ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Estimating Risk Preferences in a Portfolio Choice Experiment using Quantum Decision Theory

Master's thesis

Emina Boudemagh
D-PHYS, ETHZ

Submitted on the 13^{th} of February 2019
Revised version (22^{nd} of February 2019)

Supervisor: Prof. Dr. Didier Sornette
Co-supervisor: Giuseppe M. Ferro

Chair of Entrepreneurial Risks, D-MTEC, ETH Zürich

Abstract

In this report, a parametric formulation of Quantum Decision Theory (QDT) is developed to estimate risk preferences from experimental data on portfolio choice. The suggested parametrization extends previous works on risky binary lotteries and constitutes the first operational application of parametric QDT to non-binary choices. Risk preferences are recovered at the aggregate and individual level, using QDT based on Disappointment Aversion Theory (DA). Results are compared to the risk preferences elicited from deterministic and probabilistic versions of DA. QDT and the probabilistic version of DA both report risk and disappointment aversion at the aggregate level, while deterministic DA reports no disappointment aversion but higher risk aversion. Interestingly, QDT performs significantly better than probabilistic DA for two specific subjects, that were considered as outliers in the original experiment paper and explains their choices parametrically. An interpretation for the two introduced quantum parameters is suggested.

Contents

Acknowledgement

I am grateful to Prof. Dr. Didier Sornette for accepting to supervise this thesis and providing me with constructive feedback. I have enjoyed the opportunity of conducting my Semester project and Master's thesis at the Chair of Entrepreunarial Risks, in a studying environment fostering innovation, critical thinking and high-quality scientific research.

I also hereby show my gratitude to my co-supervisor, PhD candidate Giuseppe M. Ferro, for his most appreciated advice and support.

In a previous work on a related topic at the Chair of Entrepreneurial Risks, I have greatly benefited from the supervision of Dr. Tatyana Kovalenko, to whom I also express my gratitude.

Introduction

Most real-world decisions involve a risk component and it is well established that individuals show varying behaviour towards risk. Modeling risk attitudes is therefore a major topic in economics and finance where it finds a wide range of applications (Holt and Laury, 2014). The shortcomings of standard expected utility in describing the differing risk attitudes (Camerer, 1992) gave rise to the development of a large set of extensions to expected utility (Weber and Camerer, 1987). Empirical tests of such models show varying results in the elicited preferences and still occupies a large part of the literature (Hey and Orme, 1994). In this report, risk preferences are estimated at the aggregate and individual level from experimental portfolio choice data.

In the experiment conducted by Choi et al. (2007a), each subject faces 50 randomly generated portfolio choice tasks in which they allocate their wealth between two risky assets. The participants enter their choice through an intuitive graphical interface by clicking the corresponding point on a visualization of the portfolio budget line (Choi et al., 2007b). This experimental setting provides a rich data set of non-binary choices and allows to study preferences at the individual level. The authors of the experiment report high heterogeneity of behaviour between subjects but significant consistency with preference ordering at the individual level. Moreover, a significant fraction of subjects behave according to standard expected utility.

The standard approach to such portfolio choice problems is to test consistency with any form of utility maximization, using the framework of revealed preference theory (Samuelson, 1948). As to parametric recovery of preferences, it is performed via different methods (Halevy et al., 2018). A usual approach is to minimize the error between observed and optimal choices, where optimal choices are calculated by maximizing a deterministic utility function. Following the work of Hey and Orme (1994), some advantages were found in recovering preferences through a probabilistic model of choice using maximum likelihood estimation.

Taking advantage of the probabilistic nature of quantum physics, Quantum Decision Theory (QDT) was introduced by Yukalov and Sornette (2008) as a probabilistic model of choice. By relying on the mathematical structure of Hilbert spaces, QDT can accommodate some famous paradoxes of standard expected utility such as Allais' paradox (Yukalov and Sornette, 2010) and the conjunction fallacy (Kovalenko and Sornette, 2017). The goal of this report is to adapt the parametric formulation of Quantum Decision Theory to elicit risk preferences and describe the choices from the experimental data on portfolio choice gathered by Choi et al. (2007a). The QDT model presented in this report extends previous works on parametric recovery of preferences in risky binary lottery choices (Vincent et al., 2016) and applies QDT to a portfolio choice problem for the first time. The suggested parametrization is based on Disappointment Aversion Theory (DA) and allows to elicit risk and disappointment aversion at the individual and aggregate level. The two additional parameters of the model pertain to the quantum contribution. An interpretation of these parameters is suggested.

The report is organized as follows. First, preliminary notions of traditional decision theory are defined in Section 2, including the standard utility functions and risk aversion measures used in this study. Section 3 presents the mathematical framework of Quantum Decision Theory. Section 4 outlines the experimental setting and data from the portfolio choice experiment by Choi et al. (2007a). In Section 5, the 'classical' methods used to elicit risk preferences, namely deterministic and probabilistic versions of DA, are explained. Section 6 presents the parametrization of QDT suggested to model the portfolio choice problem considered. Section 7 reports the risk preferences elicited at the aggregate and individual level using both 'classical' and quantum methods. Parameter estimation results from the different models are discussed. Finally, a synthesis is provided in Section 8.

Preliminary notions of decision theory

A large body of literature focuses on modeling decisions under risk (Gollier, 2004). This section defines the notions of decision theory that will be used in this report. First, standard utility models and in particular Disappointment Aversion Theory are presented. An overview of probabilistic choice modeling follows.

2.1 Standard utility theory

2.1.1 Expected Utility Theory

Expected Utility Theory (EU) was formalized by Von Neumann and Morgenstern (1944). Consider a lottery $L = \{(x_i, p_i)\}_{i=1}^{N}$ with N outcomes where outcome x_i is associated with probability p_i. The expected utility of lottery L is given by:

$$EU(L) = \sum_{i}^{N} u(x_i)p(x_i) \tag{2.1}$$

where $u(x)$ is a so-called Bernoulli utility function representing how decision-makers value each outcome. The decision-maker behaves according to EU if he chooses the lottery L that maximizes $EU(L)$.

2.1.2 Disappointment Aversion Theory

Disappointment aversion theory (DA) was introduced by Gul (1991) as a generalization of EU with one additional parameter. It therefore represents one of the most parsimonious generalizations of EU. DA accounts for the possible disutility from being disappointed by an outcome. Following Abdellaoui and Bleichrodt (2007), consider a binary choice between two prospects x and y where x is received with probability p and y is received with probability $1 - p$. If the *disappointment component* is y, x is called the *elation component* and the decision-maker evaluates the prospects as follows:

$$DA(x,y) = \gamma(p) \cdot u(x) + (1-\gamma(p)) \cdot u(y) \tag{2.2}$$

with

$$\gamma(p) = \frac{p}{1+(1-p)\cdot\beta} \tag{2.3}$$

where $\beta > -1$ is the coefficient of disappointment aversion. An individual is said to be *disappointment averse* if $\beta \geq 0$ and *elation loving* if $-1 < \beta < 0$.

Since $\gamma(0) = 0$ and $\gamma(1) = 1$, $\gamma : [0,1] \rightarrow [0,1]$ satisfies the weighting probabilities conditions. Disappointment aversion is therefore a special case of Rank-Dependent Utility (Quiggin, 1982). If $\beta = 0$, standard EU is retrieved, as one can see from the reformulation below:

$$DA(x,y) = \frac{p}{1+(1-p)\cdot\beta} \cdot u(x) + \frac{(1+\beta)(1-p)}{1+(1-p)\cdot\beta} \cdot u(y) \tag{2.4}$$

Figure 2.1 shows the behavior of the weighting probability $\gamma(p)$ with respect to p for different values of β.

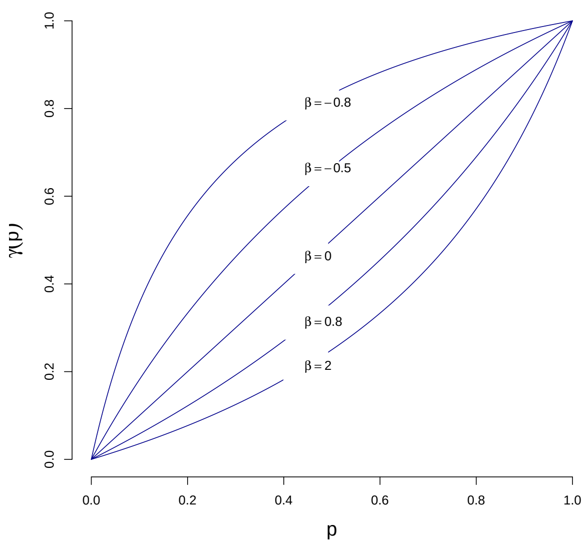
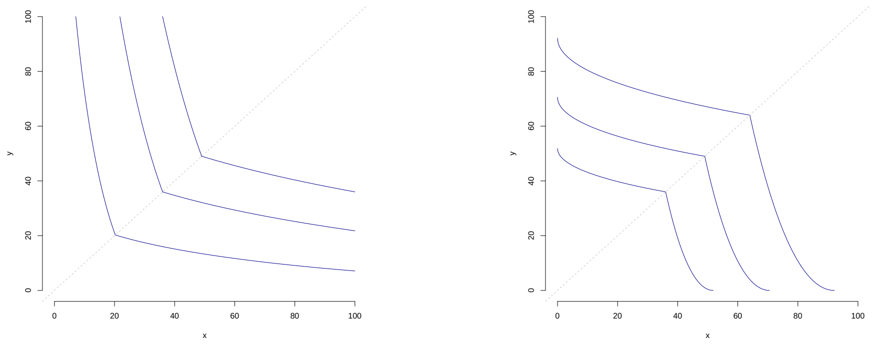


Figure 2.1: Behavior of the weighting probability $\gamma(p)$ for different values of β as a function of the probability p.

The functional form below was used in the paper by Choi et al. (2007a) to recover probabilities:

$$DA(x,y) = \min\{\alpha \cdot u(x) + u(y), \quad u(x) + \alpha \cdot u(y)\} \tag{2.5}$$



(a) Disappointment aversion ($\beta = 2$) (b) Disappointment seeking ($\beta = -0.8$)

Figure 2.2: Indifference curves of Disappointment aversion theory. There is a characteristic 'kink' at the 45-degree line if outcomes x and y are equiprobable

However, for the purpose of this report, the functional form used in Halevy et al. (2018) is preferred, as the disappointment parameter β is more tractable:

$$DA(x,y) = \gamma(p) \cdot u(\max\{x,y\}) + (1-\gamma(p)) \cdot u(\min\{x,y\}) \tag{2.6}$$

with $\gamma(p)$ defined as in 2.3, leading to:

$$DA(x,y) = \frac{p}{1+(1-p)\cdot\beta} \cdot u(\max\{x,y\}) + \frac{(1+\beta)(1-p)}{1+(1-p)\cdot\beta} \cdot u(\min\{x,y\}) \tag{2.7}$$

Figure 2.2 shows some indifference curves generated by equation 2.7. Indifference curves are the values of (x,y) for which the decision-maker is indifferent between prospect x and prospect y, *i.e.* such that $DA(x,y)$ is constant. The value function u chosen is a power function, namely the CRRA utility function equation 2.11. The CRRA specification will be detailed in the next section. If outcomes x and y are equiprobable, the indifference curve of DA shows what is commonly referred to as a 'kink' at the 45-degree line.

DA was chosen in this study because it is one of the most parsimonious generalizations of EU (only one additional parameter) and therefore provides good tractability of the results. Moreover, the retrieval of EU is straightforward ($\beta = 0$). DA was also chosen to perform parametric recovery of risk preferences in the reference paper to this study (Choi et al., 2007a), as well as as in an alternative study replicating the experiment (Halevy et al., 2018).

2.1.3 Measures of risk aversion

Risk aversion plays a central role in explaining economic decisions (Holt and Laury, 2014). The following definitions define two common measures of risk aversion (Gollier, 2004).

Constant absolute risk aversion The Arrow-Pratt Absolute Risk Aversion is defined as:

$$ARA(x) = -\frac{u''(x)}{u(x)} \tag{2.8}$$

The most commonly used utility displaying ARA is Constant Absolute Risk Aversion (CARA).

$$u_{CARA}(x) = -e^{-A \cdot x} \tag{2.9}$$

with $A > 0$.

Constant relative risk aversion The Arrow-Pratt relative risk aversion is defined as:

$$RRA(x) = -\frac{u''(x) \cdot x}{u(x)} = x \cdot ARA(x) \tag{2.10}$$

The most commonly used utility function displaying RRA is Constant Relative Risk Aversion (CRRA):

$$u_{CRRA}(x) = \begin{cases} \frac{x^{1-\rho}-1}{1-\rho} & \text{if} \quad \rho \geq 0, \rho \neq 1 \\ \ln(x) & \text{if} \quad \rho = 1 \end{cases} \tag{2.11}$$

Figure 2.3 shows the behavior of the CRRA utility function for different values of ρ.

In this report, the CRRA utility index will be used. The CARA function has also been implemented but was not retained due to computational issues and no added value to the results. Moreover, the used CRRA utility index is different from the specifications in Choi et al. (2007a) and Halevy et al. (2018). Indeed, for $\rho \neq 1$, using the specification described in Equation 2.11 rather than $u(x) = \frac{x^{1-\rho}}{1-\rho}$ ensures a smooth transition at the discontinuity point $\rho = 1$ which is important for an optimization procedure.

2.2 Probabilistic choice models

Standard utilities mainly have an ordinal purpose, *i.e.* induce an ordering of preferences. Probabilistic choice models however, provide the probability

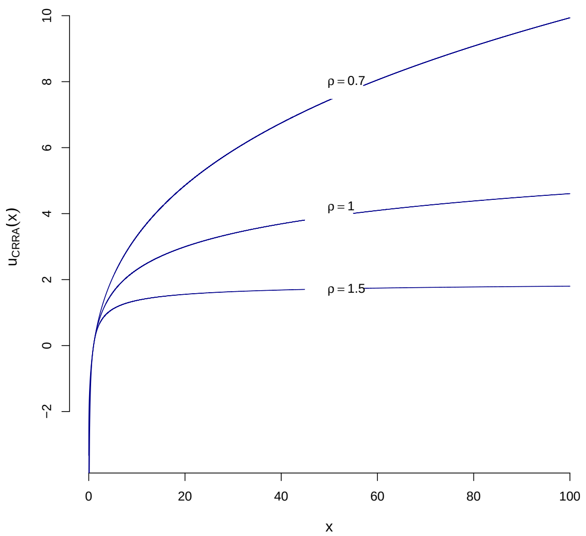


Figure 2.3: Behaviour of the CRRA function for different levels of risk aversion ρ

of choosing a prospect. While utility is traditionally at the center of economic research, the numerous evidences of varying individual behavior has triggered the development of probabilistic choice models.

2.2.1 Random utility theory

Random utility was introduced by Luce (1959) to account for deviations from standard utility. Random utility of a lottery L is expressed as follows:

$$W(L) = U(L) + \varepsilon_L \tag{2.12}$$

Where $U(L)$ is a deterministic utility such as EU or DA. The *disturbance* ε_L is a random component. Therefore, the chosen lottery might not necessarily maximize $U(L)$.

2.2.2 Multinomial logit model

It can be shown (McFadden, 1980) that random utilities are equivalent to practical logit or probit forms. In the case study of this report, the choices are not binary but multiple. Therefore, the appropriate probabilistic formulation is the multinomial logit model. According to Bunch (1987), the probability of choosing lottery L_j given a set of parameters θ and a set of explanatory variables Z is given by:

$$P(L_j|\theta, Z) = \frac{e^{v(\theta, z_j)}}{\sum_k e^{v(\theta, z_k)}} \tag{2.13}$$

where $v(\theta, z_j)$ is a non linear function of the parameters and the explanatory variables. In the probabilistic models studied in this report, the response strength $v(\cdot)$ will be identified with Disappointment Aversion Theory.

Quantum Decision Theory

Quantum Decision Theory (QDT) is a probabilistic choice model relying on the mathematics of Hilbert spaces. It interprets quantum probabilities as behavioural probabilities. QDT was first introduced in (Yukalov and Sornette, 2008) and further developed in a series of articles including (Yukalov and Sornette, 2009, 2010, 2016a). The theory has many applications such as choice reversals in risky lotteries (Vincent et al., 2016), the decoy effect (Yukalov and Sornette, 2016a) and the conjunction fallacy (Kovalenko and Sornette, 2017).

3.1 Motivation for a quantum approach to decision theory

Unresolved paradoxes in decision theory Needless to say the unresolved paradoxes in Expected Utility Theory call for the development of further theories. Among the most famous violations to standard utility theory stand the Allais (1990) and Ellsberg (1961) paradoxes. Recently, quantum-like approaches to decision theory have been developed to address such shortcomings. They however face difficulties in entering the economic literature as one, due to a high heterogeneity in their formulations. Indeed, there are many different views on the possible advantages of a quantum model in decision theory.

A 'measurement' problem In a preliminary work (Semester Project) the following motivation was highlighted. In physics, the principle of the *Received View* (Carnap, 1923) states that any concept used in a theory should be related to observables, *i.e.* a quantity that can be measured. If a parallel with decision theory was to be made, the observables in an experiment would be the choices, *i.e.* the revealed preferences. Since the utility function mainly has ordinal use, it is not possible to directly measure it. The reader is referred to (Gilboa, 2009), p.59-71, for a detailed discussion. Probabilities of choice are however directly measurable in an experiment, whether it is in the form of the frequency of choice within a society or with respect to an individual facing many decisions. The link between utility and behavioral probabilities is not straightforward and was famously axiomatized by Luce (Luce, 1959), showing the complementarity of these two notions. One may find useful to rely on the mathematical framework of quantum mechanics to develop probabilistic choice models rather than utility-based models. An example of such a quantum-like model is QDT. In particular, the notion of *inconclusive events*, detailed later in this section, comes as a natural answer to what can be interpreted as a 'measurement problem' in Economics: the difficulty to elicit utility from observed choices.

Intrinsic probabilistic nature of quantum mechanics When considering probabilistic choice models, it is only natural to make a link with the framework to quantum theory, which is probabilistic in essence. However, as discussed in Vincent et al. (2016) the stochastic nature is distinct in the two approaches. In random utility theory it is an added term while in Quantum Decision Theory, it arises from the mathematics of the Hilbert space in the form of an interference term.

The classical limit Last but not least, recovering the classical limit is an important criterion in any theory of modern physics. In the context of Quantum Decision Theory, the classical limit is attained when the interference term vanishes and one retrieves a 'classical' probabilistic model just as commonly used in economics. Therefore, Quantum Decision Theory offers great possibilities of generalization of EUT and in particular in resolving the above mentioned paradoxes also from a normative perspective.

3.2 Mathematical formalism of QDT

The basic scheme exposed below is inspired from Yukalov and Sornette (2018). For a detailed explanation of the quantum probabilities underlying the theoretical foundations of QDT, the reader is referred to Yukalov and Sornette (2016b).

Lotteries Let $\{L_n\}$ be a set of N lotteries with k outcomes such that

$$L_n = (x_i, p_n(x_i)) \quad \text{where} \quad i = 1, ..., k \text{ and } \sum_i p_n(x_i) = 1 \tag{3.1}$$

Conclusive operationally testable event The event A_n with $n = 1, .., N$ corresponds to the action of choosing lottery L_n and is represented by the state vector $|n\rangle$ pertaining to a Hilbert space $\mathcal{H}_A = \text{span}_n\{|n\rangle\}$. A_n is considered a *conclusive operationally testable event* in the sense that it is possible to observe it in experiments in the form of a revealed preference. The A_n are mutually exclusive events and the set $\{|n\rangle\}$ thus forms an orthonormal basis of $\mathcal{H}_A$.

Inconclusive event The choice of a lottery is accompanied by objective risk (the outcome is not known with certainty), and subjective uncertainty (in many possible forms: incorrect understanding of the setting, uncertainty on the outcomes probabilities, abilities of the decision-maker, emotions, biases etc.). Risk and uncertainty are both represented by a set of 'uncertain' items $\mathcal{B} = \{|\alpha\rangle : \alpha = 1, ...\}$. A so-called *inconclusive event* B is represented by a vector in the Hilbert space $\mathcal{H}_B = \text{span}_\alpha\{|\alpha\rangle\},$:

$$|B\rangle = \sum_\alpha b_\alpha |\alpha\rangle \tag{3.2}$$

where b_α are random complex numbers. Inconclusive events are not testable in experiments.

Prospect The choice of a lottery L_n is thus considered a composite event termed *prospect* consisting of a final choice A_n after deliberations involving the set of uncertain events $\mathcal{B}$. A composite prospect state is defined as an element of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \text{span}_{n\alpha}\{|n\alpha\rangle\}$ where $|n\alpha\rangle = |n\rangle \otimes |\alpha\rangle$:

$$|\pi_n\rangle = |n\rangle \otimes |B\rangle = \sum_\alpha b_\alpha |n\alpha\rangle \tag{3.3}$$

The $\{|n\alpha\rangle\}$ are called *elementary prospects* and form an orthogonal basis of $\mathcal{H}$ (not necessarily orthonormalized). The prospect operator plays the role of the observable (operators) in the theory of quantum measurement, as follows:

$$\hat{P}_n = |\pi_n\rangle \langle\pi_n| \tag{3.4}$$

The $\{\hat{P}(\pi_n)\}$ composes a positive operator-valued measure (POVM).

Strategic state of mind The decision-maker is characterized by a strategic state of mind represented by an element $|\psi\rangle$ of $\mathcal{H}$, as follows:

$$|\psi\rangle = \sum_{n,\alpha} \zeta_{n\alpha} |n\alpha\rangle \tag{3.5}$$

Probability of choice It then becomes possible to define a quantum behavioral probability for each prospect π_n:

$$p(\pi_n) = \langle\psi|\hat{P}_n|\psi\rangle = \underbrace{\sum_\alpha |\zeta_{n\alpha}b_\alpha|^2}_{f(\pi_n)} + \underbrace{\sum_{\alpha\neq\beta} \zeta_{n\beta}b_\alpha\zeta^*_{n\alpha}b^*_\beta}_{q(\pi_n)} \tag{3.6}$$

As a probability measure, the family of $p(\pi_n)$ satisfies:

$$\sum_n p(\pi_n) = 1, \quad 0 \leq p(\pi_n) \leq 1 \tag{3.7}$$

The quantum probability splits into two terms: $p(\pi_n) = f(\pi_n) + q(\pi_n)$. It is then possible to associate $p(\pi_n)$ to a behavioral probability, as it takes into account a rational utility factor and an irrational interference term.

Utility factor The utility factor f represents the classical objective utility of the prospect:

$$f(\pi_n) = \sum_\alpha |\zeta_{n\alpha}b_\alpha|^2 \tag{3.8}$$

satisfying

$$\sum_n^N f(\pi_n) = 1 \quad \text{and} \quad 0 \leq f(\pi_n) \leq 1 \tag{3.9}$$

Attraction factor The attraction factor represents the subjective attitudes towards that prospect:

$$q(\pi_n) = \sum_{\beta\neq\alpha} \zeta_{n\beta}b_\alpha\zeta^*_{n\alpha}b^*_\beta \tag{3.10}$$

Moreover, equations 3.2, 3.7 and 3.9 lead to the following properties for q:

$$\sum_n^N q(\pi_n) = 0 \quad \text{and} \quad -1 \leq q(\pi_n) \leq 1 \tag{3.11}$$

The latter property corresponding to the *alternation law*.

Preference model The prospect probability allows for preference ordering as follows:

$$\pi_1 \succeq \pi_2 \iff p(\pi_1) \geq p(\pi_2) \tag{3.12}$$

Moreover, prospect π_1 will be considered *more useful* than π_2 if $f(\pi_1) > f(\pi_2)$ and *more attractive* if $q(\pi_1) > q(\pi_2)$.

Description of the portfolio choice experiment under risk by Choi et al. (2007a)

In this section, the data set used to recover risk preferences in portfolio-choice is presented.

The experimental setting introduced by Choi et al. (2007b) is a general graphical interface for generating experimental portfolio choice data. It was used to recover risk preferences in Choi et al. (2007a) and Halevy et al. (2018), as well as to recover ambiguity preferences in Ahn et al. (2014).

4.1 Experimental setting

The experiment was conducted at the Experimental Social Science Laboratory (Xlab) at the University of California, Berkeley. Participants face 50 independent decision tasks. Each task consists in the following portfolio choice problem: the subject must allocate his wealth between two assets (risky securities) under budget constraint.

- Asset x (resp. Asset y) is an Arrow security that pays 1$[1] with probability π_x (resp. prrobability π_y). p_x (resp. p_y) is the price of asset x (resp. y).
- At each decision round, a portfolio is randomly generated by drawing the prices p_x and p_y [2].
- The participants can buy x and y units of securities x and y such that $0 \leq x \leq \frac{1}{p_x}$ and $0 \leq x \leq \frac{1}{p_x}$ satisfying the budget constraint $p_x \cdot x +$

[1] In the original experiment, the currency unit are Tokens and the payoff in $ is calculated using the correspondence 1 Token = 0.50 $. For simplicity 1 Token = 1$ in this report

[2] Additional constraint on the prices: at least one of the asset prices p_k is such that $\frac{1}{p_k} \geq 50\$$ and all asset prices p_i should satisfy $\frac{1}{p_i} \leq 100\$$

$p_y \cdot y = 1$.

- At the end of the experiment, one of the decision round will be randomly chosen to be payoff relevant and the state of the world (x or y) will be revealed. The participants get the returns $x \cdot 1\$$ if state x pays off (with probability π_x) and $y \cdot 1\$$ if state y pays off (with probability π_y).

Subjects choose a portfolio on the budget set by using a graphical interface with a simple 'point-and-click' method. Figure 4.1 below shows the graphical interface that participants use to choose a portfolio.

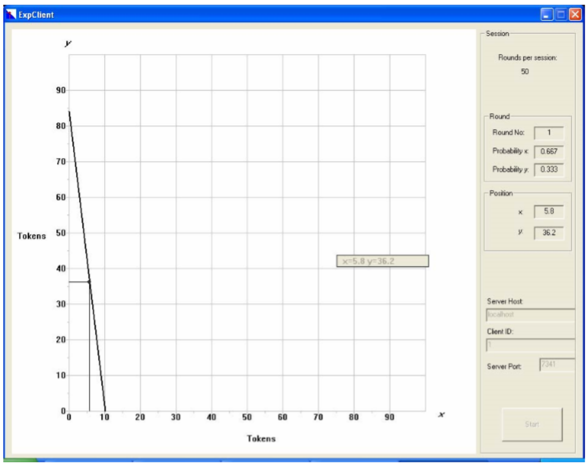


Figure 4.1: Graphical interface used to generate portfolio-choice data in the Xlab. Portfolio choice problem with two risky assets. The line defines the budget set for the subject. Figure from the appendix of Choi et al. (2007a).

Variations The authors offer treatment variations as described below. This study focuses on the symmetric treatment under risk[3].

- *Risk (symmetric)* Allocation between two Arrow securities x and y with probabilities of states $\pi_x = \pi_y = \frac{1}{2}$ (see Choi et al. (2007a))
- *Risk (asymmetric)* Allocation between two Arrow securities with probabilities of states $\pi_x = \frac{1}{3}$ and $\pi_y = \frac{2}{3}$ (see Choi et al. (2007a))
- *Uncertainty* Allocation between three Arrow securities. Security x pays off with probability $\pi_x = \frac{1}{3}$ and π_y and π_z are unknown ($\pi_2 + \pi_3 = \frac{2}{3}$.) (see Ahn et al. (2014))

[3]The symmetric treatment under risk of the experiment by Choi et al. (2007a) is also the treatment studied in Halevy et al. (2018)

4.2 Main features of the data

Some figures When taking into account both symmetric and asymmetric risky treatments, there are 93 subjects facing 50 independent decisions leading to a total of 4650 available observation points. In the scope of this study, only the symmetric treatment under risk will be considered. Therefore the analysis relies on **47 subjects** facing **50 decisions** each, for a total of **2350 observation points**.

Typical portfolio choices Figure 4.2 below illustrates examples of typical portfolios a subject could choose. Point A (resp. point B) corresponds to an individual investing all his wealth in the asset that pays in state x (resp. y) and are called *boundary* portfolios. Point C corresponds to the intersection of the budget line with the 45-degree line. In the symmetric treatment (where $\pi_x = \pi_y = \frac{1}{2}$) point C corresponds to a *safe* portfolio since the return $x = y$ is certain. Any portfolio that is not *boundary* or *safe* is called *intermediate*.

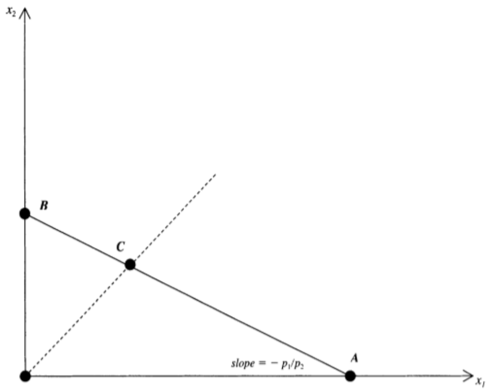


Figure 4.2: Figure from Choi et al. (2007a). Portfolio budget line with slope $-\frac{p_x}{p_y}$. Point A and B correspond to *boundary* portfolios. In the symmetric treatment, point C corresponds to a *safe* portfolio with certain return. A portfolio that is not *boundary* or *safe* is called *intermediate*.

Oberved choices Figure 4.3 describes the relationship between the log-price ratio $\ln(\frac{p_x}{p_y})$ and the relative demand $\frac{x}{x+y}$ for a selected set of representative subjects[4]. The following description is reported by Choi et al. (2007a). Some subjects show a stylized behaviour whereas for others, the consistency in behaviour is less precise. In particular, ID 304 always chooses safe portfolios with certain payoff $x = y$. This is consistent with infinite risk aversion. Alternatively, subject ID 303 is the only subject, who made almost

[4]The set of selected subjects is the same as in Choi et al. (2007a).

always equal expenditures $p_x \cdot x = p_y \cdot y$. Subject ID 307 would be consistent with risk neutrality. The subject allocated all his tokens to the cheapest security. When prices were approximately equal, the subject chose safe portfolios. The switch in behaviours manifests loss or disappointment aversion. Subjects ID 216 and 318 do not behave according to such clear patterns and combines safe, intermediate and boundary portfolios.

Main results The authors report high heterogeneity at the aggregate level but high consistency with preference ordering at the individual level. Indeed, the homogeneity of choices is statistically significant at the individual level. Figure 4.3 shows a set of selected representative subjects. They show a high consistency in their choices but display very different behavior from one another. Interestingly, the authors report that almost the majority of subjects behave according to EU. The authors further report that the preferences are well represented by Disappointment Aversion Theory due to the 'kink' in the model (see indifference curves of DA in Figure 2.2). The authors conclude that there is no clear taxonomy that allows to classify all subjects. However subjects seem to behave along a mixture of stylized facts.

4.3 Advantages of studying the data set

This data set was chosen for the present study because it presents significant advantages. Advantages were found with respect to the economic literature of risk preference elicitation from experimental data but also given the current state of the parametric analysis of data sets in Quantum Decision Theory.

- The authors present a standard decision problem that can be interpreted either as a portfolio-choice problem (allocation of wealth between assets) or as a consumer decision problem (selection of a bundle of state-contingent commodities with budget constraint.)
- The authors claim portfolio choice under budget constraint gives more information than binary choices. Indeed, in the present experiment, a revealed preference is considered preferred to all other preferences ad not only to one other alternative.
- As pointed out by the authors of Choi et al. (2007b), the majority of experimental studies of violations of EUT reviewed in Camerer (1992) focus on binary choices and specific experimental designs studying violated axioms of EUT. The presente framework is less specific. Moreover, there are not many data sets that allow to assess the heterogeneity of preferences at the individual level.
- The unified framework of the graphical interface for portfolio choice allows for many variations and is therefore easily adaptable to cus-

tomized experiments. Also on the descriptive side, it is possible to test many theories against the same data set.

- The experiment is very well cited in the literature of experimental studies of decisions under risk and uncertainty. The quality of data and instructions meets the expectations of top publication journals in economics. Experimental design is well argued with theoretical arguments of revealed preference theory (Choi et al., 2007b).
- The data set is available online. Reproducibility of results allows for different researchers to study the same data set or experimental framework (Halevy et al., 2018) with different research questions and methods.
- For QDT in particular, it is the occasion to model a portfolio-choice experiment for the first time. In the future, it will be possible to vary the treatments under study (asymmetric, uncertainty) for a better refinement and interpretation of the parametric calibrations.
- The fact that almost a majority of subjects behaves according to EUT, but that outlier stylized behaviours are observed allows for a good starting point for a parametric model calibration.

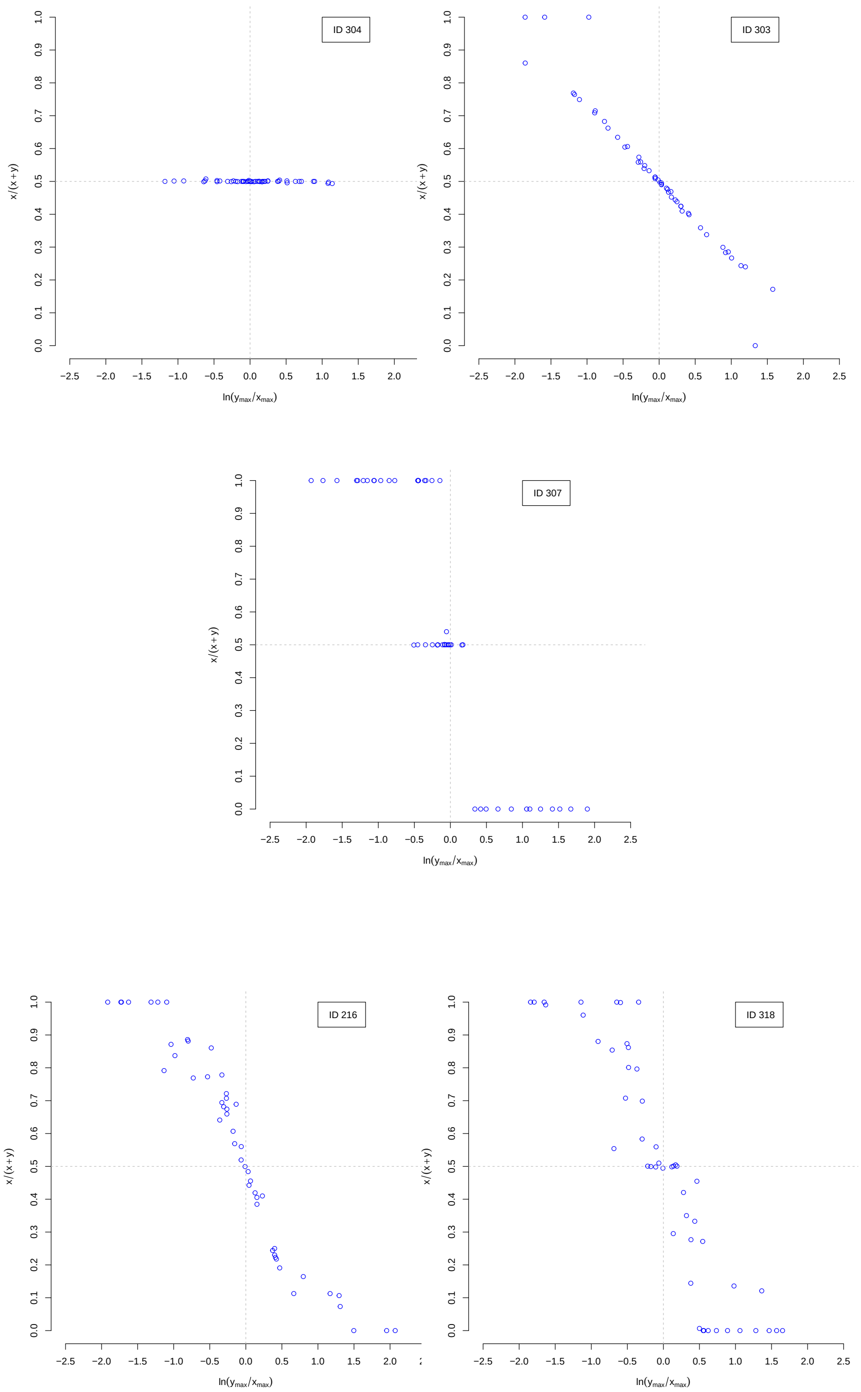


Figure 4.3: Portfolio-choices for a set of representative subjects. Relationship between the log-price ratio $\ln(\frac{p_x}{p_y})$ and the relative demand $\frac{x}{x+y}$. Selection of representative subjects showing high regularity of choices in the symmetric treatment ($\pi_x = \pi_y$).

Recovering preferences with 'classical' choice models

The classical estimation methods used to retrieve risk preferences from the data set are presented in this section. First of all, the error minimization between observed and optimal choices prescribed by deterministic DA utility is explained. Then, the maximum likelihood estimation using a probabilistic version of DA is detailed.

5.1 Differences with the original paper

Parameter estimation methods Both estimation methods, namely non-linear least squares and maximum likelihood estimation described below were also performed in the reference paper by Choi et al. (2007a) but some differences are to be noted. In the present study, the optimal demands are derived in a different manner. Moreover, the present work assumes a multinomial logit model while Choi et al. (2007a) work directly with a random utility formulation. Once again, we stress that the specification of DA used was the one from Halevy et al. (2018) (Equation 2.6) rather than the one in Choi et al. (2007a) (Equation 2.5). This allows us to directly estimate the parameter β for disappointment aversion. Moreover, the CRRA specification used (Equation 2.11) is also sightly different but ensures a smooth transition at the discontinuity point $\rho = 1$ which is important for an optimization porcedure.

Notations In the following the *maximal demand* (x_{max} and y_{max}) will be preferred to the prices p_x and p_y. The relationships between maximal demands and prices are $x_{max} = \frac{1}{p_x}$ and $y_{max} = \frac{1}{p_y}$. Moreover, the demand in asset y will be substituted by $b(x)$, a function of x satisfying the budget constraint $p_x \cdot x + p_y \cdot y = 1$ such that:

$$y = b(x) = y_{max} \cdot (1 - \frac{x}{x_{max}}) \tag{5.1}$$

5.2 Deterministic Disappointment Aversion Theory

In a first step, risk preferences will be estimated using deterministic DA with the CRRA specification as described in equations 2.6 and 2.11.

5.2.1 Optimal demand

The optimal demand is such that :

$$x^* = \arg\max_x \quad DA(x) \tag{5.2}$$

or, equivalently:

$$\frac{d}{dx} DA(x)\Big|_{x=x^*} = 0 \tag{5.3}$$

Recall that :

$$DA(x) = \gamma(p) \cdot u\left(\max\{x, b(x)\}\right) + (1 - \gamma(p)) \cdot u\left(\min\{x, b(x)\}\right) \tag{5.4}$$

with $\gamma(p) = \frac{p}{1+(1-p)\cdot\beta}$ where p is the probability of occurrence of state x and β the parameter of disappointment aversion. The allocation y has been substituted using the budget constraint, such that

$$y = b(x) = y_{max} \cdot (1 - \frac{x}{x_{max}}) \tag{5.5}$$

When using the CRRA specification, equation 2.6 rewrites:

- For $\rho \neq 1$

$$\begin{aligned} DA(x) &= \frac{\gamma(p)}{1-\rho} \cdot \left(\max\{x, b(x)\} - 1\right)^{1-\rho} \\ &+ \frac{(1-\gamma(p))}{1-\rho} \cdot \left(\min\{x, b(x)\} - 1\right)^{1-\rho} \end{aligned} \tag{5.6}$$

- For $\rho = 1$

$$\begin{aligned} DA(x) &= \gamma(p) \cdot \ln\left(\max\{x, b(x)\}\right) \\ &+ (1-\gamma(p)) \cdot \ln\left(\min\{x, b(x)\}\right) \end{aligned} \tag{5.7}$$

For the optimal demand, the following four cases hold:

- If $x^* \leq b(x^*)$ and $\rho \neq 1$:

$$x^* = \frac{x_{max} \cdot C_-}{x_{max} + C_-} \quad \text{where} \quad C_- = \left(\frac{\gamma(p)}{1-\gamma(p)} \cdot \frac{{y_{max}}^{1-\rho}}{x_{max}} \right)^{-\frac{1}{\rho}} \tag{5.8}$$

- If $x^* > b(x^*)$ and $\rho \neq 1$:

$$x^* = \frac{x_{max} \cdot C_+}{x_{max} + C_+} \quad \text{where} \quad C_+ = \left(\frac{1-\gamma(p)}{\gamma(p)} \cdot \frac{{y_{max}}^{1-\rho}}{x_{max}} \right)^{-\frac{1}{\rho}} \tag{5.9}$$

- If $x^* \leq b(x^*)$ and $\rho = 1$:

$$x^* = (1 - \gamma(p)) \cdot x_{max} \tag{5.10}$$

- If $x^* > b(x^*)$ and $\rho = 1$:

$$x^* = \gamma(p) \cdot x_{max} \tag{5.11}$$

5.2.2 Non-Linear Least Squares

The set of parameters $\hat{\theta}$ that best represents the data is estimated for each individual by minimizing the squared distance between the observed choice x_j^{obs} and the optimal choice x_j^* for each deision task $j = 1, ..., N_d$, where N_d is the number of decisions that individuals face.
At the individual level, for each subject i, θ_i is estimated such that:

$$\hat{\theta}_i = \underset{\theta_i}{\arg\min} \quad d(\theta_i) \tag{5.12}$$

with

$$d(\theta_i) = \sum_{j=1}^{Nd} \| x_{i,j}^{obs} - x^*(\theta_i; x_{max,i,j}, y_{max,i,j}) \|^2 \tag{5.13}$$

At the aggregate level:

$$\hat{\theta} = \underset{\theta}{\arg\min} \quad d(\theta) \tag{5.14}$$

with

$$d(\theta) = \sum_{i=1}^{N_S} \sum_{j=1}^{Nd} \| x_{i,j}^{obs} - x^*(\theta; x_{max,i,j}, y_{max,i,j}) \|^2 \tag{5.15}$$

5.3 Probabilistic Disappointment Aversion Theory

In a second step, risk preferences will be retrieved using a probabilistic choice model which consists in a probabilistic version of Disappointment Aversion Theory.

5.3.1 Multinomial logit model

According to Bunch (1987), the probability of choosing allocation x_j given a set of parameters θ and a set of explanatory variables Z is given by:

$$P(x_j|\theta, Z) = \frac{e^{v(\theta,z_j)}}{\sum_k e^{v(\theta,z_k)}} \tag{5.16}$$

where $v(\theta, z_j)$ is a non linear function of the parameters θ and the elements z of the set of explanatory variables Z.

$$P(x_j|\theta, Z) = \frac{e^{DA(\theta,z_j)}}{\sum_k e^{DA(\theta,z_k)}} \tag{5.17}$$

The scale function $v(\cdot)$ is identified to Disappointment Aversion Theory with the CRRA utility index, and it is further assumed that $\theta = \{\beta, \rho\}$ and $Z = \{x_j, x_{max}, y_{max}\}$.

- If $\rho \neq 1$, one has :

$$\begin{aligned} v(\beta, \rho; x_j, x_{max}, y_{max}) &= DA_{CRRA}(x_j) \\ &= \frac{\gamma(p)}{1-\rho} \cdot \left(\max\left\{x_j, b(x_j)\right\}\right)^{1-\rho} \\ &\quad + \frac{1-\gamma(p)}{1-\rho} \cdot \left(\min\left\{x_j, b(x_j)\right\}\right)^{1-\rho} \end{aligned} \tag{5.18}$$

  with $\gamma(p) = \frac{p}{1+(1-p)\cdot\beta}$ where p is the probability of occurrence of state x, β the parameter of disappointment aversion and ρ the parameter of risk aversion. x_j is the observed choice, namely the demand in security x. The demand in security y has been substituted using the budget constraint, such that $y = b(x) = y_{max} \cdot (1 - \frac{x}{x_{max}})$.

- If $\rho = 1$:

$$\begin{aligned} v(\beta, \rho; x_j, x_{max}, y_{max}) &= DA_{CRRA}(x_j) \\ &= \gamma(p) \cdot \ln\left(\max\left\{x_j, b(x_j)\right\}\right) \\ &\quad + (1-\gamma(p)) \cdot \ln\left(\min\left\{x_j, b(x_j)\right\}\right) \end{aligned} \tag{5.19}$$

The term $\sum_k e^{DA(\theta,z_k)}$ is calculated by discretizing the space of demands as follows. The choice resolution of the experiment is $r = 0.1\$$. Therefore, for each decision task j, the individual can choose a demand x_k of asset x such that $0 \leq x_k \leq x_{max,j}$ with $k = 1, 2, ..., N_j$ where $N_j = \left\lfloor \frac{x_{max,j}}{r} \right\rfloor$. For each decision task j, the sum is then computed over all possible demands x_k.

5.3.2 Maximum Likelihood Estimation

The maximum likelihood method used to estimate the parameters of the above presented model, is described below for the aggregate level. The individual level estimation is performed by setting the number of subjects N_s to 1.

The set of parameters $\hat{\theta}$ that best represents the observed data maximizes the likelihood function $l(\theta)$ as follows:

$$\hat{\theta} = \underset{\theta}{\arg\max} \quad l(\theta) \tag{5.20}$$

with $l(\theta)$ the likelihood function defined below:

$$\begin{aligned} l(\theta) &= \prod_{i=1}^{N_s} \prod_{j=1}^{N_d} P(L_j^i | \theta, Z) \\ &= \prod_{i=1}^{N_s} \prod_{j=1}^{N_d} \frac{e^{v(\theta, z_j^i)}}{\sum_k e^{v(\theta, z_{k,j}^i)}} \end{aligned} \tag{5.21}$$

With N_s the number of subjects and N_d the number of decisions. The problem is then equivalent to the following:

$$\hat{\theta} = \underset{\theta}{\arg\max} \quad \ln\left(l(\theta)\right) = \underset{\theta}{\arg\max} \quad \mathcal{L}(\theta) \tag{5.22}$$

where

$$\mathcal{L}(\theta) = \ln\left(\prod_{i=1}^{N_s} \prod_{j=1}^{N_d} P(L_j^i | \theta, Z)\right) = \sum_{i=1}^{N_s} \sum_{j=1}^{N_d} \ln\left(P(L_j^i | \theta, Z)\right) \tag{5.23}$$

is the log-likelihood. The function to maximize is then rewritten as:

$$\mathcal{L}(\theta) = \sum_{j=1}^{N_{obs}} \left(v(\theta, z_j) - \ln\left(\sum_k e^{v(\theta, z_{k,j})}\right)\right) \tag{5.24}$$

With $N_{obs} = N_d \cdot N_s$ the total number of observations.
When considering DA with CRRA, the set of parameters to estimate is $\theta = \{\beta, \rho\}$.

The Nelder and Mead (1965) algorithm[1] is used through the `optim()` function in `R` to optimize the log-likelihood $\mathcal{L}(\theta)$ as an objective function with respect to the set of parameters $\theta = \{\beta, \rho\}$.

[1]The method that is commonly used to perform optimization with bounded constraints is L-BFGS-B. However, in our situation, Nelder-Mead showed higher efficiency and stability. The constraints on the parameters were directly implemented in the code.

Recovering preferences with Quantum Decision Theory

The goal of this report is to adapt Quantum Decision Theory (QDT) to parametric recoverability of preferences in the case of non-binary choices under risk. Therefore, the developed parametrization is outlined below. In a second step, the parameter estimation method when using QDT, namely maximum likelihood estimation, is presented.

6.1 Model specification

As explained in Section 3, Quantum decision Theory provides the probability $p(\pi_j)$ of choosing prospect π_j. To apply QDT to the present data, prospects π_j are identified with the demands x_j.

The choice resolution of demands in the experiment is of r =0.1 \$. Therefore, it is possible to discretize the demand such that for each decision task $j = 1, ..., N_d$, the individual chooses a demand x_k of asset x with $0 \leq x_k \leq x_{max,j}$, such that for each decision task j, one has $k = 1, 2, ..., N_j$ where $N_j = \left\lfloor \frac{x_{max,j}}{r} \right\rfloor$. N_d is the number of decisions.

The following formulation is suggested for the portfolio choice experiment under consideration.

Prospects

$$\begin{aligned} |\pi_k\rangle &= |x_k\rangle \otimes \{ \, b_1 \, |b_1\rangle + b_2 \, |b_2\rangle \} \\ &= b_1 \, |x_k b_1\rangle + b_2 \, |x_k b_2\rangle \end{aligned}$$

from which the measurement operator $\hat{P}_j = |\pi_j\rangle \langle \pi_j|$ can be derived.
As explained in Section 3, $|x_k\rangle$ represents the event of choosing demand x_k, which is an *operationally testable event*, *i.e.* the observed revealed preference

in the laboratory experiment.
The terms $|b_1\rangle$ and $|b_2\rangle$ represent the *inconclusive events* pertaining to the Hilbert space $\mathcal{H}_\mathcal{B}$ and representing any influence on the decision that cannot be observed in the laboratory experiment (emotional biases, incorrect understanding of the experiment,...etc.). Without loss of generality, the cardinality of $\mathcal{H}_\mathcal{B}$ was chosen to be 2 to allow for the emergence of interference terms (see Section 3.2 for further details). This choice was also motivated by the fact that this study is a first step towards a parametrization of the considered portfolio choice experiment and allows for more tractability of the quantum contribution. As prescribed by QDT, the coefficients $b_{1,2}$ are random complex numbers.

Strategic state of mind

$$|\psi\rangle = \sum_k^{N_j} (c_{k1}\,|x_k b_1\rangle + c_{k2}\,|x_k b_2\rangle) \tag{6.1}$$

Leading to the following expression for the probability of choosing prospect $|\pi_j\rangle$.

$$p(\pi_j) = \langle\psi|\hat{P}_j|\psi\rangle = |c_{j1}\alpha_1|^2 + |c_{j2}\alpha_2|^2 + c_{j1}\alpha_2^* c_{j2}^*\alpha_1 + c_{j1}^*\alpha_2 c_{j2}\alpha_1^* \tag{6.2}$$

The utility factor f and attraction factor q are given by:

$$f(\pi_j) = \underbrace{|c_{j1}\alpha_1|^2}_{f_{j1}} + \underbrace{|c_{j2}\alpha_2|^2}_{f_{j2}} \tag{6.3}$$

and

$$q(\pi_j) = c_{j1}^*\alpha_1^* c_{j2}\alpha_2 + c_{j1}\alpha_1 c_{j2}^*\alpha_2^* = 2\Re(c_{j1}^*\alpha_1^* c_{j2}\alpha_2) \tag{6.4}$$

The expression for q can be reworked in a similar way than in Vincent et al. (2016) by using

$$f = f_{j1} + f_{j2} \Rightarrow f_{j1} = t \cdot f \quad \text{and} \quad f_{j2} = (1-t)\cdot f \tag{6.5}$$

with $0 \leq t \leq 1$ to obtain:

$$q(\pi_j) = 2 \cdot f(\pi_j) \cdot \sqrt{t \cdot (t-1)} \cdot \cos(\Delta_j) \tag{6.6}$$

With Δ_j the *uncertainty angle*.

The derived attraction factor is thus found to have a similar form than in the case of binary choices reported by Vincent et al. (2016). The parametrization suggested below is however different.

For simplicity, x_j replaces π_j in the remainder of this report. The following conditions have to be satisfied by the paramerization:

$$\sum_j p(x_j) = 1 \tag{6.7}$$

$$\sum_j f(x_j) = 1 \tag{6.8}$$

$$\sum_j q(x_j) = 0 \quad \text{and} \quad -1 \leq q(x_j) \leq 1 \tag{6.9}$$

$$q(x_j) \to 0 \Rightarrow p(x_j) \to f(x_j) \tag{6.10}$$

6.1.1 Utility factor

Following previous operational applications of QDT (Vincent et al., 2016), the utility factor is associated with a classical probabilistic choice model. In this study, it is the DA-based probabilistic choice model previously used. Explicitly, one has:

$$f(x_j) = \frac{e^{DA(\theta, z_j)}}{\sum_k e^{DA(\theta, z_k)}} \tag{6.11}$$

This parametrization satisfies conditions 6.8 and 6.10.

The term $\sum_k e^{DA(\theta, z_k)}$ is calculated in the same manner than for the probabilistic version of DA by discretizing the space of demands. For each decision task $j = 1, ..., N_d$, the sum is then computed over all possible demands x_k with $k = 1, ..., N_j$.

The specification of the utility factor is the same than in Section 5.3.1. The CRRA specification is used and $\theta = \{\beta, \rho\}$. Again, it is assumed that $Z = \{x_j, x_{max}, y_{max}\}$. If $\rho \neq 1$, $DA(\theta, Z)$ is identified with Equation 5.18. If $\rho = 1$, $DA(\theta, Z)$ is identified with Equation 5.19.

6.1.2 Attraction factor

The suggested parametrization of the attraction factor q in the case of the experiment under study is based on Equation 6.6 and is as follows:

$$q(x_j) = 2 \cdot \min\{f(x_j), 1 - f(x_j)\} \cdot \sqrt{t(t-1)} \cdot \cos\left(\delta \cdot \left(EU\left(x_{max}\right) - EU\left(y_{max}\right)\right)\right) \tag{6.12}$$

where $EU(x) = \frac{1}{2} \cdot x + \frac{1}{2} \cdot y$ is the standard linear expected utility.

Therefore, one has:

$$q(x_j) = 2 \cdot \min \{f(x_j), 1 - f(x_j)\} \cdot \sqrt{t(t-1)} \cdot \cos \left(\delta \cdot \frac{1}{2} \cdot (x_{max} - y_{max}) \right) \tag{6.13}$$

This parametrization ensures conditions 6.9 and 6.7. Indeed, the term $\min \{f(x_j), 1 - f(x_j)\}$ ensures the restriction $|q(x_j)| \leq 1 - f(x_j)$ such that $p(x_j) = f(x_j) + q(x_j) \leq 1$ as well as the restriction $|q(x_j)| \leq f(x_j)$ such that $p(x_j) = f(x_j) + q(x_j) \geq 0$.

The motivation for the parametrization of the cosine term comes from the heuristic that an individual would take into account the prices when buying a security [1]. More precisely, a decision-maker would mentally process and weight the expected returns of the boundary portfolios, *i.e.* if allocating his wealth in security x exclusively or in security y exclusively. This mental representation may *attract* him towards asset x or y, giving all its meaning to q, termed the *attraction factor*. The parameter δ measures one's sensitivity to the introduced heuristic on boundary portfolios.
Parameter t arises from the theoretical formulation and weights the contributions of the inconclusive events b_1 and/or b_2.

The reparametrization $\alpha = \sqrt{t \cdot (t-1)}$ was found to be better performing in estimations leading to the expression for the attriction factor:

$$q(x_j) = 2 \cdot \min \{f(x_j), 1 - f(x_j)\} \cdot \alpha \cdot \tanh \left(\delta \cdot \frac{1}{2} \cdot (x_{max} - \cdot y_{max}) \right) \tag{6.14}$$

where $0 \leq \alpha \leq \frac{1}{2}$ and $\delta \geq 0$.

6.2 Maximum Likelihood Estimation

The maximum likelihood method used to estimate the parameters is described below for the aggregate-level. The individual-level estimation is performed by setting the number of subjects N_s to 1.

The set of parameters $\hat{\theta}$ that best represents the observed data maximizes the likelihood function $l(\theta)$ as follows:

$$\hat{\theta} = \arg\max_{\theta} \quad l(\theta) \tag{6.15}$$

[1]This heuristic was suggested by Prof. D. Sornette.

with $l(\theta)$ the likelihood function defined below:

$$\begin{aligned}
l(\theta) &= \prod_{i=1}^{N_s}\prod_{j=1}^{N_d} p(x_j^i) \\
&= \prod_{i=1}^{N_s}\prod_{j=1}^{N_d} \left(f(x_j^i) + q(x_j^i) \right) \\
&= \prod_{i=1}^{N_s}\prod_{j=1}^{N_d} \left(\frac{e^{DA(\theta, z_j^i)}}{\sum_k e^{DA(\theta, z_{k,j}^i)}} \right. \\
&+ \left. 2 \cdot \min\left\{ f(x_j^i), 1 - f(x_j^i) \right\} \cdot \alpha \cdot \tanh\left(\delta \cdot \frac{1}{2} \cdot \left(x_{max,j}^i - y_{max,j}^i \right) \right) \right)
\end{aligned}$$

With N_s the number of subjects and N_d the number of decisions. The problem is then equivalent to the following:

$$\hat{\theta} = \arg\max_{\theta} \ln\left(l(\theta)\right) = \arg\max_{\theta} \mathcal{L}(\theta) \tag{6.16}$$

where

$$\mathcal{L}(\theta) = \ln\left(\prod_{i=1}^{N_s}\prod_{j=1}^{N_d} p(x_j^i) \right) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_d} \ln\left(p(x_j^i) \right) \tag{6.17}$$

is the log-likelihood to maximize.

When considering DA utility with the CRRA utility index, the set of parameters to estimate is $\theta = \{\beta, \rho, \alpha, \delta\}$

Again, the Nelder-Mead algorithm is used through the `optim()` function in `R` to maximize the log-likelihood $\mathcal{L}(\theta)$ as an objective function with respect to the set of parameters $\theta = \{\beta, \rho, \alpha, \delta\}$.

Parameter estimation results

Risk preferences were recovered at the aggregate and individual level using Disappointment Aversion Theory (DA), the probabilistic version of Disappointment Aversion Theory (logit-DA)[1] and Quantum Decision Theory based on Disappointment Aversion Theory (DA-QDT).

7.1 Aggregate level

The table below provides the estimated parameters at the aggregate-level for deterministic (DA) and probabilistic parameter estimation methods (logit-DA and DA-based QDT.)

	$\hat{\beta}$	$\hat{\rho}$	$\hat{\alpha}$	$\hat{\delta}$
DA	0.000	0.706	-	-
logit-DA	0.817	0.237	-	-
DA-based QDT	0.817	0.237	0.019	1.331

Table 7.1: Estimated risk preferences at the aggregatee level for each of the investigated models

One sees that both logit-DA and DA-based QDT provide the same estimations for the risk and disappointment aversion parameters at the aggregate-level. According to these estimates, the sample is generally risk and disappointment averse at the aggregate level, which is in good accordance with the literature (Halevy et al., 2018; Haga and Rivenæs, 2016).

The quantum parameters $\hat{\alpha}$ and $\hat{\delta}$ of the DA-QDT model are however not null at the aggregate level, although the estimations for β and ρ are the

[1] The appropriate notation would be logistic-DA, however due to space constraints, the probabilistic version of DA is referred to as logit-DA (l-DA is also used).

same than with the two-parameter estimation in logit-DA. This leads to the interpretation that they have a role to play in choices although do not interfere with the aggregate risk and disappointment preferences, or at least not in a straightforward manner.

On the other hand, the estimations from the deterministic DA are drastically different, showing no disappointment aversion at the aggregate-level. A higher-level of risk aversion is however observed. This trade-off between risk and disappointment aversion is also visible at the individual-level in table 7.2 and figure 7.2. The DA estimation seems to systematically provide lower disappointment aversion and higher risk aversion at the individual-level. Choi et al. (2007a) also report similar differences between the deterministic and probabilistic approaches.

7.2 Individual level

The estimated parameters at the individual level for all three models are presented in the the appendix. The table below provides the statistical distribution of the estimated parameters for each of the investigated models at the individual-level.

	$\hat{\beta}_{DA}$	$\hat{\rho}_{DA}$	$\hat{\beta}_{l-DA}$	$\hat{\rho}_{l-DA}$	$\hat{\beta}_{QDT}$	$\hat{\rho}_{QDT}$	$\hat{\alpha}_{QDT}$	$\hat{\delta}_{QDT}$
Mean	0.036	1.012	2.999	0.180	1.249	0.260	0.122	0.560
Maximum	0.314	9.672	48.332	0.771	13.928	0.762	0.500	14.563
3^{rd} quartile	0.088	0.882	1.973	0.283	1.423	0.480	0.176	0.069
Median	0.000	0.605	1.093	0.147	0.140	0.164	0.049	0.016
1^{st} quartile	0.000	0.536	0.580	0.009	0.259	0.066	0.000	0.000
Minimum	-0.190	0.060	-0.461	0.000	-0.717	0.000	0.000	0.000

Table 7.2: Summary statistics for parameter estimation at the individual level for each of the models investigated. Subjects 205, 206, 218, 304 and 320 were removed from the logit-DA statistics due to high values of β

Figure 7.2 allows further examination of the individual estimated parameters from the three different models. Subjects 205, 206, 218, 304 and 320 were removed from the logit-DA statistics due to high values of β. DA-based QDT seems to provide less aberrant values for risk and disappointment aversion than logit-DA and therefore might handle outlier cases better while generally providing similar results for the rest of the sample. Again, the trade-off between values of risk and disappointment aversion in the probabilistic and deterministic approaches is visible.

Figure 7.1 shows the distribution of the quantum parameters. Subjects ID

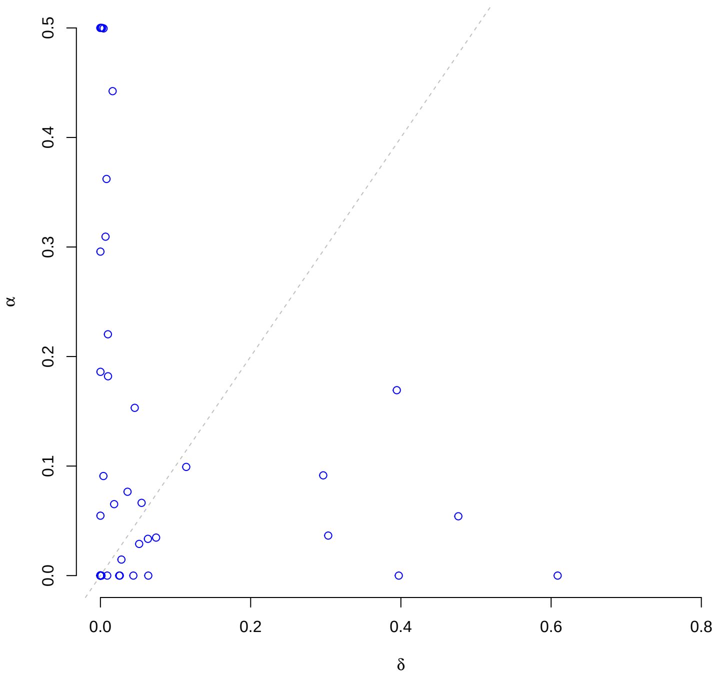


Figure 7.1: Estimated quantum parameters in the DA-QDT model. Subjects ID 308, 309, 317 and 323 with values of δ higher than 0.8 were excluded for better visibility.

308, 309, 317 and 323 with values of δ higher than 0.8 were excluded for better visibility. While α covers the whole range of values between the parameter bounds $0 \leq \alpha \leq \frac{1}{2}$. The parameter δ is generally more concentrated around 0 and seems to play a role only for a few individuals.

7.3 Model selection

The similar results obtained both at the aggregate and individual level with DA-based QDT and logit-DA call for further investigation. In order to assess their performance, we perform two common statistical tests: the Akaike Information Criterion (Sakamoto et al., 1986) and a nested hypothesis testing using Wilks (1938) theorem.

7.3.1 Akaike Information Criterion

It is possible that the likelihood increases from one model to another solely due to adding more parameters. The Akaike Information Criterion (AIC) is a common method for model selection that accounts for this fact by penalizing the number of parameters. The model with the lower AIC is chosen. The AIC is computed for each model as follows:

$$AIC = -2 \cdot \mathcal{L} + 2 \cdot k \tag{7.1}$$

where $\mathcal{L}$ is the log-likelihood and k the number of parameters of the model. The table below reports the AIC values for the classical and quantum models investigated at the aggregate level.

	Aggregate-level AIC
logit-DA	28538.722
DA-based QDT	28539.374

Table 7.3: Values of the Akaike Information Criterion for the probabilistic approaches logit-DA and DA-QDT at the aggregate level.

The logit-DA model shows a lower AIC value than DA-based QDT. However, it is more suitable to consider the difference between the two AIC values rather than the absolute values. Let the difference in AIC values be defined as:

$$\Delta AIC = AIC_{logit-DA} - AIC_{QDT} \tag{7.2}$$

such that $\Delta AIC \geq 0$ if $AIC_{QDT} \leq AIC_{logit-DA}$, *i.e.* when DA-based QDT is selected.

The AIC value for DA-based QDT is slightly smaller ($\Delta AIC = -0.652$), meaning that the cost of the additional parameters in DA-based QDT is not compensated by the gain in likelihood at the aggregate level.

The individual computation of the AIC is relegated to the Appendix.

7.3.2 Nested hypothesis testing

Models are said to be nested when the nested model can be obtained by adding linear restrictions to certain parameters of the nesting model. In this instance the logit-DA model is clearly a nested version of DA-QDT with α and δ being restricted to 0.

The aim of a likelihood ratio is to compare the goodness of fit of two different models. More specifically, it tests a null model against an alternative one. Wilks (1938) has shown that, under the assumption of the null model, the doubled difference between the log-likelihoods of a nested (null model) and its nesting model (alternative model) follows a chi-square distribution with a degree of freedom equal to the difference in the number of parameters such that:

$$-2\mathcal{L}(\theta_{logit\text{-}DA}) + 2\mathcal{L}(\theta_{DA\text{-}QDT}) \sim \chi^2(k_{DA\text{-}QDT} - k_{logit\text{-}DA}) \tag{7.3}$$

with k being the number of parameters used in the model and $\mathcal{L}(\theta)$ being the log-likelihood.

From there, the p-value can be computed as the probability of observing such or more extreme evidence under the null hypothesis. At a significant level of 5 %, the null hypothesis is rejected when the p-value is smaller than 0.05.

	p-value
logit-DA vs. DA-QDT	0.187

Table 7.4: p-value from the nested hypothesis testing with logit-DA being nested in DA-QDT.

The p-value shows that the null hypothesis of logit-DA being the true model cannot be rejected at the 5% confidence level. However, this p-value indicates that under the null hypothesis there is a 18.7% probability of observing an improvement in likelihood when using the DA-QDT model by chance. This result indicates that DA-QDT shows better fit at the aggregate level, however not significantly using the conventional threshold of a 0.05 p-value.

The individual computations of the p-value are relegated to the Appendix.

7.4 Subjects represented by QDT

After performing both the Akaike Information Criterion and Wilk's theorem at the individual-level (see Table with individual values in the Appendix), it was found that DA-based QDT is selected for two subjects ID: 205 and 320. The results of the statistical tests are provided in Table 7.5. The p-value of the nested hypothesis testing shows that for both subjects, the null hypothesis of logit-DA being the true model is rejected at the 5% confidence level. Moreover, the AIC values of DA-QDT were found to be lower than the AIC values of the logit-DA model. Both these results indicate that DA-QDT represents Subjects ID 205 and 320 significantly better than logit-DA.

Interestingly, these are both subjects that are systematically considered as outliers in the article from Choi et al. (2007a). These subjects show a stylized

	p-value	ΔAIC
ID 205	0.001	10.697
ID 320	0.008	5.772

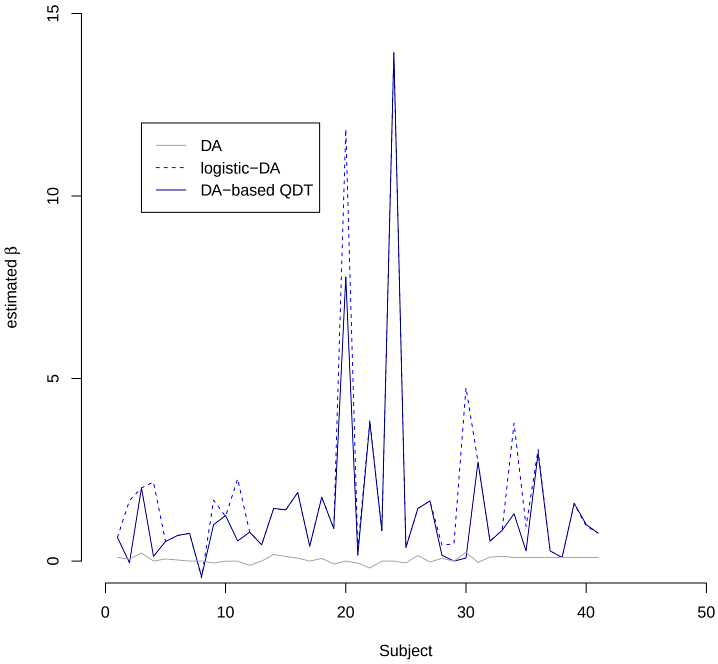
Table 7.5: p-value from the nested hypothesis testing and value of the Akaike Information Criterion of two subjects (ID 205 and 320) for whom both statistical tests imply they are represented by DA-QDT

behaviour in that they always choose to buy a minimum of 10 $ in one of the securities. QDT explains this behaviour with the parameters presented in table 7.6:

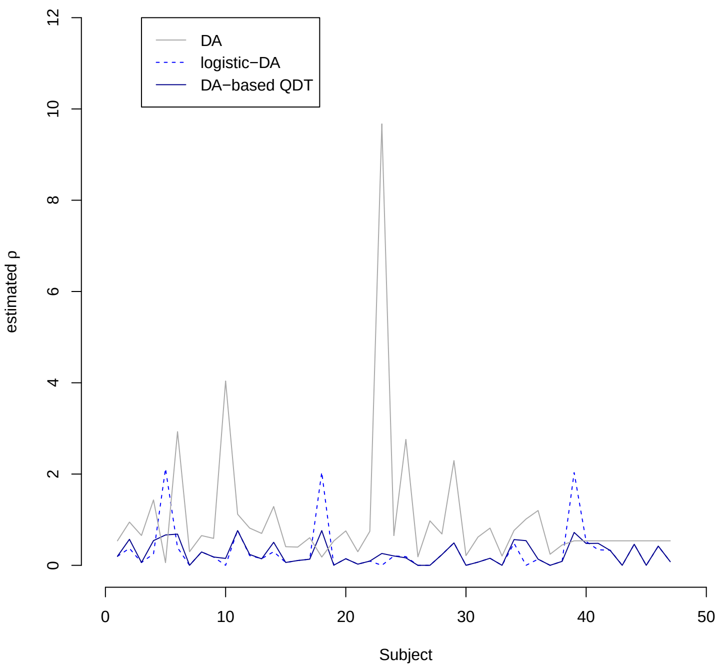
	$\hat{\beta}_{logit-DA}$	$\hat{\rho}_{logit-DA}$	$\hat{\beta}_{QDT}$	$\hat{\rho}_{QDT}$	$\hat{\alpha}_{QDT}$	$\hat{\delta}_{QDT}$
ID 205	11650.430	2.120	-0.716	0.665	0.000	0.608
ID 320	320	6036.808	-0.698	0.721	0.442	0.016

Table 7.6: Parameter estimations in the probabilistic approaches (logit-DA and DA-QDT) for subjects ID 205 and 320 for whom both statistical tests imply they are represented by DA-QDT

Indeed, one sees that the estimations from logit-DA are off-track. QDT instead provides the explanation that the subjects are both disappointment loving while both risk averse. Moreover, given the suggested interpretation of the quantum parameters α and δ, it is possible to interpret the choices with respect to the quantum contribution. Subject ID 205 might have used the heuristic of the mental weighting between the two boundary portfolios since the sensitivity parameter to this heuristic (δ) is high as compared to α. On the contrary, Subject ID 320 might have acted due to some unobservable 'reasons', as encoded in the inconclusive (unobservable) events represented by the parameter α (emotions, biases, incorrect understanding of the experimental setting etc.).



(a) Disappointment parameter $\hat{\beta}$



(b) Risk aversion parameter $\hat{\rho}$

Figure 7.2: Estimated parameters of risk and disappointment aversion for each investigated model at the individual level

Synthesis

In the present work, the framework of Quantum Decision Theory (QDT) for parametric recovery of preferences has been adapted to the case of non-binary choices in a portfolio choice problem for the first time. The experimental data set gathered by Choi et al. (2007a) reporting portfolio choices under risk, allowed for an analysis of preferences at the aggregate and individual level.

In the developed specification of QDT (DA-QDT), the utility factor was identified with a probabilistic version of Disappointment Aversion Theory, based on a multinomial logit model of choice. The attraction factor derived from the theoretical framework of QDT was found to have a similar form than in previous works on binary choices by Vincent et al. (2016). However, the suggested parametrization is different. The first parameter introduced measures the sensitivity to a heuristic that assumes subjects mentally process and weight the linear expected utility of choosing boundary portfolios (meaning they pretend to allocate their wealth exclusively in either one of the two risky assets). The second parameter arises from the theoretical derivation of the attraction factor and reflects the contribution of the unobservable events assumed by QDT (emotions, biases, incorrect understanding of the experimental setting etc.).

To allow for comparison with 'classical' models, preferences were also elicited at the aggregate and individual level using both deterministic (DA) and probabilistic versions of Disappointment Aversion Theory (logit-DA).

The parametric recovery of risk preferences using all three models provided the following findings. The sample was found to be risk and disappointment averse at the aggregate-level and on average at the individual level by logit-DA and DA-QDT which is in good accordance with the literature (Haga and Rivenæs, 2016; Halevy et al., 2018). Both logit-DA and DA-based QDT provide the same estimations for the risk and disappointment aversion parameters at the aggregate-level. However, the quantum parameters are not found to be null and thus play a role, also at the aggregate level. At the individual level, an examination of the estimated risk preferences leads to the conclusion that DA-based QDT seems to provide less aberrant values for risk and disappointment aversion and therefore might handle outlier cases better, while still describing the rest of the individuals similarly to logit-DA.

When using deterministic DA, the sample did not show any disappointment aversion at the aggregate-level but higher risk aversion. This observed trade-off between risk and disappointment aversion is also visible at the individual level. This systematic difference was also reported in the analysis by Choi et al. (2007a).

The similar results obtained with DA-based QDT and logit-DA was further investigated with statistical tests. At the aggregate-level, the value for the Akaike Information Criterion (AIC) of the logit-DA model is slightly lower than the AIC value of DA-based QDT, but the difference is small and does not allow for firm model selection. The p-value of the nested hypothesis testing (0.187) showed that the null hypothesis of logit-DA being the true model cannot be rejected at the 5% confidence level. Interestingly, the statistical tests at the individual level reveal that QDT performs significantly better for two subjects (ID 205 and 320). These two subjects were often considered among the outliers of the analysis by Choi et al. (2007a) due to their stylized behaviour, always securing a minimum of 10$ in the cheapest security. QDT explains these choices parametrically with plausible values of disappointment and risk aversion and differing values of the quantum parameters, while logit-DA gives aberrant estimates.

Other possible directions of research may include further statistical analysis of the models and the parameter distributions. It would also be possible to test the validity of EU-based QDT or other generalizations of EU as a basis for QDT. Given the rich experimental data offered by the experimental framework of Choi et al. (2007b), it would be interesting to perform a similar analysis in other treatments (asymmetric risk treatment, uncertainty treatment) for a better refinement and interpretation of the parametric calibrations, in particular regarding the role of the quantum contribution.

Bibliography

Mohammed Abdellaoui and Han Bleichrodt. Eliciting gul's theory of disappointment aversion by the tradeoff method. *Journal of Economic Psychology*, 28(6):631–645, 2007.

David Ahn, Syngjoo Choi, Douglas Gale, and Shachar Kariv. Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics*, 5(2):195–223, 2014.

Maurice Allais. Allais paradox. In *Utility and probability*, pages 3–9. Springer, 1990.

David S Bunch. Maximum likelihood estimation of probabilistic choice models. *SIAM Journal on Scientific and Statistical Computing*, 8(1):56–70, 1987.

Colin F Camerer. Recent tests of generalizations of expected utility theory. In *Utility theories: Measurements and applications*, pages 207–251. Springer, 1992.

Rudolf Carnap. Über die aufgabe der physik. *Kant-Studien*, 28(1-2):90–107, 1923.

Syngjoo Choi, Raymond Fisman, Douglas Gale, and Shachar Kariv. Consistency and heterogeneity of individual behavior under uncertainty. *The American Economic Review*, 97(5):1921–1938, 2007a. ISSN 00028282. URL http://www.jstor.org/stable/30034591.

Syngjoo Choi, Raymond Fisman, Douglas M. Gale, and Shachar Kariv. Revealing preferences graphically: An old method gets a new tool kit. *American Economic Review*, 97(2):153–158, May 2007b. doi: 10.1257/aer.97.2.153. URL http://www.aeaweb.org/articles?id=10.1257/aer.97.2.153.

Daniel Ellsberg. Risk, ambiguity, and the savage axioms. *The quarterly journal of economics*, pages 643–669, 1961.

Itzhak Gilboa. *Theory of decision under uncertainty*, volume 1. Cambridge university press Cambridge, 2009.

Christian Gollier. *The economics of risk and time*. MIT press, 2004.

Faruk Gul. A theory of disappointment aversion. *Econometrica: Journal of the Econometric Society*, pages 667–686, 1991.

Emil Haga and André Waage Rivenæs. Analysing risk preferences among insurance customers: expected utility theory versus disappointment aversion theory. Master's thesis, 2016.

Yoram Halevy, Dotan Persitz, and Lanny Zrill. Parametric recoverability of preferences. *Journal of Political Economy*, 126(4):1558–1593, 2018.

John D Hey and Chris Orme. Investigating generalizations of expected utility theory using experimental data. *Econometrica: Journal of the Econometric Society*, pages 1291–1326, 1994.

Charles A Holt and Susan K Laury. Assessment and estimation of risk preferences. In *Handbook of the economics of risk and uncertainty*, volume 1, pages 135–201. Elsevier, 2014.

Tatyana Kovalenko and Didier Sornette. The conjunction fallacy in quantum decision theory. *Working Paper*, 2017.

Duncan R Luce. Individual choice behavior. 1959.

Daniel McFadden. Econometric models for probabilistic choice among products. *Journal of Business*, pages S13–S29, 1980.

John A Nelder and Roger Mead. A simplex method for function minimization. *The computer journal*, 7(4):308–313, 1965.

John Quiggin. A theory of anticipated utility. *Journal of Economic Behavior & Organization*, 3(4):323–343, 1982.

Yosiyuki Sakamoto, Makio Ishiguro, and Genshiro Kitagawa. Akaike information criterion statistics. *Dordrecht, The Netherlands: D. Reidel*, 81, 1986.

Paul A Samuelson. Consumption theory in terms of revealed preference. *Economica*, 15(60):243–253, 1948.

Sabine Vincent, Tatyana Kovalenko, Vyacheslav I Yukalov, and Didier Sornette. Calibration of quantum decision theory, aversion to large losses and predictability of probabilistic choices. 2016.

John Von Neumann and Oskar Morgenstern. *Theory of games and economic behavior*. Princeton university press, 1944.

Martin Weber and Colin Camerer. Recent developments in modelling preferences under risk. *Operations-Research-Spektrum*, 9(3):129–151, 1987.

Samuel S Wilks. The large-sample distribution of the likelihood ratio for testing composite hypotheses. *The Annals of Mathematical Statistics*, 9(1): 60–62, 1938. doi: 10.1214/aoms/1177732360.

Vyacheslav I Yukalov and Didier Sornette. Quantum decision theory as quantum theory of measurement. *Physics Letters A*, 372(46):6867–6871, 2008.

Vyacheslav I Yukalov and Didier Sornette. Processing information in quantum decision theory. *Entropy*, 11(4):1073–1120, 2009.

Vyacheslav I Yukalov and Didier Sornette. Mathematical structure of quantum decision theory. *Advances in Complex Systems*, 13(05):659–698, 2010.

Vyacheslav I Yukalov and Didier Sornette. Inconclusive quantum measurements and decisions under uncertainty. *Frontiers in Physics*, 4:12, 2016a.

Vyacheslav I Yukalov and Didier Sornette. Quantum probability and quantum decision-making. *Phil. Trans. R. Soc. A*, 374(2058):20150100, 2016b.

Vyacheslav I Yukalov and Didier Sornette. Quantitative predictions in quantum decision theory. *IEEE Transactions on systems, man, and cybernetics: systems*, 48(3):366–381, 2018.

Appendix

Table of estimated risk preferences at the individual-level

ID	$\hat{\beta}_{DA}$	$\hat{\rho}_{DA}$	$\hat{\beta}_{l-DA}$	$\hat{\rho}_{l-DA}$	$\hat{\beta}_{QDT}$	$\hat{\rho}_{QDT}$	$\hat{\alpha}_{QDT}$	$\hat{\delta}_{QDT}$	p-value	ΔAIC
201	0.101	0.537	0.649	0.198	0.649	0.198	0	0.064	1	-4
202	0.056	0.948	1.658	0.373	-0.046	0.57	0	0	57615124.994	-39.739
203	0.223	0.654	2.005	0.062	2.008	0.062	0.029	0.052	0.876	-3.734
204	0	1.433	2.165	0.25	0.131	0.541	0	0	2425452410559.23	-61.034
205	0.095	0.06	11650.435	2.12	-0.717	0.666	0	0.609	0.001	10.697
206	0	2.929	8470.064	0.388	0.241	0.684	0	0.001	2405458676604299776	-88.649
207	0.058	0.299	0.538	0.001	0.537	0.001	0.035	0.074	0.99	-3.979
208	0.031	0.652	0.7	0.294	0.7	0.293	0.362	0.008	0.783	-3.512
209	0	0.591	0.761	0.183	0.761	0.183	0.091	0.004	0.993	-3.986
210	0.113	4.039	48.332	0.001	1.208	0.151	0.5	0	9.4174107211262e+28	-137.43
211	0	1.12	-0.461	0.771	-0.422	0.762	0	0.026	1.012	-4.023
212	-0.053	0.817	1.668	0.213	0.997	0.239	0.5	0	42.214	-11.486
213	-0.003	0.699	1.222	0.147	1.248	0.143	0.153	0.046	0.347	-1.881
214	0	1.288	2.252	0.299	0.551	0.505	0	0	8592302.749	-35.933
215	-0.115	0.409	0.792	0.063	0.793	0.063	0.22	0.01	0.825	-3.616
216	0	0.4	0.445	0.105	0.445	0.106	0.092	0.297	0.476	-2.515
217	0.181	0.605	1.443	0.135	1.443	0.134	0.186	0	1	-4
218	0	0.182	107831.561	2.04	-0.69	0.76	0.296	0	0.139	-0.061
219	0.126	0.534	1.398	0.004	1.402	0.003	0.015	0.028	0.928	-3.85
301	0.082	0.754	1.878	0.145	1.879	0.145	0.182	0.01	0.901	-3.792
302	0	0.298	0.406	0.026	0.406	0.026	0.099	0.114	0.533	-2.74
303	0.073	0.745	1.751	0.093	1.751	0.093	0	0.397	1	-4
304	0.313	9.672	17508214.947	0.001	0.682	0.26	0.5	0.004	1.03909898118178e+53	-248.151
305	-0.08	0.652	0.893	0.205	0.892	0.205	0.309	0.007	0.843	-3.657
306	0	2.758	11.829	0.191	7.784	0.164	0.5	0.001	1.154	-4.287
307	-0.055	0.188	0.335	0.001	0.161	0.001	0	0	591.562	-16.766
308	-0.19	0.975	3.862	0.001	3.8	0.001	0.05	14.563	1.923	-5.308
309	0	0.687	0.833	0.236	0.834	0.236	0.055	0	1	-4
310	0	2.295	13.9	0.492	13.928	0.492	0	3.336	1	-4
311	-0.052	0.213	0.367	0.001	0.369	0.001	0	0.025	1.001	-4.001
312	0.151	0.62	1.439	0.071	1.443	0.07	0.077	0.036	0.884	-3.753
313	-0.024	0.816	1.647	0.155	1.647	0.155	0.065	0.018	0.949	-3.895
314	0.075	0.204	0.437	0.001	0.161	0.001	0	0	3987941.117	-34.398
315	0	0.765	0.474	0.484	-0.001	0.564	0	0	10.844	-8.767
316	0.24	1.015	4.749	0.001	0.084	0.541	0	0	2.27394704990749e+41	-194.455
317	-0.034	1.201	2.717	0.135	2.716	0.135	0.06	3.645	0.697	-3.278
318	0.111	0.244	0.557	0.001	0.545	0.001	0	0.009	0.981	-3.961
319	0.13	0.446	0.839	0.087	0.839	0.087	0.034	0.063	0.939	-3.875
320	0.000	0.025	6036.808	2.031	-0.698	0.721	0.442	0.016	0.008	5.772
321	0.000	1.409	3.779	0.521	1.296	0.479	0.5	0.002	19.519	-9.943
322	0.000	0.794	0.95	0.341	0.277	0.482	0	0	87.315	-12.939
323	0.000	1.390	3.053	0.331	2.955	0.326	0.035	1.582	1.255	-4.454
324	0.000	0.362	0.278	0.001	0.28	0.001	0.066	0.055	0.813	-3.587
325	0.000	0.650	0.098	0.462	0.098	0.462	0	0.044	1	-4
326	0.159	0.674	1.567	0.001	1.585	0.001	0.037	0.303	0.588	-2.939
327	0.000	0.903	0.964	0.42	1.002	0.422	0.169	0.395	0.1	0.61
328	0.004	0.585	0.766	0.079	0.766	0.079	0.054	0.476	0.76	-3.45

Table 1: Table presenting the estimated parameters using the three different methods: DA, logit-DA and QDT. The individual computations of the statistical tests between QDT and logit-DA are also reported.