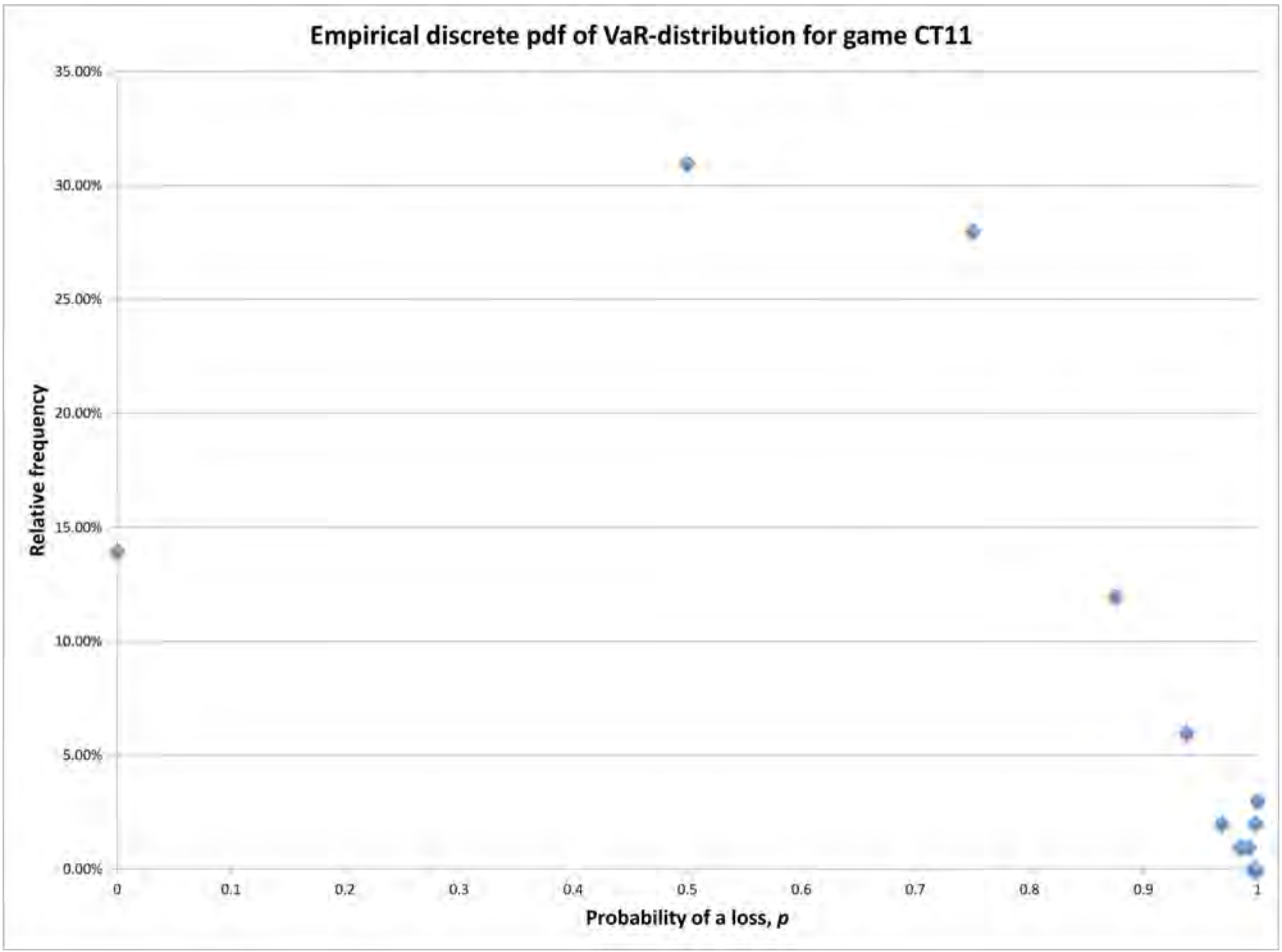

Figure 24: Relative frequency of each bin b_p with p probability of loss, for the game cut off at 1024 (CT11)

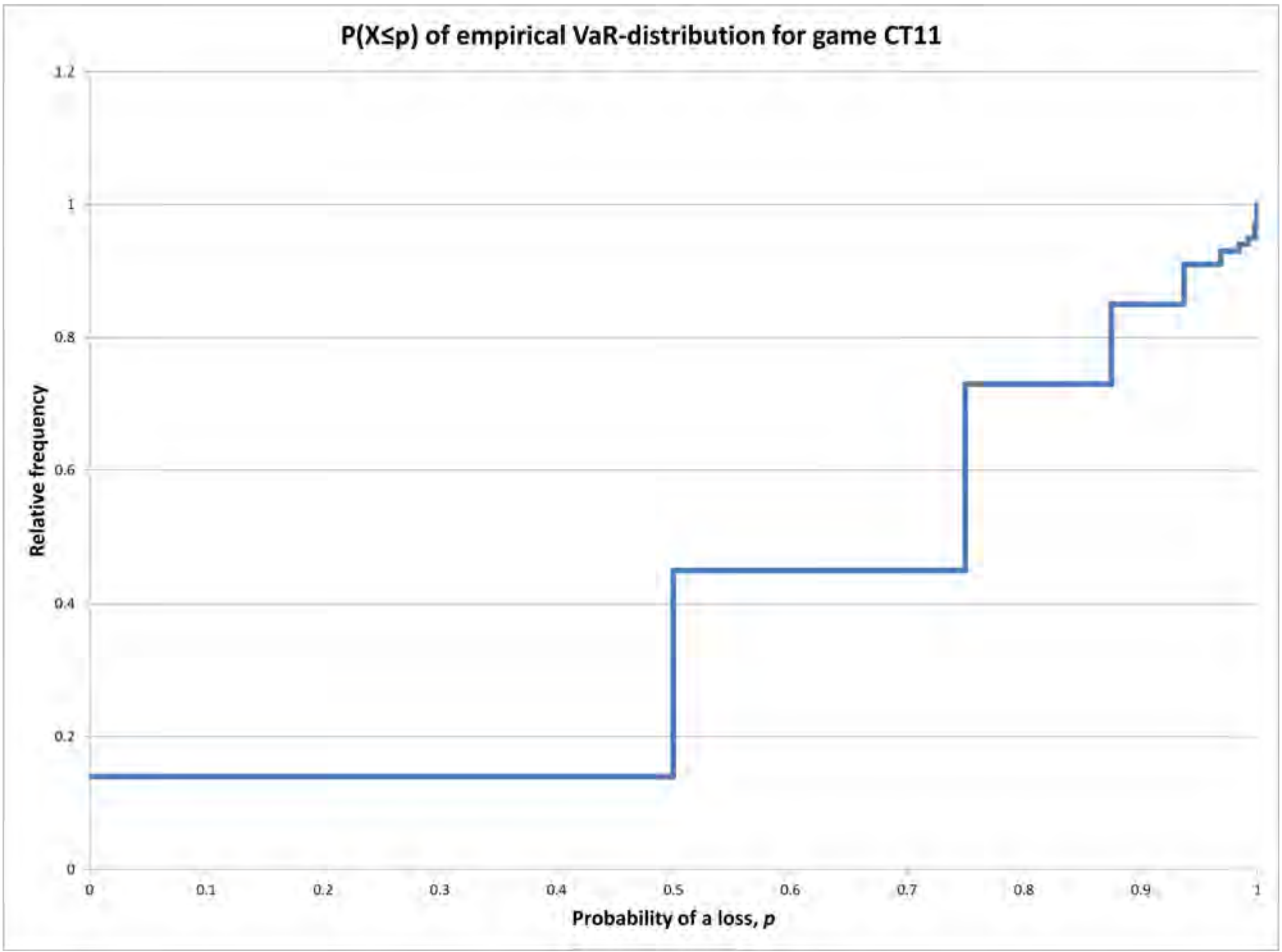

Figure 25: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game cut off at 1024 (CT11).

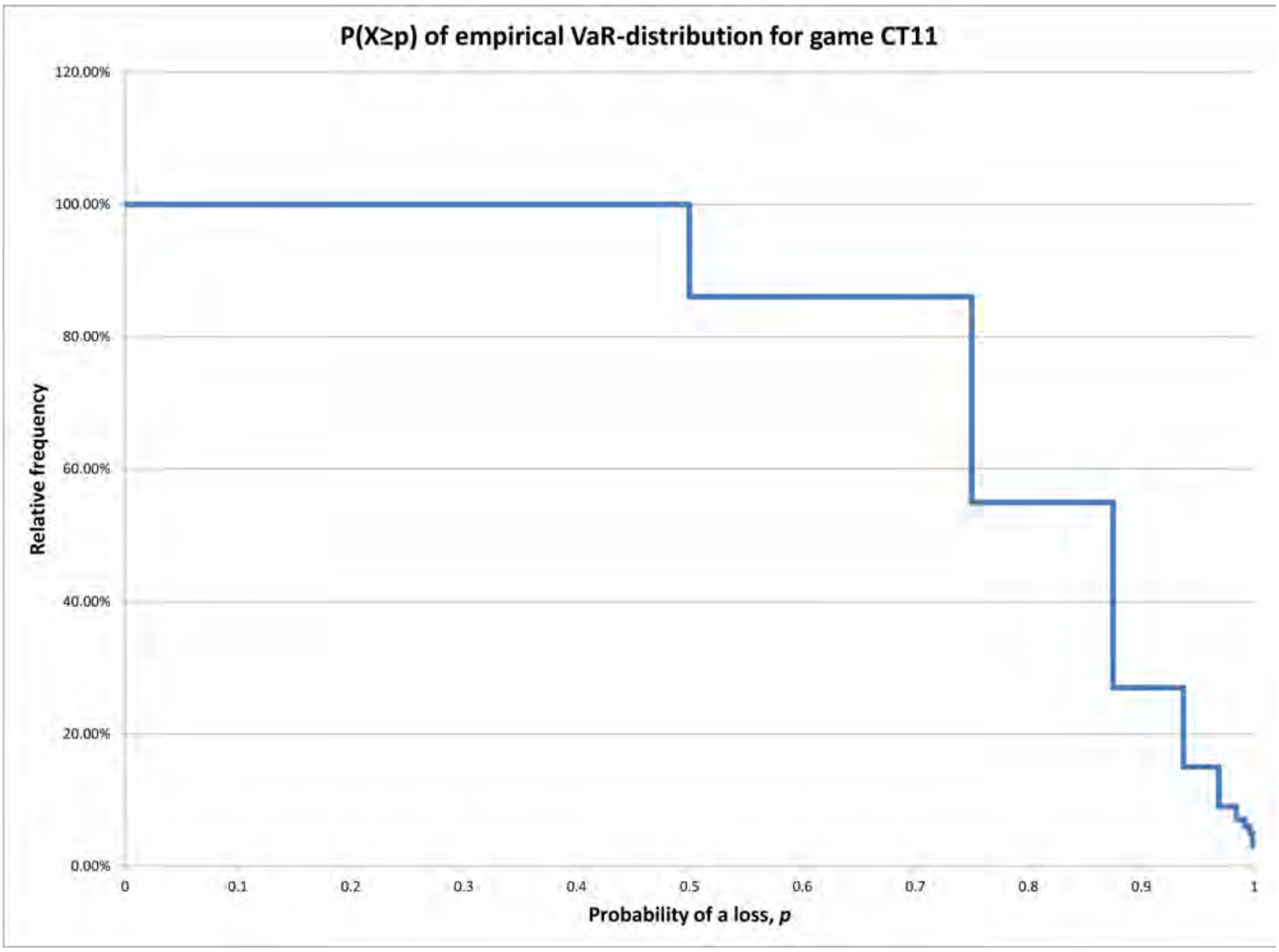

Figure 26: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game cut off at 1024 (CT11).

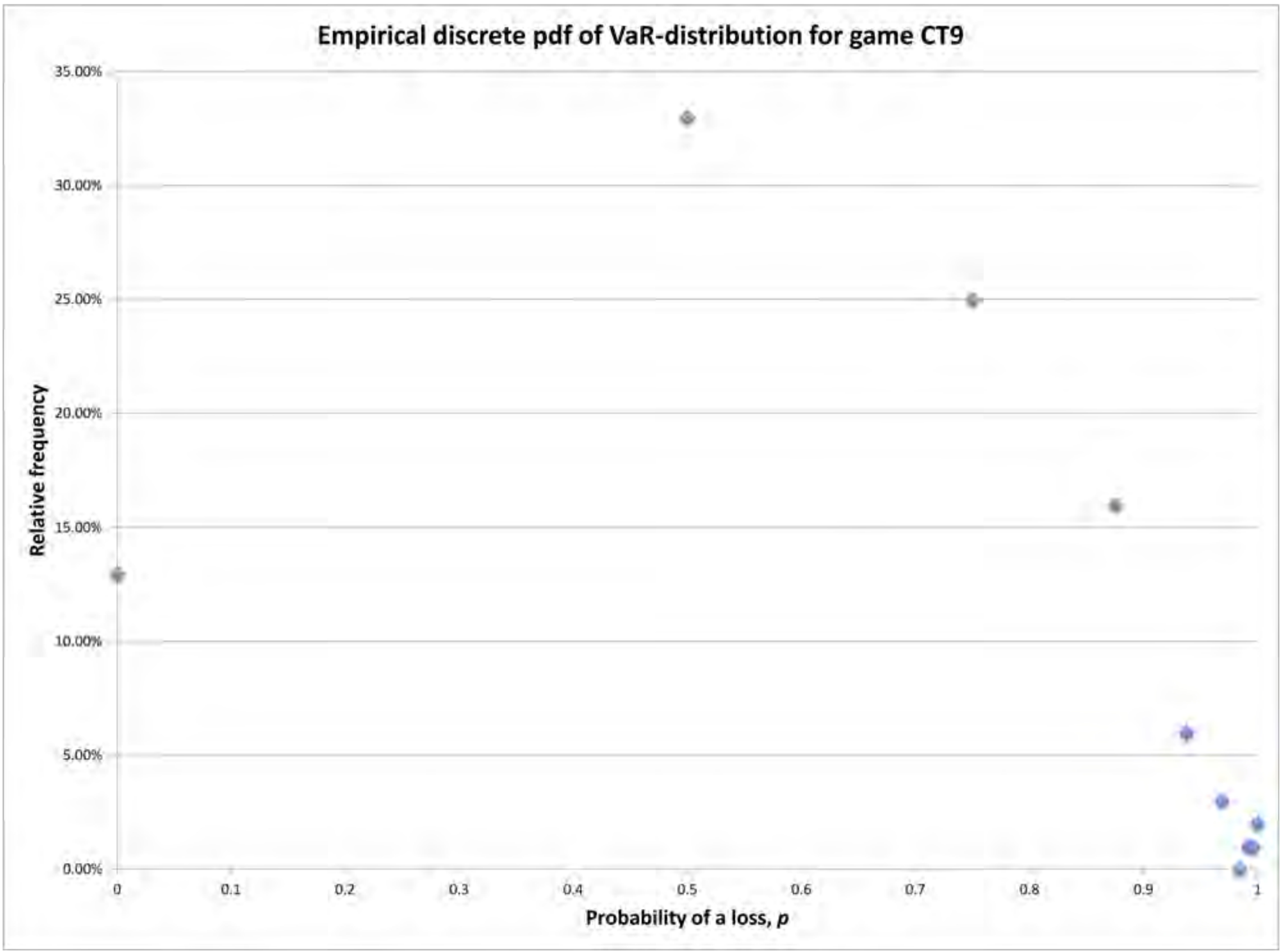

Figure 27: Relative frequency of each bin b_p with p probability of loss, for the game cut off at 256 (CT9)

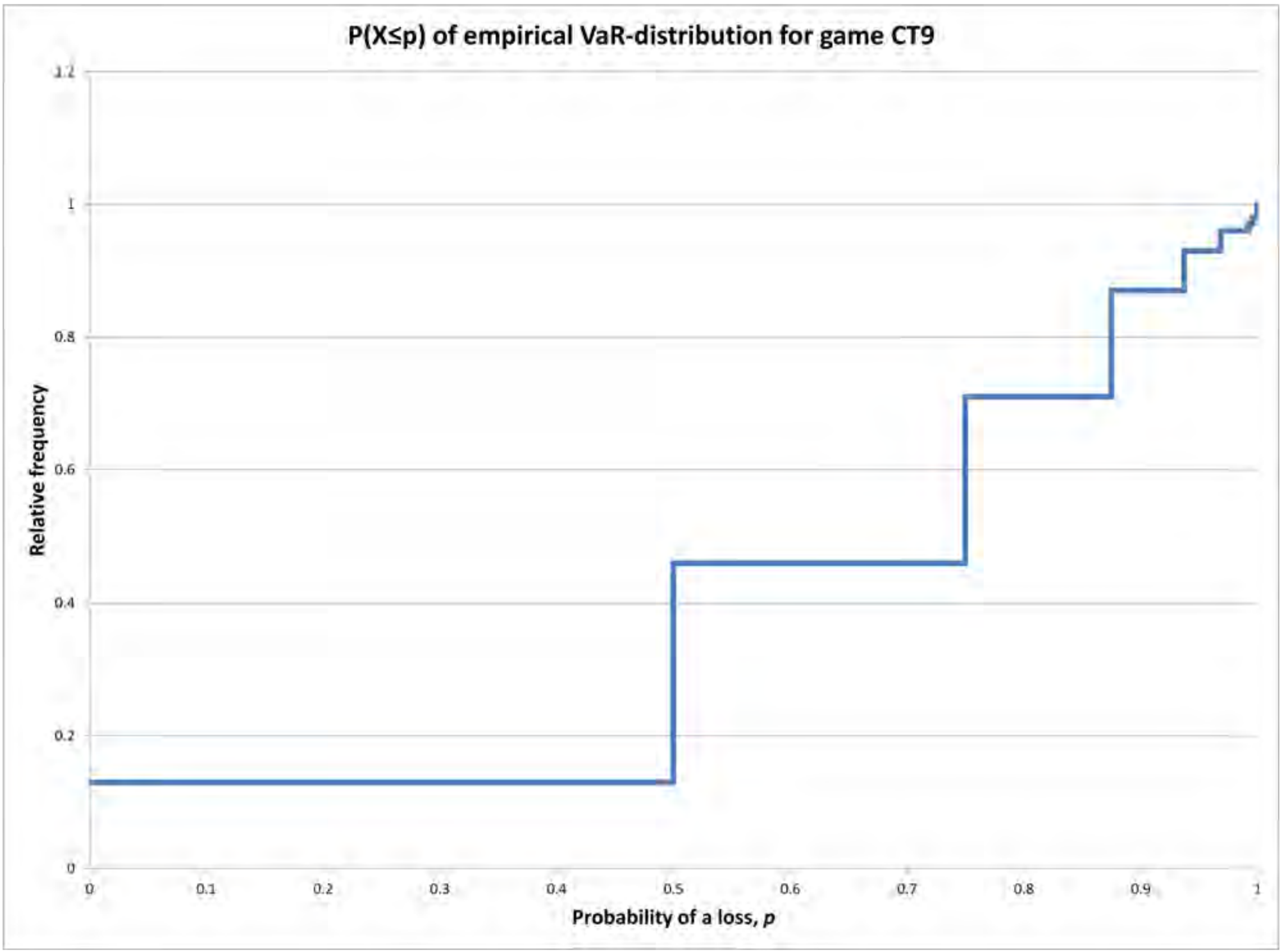

Figure 28: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game cut off at 256 (CT9).

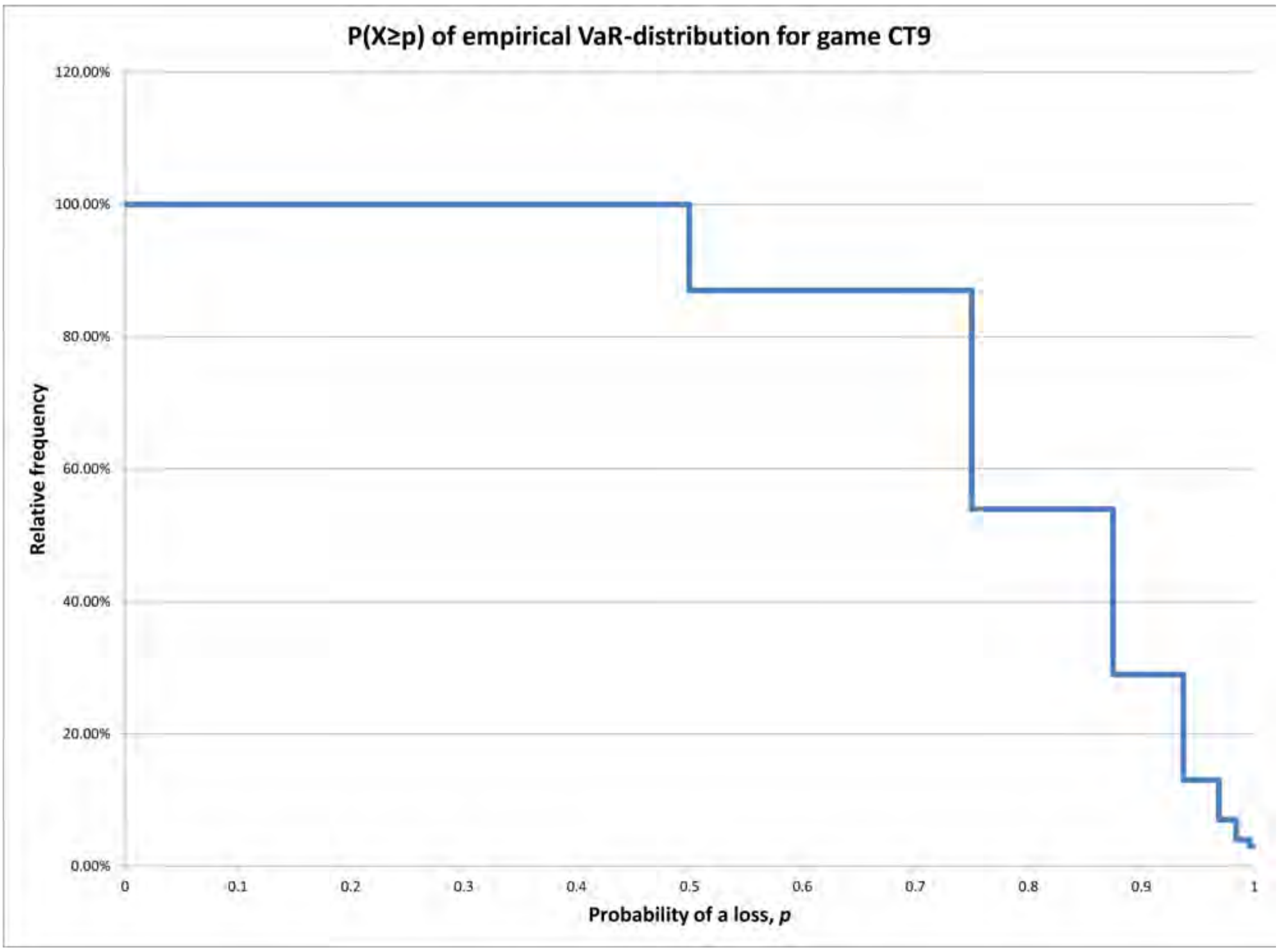

Figure 29: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game cut off at 256 (CT9).

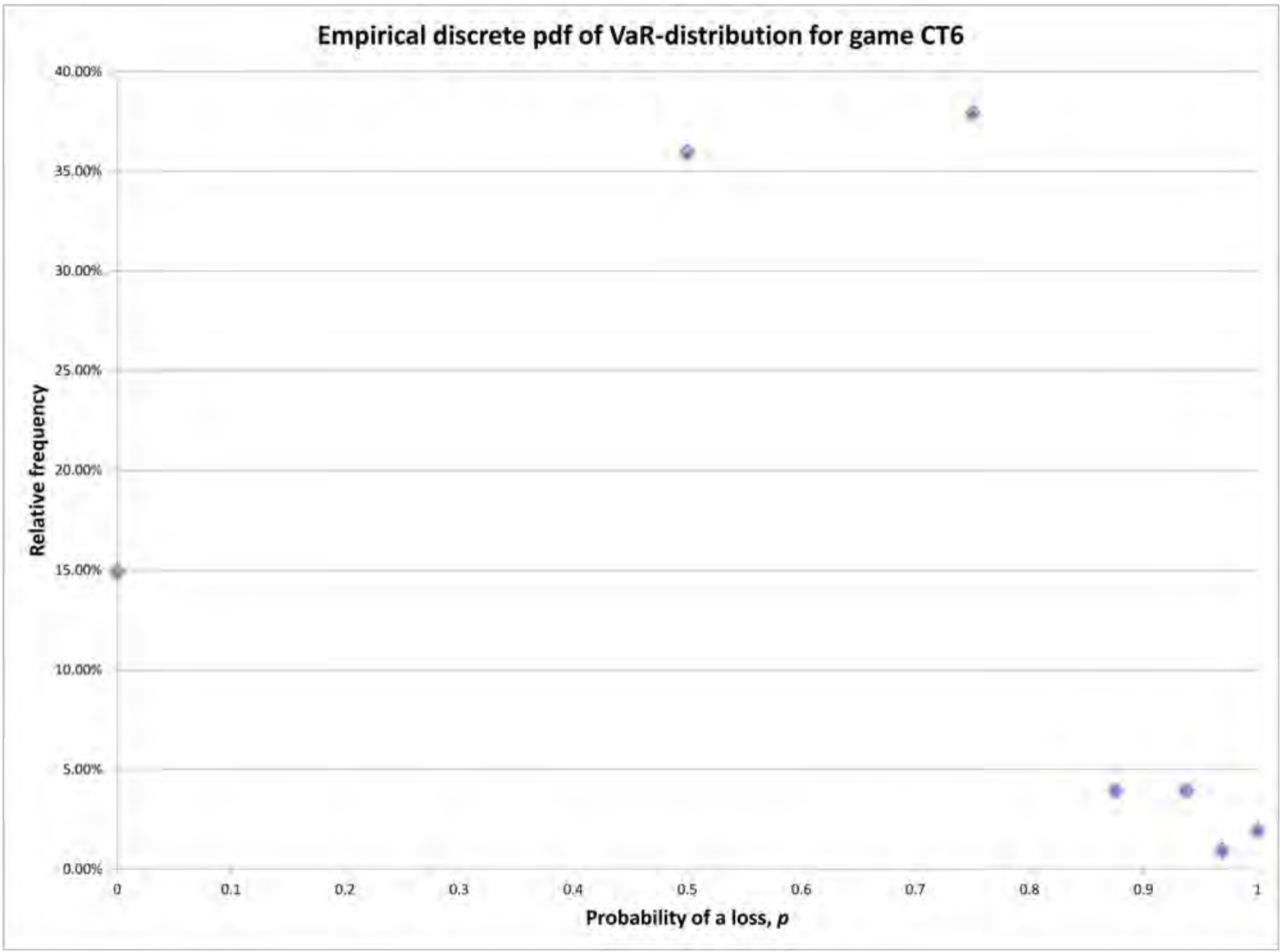

Figure 30: Relative frequency of each bin b_p with p probability of loss, for the game cut off at 32 (CT6)

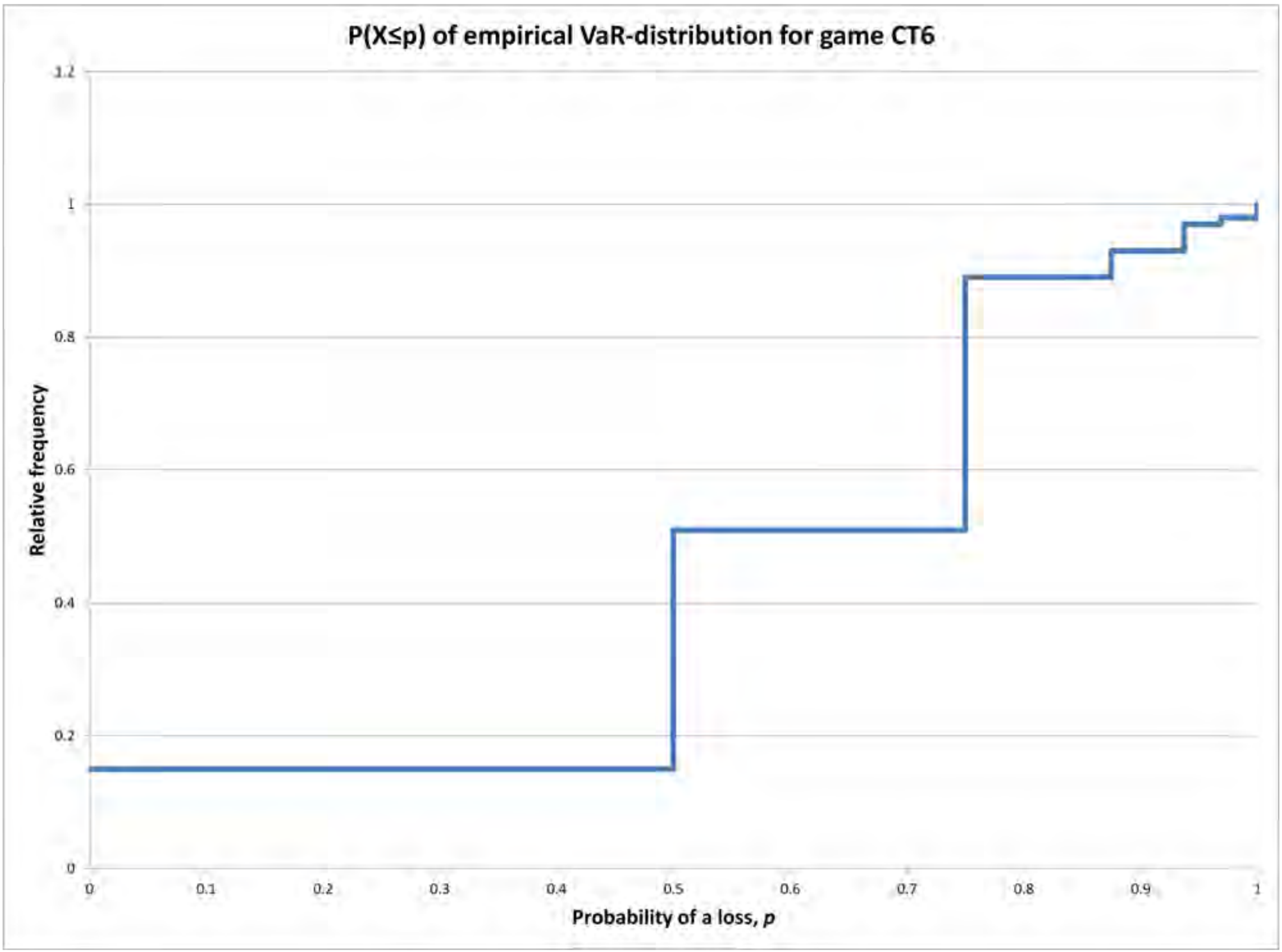

Figure 31: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game cut off at 32 (CT6).

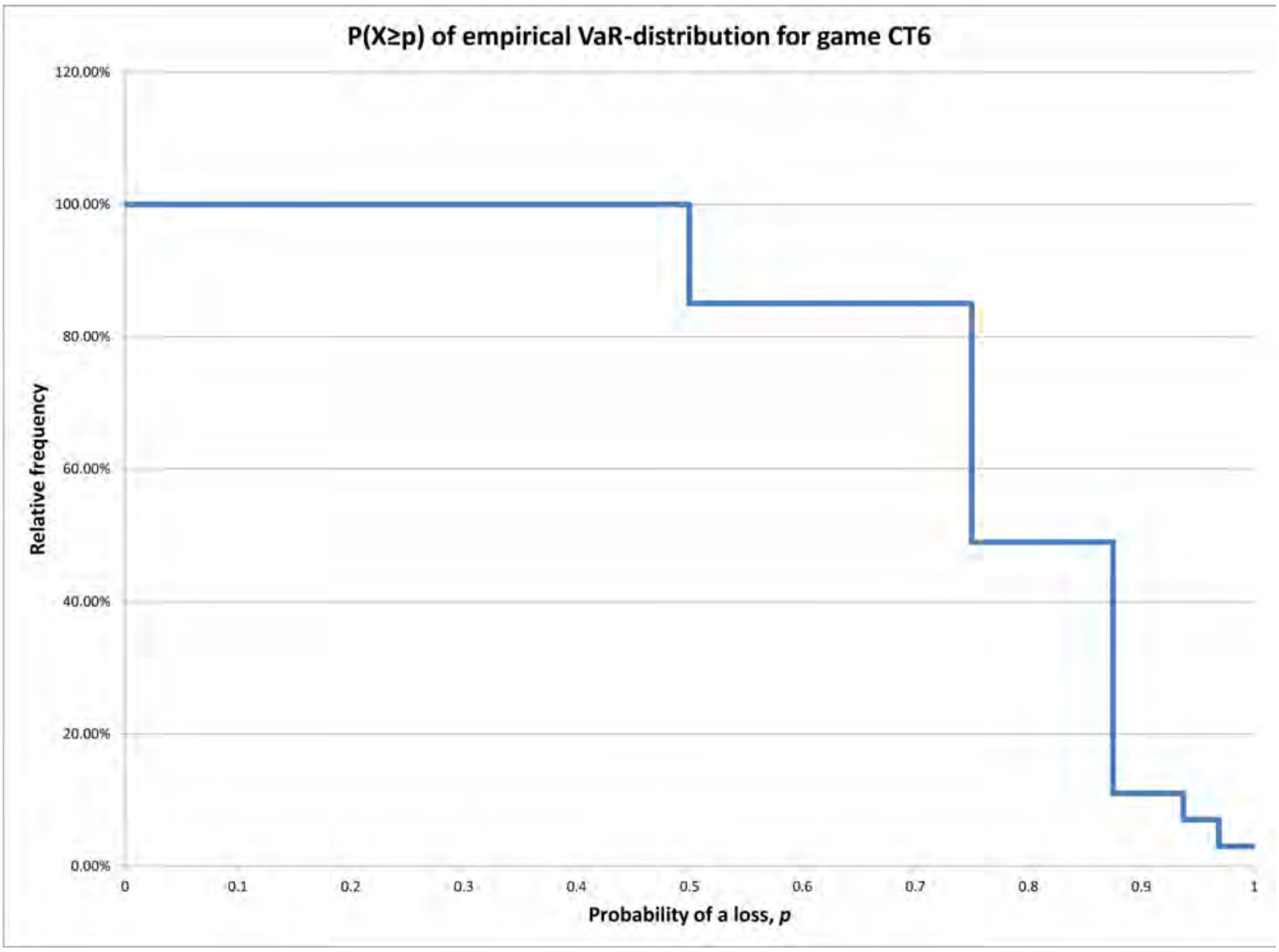

Figure 32: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game cut off at 32 (CT6).

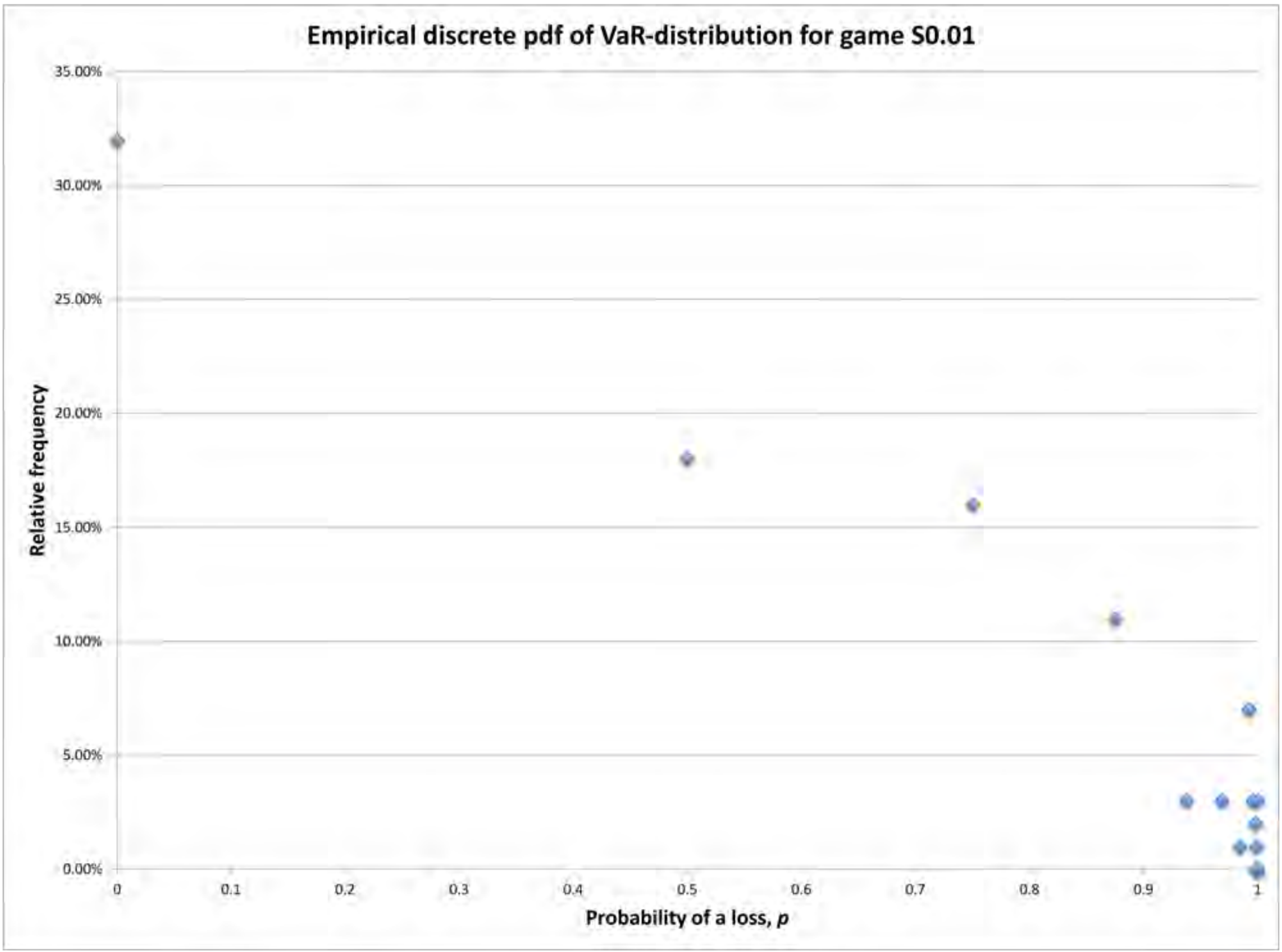

Figure 33: Relative frequency of each bin b_p with p probability of loss, for the game that starts at 0.01 (S0.01)

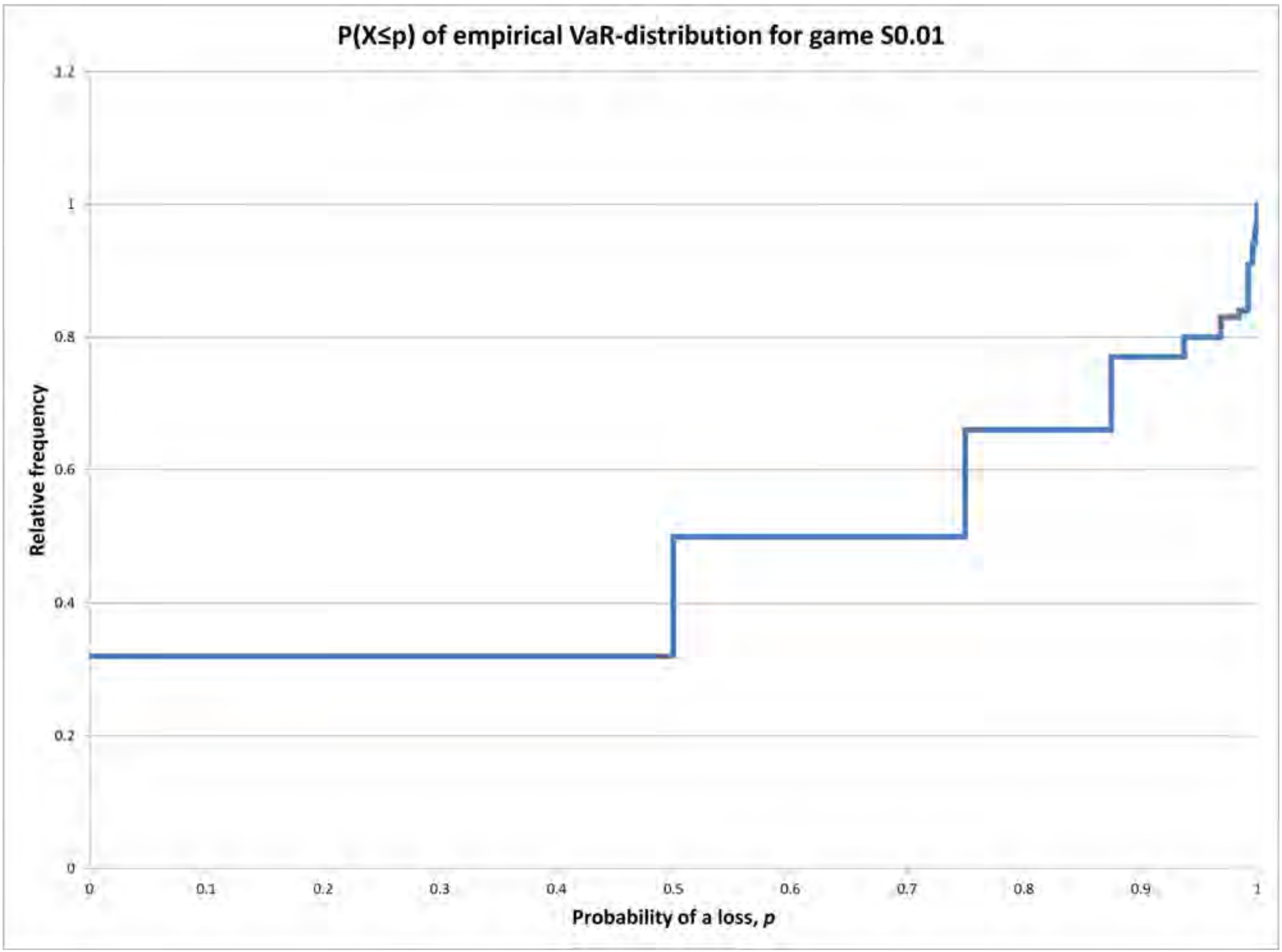

Figure 34: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game that starts at 0.01 (S0.01).

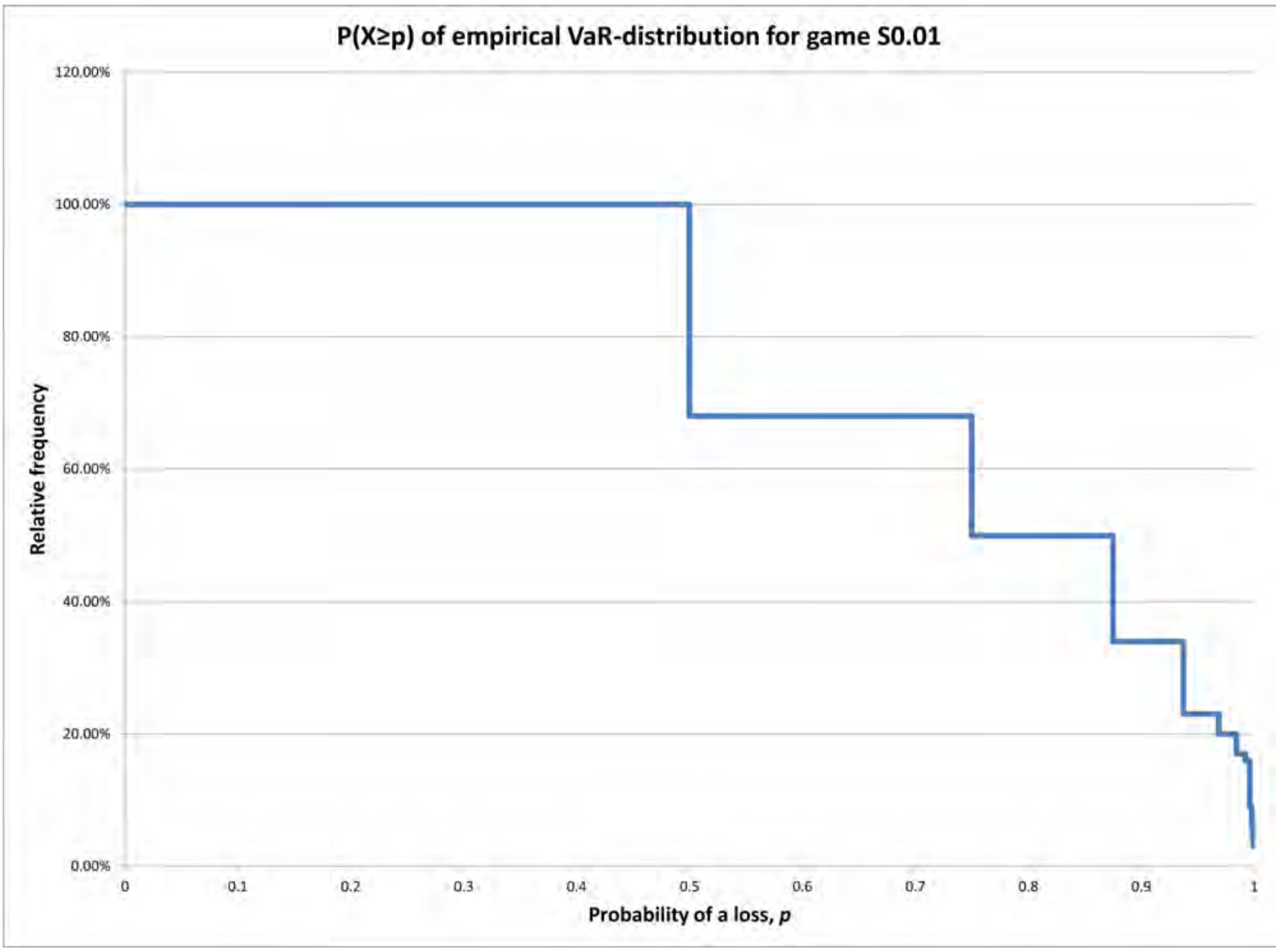

Figure 35: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game that starts at 0.01 (S0.01).

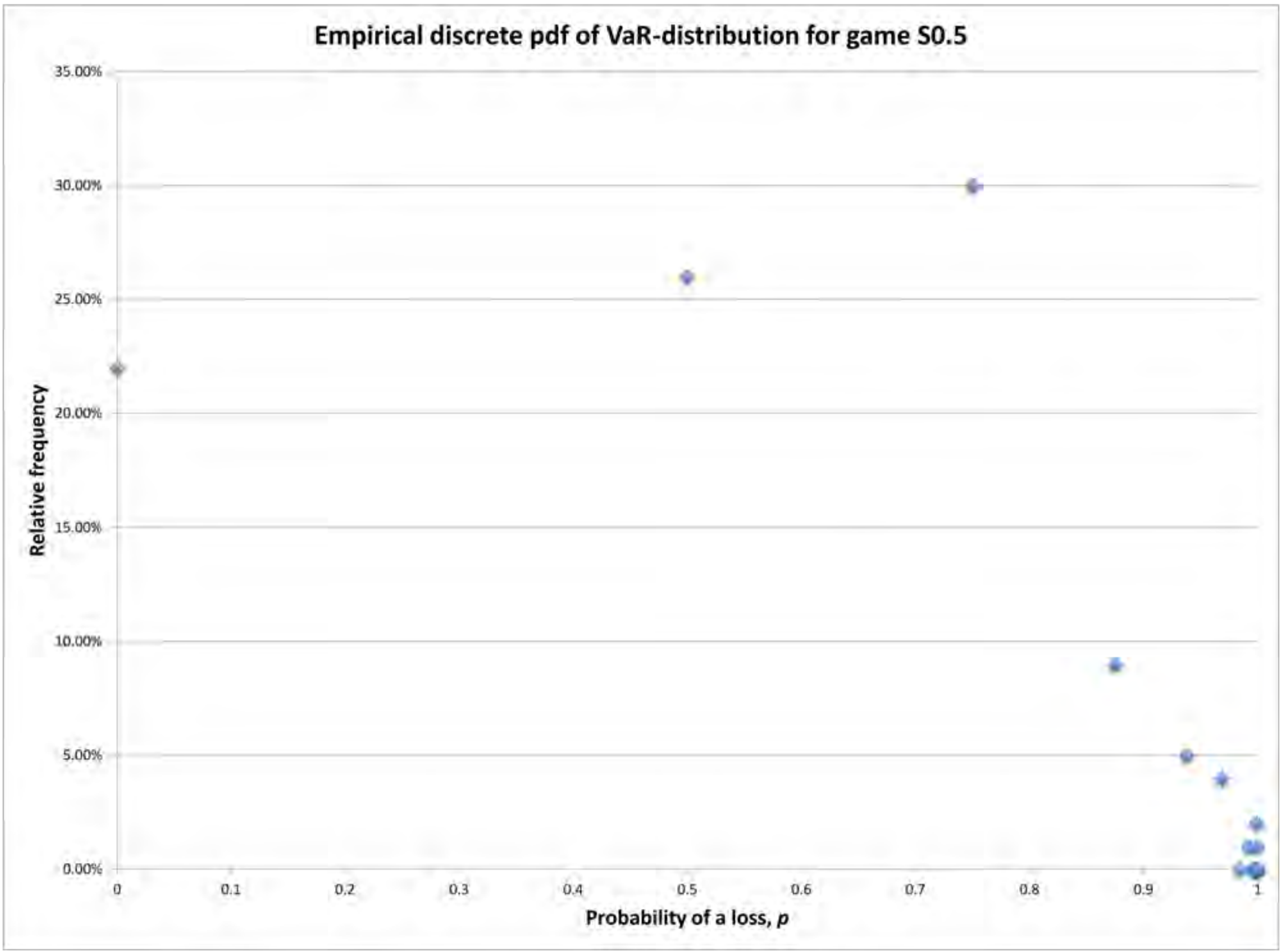

Figure 36: Relative frequency of each bin b_p with p probability of loss, for the game that starts at 0.5 (S0.5)

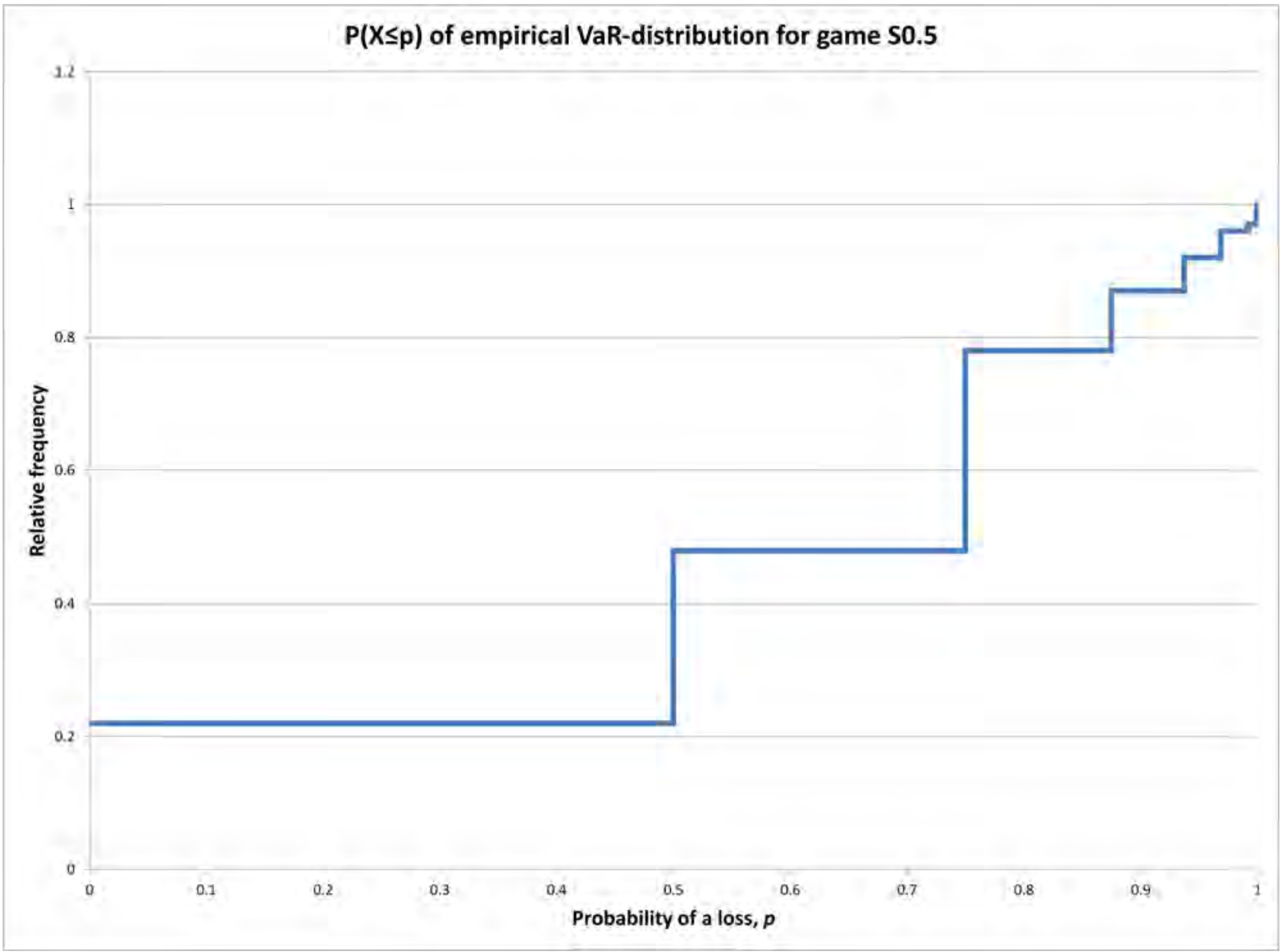

Figure 37: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game that starts at 0.5 (S0.5).

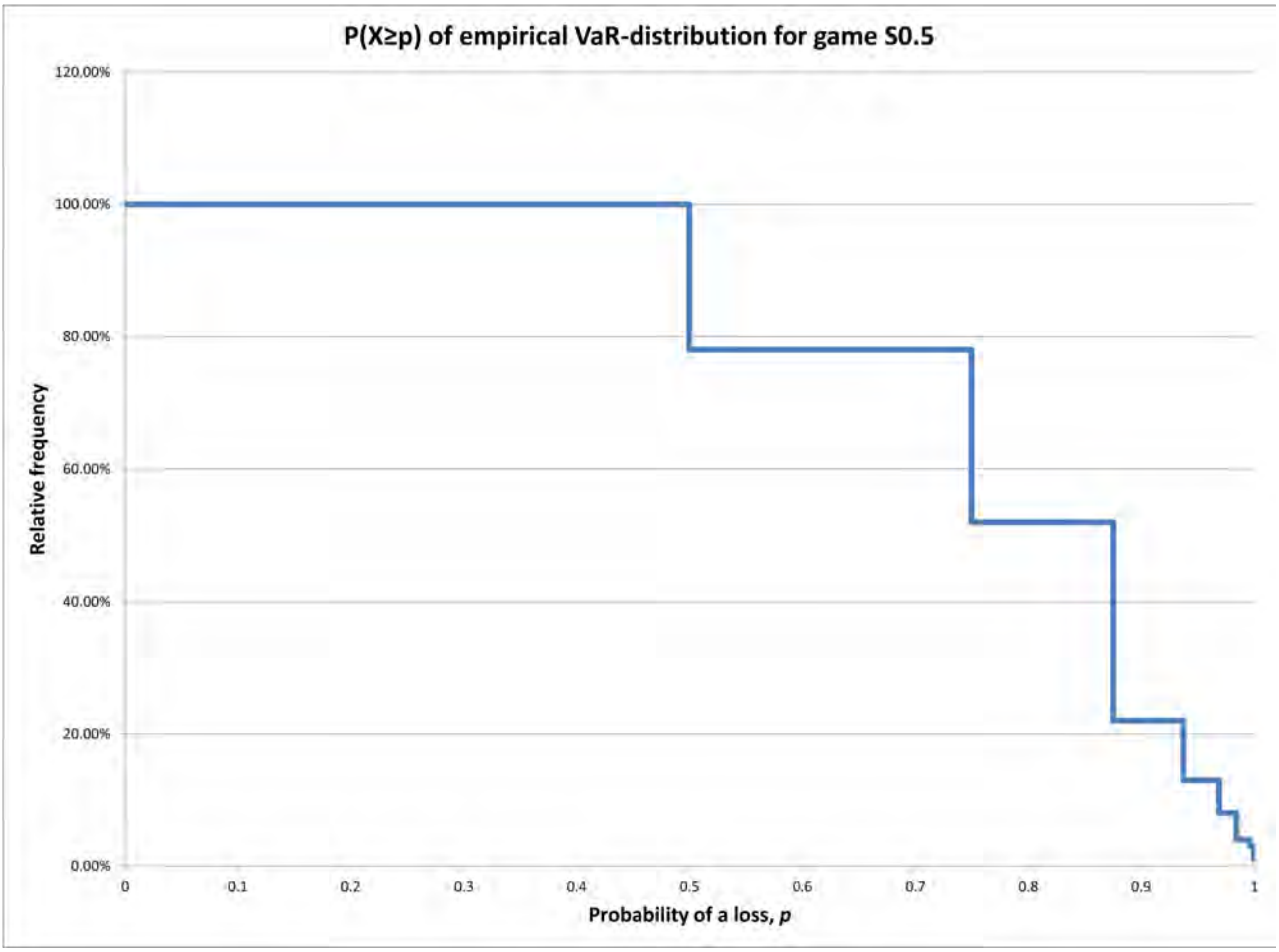

Figure 38: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game that starts at 0.5 (S0.5).

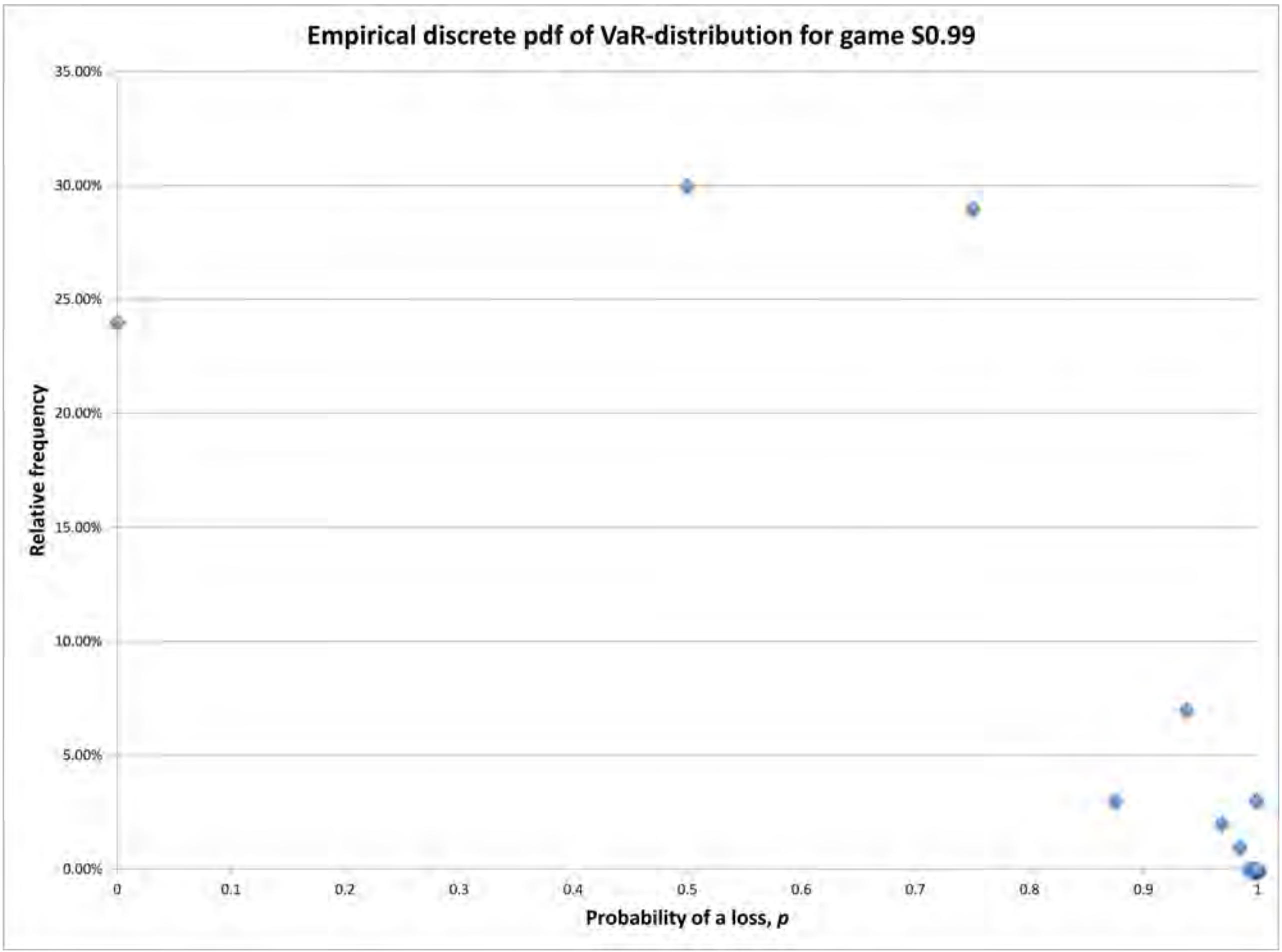

Figure 39: Relative frequency of each bin b_p with p probability of loss, for the game that starts at 0.99 (S0.99)

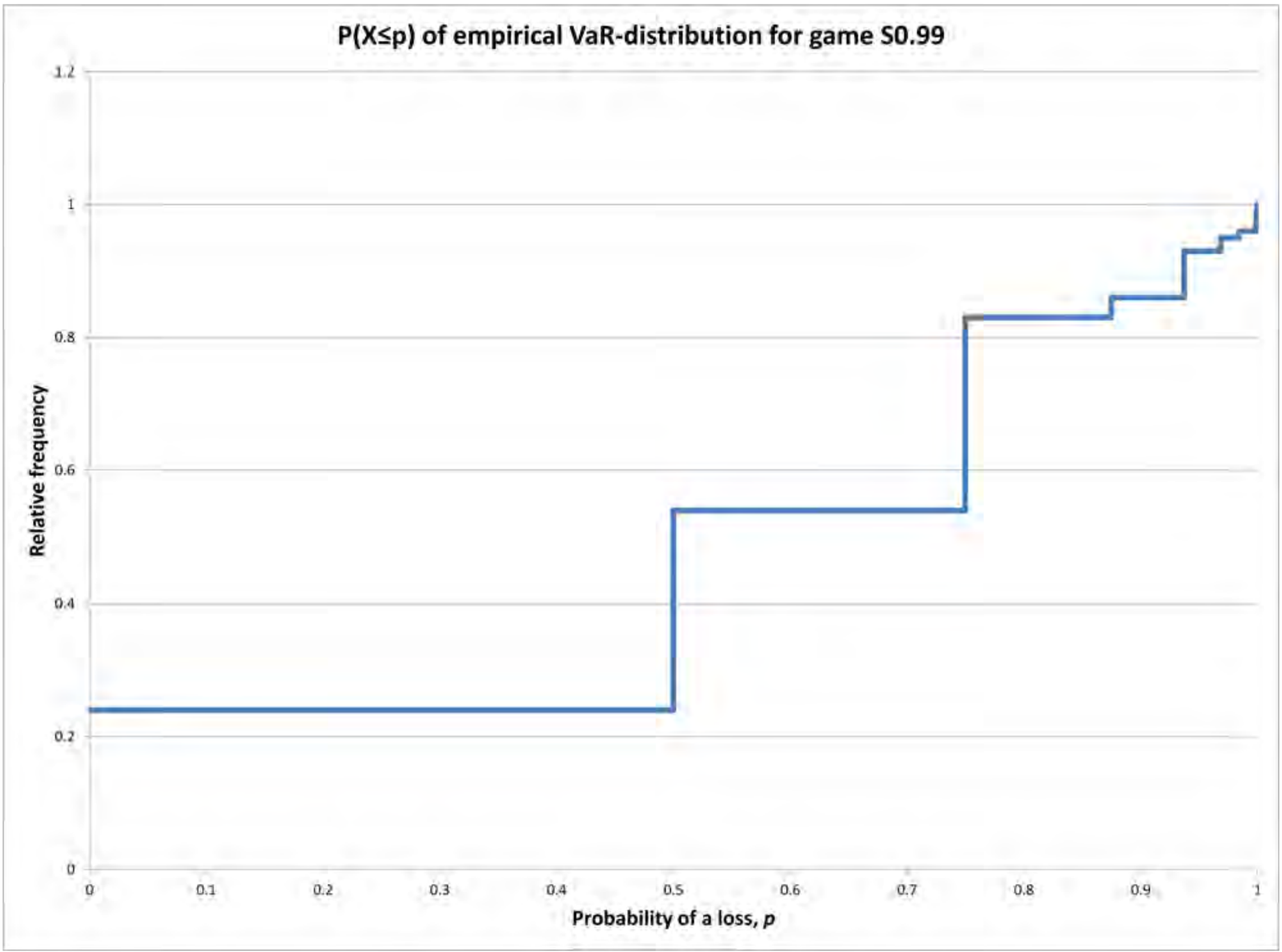

Figure 40: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game that starts at 0.99 (S0.99).

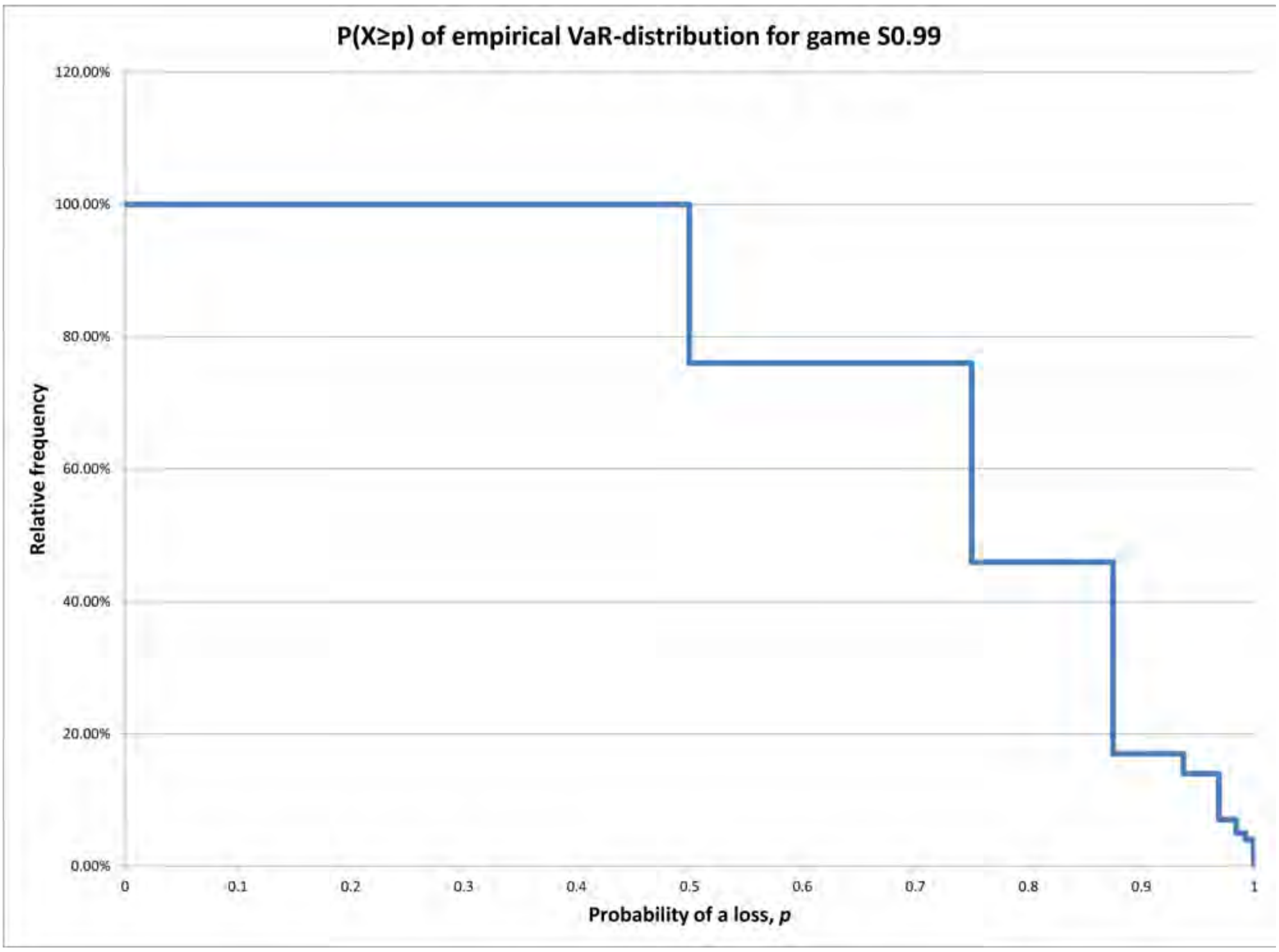

Figure 41: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game that starts at 0.99 (S0.99).

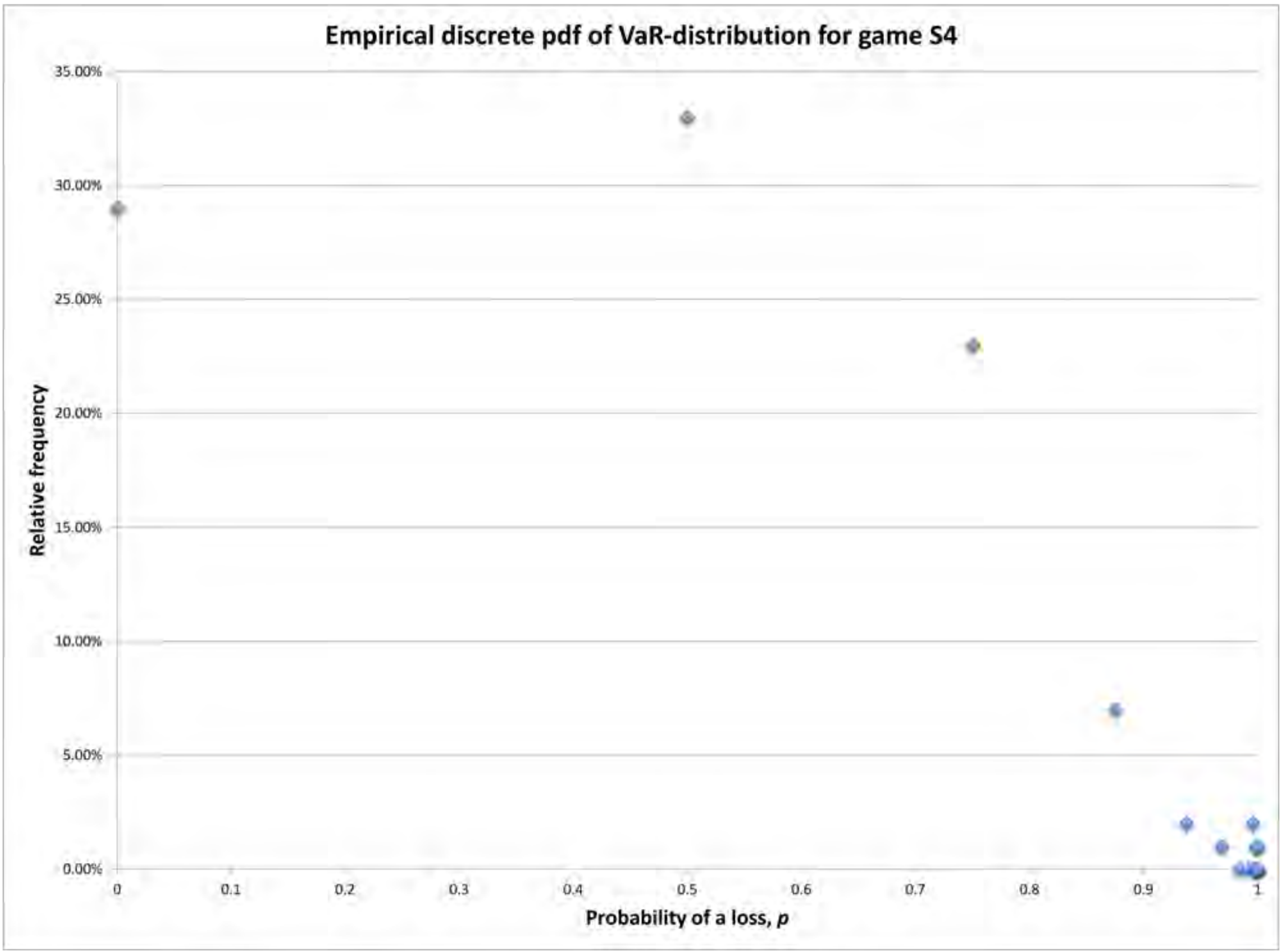

Figure 42: Relative frequency of each bin b_p with p probability of loss, for the game that starts at 4 (S4)

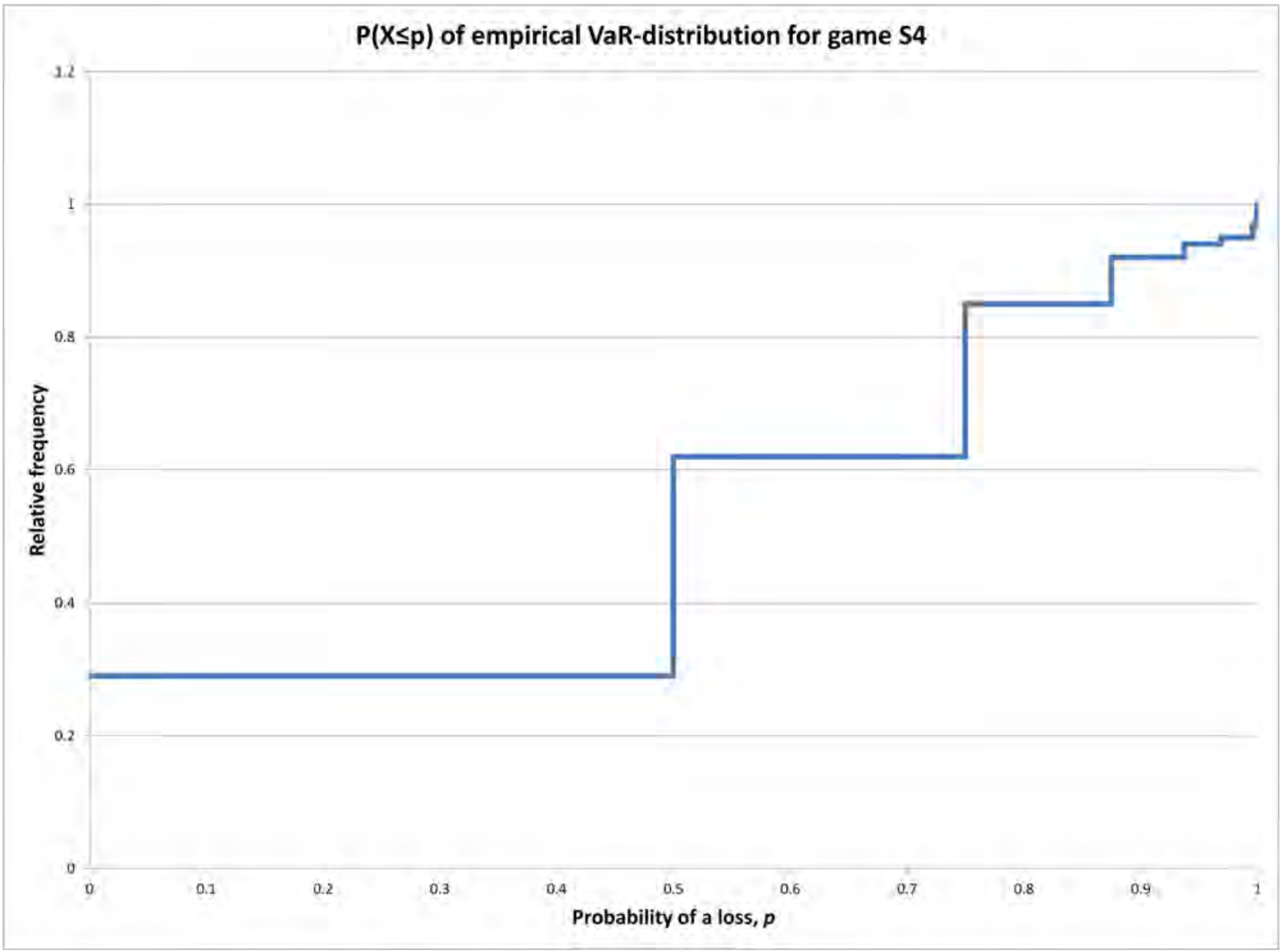

Figure 43: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game that starts at 4 (S4).

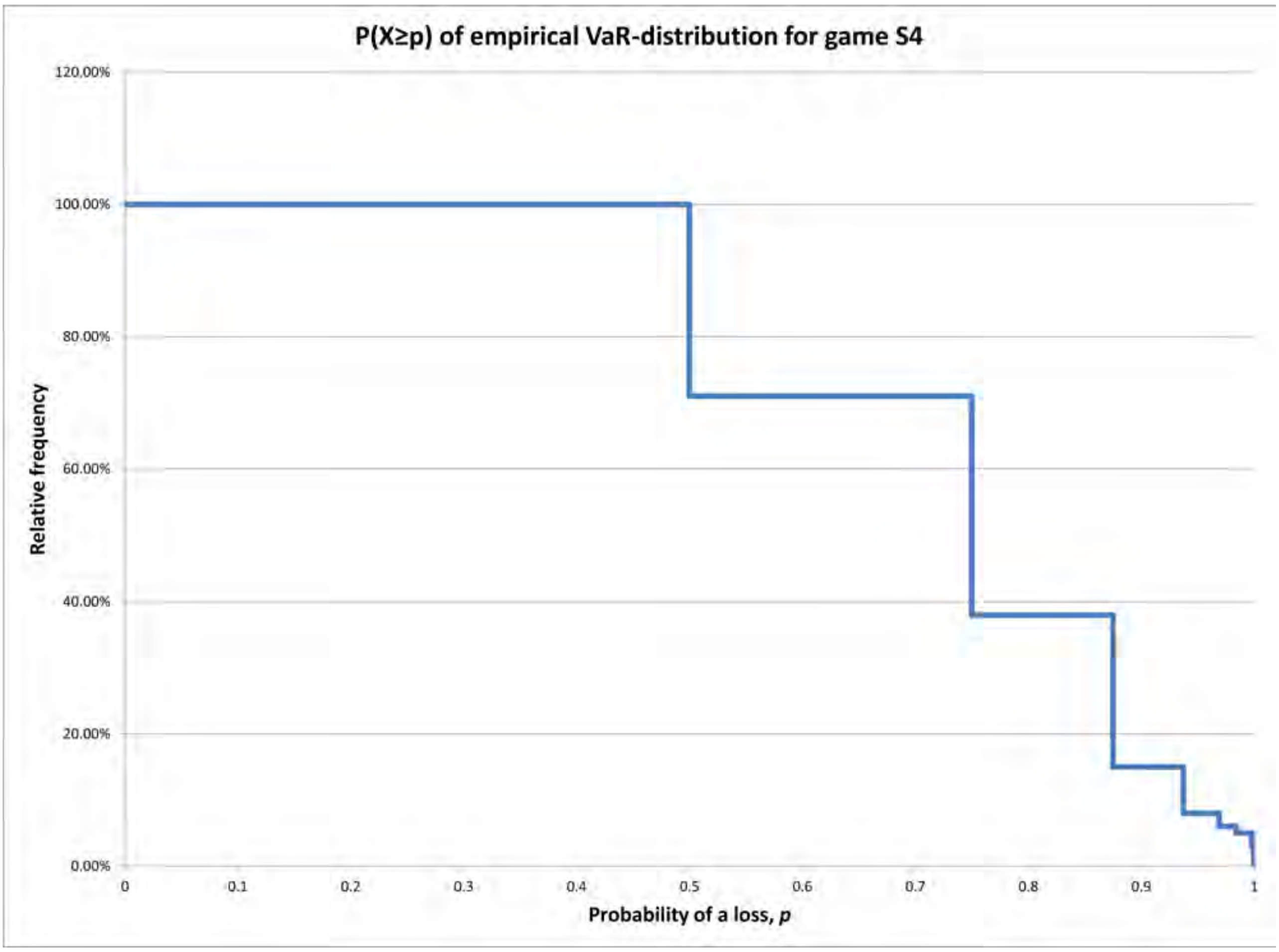

Figure 44: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game that starts at 4 (S4).

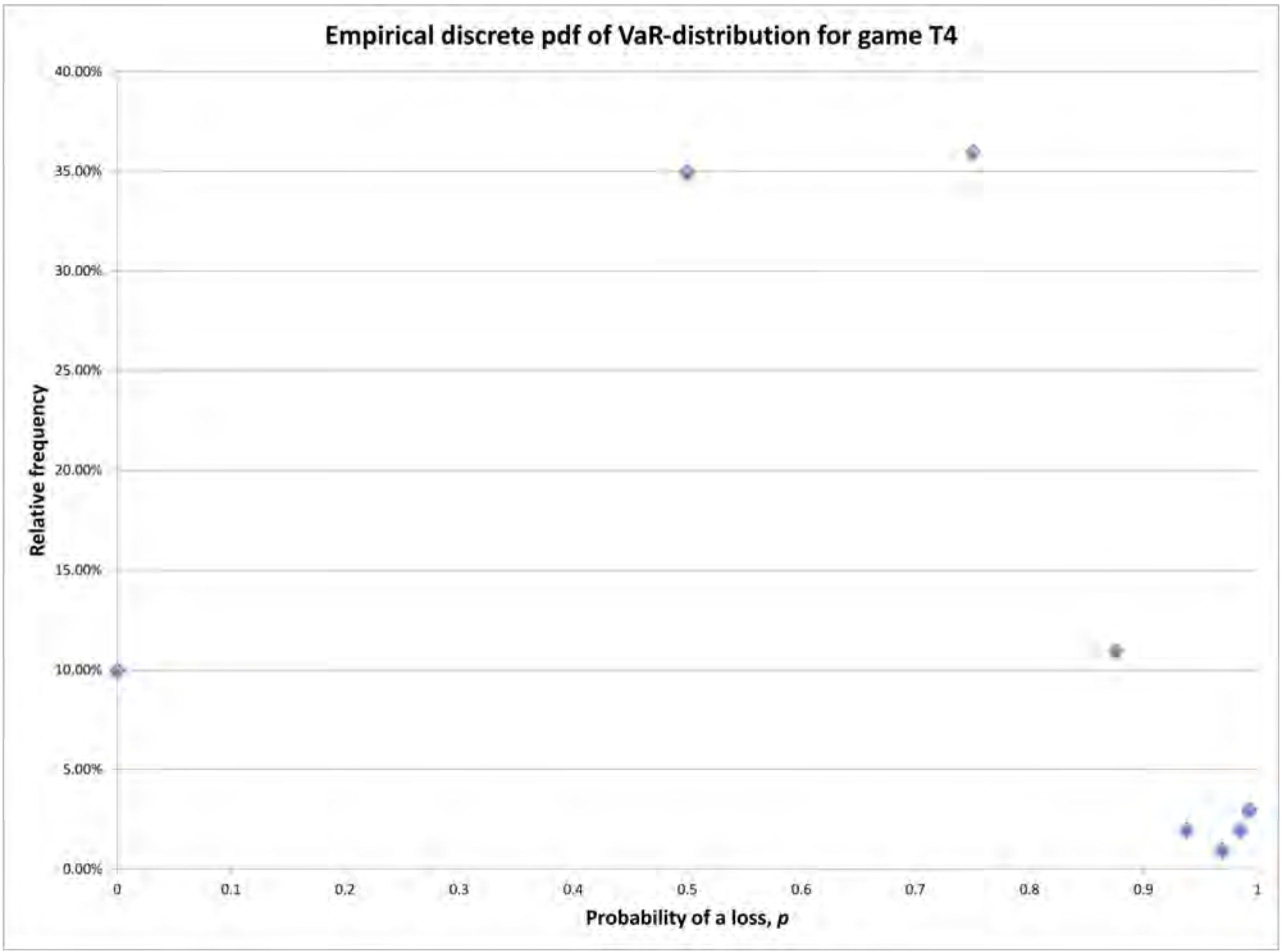

Figure 45: Relative frequency of each bin b_p with p probability of loss, for the game with a power of return of 4 (T4)

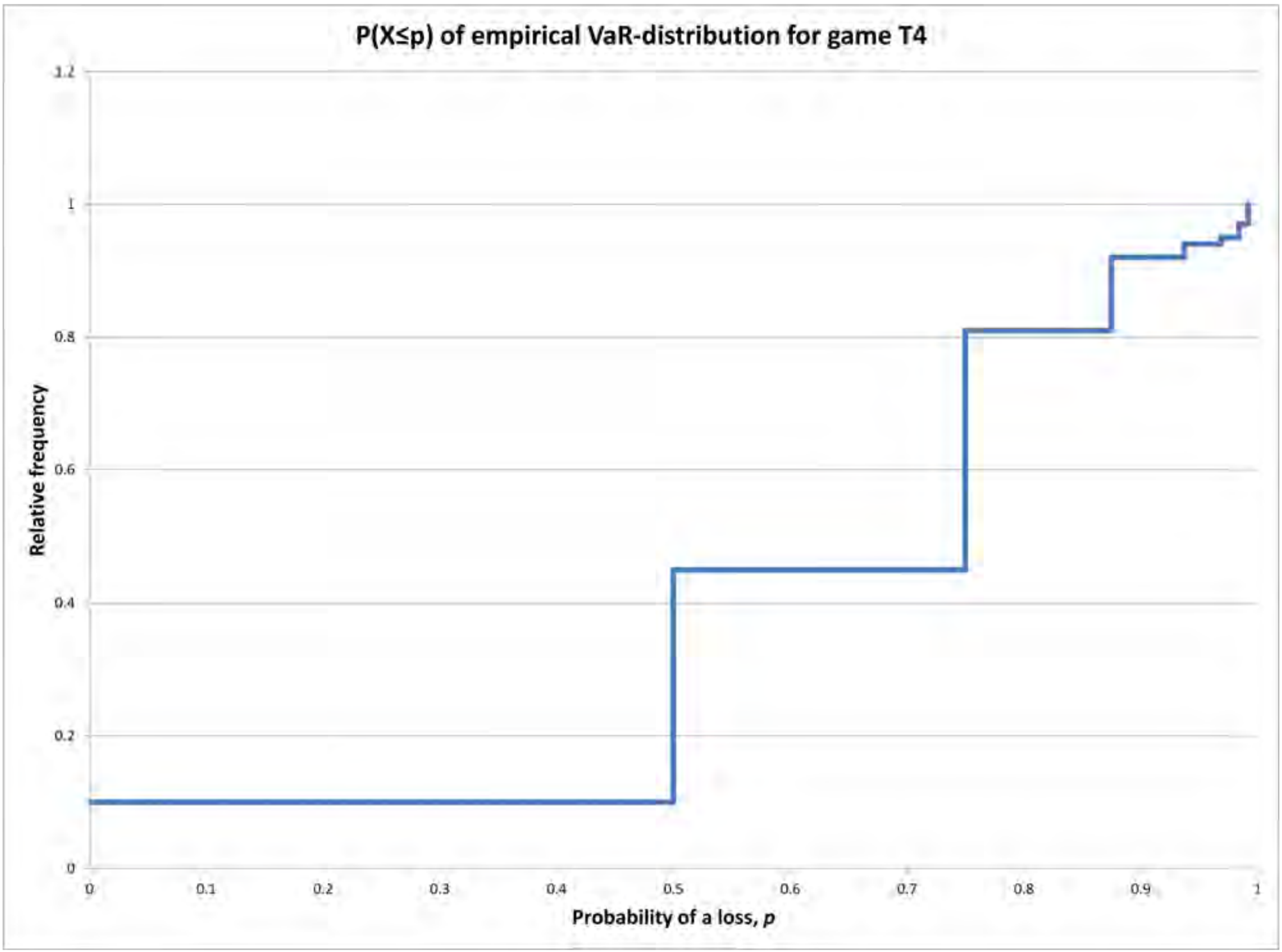

Figure 46: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the game with a power of return of 4 (T4).

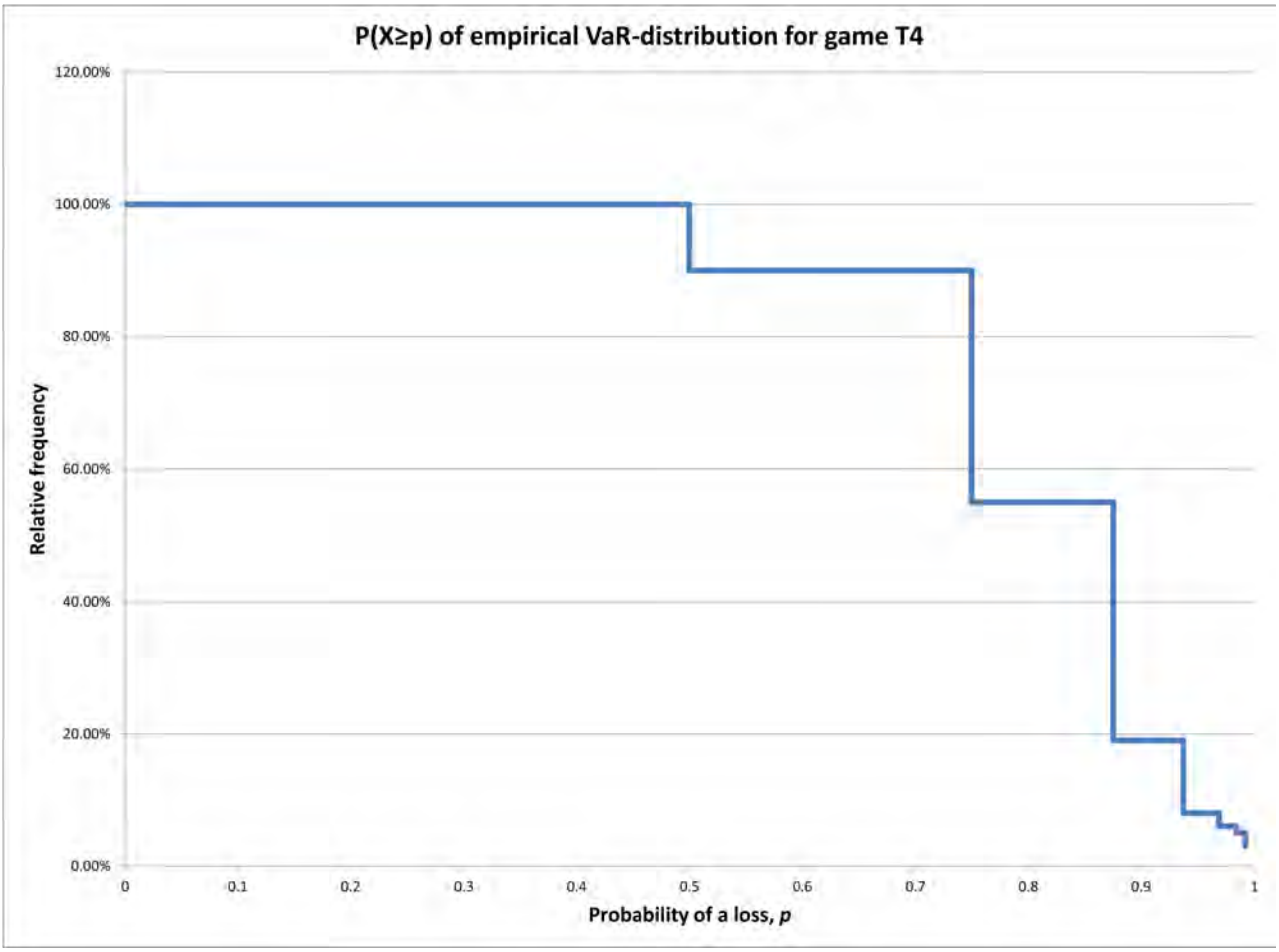

Figure 47: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the game with a power of return of 4 (T4).

B Our Survey

Tables and Figures from our survey

Empirical distribution of the basic coin game in our survey (C B)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0	2	4%	4%	100%
$b_{0.5}$	$2 - 4^-$	0.5	14	25%	29%	96%
$b_{0.75}$	$4 - 8^-$	0.75	13	24%	53%	71%
$b_{0.875}$	$8 - 16^-$	0.875	12	22%	75%	47%
$b_{0.9375}$	$16 - 32^-$	0.9375	5	9%	84%	25%
$b_{0.96875}$	$32 - 64^-$	0.96875	3	5%	89%	16%
$b_{0.984375}$	$64 - 128^-$	0.984375	0	0%	89%	11%
$b_{0.9921875}$	$128 - 256^-$	0.9921875	2	4%	93%	11%
...	...	...	...	...	...	...

Table 29: Value at Risk distribution for the basic coin game in our survey, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the coin game, cut off at 1024, in our survey (C CT10)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0	3	5%	5%	100%
$b_{0.5}$	$2 - 4^-$	0.5	18	32%	38%	95%
$b_{0.75}$	$4 - 8^-$	0.75	12	21%	59%	63%
$b_{0.875}$	$8 - 16^-$	0.875	20	36%	95%	41%
$b_{0.9375}$	$16 - 32^-$	0.9375	0	0%	95%	5%
$b_{0.96875}$	$32 - 64^-$	0.96875	2	4%	98%	5%
$b_{0.984375}$	$64 - 128^-$	0.984375	0	0%	98%	2%
$b_{0.9921875}$	$128 - 256^-$	0.9921875	1	2%	100%	2%
...	...	...	...	...	...	...

Table 30: Value at Risk distribution for the coin game in our survey, cut off at 1024, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the coin game, cut off at 32, in our survey (C CT5)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0	6	11%	11%	100%
$b_{0.5}$	$2 - 4^-$	0.5	22	41%	52%	89%
$b_{0.75}$	$4 - 8^-$	0.75	18	33%	85%	48%
$b_{0.875}$	$8 - 16^-$	0.875	6	11%	96%	15%
$b_{0.9375}$	$16 - 32^-$	0.9375	1	2%	98%	4%
b_1	$32-$	1	1	2%	100%	2%

Table 31: Value at Risk distribution for the coin game in our survey, cut off at 32, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the coin game, with a power of return of 4, in our survey (C T4)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0	0	0%	0%	100%
$b_{0.5}$	$2 - 8^-$	0.5	16	29%	29%	100%
$b_{0.75}$	$8 - 32^-$	0.75	19	35%	64%	71%
$b_{0.875}$	$32 - 128^-$	0.875	9	16%	80%	36%
$b_{0.9375}$	$128 - 512^-$	0.9375	7	13%	93%	20%
$b_{0.96875}$	$512 - 2048^-$	0.96875	2	4%	96%	7%
$b_{0.984375}$	$2048 - 8192^-$	0.984375	0	0%	96%	4%
$b_{0.9921875}$	$8192 - 32768^-$	0.9921875	0	0%	96%	4%
...	...	...	...	...	...	...

Table 32: Value at Risk distribution for the coin game in our survey, with a power of return of 4, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the basic six-sided dice game in our survey (6D B)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	2	4%	4%	100%
$b_{0.17}$	$2 - 4^-$	0.17	5	9%	13%	96%
$b_{0.31}$	$4 - 8^-$	0.31	3	6%	19%	87%
$b_{0.42}$	$8 - 16^-$	0.42	10	19%	37%	81%
$b_{0.52}$	$16 - 32^-$	0.52	11	20%	57%	63%
$b_{0.60}$	$32 - 64^-$	0.60	9	17%	74%	43%
$b_{0.67}$	$64 - 128^-$	0.67	6	11%	85%	26%
$b_{0.72}$	$128 - 256^-$	0.72	3	6%	91%	15%
$b_{0.77}$	$256 - 512^-$	0.77	3	6%	96%	9%
$b_{0.81}$	$512 - 1024^-$	0.81	1	2%	98%	4%
...	...	...	...	...	...	...

Table 33: Value at Risk distribution for the basic six-sided dice game in our survey, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the six-sided dice game, cut off at 1024, in our survey (6D CT10)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	3	5.77%	6%	100%
$b_{0.17}$	$2 - 4^-$	0.17	4	7.69%	13%	94%
$b_{0.31}$	$4 - 8^-$	0.31	3	5.77%	19%	87%
$b_{0.42}$	$8 - 16^-$	0.42	12	23.08%	42%	81%
$b_{0.52}$	$16 - 32^-$	0.52	10	19.23%	62%	58%
$b_{0.60}$	$32 - 64^-$	0.60	9	17.31%	79%	38%
$b_{0.67}$	$64 - 128^-$	0.67	8	15.38%	94%	21%
$b_{0.72}$	$128 - 256^-$	0.72	2	3.85%	98%	6%
$b_{0.77}$	$256 - 512^-$	0.77	0	0.00%	98%	2%
$b_{0.81}$	$512 - 1024^-$	0.81	0	0.00%	98%	2%
b_1	$1024-$	1	1	1.92%	100%	2%

Table 34: Value at Risk distribution for the six-sided dice game in our survey, cut off at 1024, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the six-sided dice game, cut off at 32, in our survey (6D CT5)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	4	7.55%	8%	100%
$b_{0.17}$	$2 - 4^-$	0.17	10	18.87%	26%	92%
$b_{0.31}$	$4 - 8^-$	0.31	11	20.75%	47%	74%
$b_{0.42}$	$8 - 16^-$	0.42	18	33.96%	81%	53%
$b_{0.52}$	$16 - 32^-$	0.52	7	13.21%	94%	19%
b_1	$32-$	1	3	5.66%	100%	6%

Table 35: Value at Risk distribution for the six-sided dice game in our survey, cut off at 32, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the six-sided dice game, with a power of return of 4, in our survey (6D T4)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	1	2%	2%	100%
$b_{0.17}$	$2 - 8^-$	0.17	4	8%	9%	98%
$b_{0.31}$	$8 - 32^-$	0.31	12	23%	32%	91%
$b_{0.42}$	$32 - 128^-$	0.42	7	13%	45%	68%
$b_{0.52}$	$128 - 512^-$	0.52	13	25%	70%	55%
$b_{0.60}$	$512 - 2048^-$	0.60	12	23%	92%	30%
$b_{0.67}$	$2048 - 8192^-$	0.67	2	4%	96%	8%
$b_{0.72}$	$8192 - 32768^-$	0.72	0	0%	96%	4%
...	...	...	...	...	...	...

Table 36: Value at Risk distribution for the six-sided dice game in our survey, with a power of return of 4, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the basic ten-sided dice game in our survey (10D B)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	2	4%	4%	100%
$b_{0.1}$	$2 - 4^-$	0.10	3	6%	9%	96%
$b_{0.19}$	$4 - 8^-$	0.19	1	2%	11%	91%
$b_{0.27}$	$8 - 16^-$	0.27	5	9%	20%	89%
$b_{0.34}$	$16 - 32^-$	0.34	9	17%	37%	80%
$b_{0.41}$	$32 - 64^-$	0.41	7	13%	50%	63%
$b_{0.47}$	$64 - 128^-$	0.47	7	13%	63%	50%
$b_{0.52}$	$128 - 256^-$	0.52	4	7%	70%	37%
$b_{0.57}$	$256 - 512^-$	0.57	4	7%	78%	30%
$b_{0.61}$	$512 - 1024^-$	0.61	7	13%	91%	22%
$b_{0.65}$	$1024 - 2048^-$	0.65	1	2%	93%	9%
$b_{0.69}$	$2048 - 4096^-$	0.69	0	0%	93%	7%
$b_{0.72}$	$4096 - 8192^-$	0.72	2	4%	96%	7%
...	...	...	...	...	...	...

Table 37: Value at Risk distribution for the basic ten-sided dice game in our survey, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the 10-sided dice game, cut off at 1024, in our survey (10D CT10)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	2	4%	4%	100%
$b_{0.1}$	$2 - 4^-$	0.10	3	6%	10%	96%
$b_{0.19}$	$4 - 8^-$	0.19	1	2%	12%	90%
$b_{0.27}$	$8 - 16^-$	0.27	12	24%	35%	88%
$b_{0.34}$	$16 - 32^-$	0.34	7	14%	49%	65%
$b_{0.41}$	$32 - 64^-$	0.41	6	12%	61%	51%
$b_{0.47}$	$64 - 128^-$	0.47	6	12%	73%	39%
$b_{0.52}$	$128 - 256^-$	0.52	8	16%	88%	27%
$b_{0.57}$	$256 - 512^-$	0.57	4	8%	96%	12%
$b_{0.61}$	$512 - 1024^-$	0.61	2	4%	100%	4%
b_1	$1024-$	1.00	0	0%	100%	0%

Table 38: Value at Risk distribution for the ten-sided dice game in our survey, cut off at 1024, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the ten-sided dice game, cut off at 32, in our survey (10D CT5)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	4	8%	8%	100%
$b_{0.1}$	$2 - 4^-$	0.10	7	13%	21%	92%
$b_{0.19}$	$4 - 8^-$	0.19	6	12%	33%	79%
$b_{0.27}$	$8 - 16^-$	0.27	11	21%	54%	67%
$b_{0.34}$	$16 - 32^-$	0.34	23	44%	98%	46%
b_1	$32-$	1.00	1	2%	100%	2%

Table 39: Value at Risk distribution for the ten-sided dice game in our survey, cut off at 32, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

Empirical distribution of the ten-sided dice game, with a power of return of 4, in our survey (10D T4)						
Bins	**Bid range**	**P(Loss)=p**	**Frequency**	**Relative Frequency**	$P(X \leq p)$	$P(X \geq p)$
b_0	$0 - 2^-$	0.00	2	4%	4%	100%
$b_{0.1}$	$2 - 8^-$	0.10	3	6%	10%	96%
$b_{0.19}$	$8 - 32^-$	0.19	5	10%	20%	90%
$b_{0.27}$	$32 - 128^-$	0.27	14	28%	48%	80%
$b_{0.34}$	$128 - 512^-$	0.34	4	8%	56%	52%
$b_{0.41}$	$512 - 2048^-$	0.41	10	20%	76%	44%
$b_{0.47}$	$2048 - 8192^-$	0.47	6	12%	88%	24%
$b_{0.52}$	$8192 - 32768^-$	0.52	3	6%	94%	12%
$b_{0.57}$	$32768 - 131072^-$	0.57	1	2%	96%	6%
...	...	...	...	...	...	...

Table 40: Value at Risk distribution for the ten-sided dice game in our survey, with a power of return of 4, 55 total bids. Frequency is the number of bids in respective bin. Relative frequency is number of bids weighted over total number of bids. $P(X \leq p)$ is number of bids in bins with with loss probability p or lower. $P(X \geq p)$ is number of bids in bins with with loss probability p or higher.

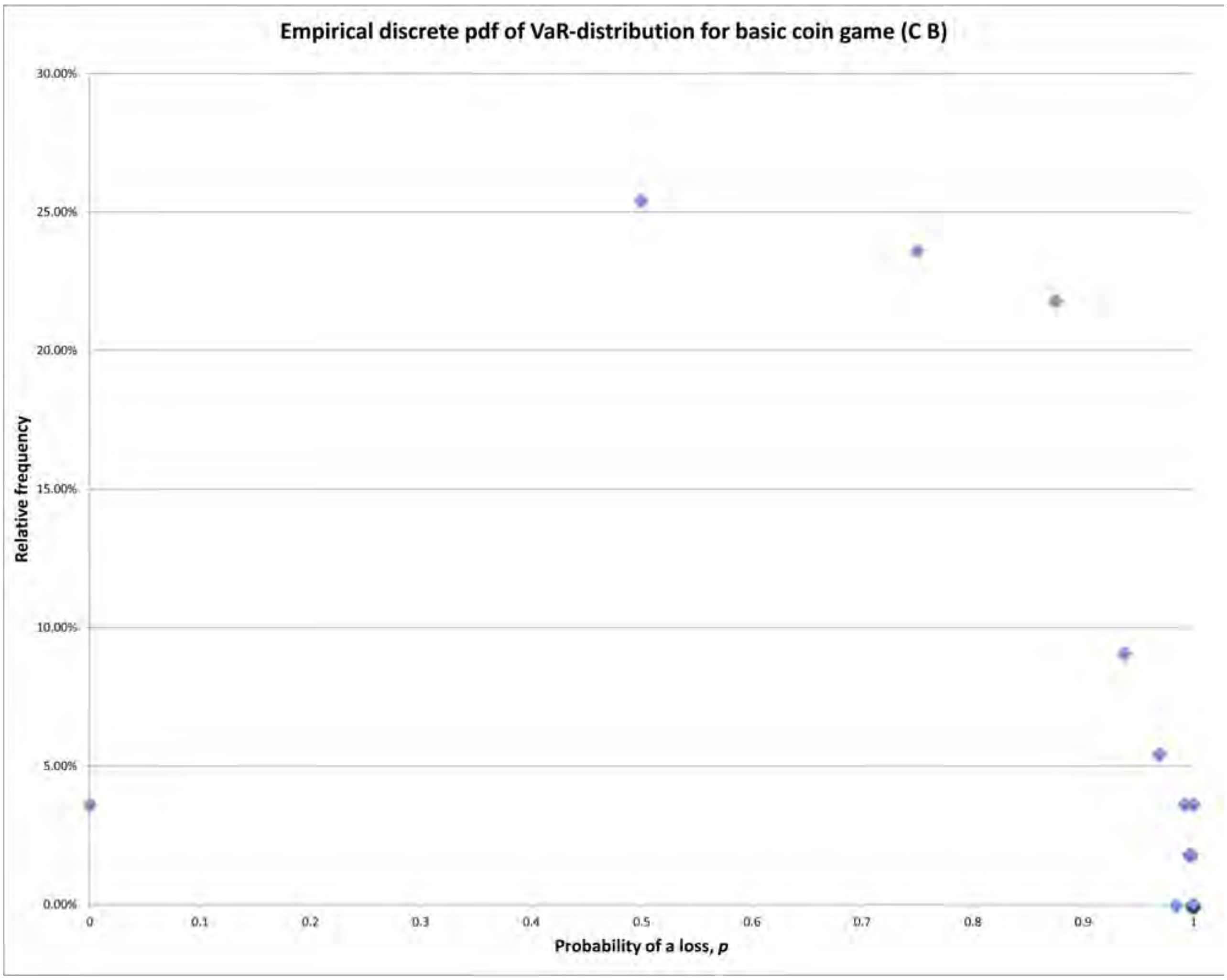

Figure 48: Relative frequency of each bin b_p with p probability of loss, for the basic coin game game in our survey (C B).

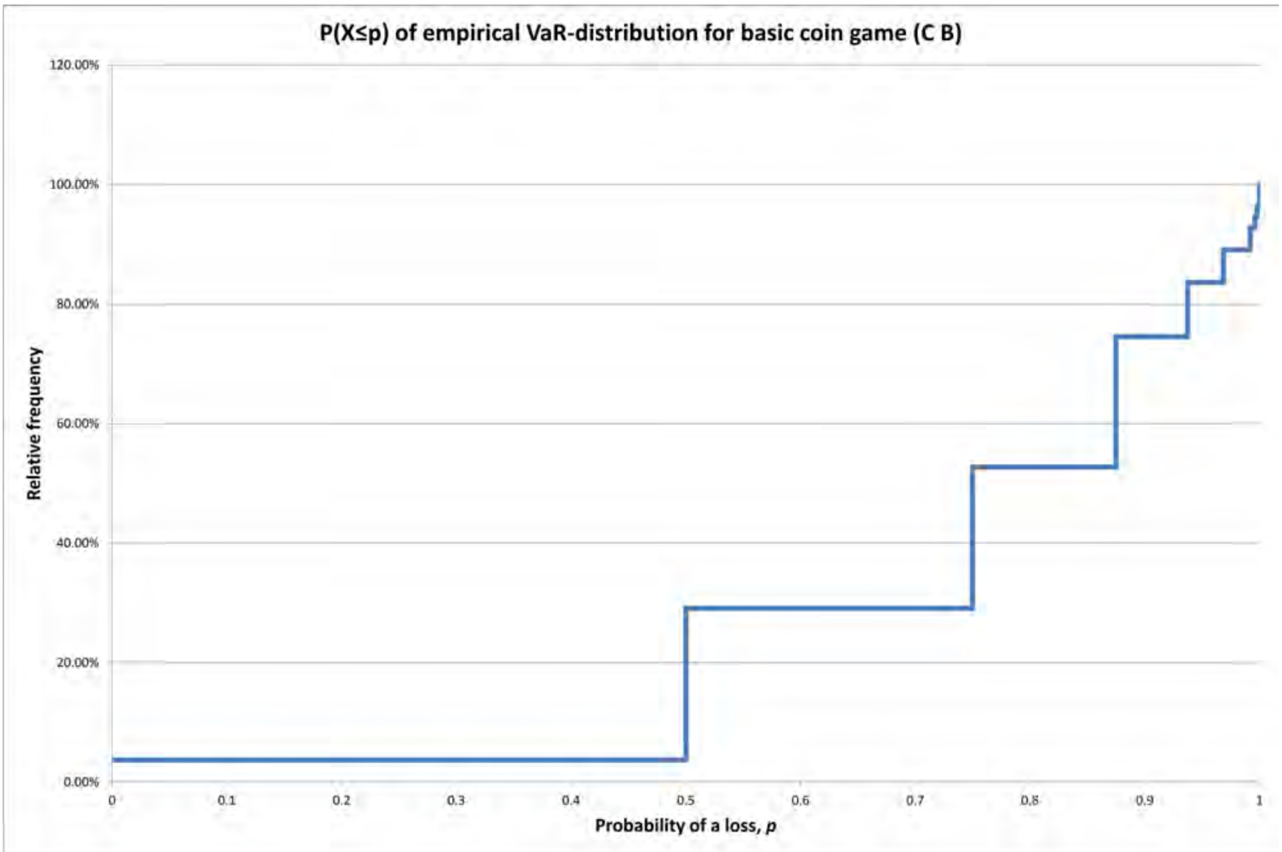

Figure 49: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the basic coin game in our survey (C B).

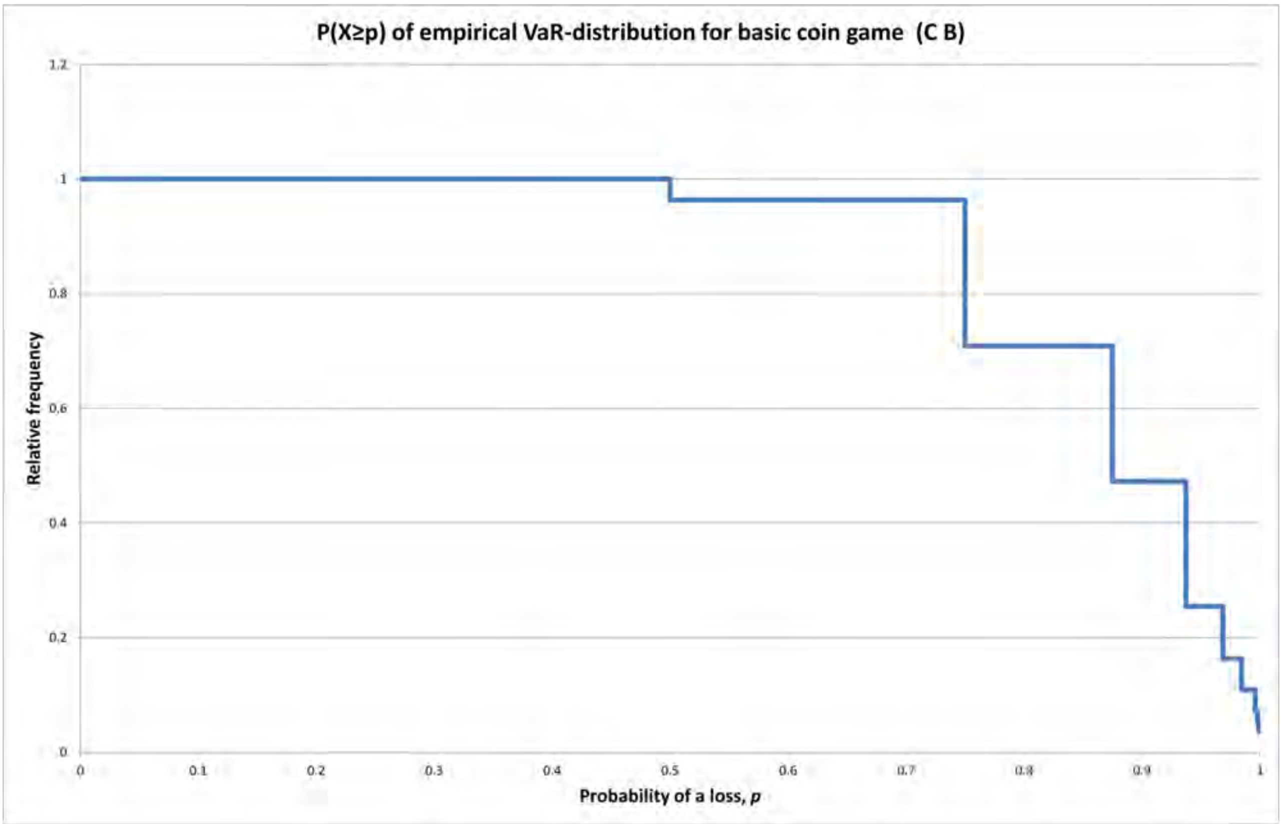

Figure 50: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the basic coin game in our survey (C B).

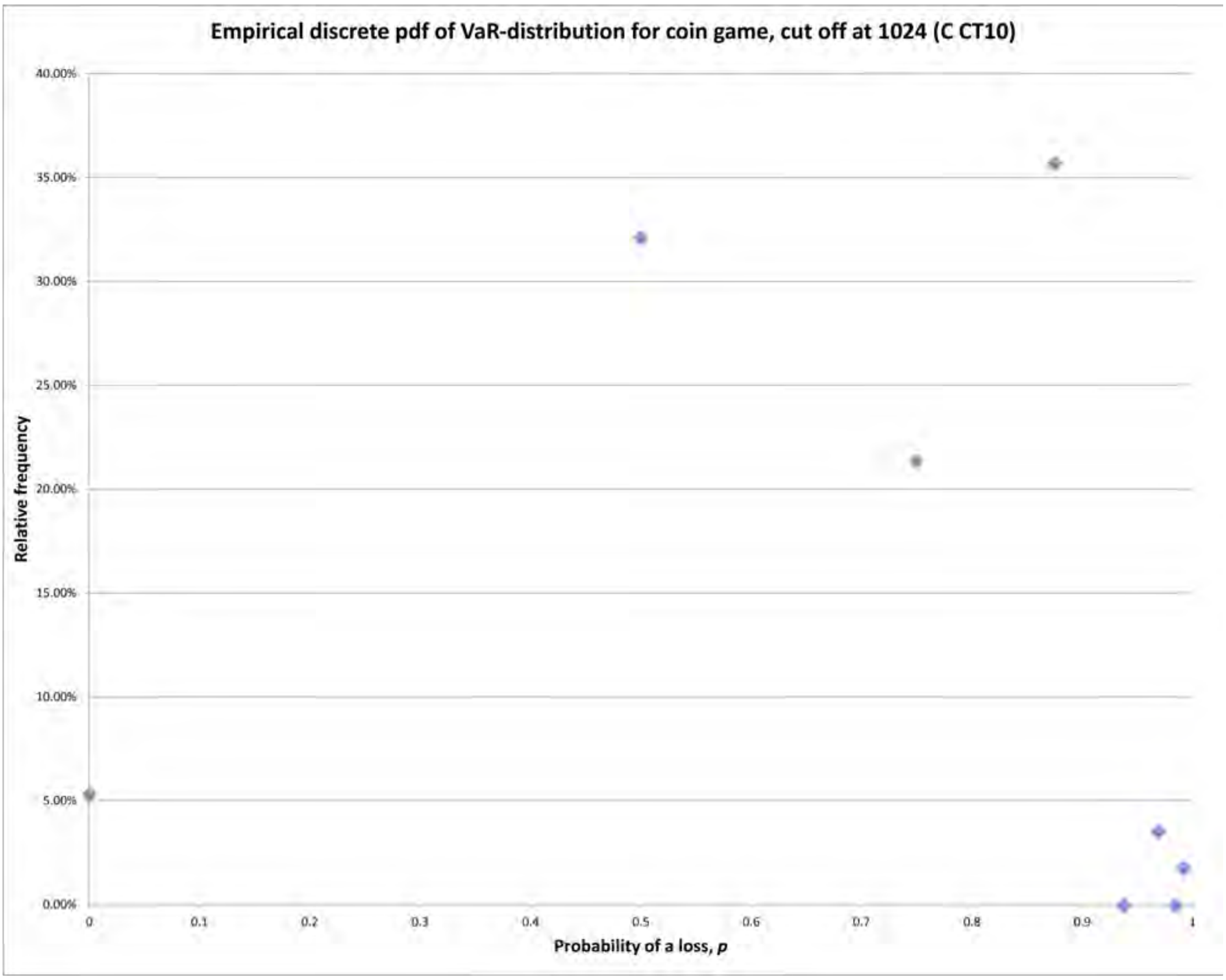

Figure 51: Relative frequency of each bin b_p with p probability of loss, for the coin game in our survey, cut off at 1024 (C CT10).

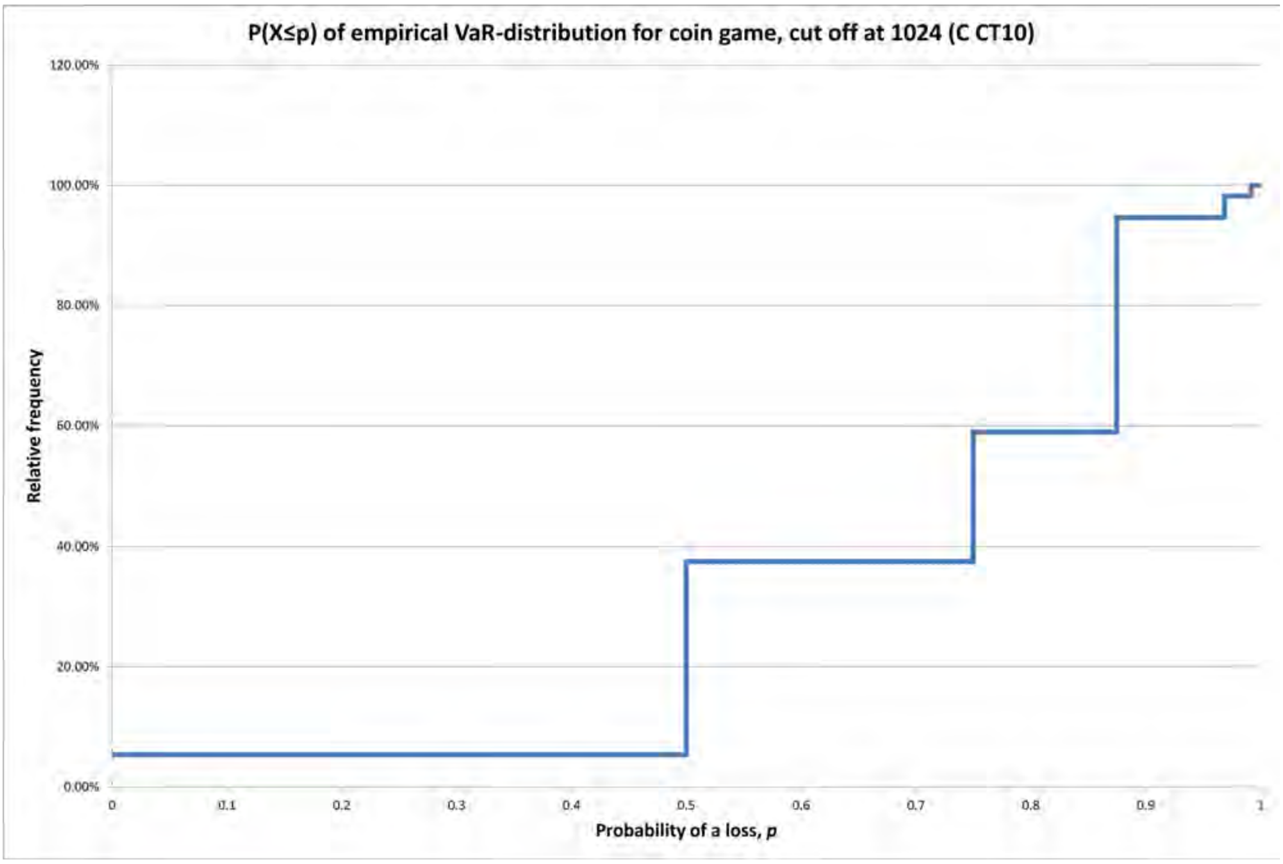

Figure 52: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the coin game in our survey, cut off at 1024 (C CT10).

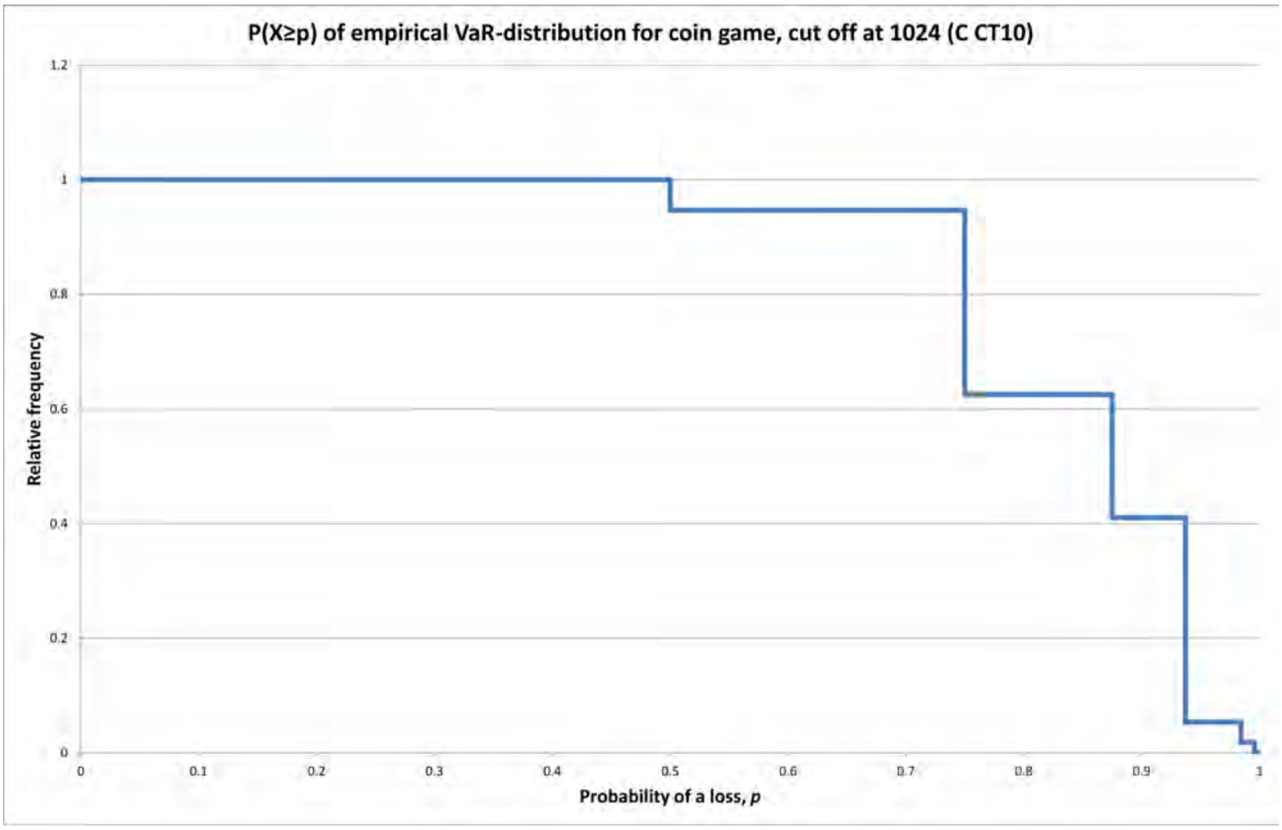

Figure 53: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the coin game in our survey, cut off at 1024 (C CT10).

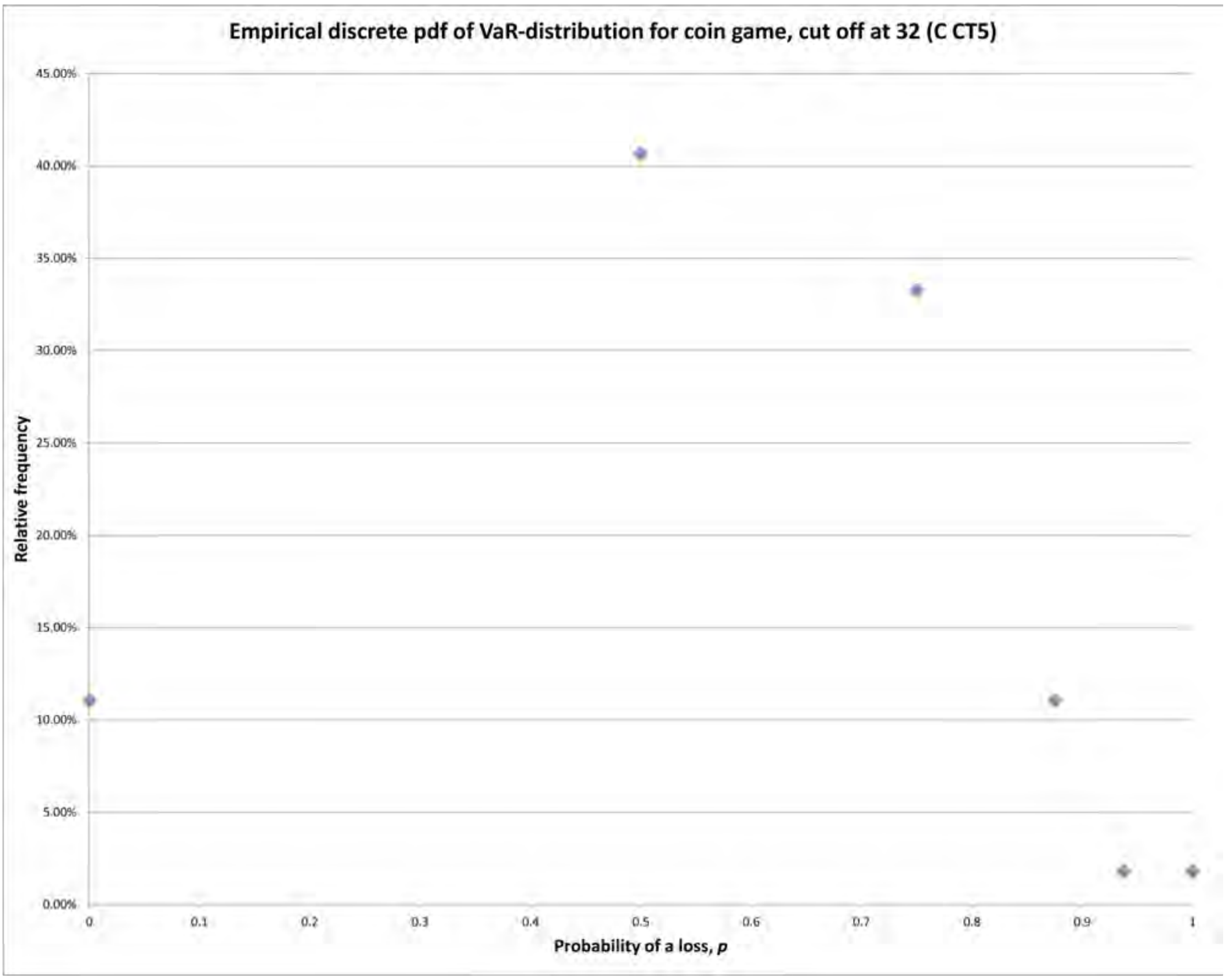

Figure 54: Relative frequency of each bin b_p with p probability of loss, for the coin game in our survey, cut off at 32 (C CT5).

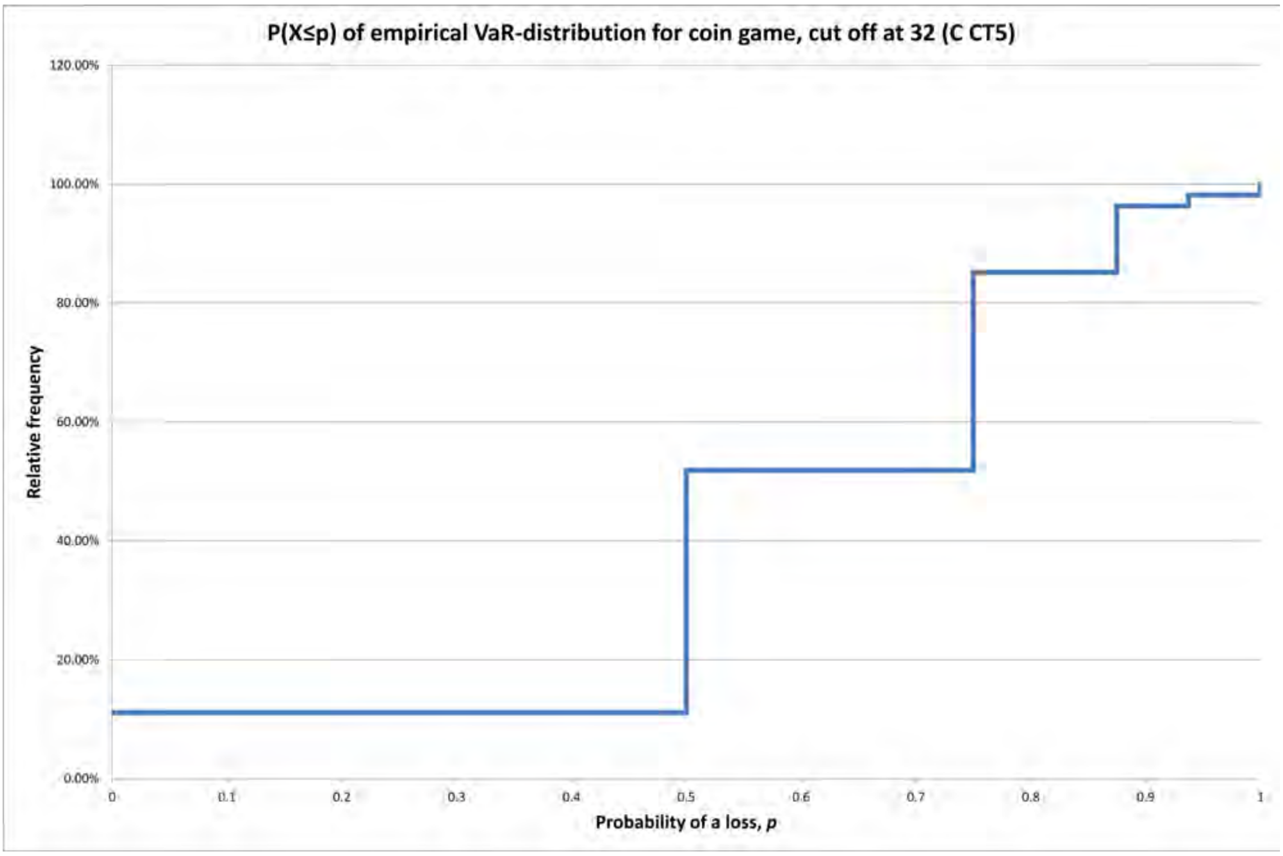

Figure 55: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the coin game in our survey, cut off at 32 (C CT5).

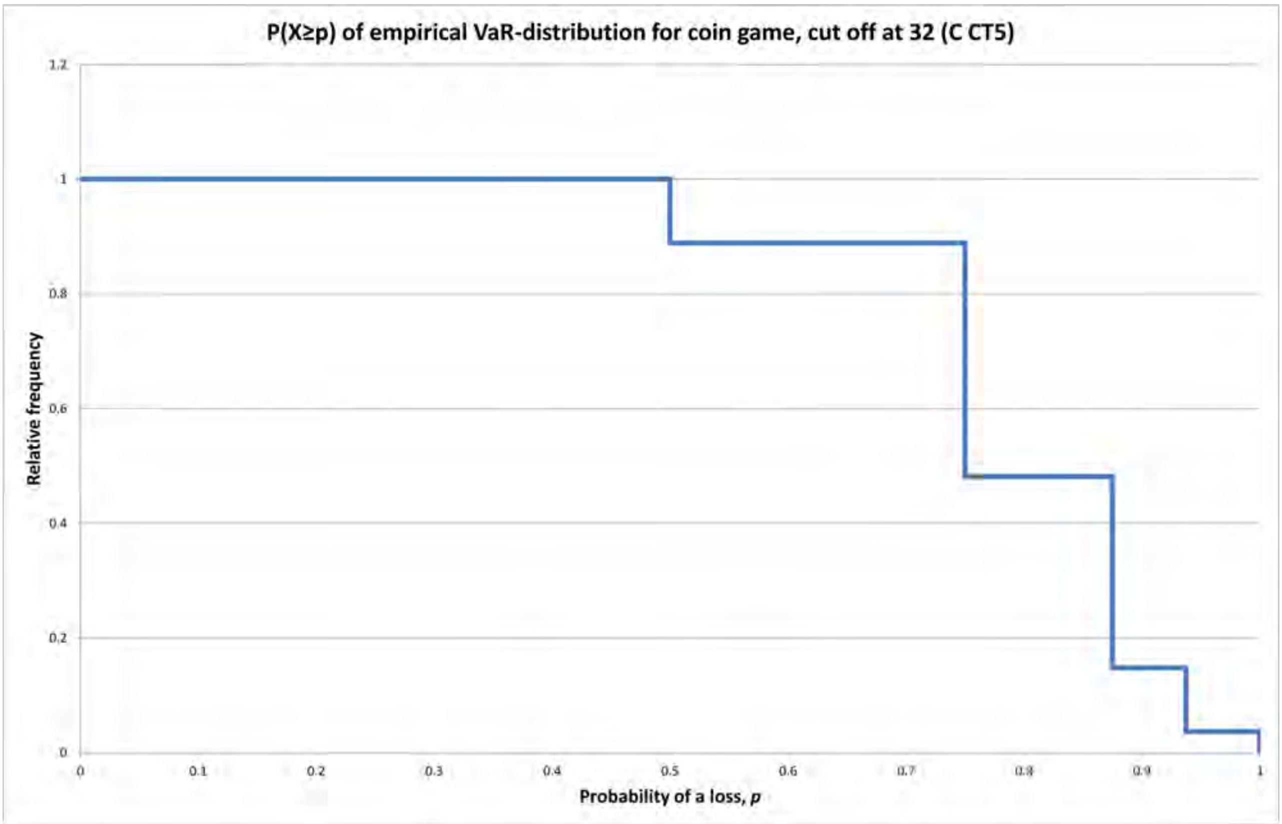

Figure 56: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the coin game in our survey, cut off at 32 (C CT5).

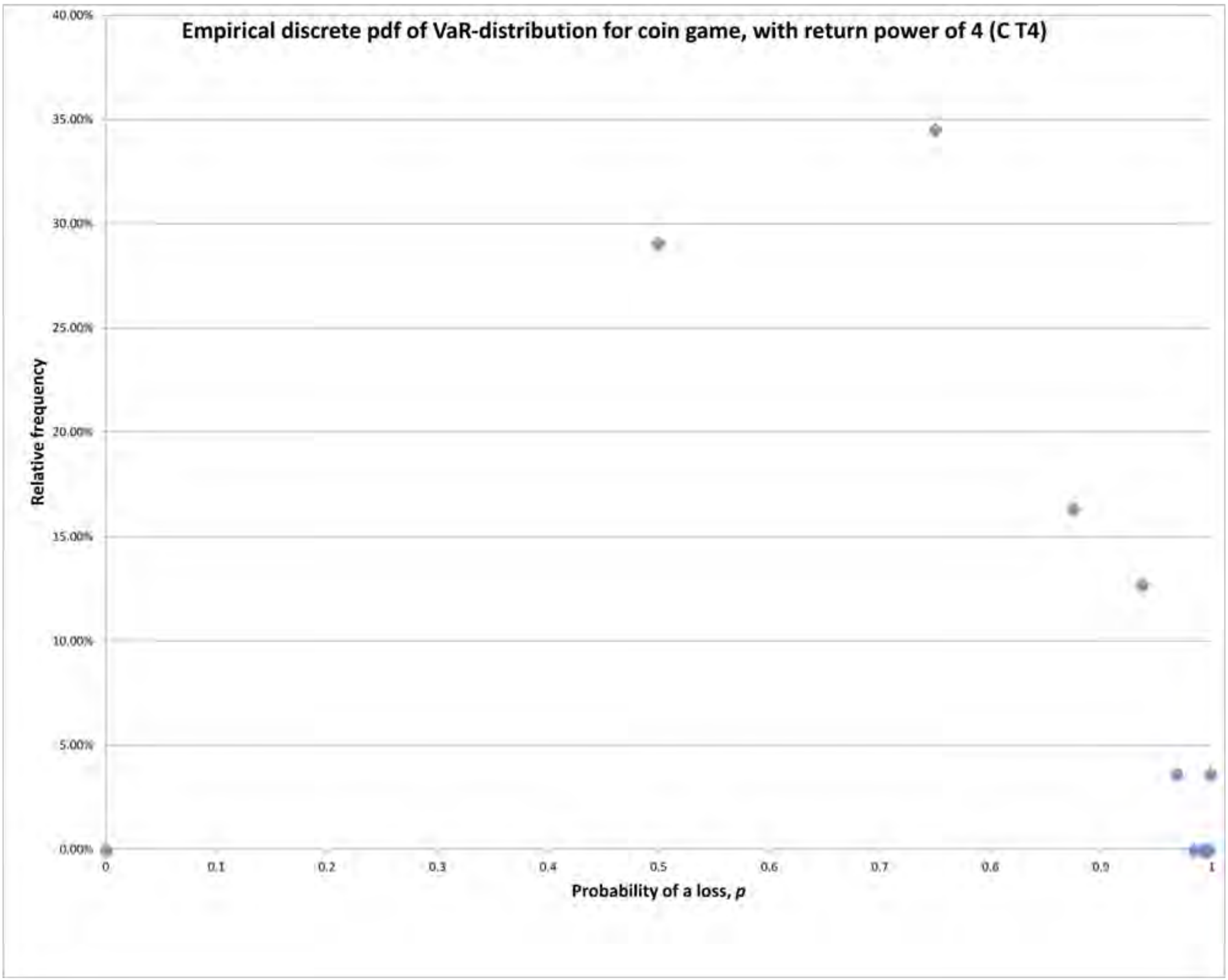

Figure 57: Relative frequency of each bin b_p with p probability of loss, for the coin game in our survey, with a power of return of 4 (C T4).

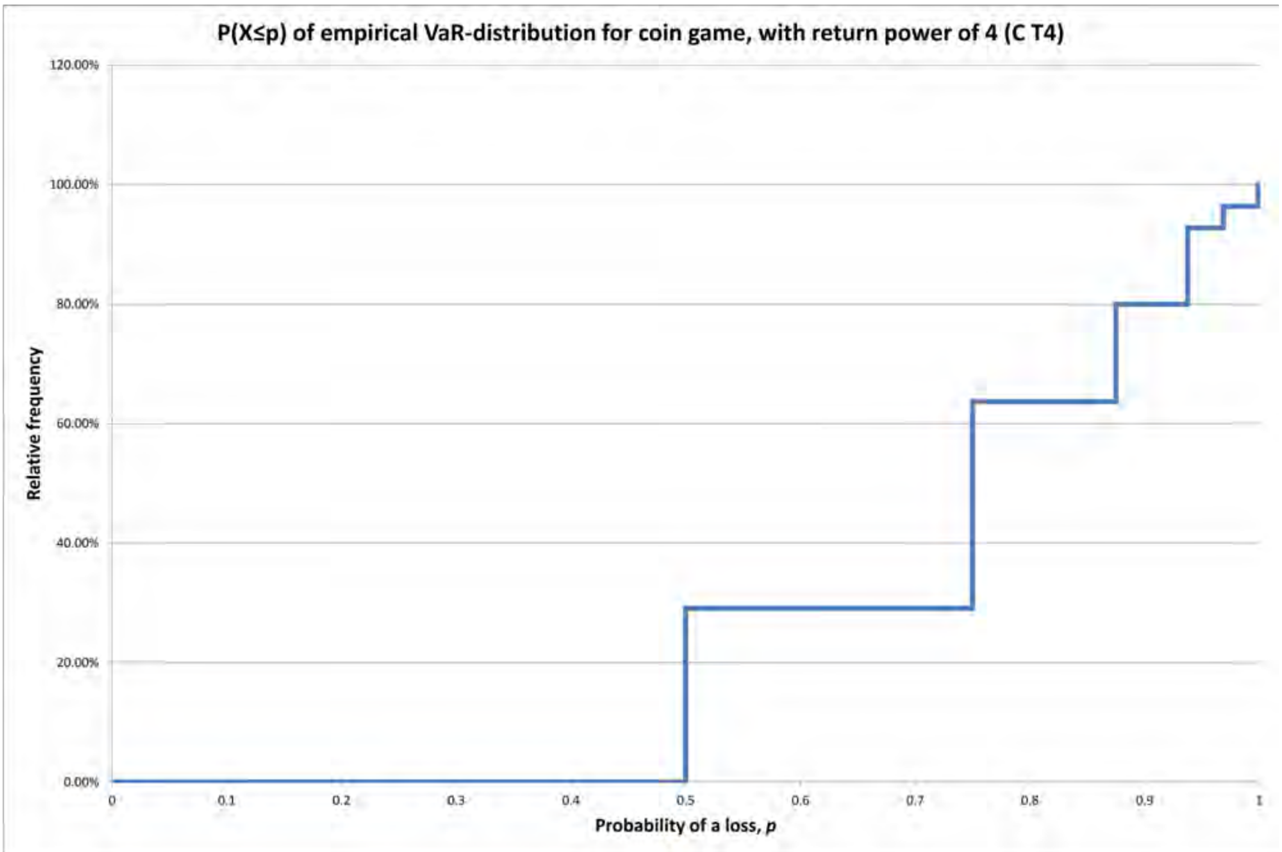

Figure 58: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the coin game in our survey, with a power of return of 4 (C T4).

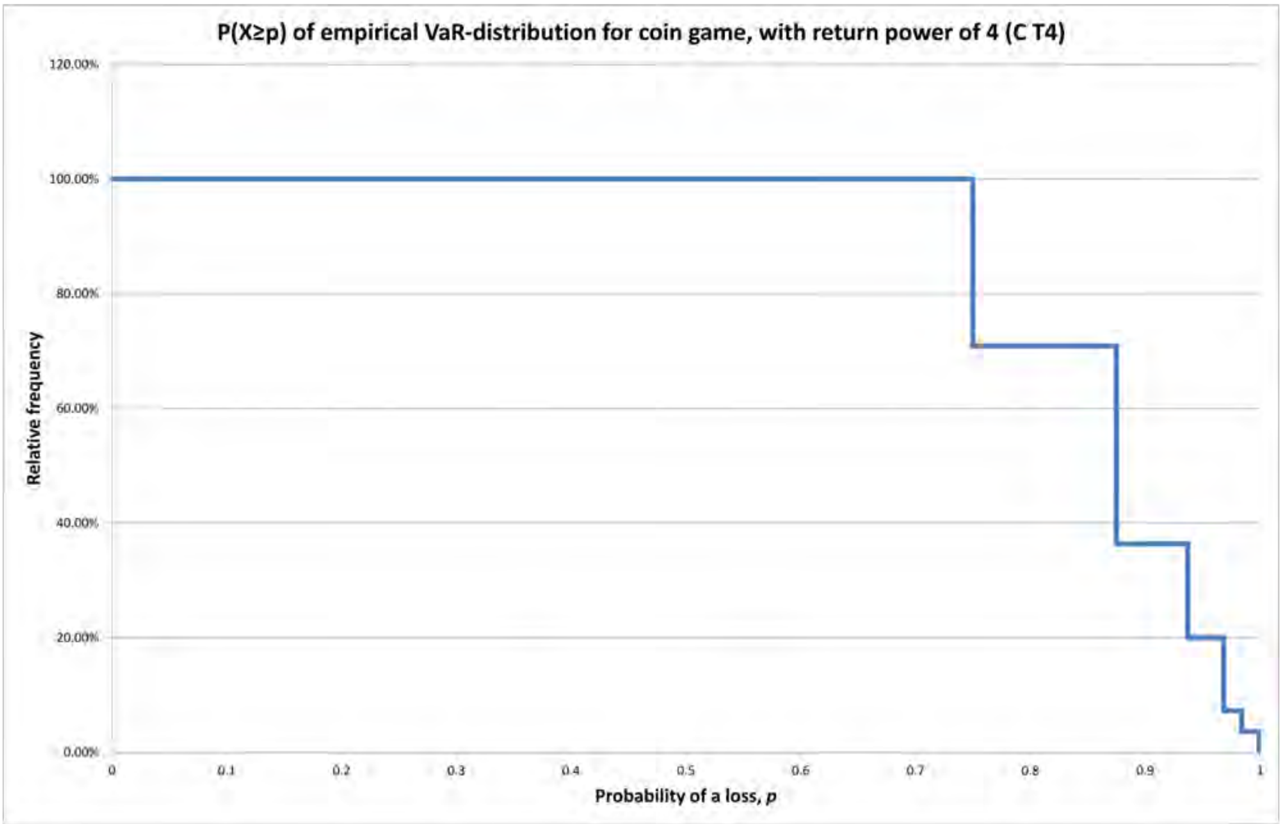

Figure 59: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the coin game in our survey, with a power of return of 4 (C T4).

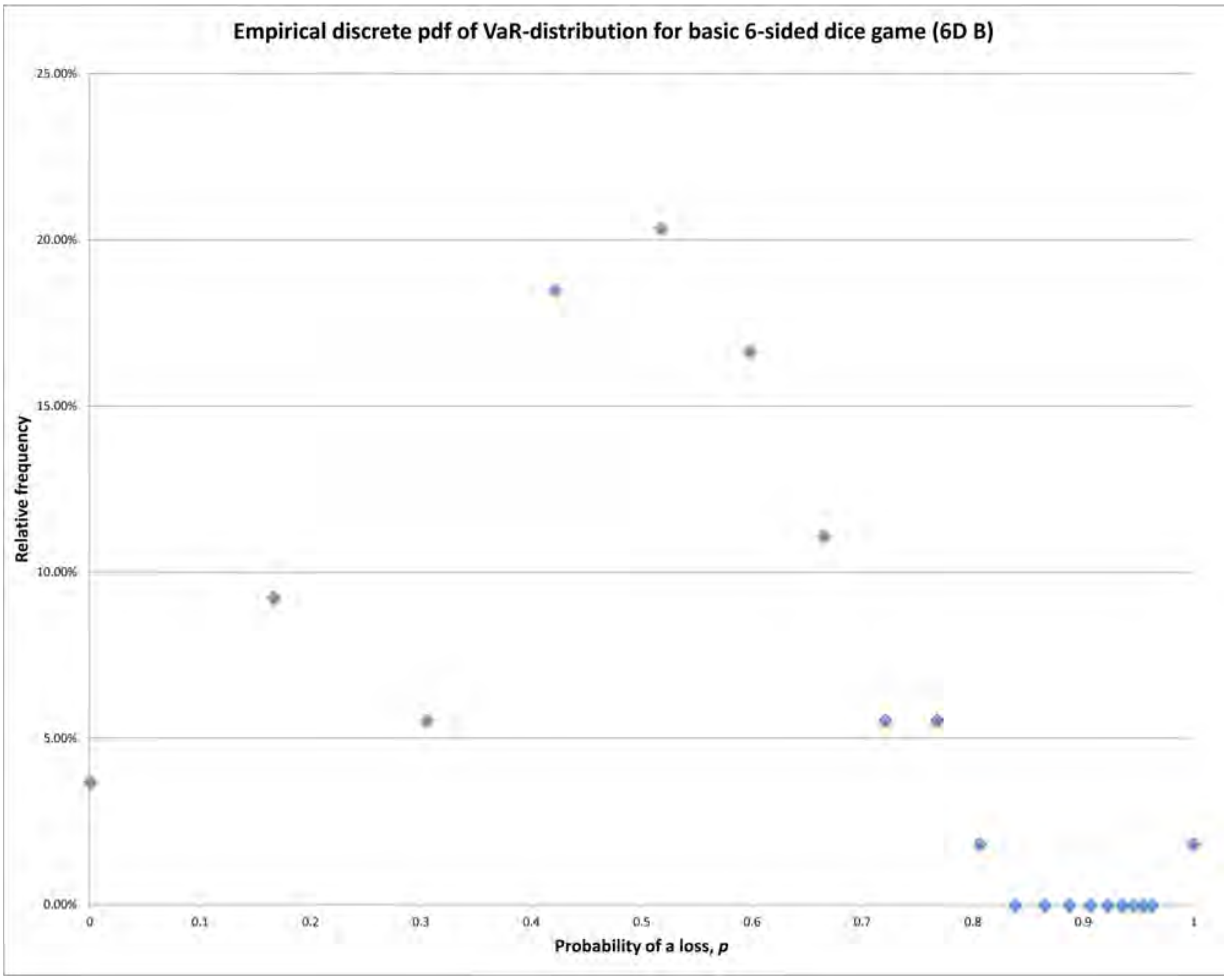

Figure 60: Relative frequency of each bin b_p with p probability of loss, for the basic 6-sided dice game game in our survey (6D B).

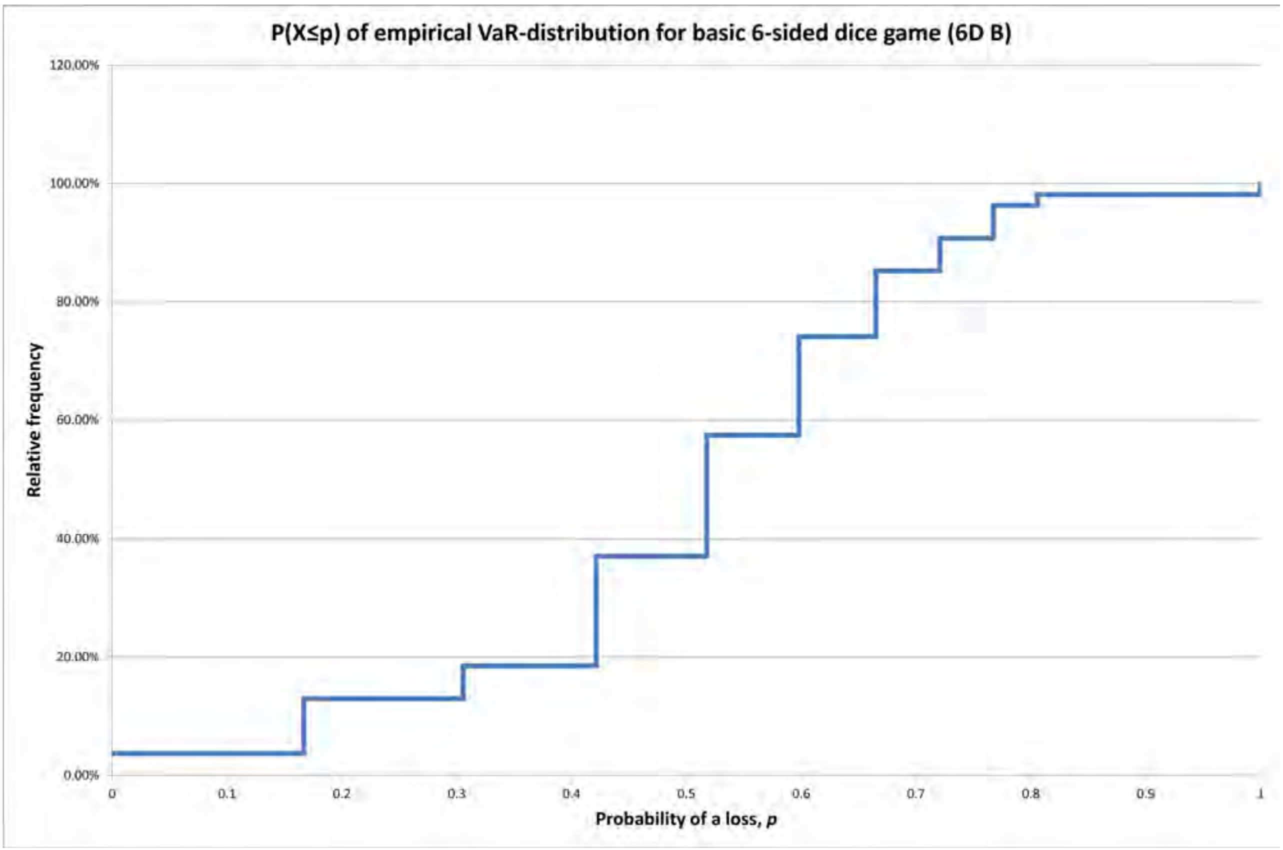

Figure 61: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the basic 6-sided dice game in our survey (6D B).

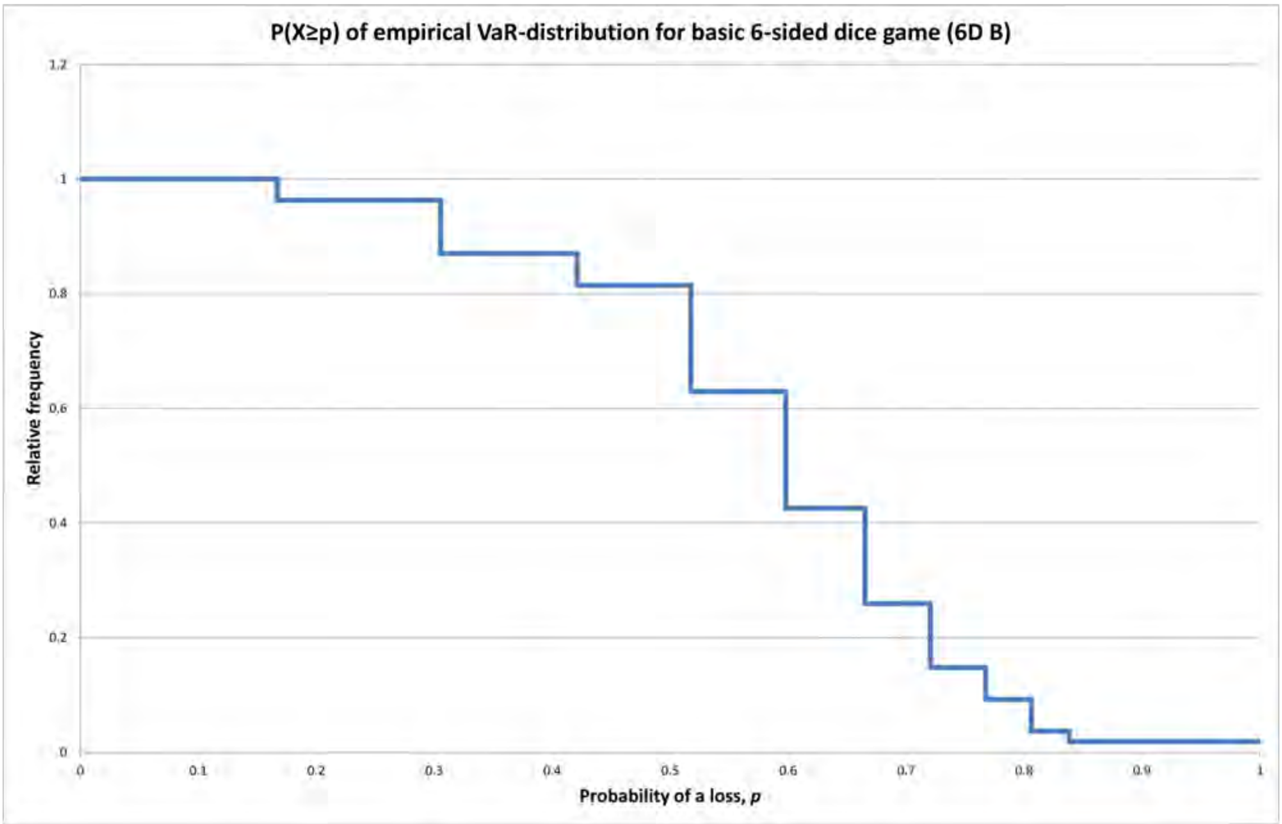

Figure 62: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the basic 6-sided dice game in our survey (6D B).

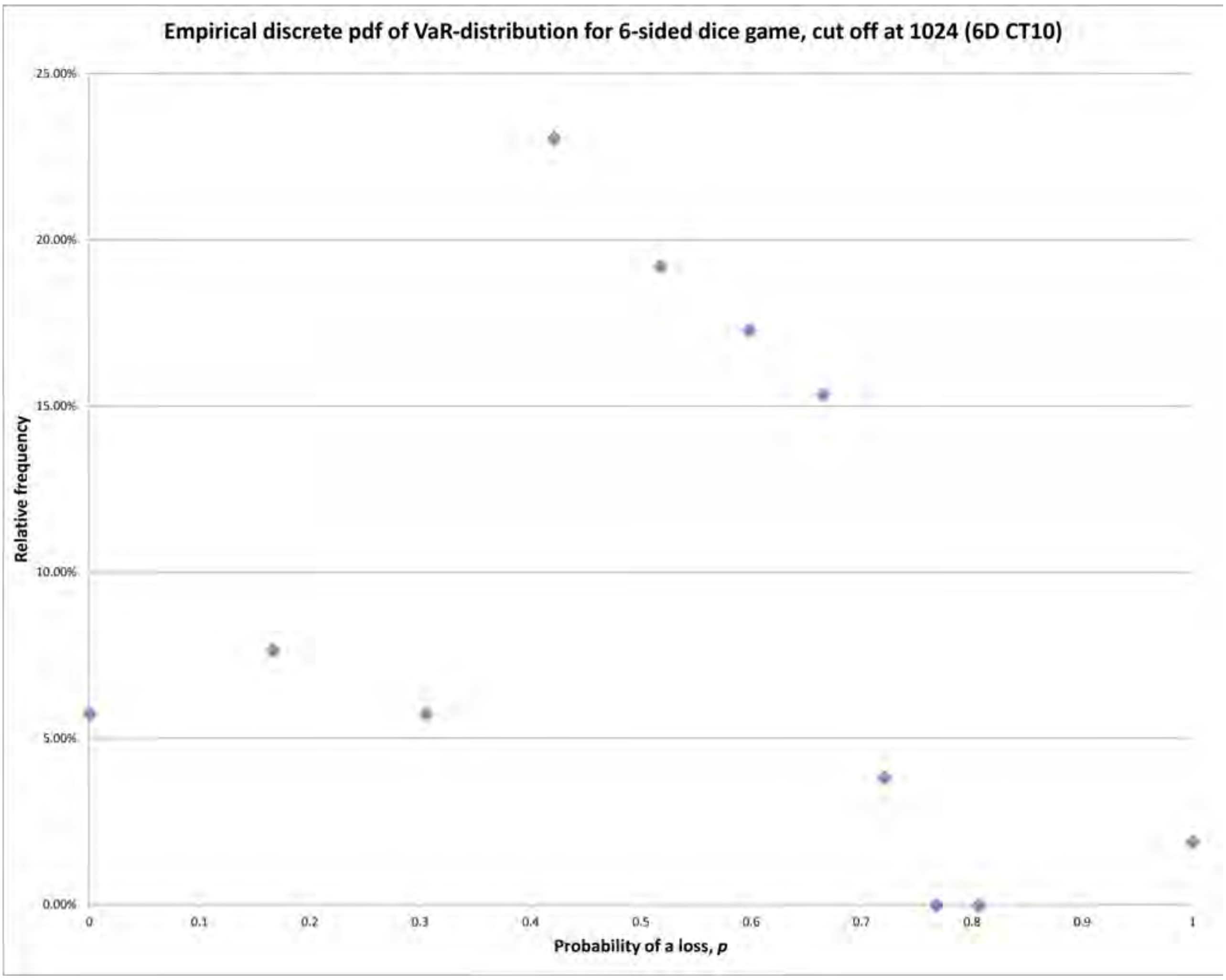

Figure 63: Relative frequency of each bin b_p with p probability of loss, for the 6-sided dice game in our survey, cut off at 1024 (6D CT10).

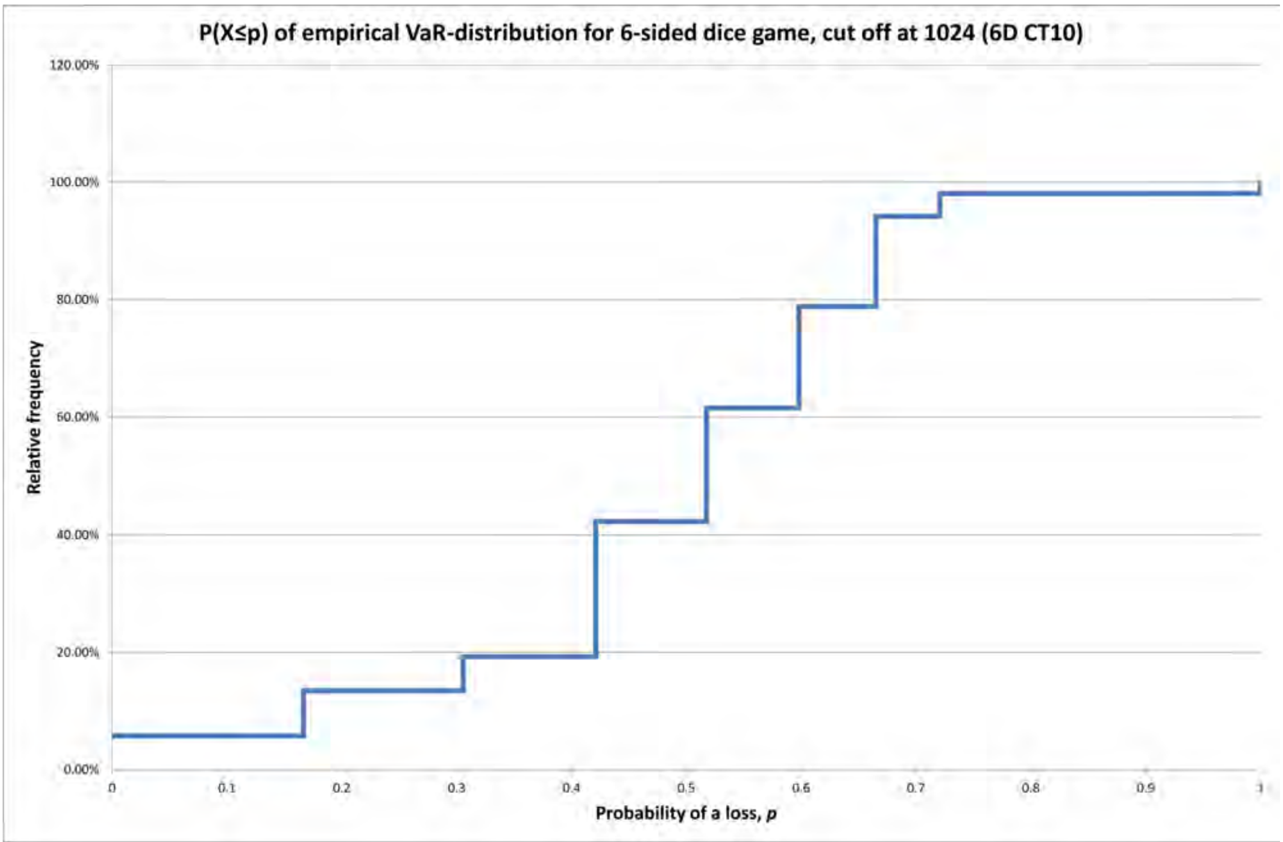

Figure 64: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 6-sided dice game in our survey, cut off at 1024 (6D CT10).

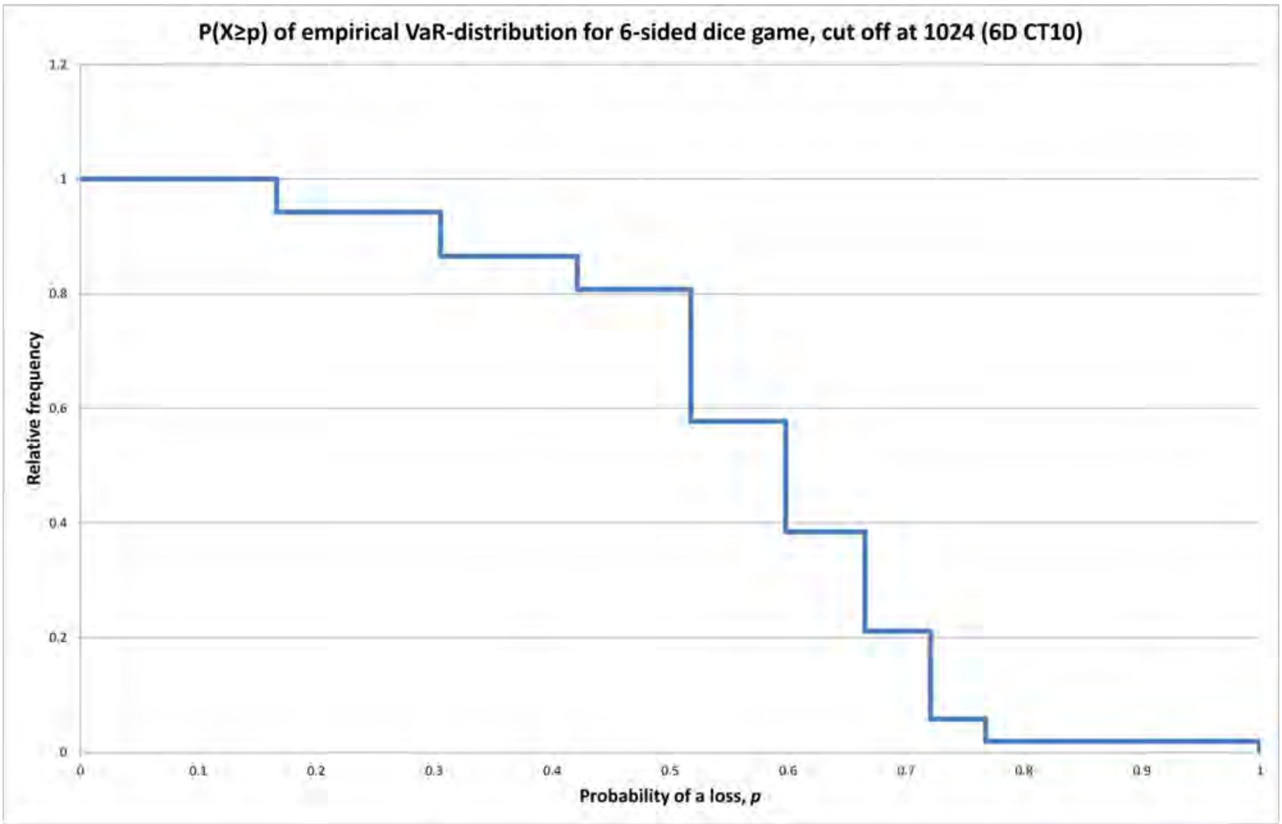

Figure 65: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 6-sided dice game in our survey, cut off at 1024 (6D CT10).

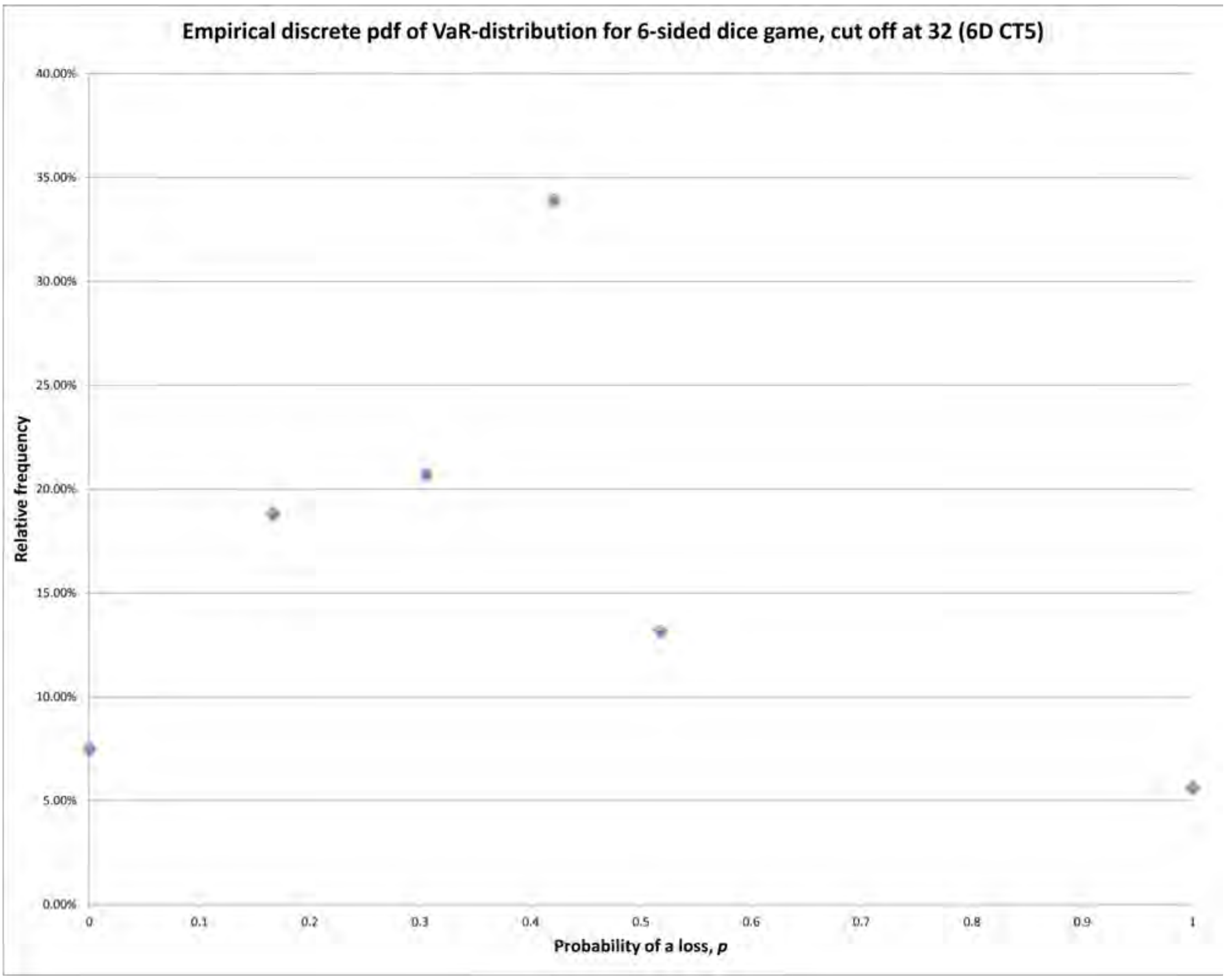

Figure 66: Relative frequency of each bin b_p with p probability of loss, for the coin game in our survey, cut off at 32 (6D CT5).

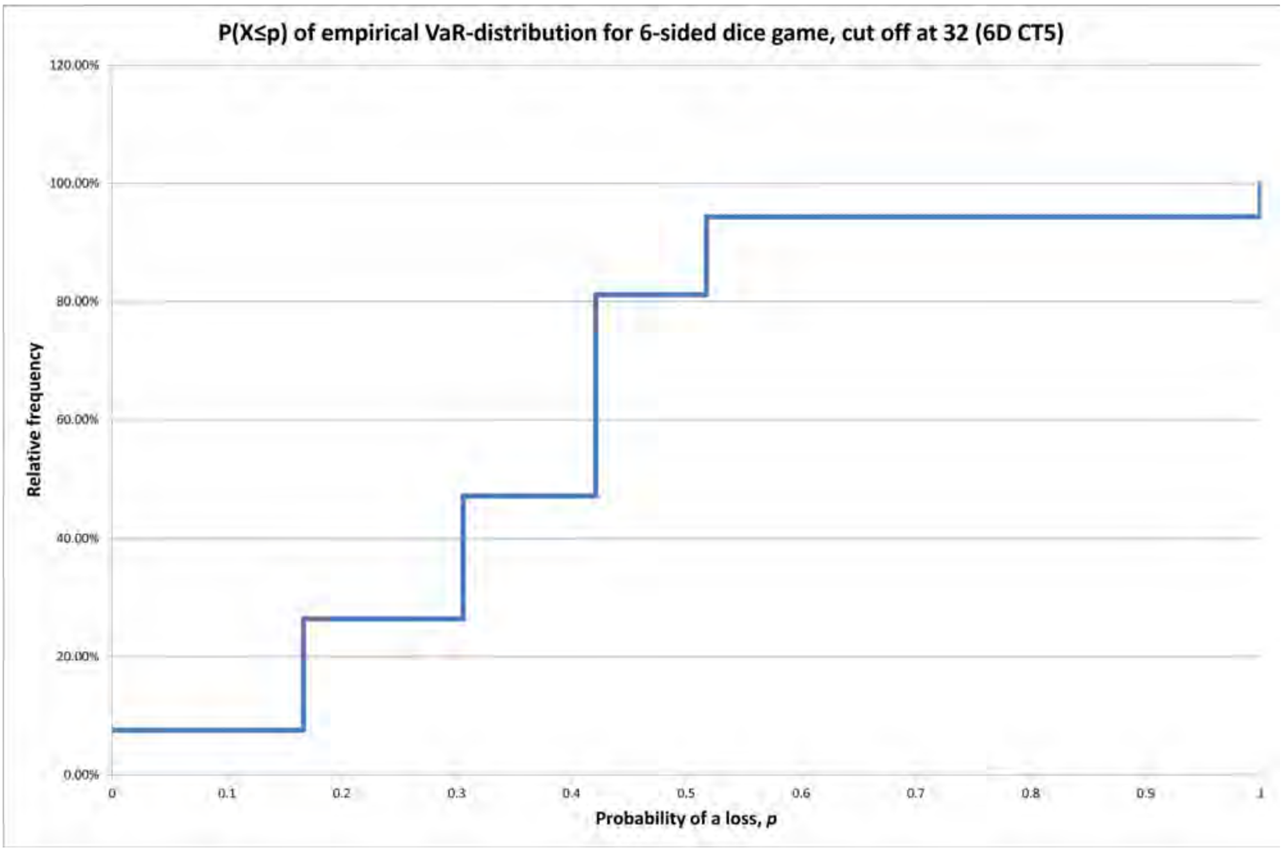

Figure 67: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 6-sided dice game in our survey, cut off at 32 (6D CT5).

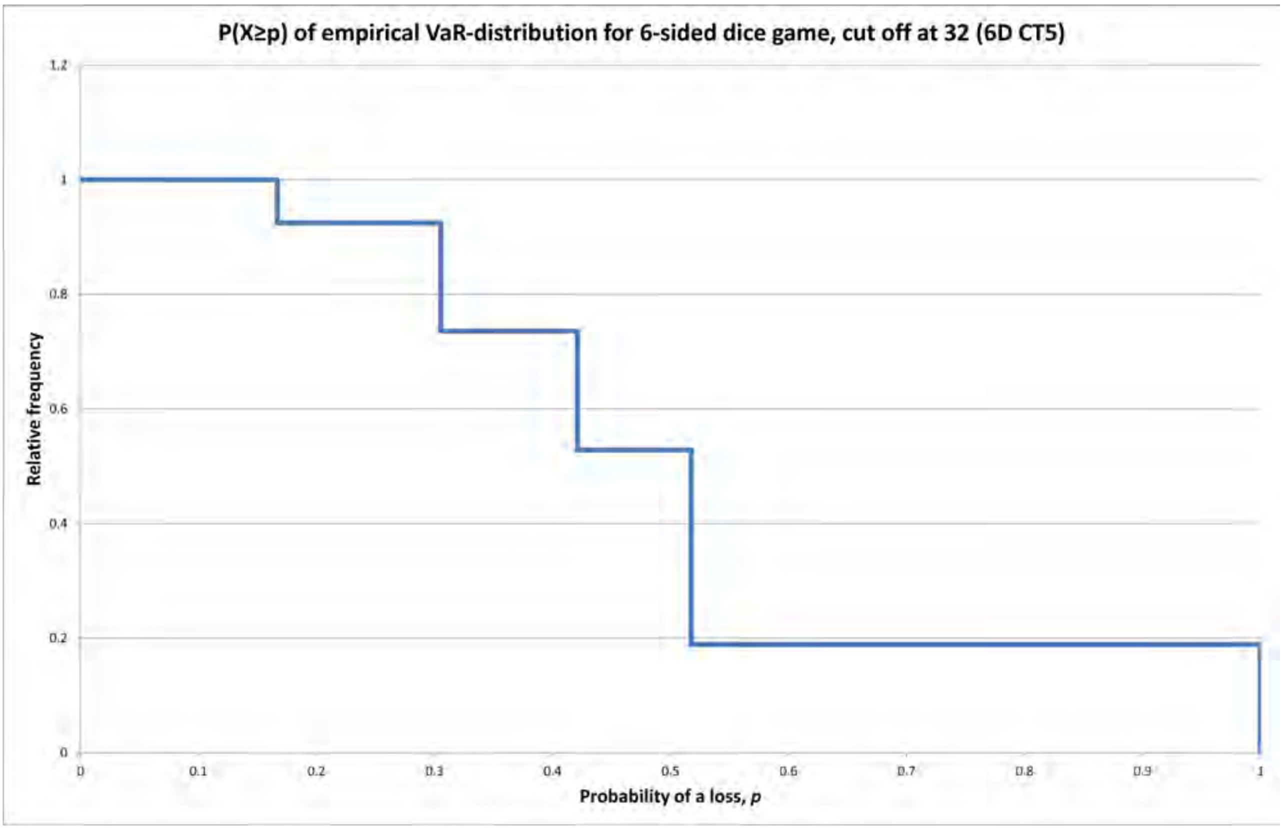

Figure 68: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 6-sided dice game in our survey, cut off at 32 (6D CT5).

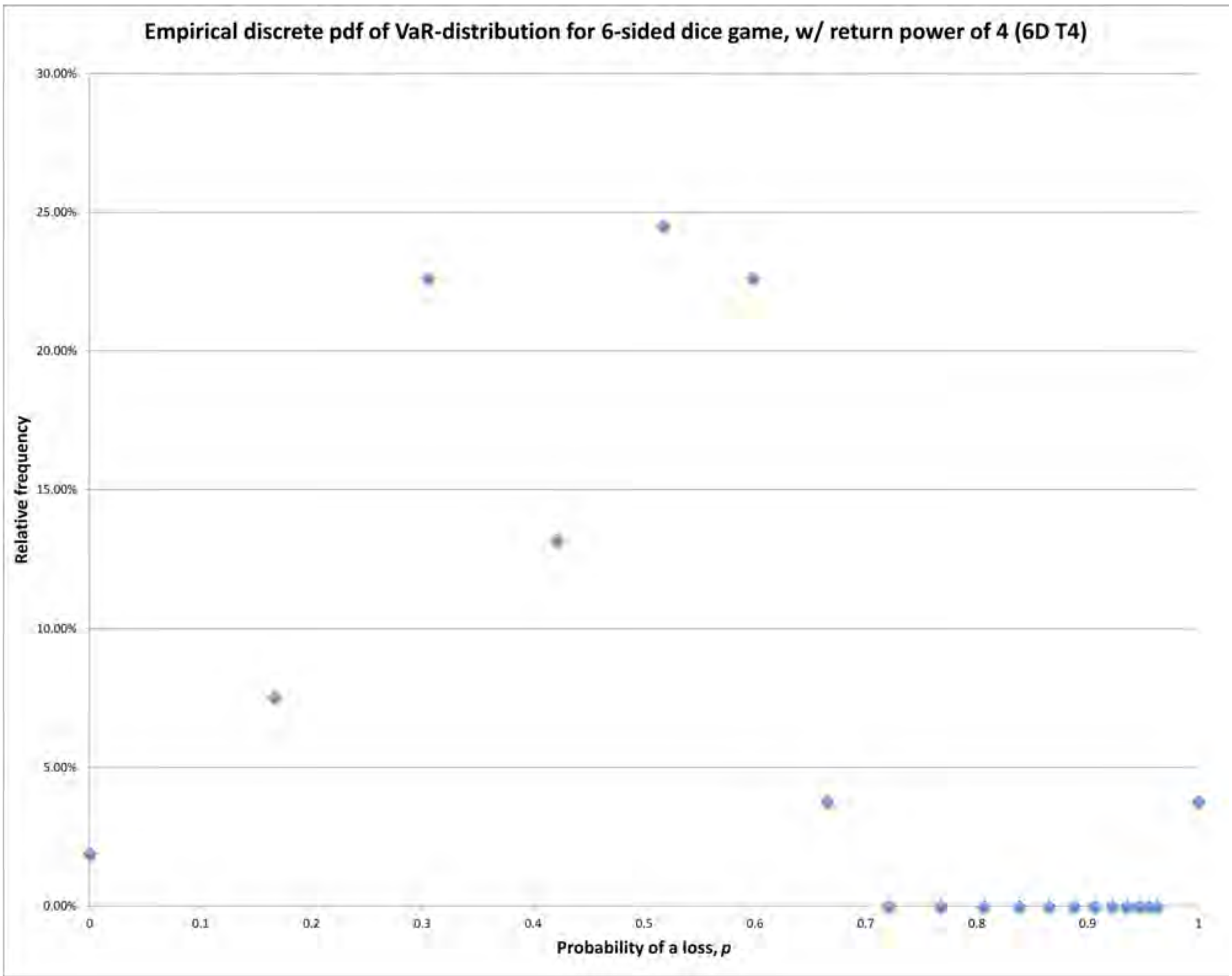

Figure 69: Relative frequency of each bin b_p with p probability of loss, for the 6-sided dice game in our survey, with a power of return of 4 (6D T4).

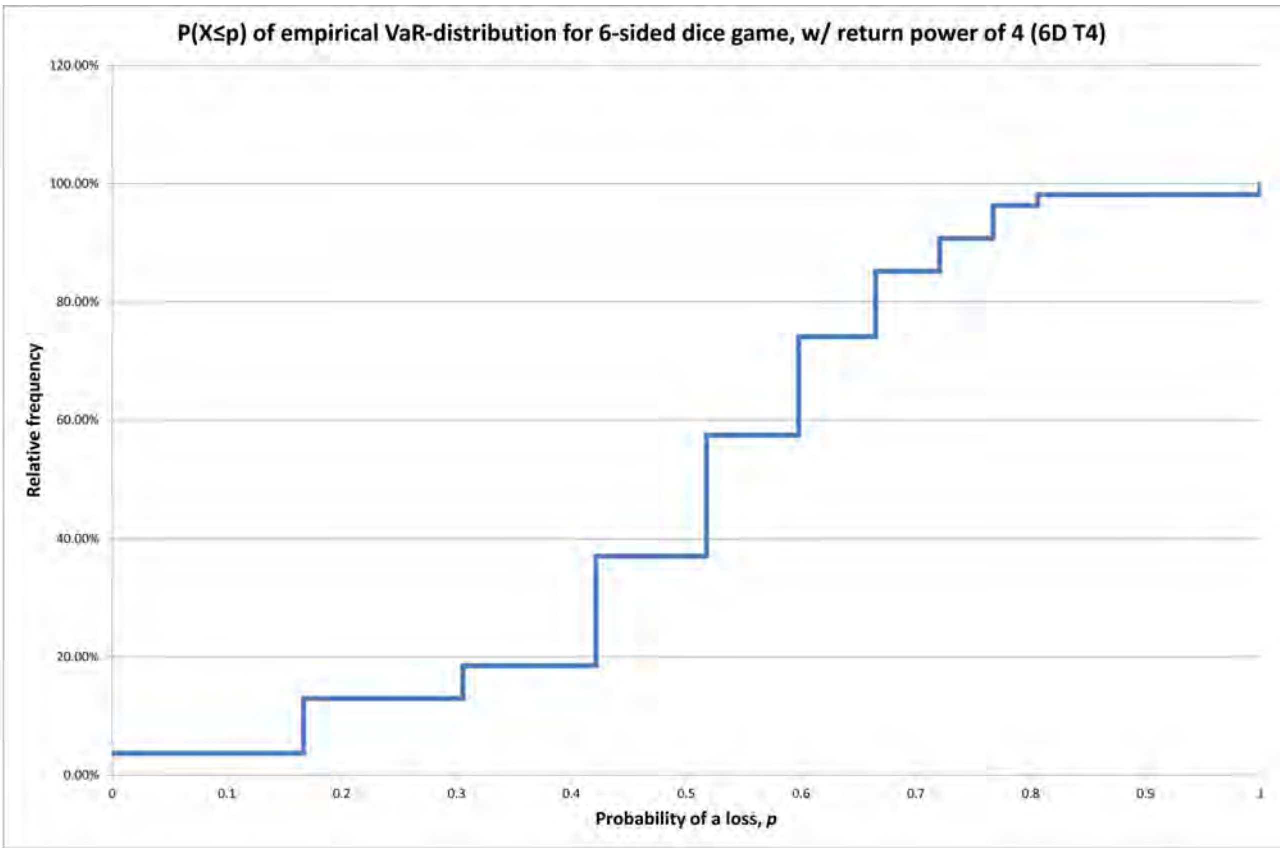

Figure 70: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 6-sided game in our survey, with a power of return of 4 (6D T4).

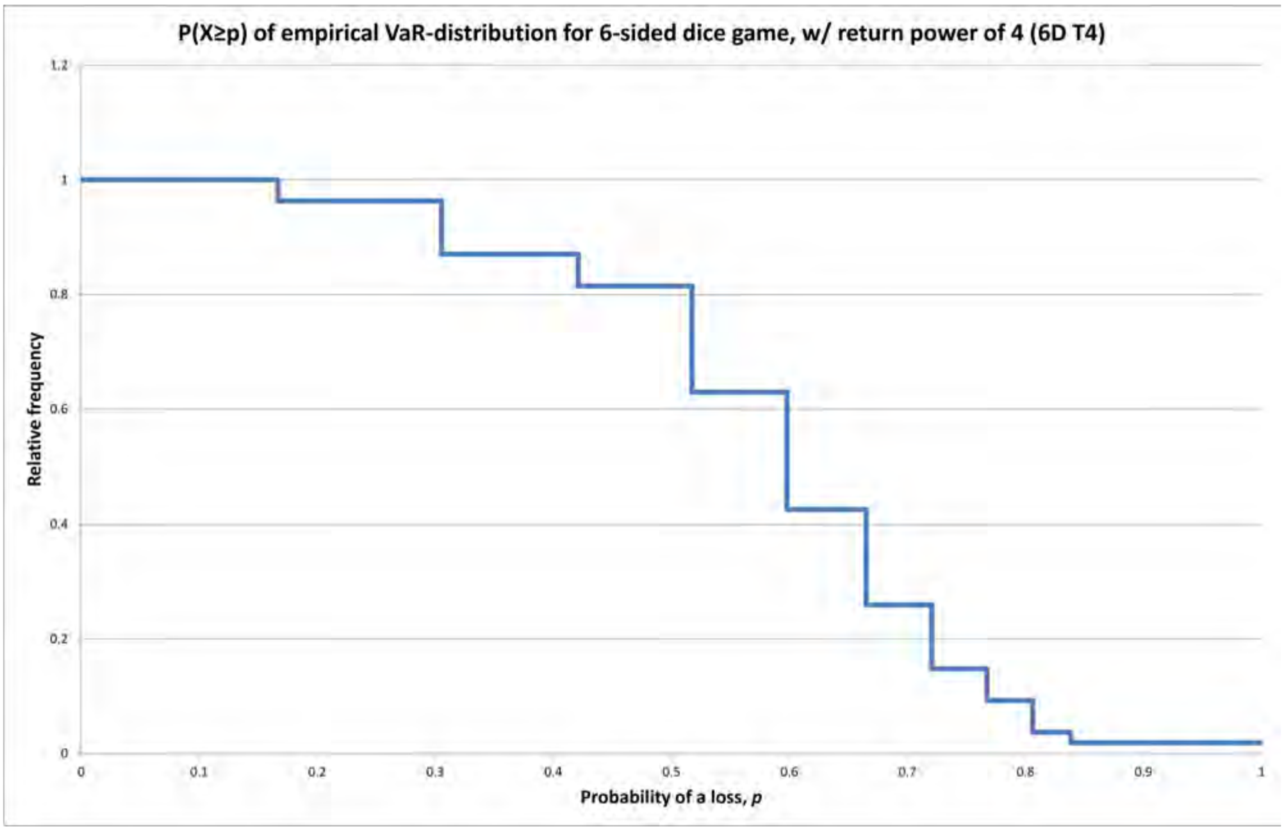

Figure 71: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 6-sided dice game in our survey, with a power of return of 4 (6D T4).

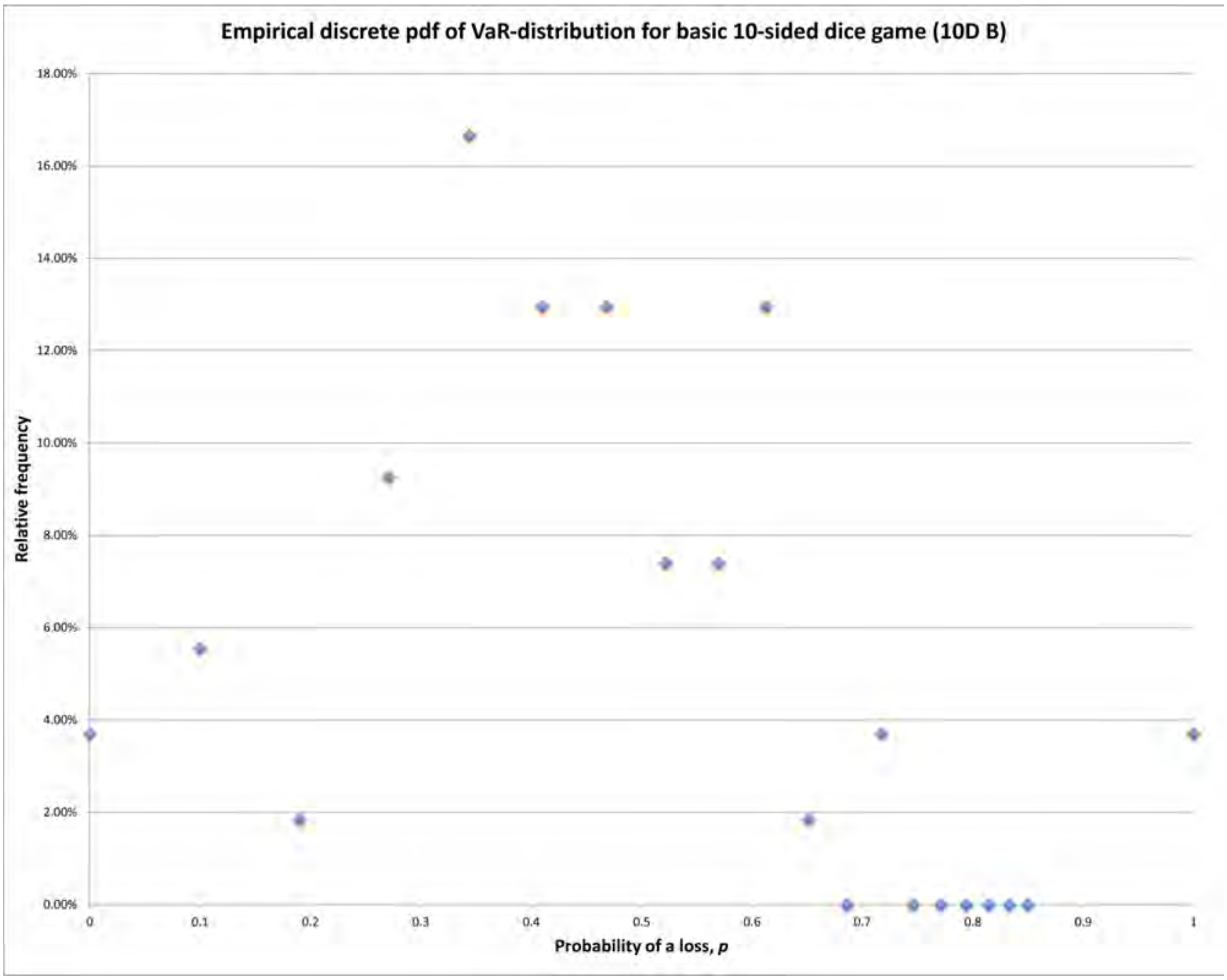

Figure 72: Relative frequency of each bin b_p with p probability of loss, for the basic 10-sided dice game game in our survey (10D B).

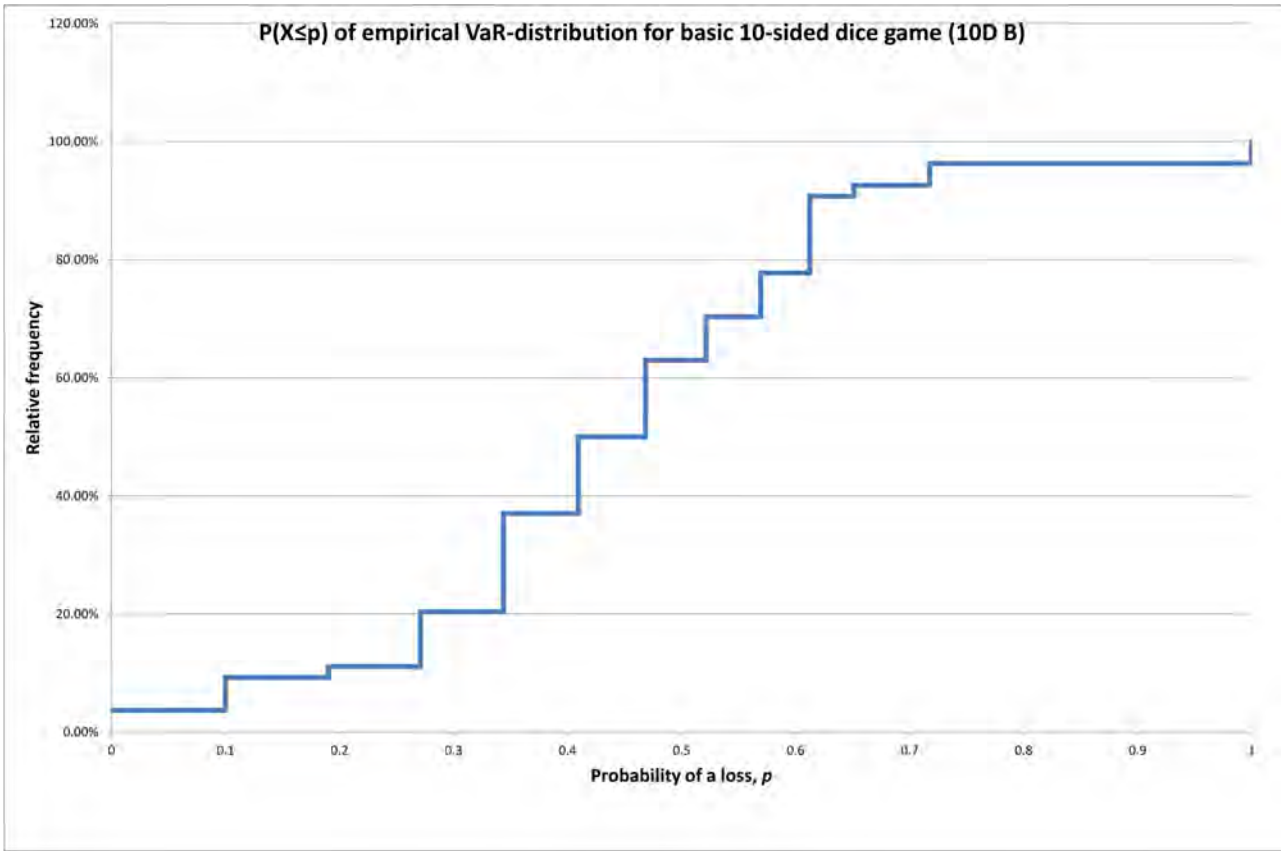

Figure 73: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the basic 10-sided dice game in our survey (10D B).

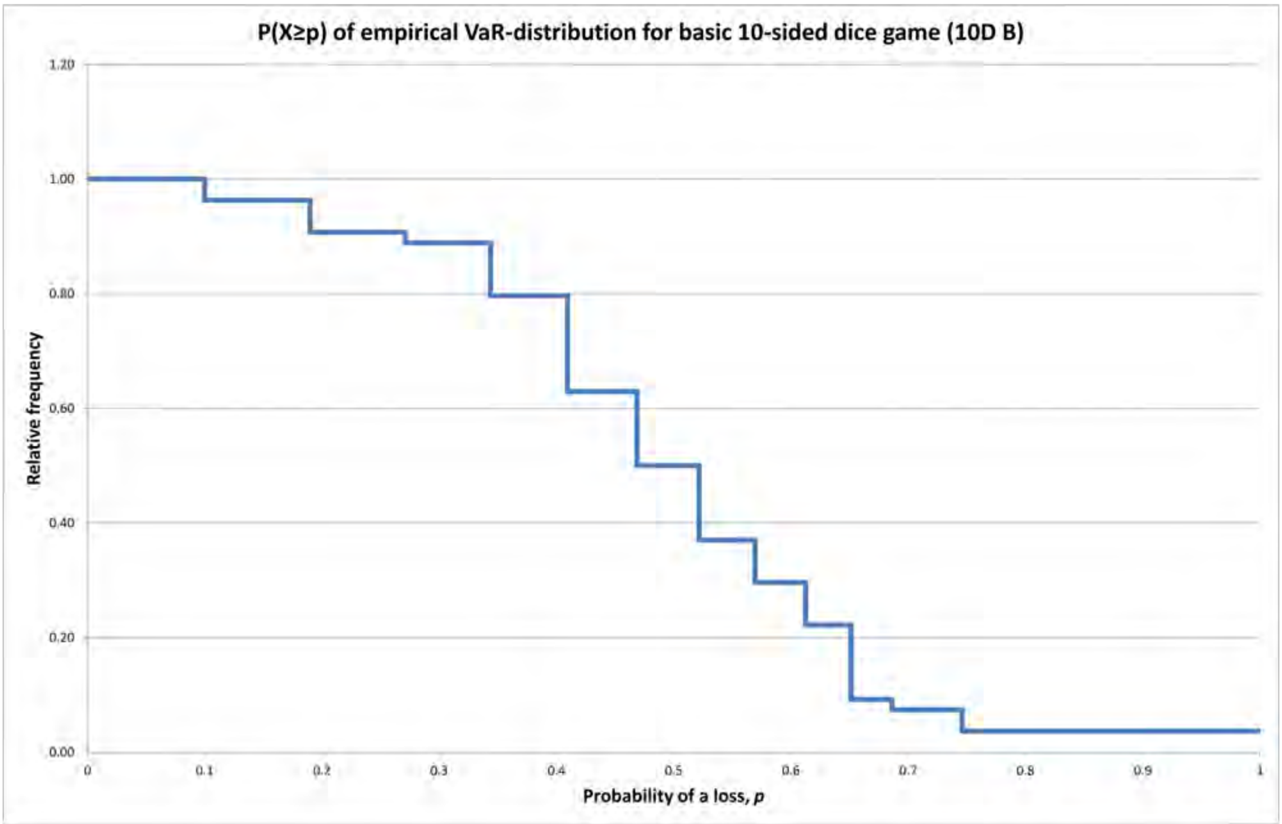

Figure 74: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the basic 10-sided dice game in our survey (10D B).

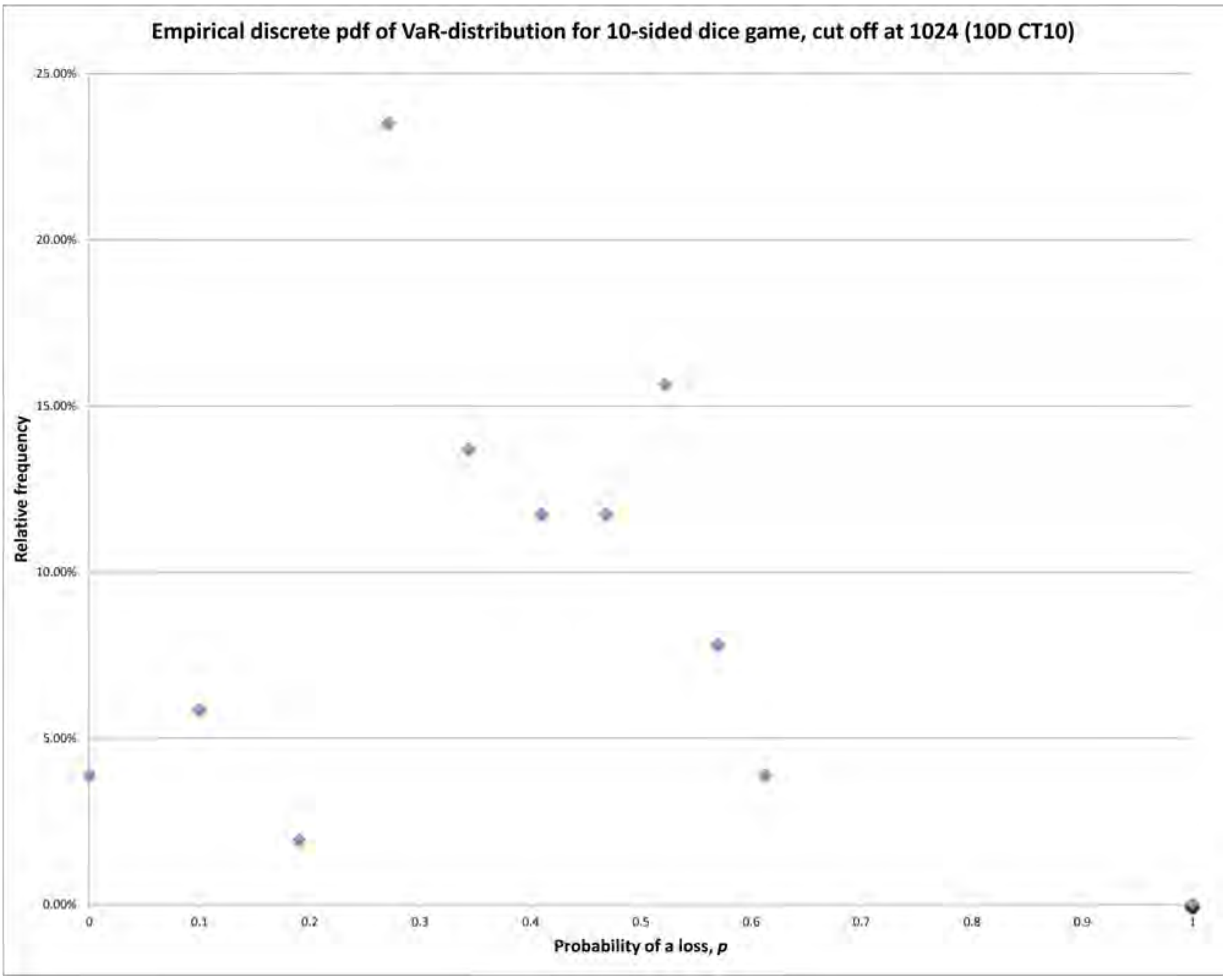

Figure 75: Relative frequency of each bin b_p with p probability of loss, for the 10-sided dice game in our survey, cut off at 1024 (10D CT10).

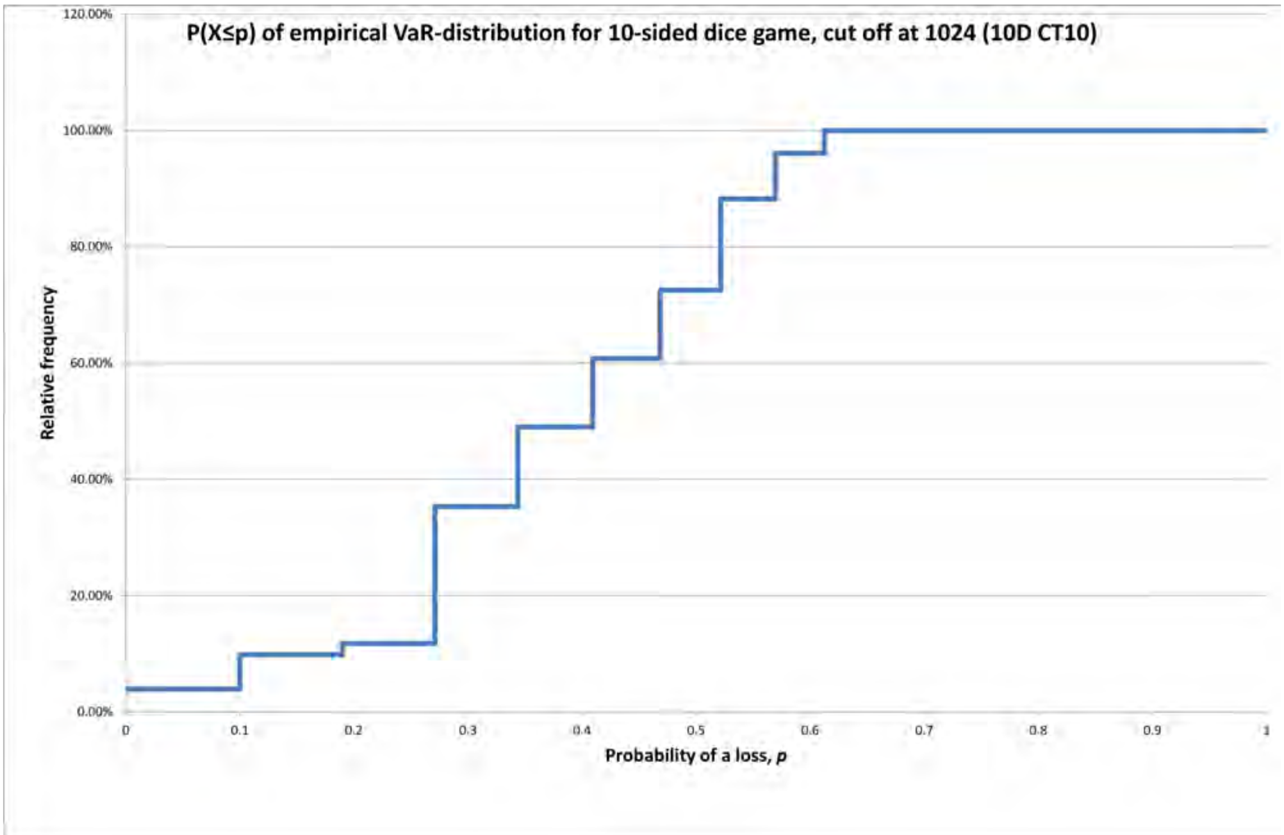

Figure 76: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 6-sided dice game in our survey, cut off at 1024 (10D CT10).

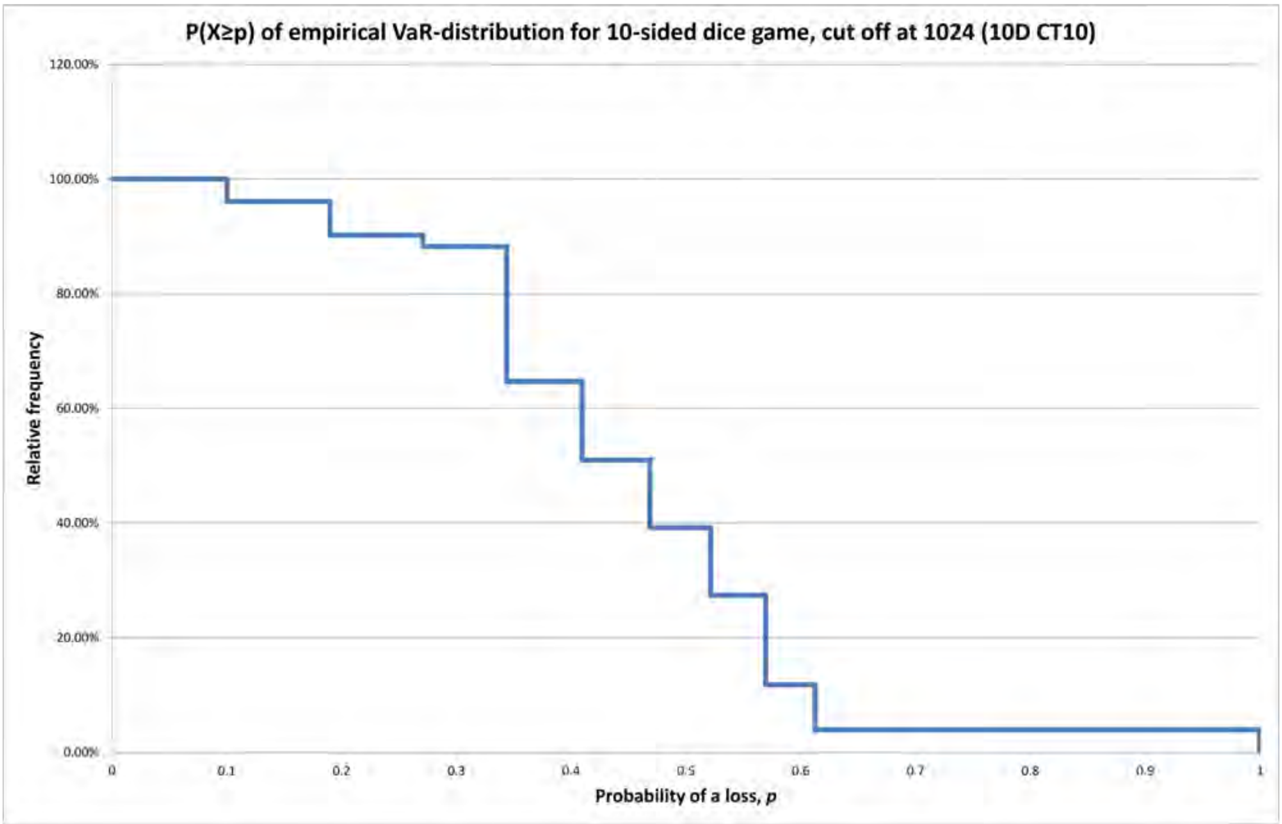

Figure 77: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 10-sided dice game in our survey, cut off at 1024 (10D CT10).

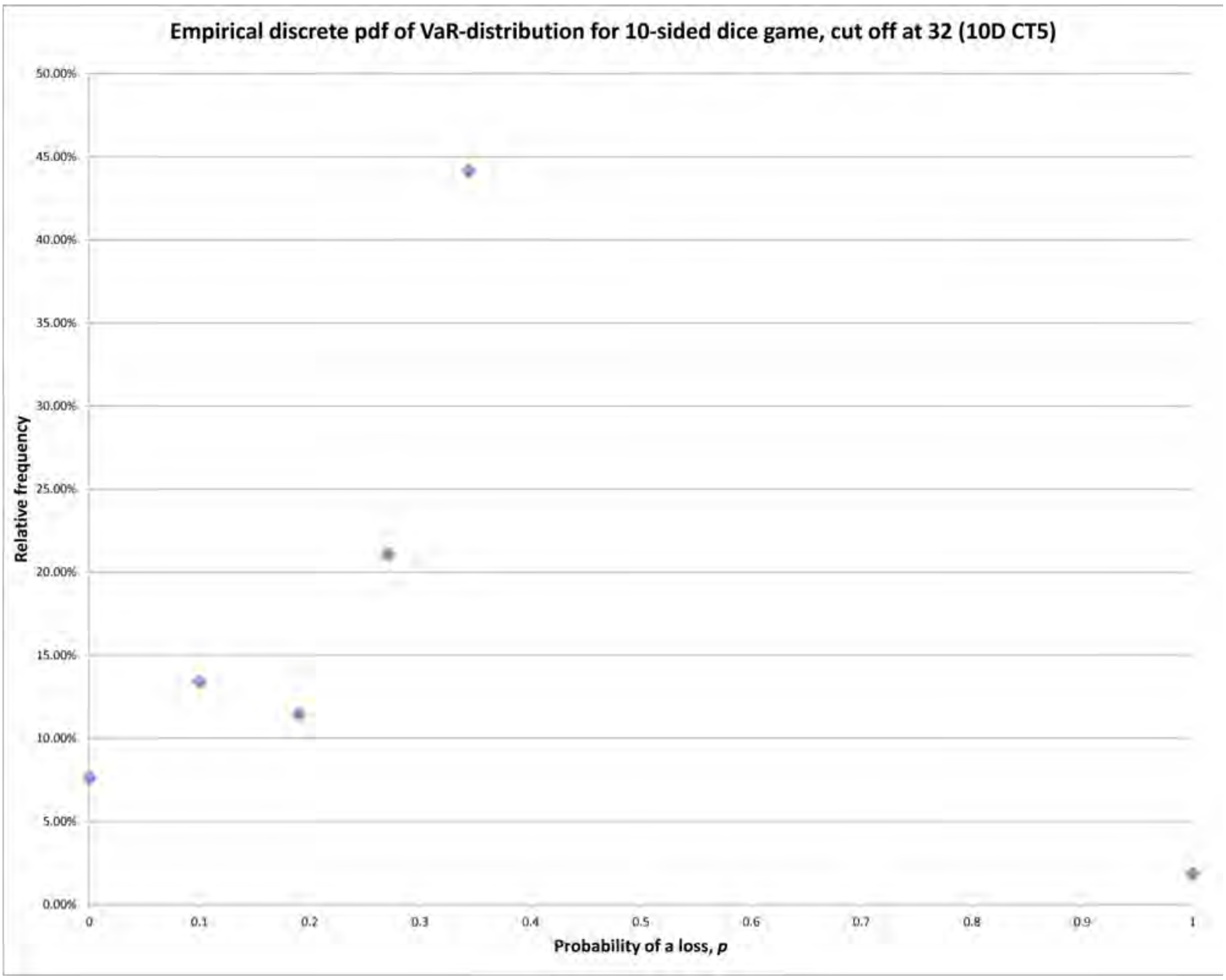

Figure 78: Relative frequency of each bin b_p with p probability of loss, for the coin game in our survey, cut off at 32 (10D CT5).

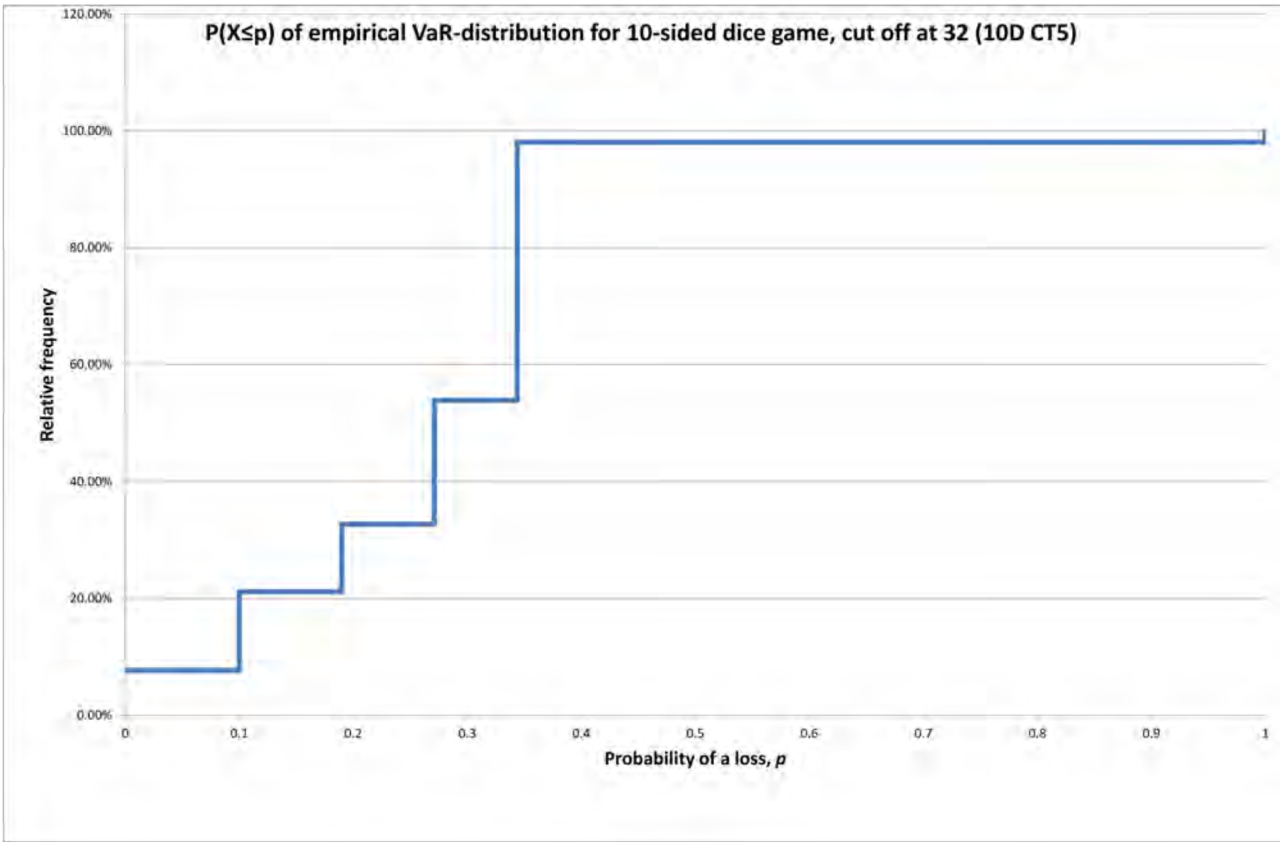

Figure 79: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 10-sided dice game in our survey, cut off at 32 (10D CT5).

Figure 80: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 10-sided dice game in our survey, cut off at 32 (10D CT5).

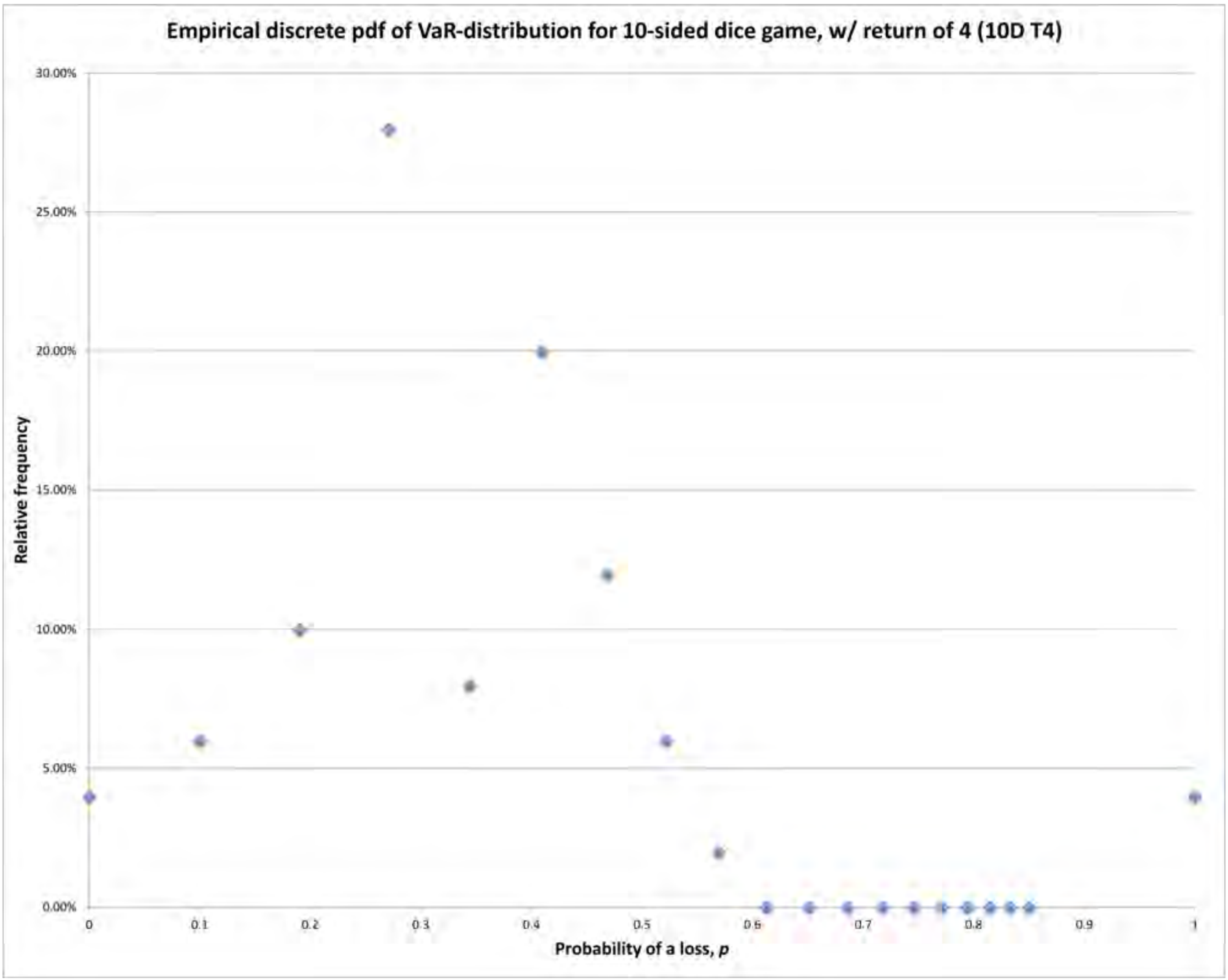

Figure 81: Relative frequency of each bin b_p with p probability of loss, for the 10-sided dice game in our survey, with a power of return of 4 (10D T4).

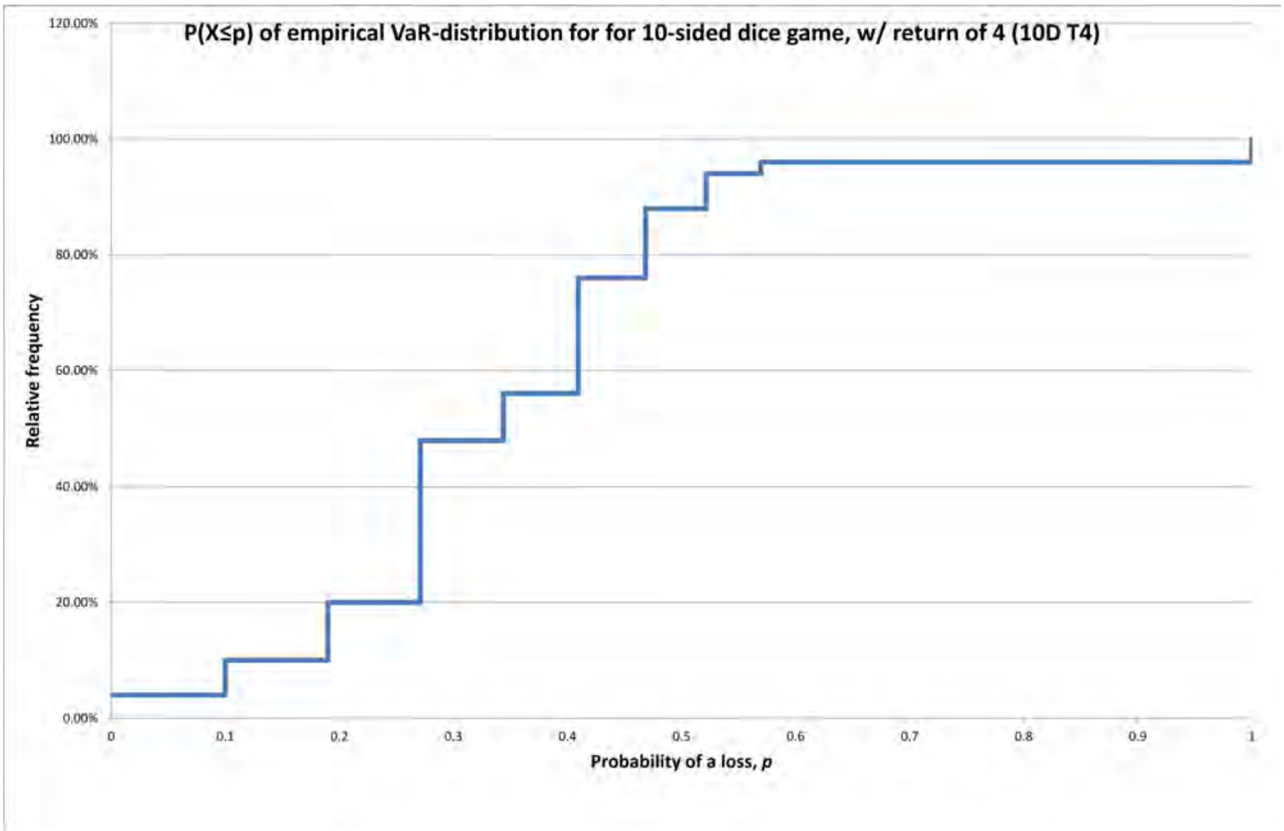

Figure 82: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \leq p$ with p probability of loss, for the 10-sided game in our survey, with a power of return of 4 (10D T4).

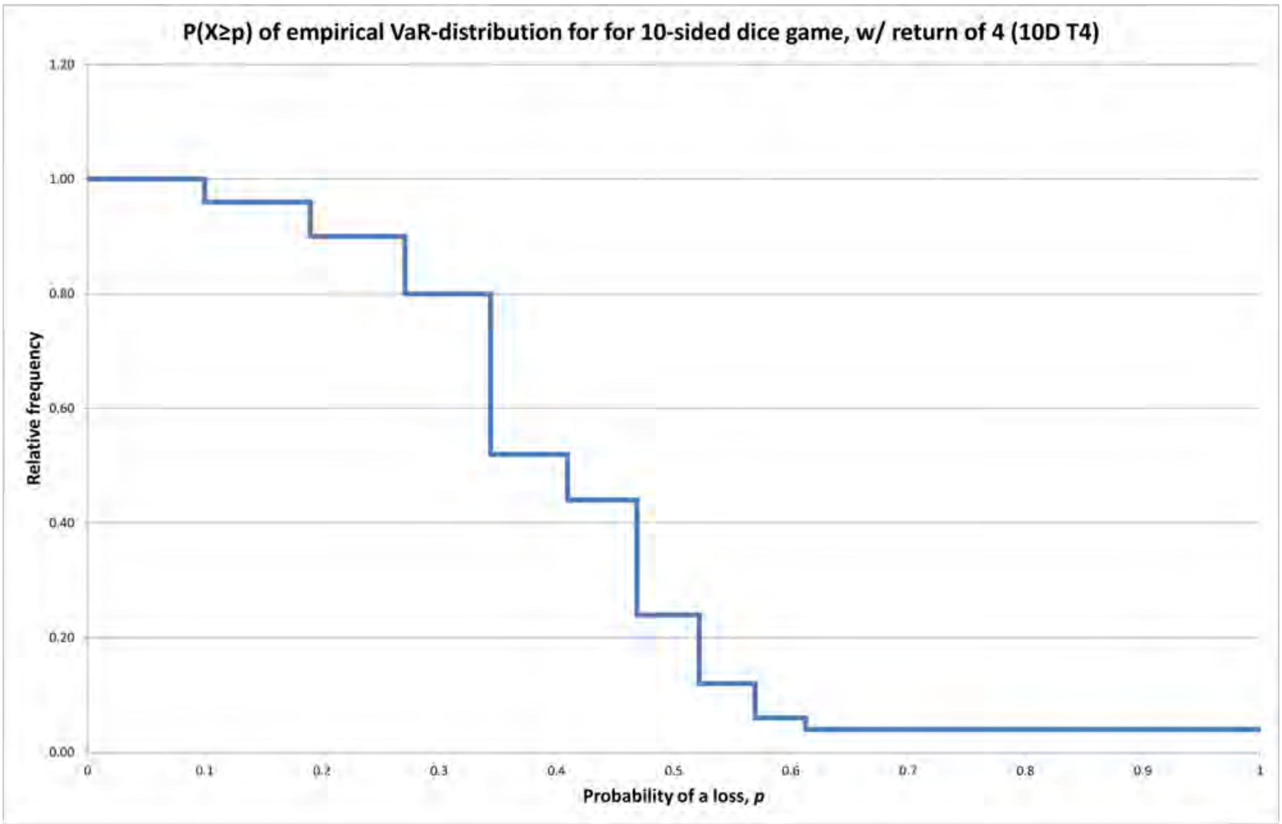

Figure 83: Relative frequency of the summed bids in bins, $\sum b_i$, for $i \geq p$ with p probability of loss, for the 6-sided dice game in our survey, with a power of return of 4 (10D T4).