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ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Calibration of Quantum Decision Theory, analysis of parameters, gender difference and risk aversion.

Master Thesis

Morgan Siffert

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Supervisor: Prof. Didier Sornette
Prof. László Forró
Tatyana Kovalenko

Chair of Entrepreneurial Risk, ETH Zürich

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Introduction

In the modern world, decisions are at the center of our everyday life. This is the reason why decision theory is one of the cornerstone of most economical theories. Indeed, a lot of research has been conducted and it has been found that models of decision makers are not perfect. For instance, they contain paradoxes, (Allais, 1953; Al-Najjar and Weinstein, 2009), and there are signs of an intrinsic limit of predictability (Vincent et al., 2017). Therefore, investigating this limit and acquiring new insides on how decisions are made motivate us strongly.

Because of the use of Hilbert space from quantum mechanics, Quantum decision theory (QDT) takes intrinsically a probabilistic approach. In the mathematical broad framework of QDT (Yukalov and Sornette, 2008, 2009, 2010, 2015b), some other theories can be incorporated/combined. QDT gives new ways to understand how decision makers behave.

"Divide and conquer" will be our maxim. It could first be applied to subjects that are separated according to their risk-aversion and also to their gender (all of this in order to distinguish special behaviors and decrease the limit of predictability), but as well to separate data types. Thereby, it will give us information on the type of model needed. Moreover, it describes well one of the main results of QDT: the separation of the probability of choice p in a utility and attraction factor.

The first section of this thesis (2) presents theoretical background (QDT, logistic-CPT). Subsequently, sections 3 and 4 describe two empirical datasets and analyses between-subjects differences, such as partitioning of subjects (probabilistic choice, gender, attitude to risk). Parametrization of QDT and logistic-CPT is in section 5, in which we introduce for QDT, a novel f-factor (based on stochastic indifference values) and "safe" q-factor (based on risk measure). In the final section (6), logistic-CPT is deeply examined.

State of the art in decision theory

2.1 Brief introduction to Quantum decision theory

Quantum decision theory (hereafter referred to as QDT) is an intrinsically probabilistic framework, using mathematics from the Hilbert space to generalize probability and to account for interferences within choice options. This alternative to existing theories was developed by Yukalov and Sornette in a series of articles, including (Yukalov and Sornette, 2008, 2009, 2010, 2015b).

A brief summary of QDT can be found in the appendix, which is borrowed from (Vincent et al., 2017). Following subsections explain basic concepts.

General formula of prospect probability

General formula of prospect's (π_n) probability

$$p(\pi_n) = f(\pi_n) + q(\pi_n) \tag{2.1}$$

can be used to parametrize QDT. "f" is the utility factor which is bounded between 0 and 1. "q" attraction factor is the interfering part and it is bounded between -1 and 1. It should as well satisfy the constrain,

$$f + q = p \in [0, 1] \tag{2.2}$$

and the alternation law

$$\sum_n q(\pi_n) = 0. \tag{2.3}$$

We will use, in this thesis, empirical data (from two different datasets) to calculate p. QDT utility factor f can be parametrized with different models from the decision theory literature. Ratio of utilities is presented in section 2.1 and used in 4.1, 4.3.1 and 5.3. Logistic-CPT is presented in section 2.2.4 and used in 3.3 and 6. Section 5.2 presents both QDT attraction factor utilized -"high loss aversion" from (Vincent et al., 2017) and "safe-q" respectively used in sections 3.3 and 5.3.

Quarter law

In article (Yukalov and Sornette, 2014), the quarter law is derived from non-informative prior on the distribution of q:

$$\frac{1}{N}\sum_{n=1}^{N}|q(\pi_n)| = \frac{1}{4}. \tag{2.4}$$

In article (Yukalov and Sornette, 2014) they proved the quarter law for different families of distribution $\varphi(q)$

For a choice between two prospects, it reduces to:

$$q_+ = \frac{1}{4}, \;\; q_- = -\frac{1}{4}. \tag{2.5}$$

As mentioned in (Yukalov and Sornette, 2016): "In simple cases, signs of the attraction factors can be prescribed by the principle of ambiguity aversion. In more complicated situations, a criterion has been suggested (Yukalov and Sornette, 2014) and applied to lotteries with gains."

Utility f-factor

The ratio of utility is used to compute a f-factor or a probability. It goes between 0 and 1 and is easy to calculate. Even if it is not perfect, it could be seen as a zero's order approximation.

$$f_B = \frac{U_B}{U_A + U_B} \tag{2.6}$$

This parametrization is used in (Favre et al., 2016), (Yukalov and Sornette, 2015a) and (2016).

Attraction q-factor

The attraction factor accounts for interference within QDT. The following comment came out about q-factor in (Yukalov and Sornette, 2011): "The second term characterizes the prospect attractiveness for this decision maker, or a subjectively defined prospect quality. Therefore, the quantity $q(\pi_j)$ can be called the *attraction factor* or *quality factor*. As it has been stressed several times throughout this article, this reflects the fact that the interference terms are embodying subjective feelings and emotions of the decision maker."

Since feelings and emotions are not singular, we can imagine there are maybe different attraction factors representing aversion to losses (equation 5.3), uncertainty (equation 5.6) or others.

2.2 Logistic-CPT

The origin of the expected utility comes from (Bernoulli, 1738). It only modifies payoff of lottery with a value function. Prospect theories is a widely used alternative to classical expected utility theory (Kahneman and Tversky, 1979). In addition to value function for payoffs, which are separated in gains and losses by the introduction of a reference point, it also includes subjective probability weighting function. In cumulative prospect theory (hereafter referred to as CPT) (Tversky and Kahneman, 1992), the subjective probability weights are transformed to cumulative rather than separable. With this change, only unlikely and extreme events are overestimated while for prospect theory all unlikely events are overestimated.

2.2.1 Deterministic choice model

Available parameters for a lottery are probabilities "p_i" of each payoff "V_i". From these parameters, a utility is computed for each lottery. For deterministic choice model, the lottery with the largest utility is chosen.

Utility function

Weighting and value functions are combined with the equation 2.7, which purpose are to give a utility value to a lottery. This form is needed in order to have the probability for a payoff of the same sign that sums to one (Tversky and Kahneman, 1992).

$$U = \begin{cases} w(p_1^A)v(V_1^A) + (1 - w(p_1^A))v(V_2^A) & \text{only gains or losses} \\ w(p_1^A)v(V_1^A) + w(p_2^A)v(V_2^A) & \text{mixed} \end{cases} \tag{2.7}$$

Where V_1^A is the largest payoff in absolute value.

2.2.2 Value function: power function

Gain and loss are different. To give more value to losses, the kink λ should be larger than 1. Otherwise, (if the power parameter $\alpha(> 0)$ is smaller than 1) it implements decreasing marginal value of money.

$$v(x) = \begin{cases} x^{\alpha} & x \geq 0, \\ -\lambda(-x)^{\alpha} & x < 0 \end{cases} \tag{2.8}$$

2.2.3 Subjective probability weigthing function

"Probability is the measure of the likelihood an event will occur" (Webster, 1913). The obvious interest of humans for jackpot lotteries, where the price of a ticket is more expensive than the expected value of the gain, definitely shows the fact that humans have a subjective view of a probability.

Experiment description

In the founding article of CPT (Tversky and Kahneman, 1992), an experiment with 25 graduated students from Berkeley and Stanford universities was conducted to compute subjective probability (weight function) (figure 2.1). Those subjects had to choose between a gain lottery (e.g. 25% chance to win \$150 and 75% chance to win \$0) and a sure-amount. Then, "The certainty equivalent of a pair of lotteries was estimated by the midpoint between the lowest accepted value and the highest rejected value in the second".

To get the shape of the probability weighting function, the median among subjects of certainty equivalents divided by the value of the gain was plotted on figure 2.1.

Weighting function: two-part power function

In the original article of CPT (Tversky and Kahneman, 1992), subjective probability is computed with this weighting function:

$$w(p) = \frac{p^c}{(p^c + (1-p)^c)^{1/c}}\ , \quad 0 < c < 1 \tag{2.9}$$

where $c = \gamma$ (resp. δ) for positive or respectively for negative payoff. They used this function to represent the inverse S-shape they observed for subjective probability.

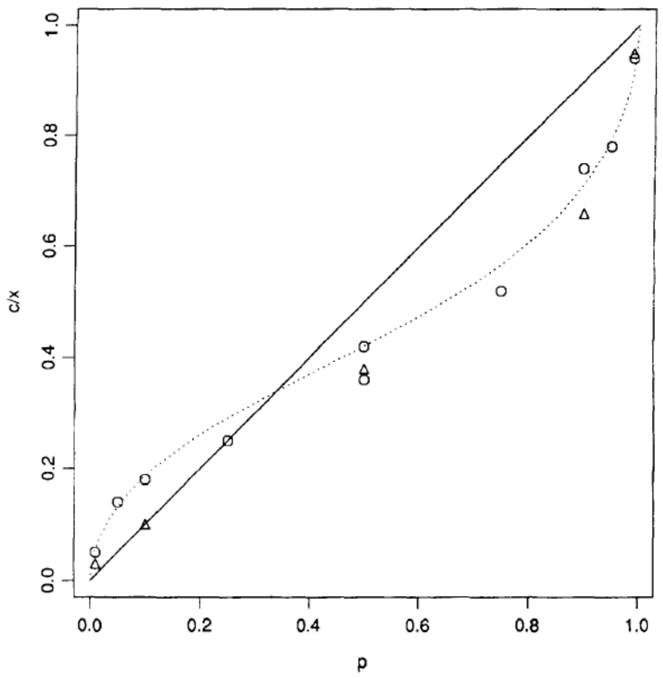


Figure 2.1: Graph to estimate the shape of the probability weighting function in the founding article of CPT (Tversky and Kahneman, 1992). Median of c/x for all gain lotteries $\{x; p\}$, where c stands for the certainty equivalent sure-amount. Triangles represent the values of x which lies above 200 and, in contrast, circles for values below 200.

Weighting function: Prelec II

(Prelec, 1998) derives an axiomatically weighting function. (A shorter derivation can be found in (Al-Nowaihi and Dhami, 2006)). This formulation offers more freedom on the shape the function may take. Parameter δ controls elevation of the probability weighting function and adjusts the height compared to $f(x) = x$ (smaller δ raise the function).

Parameter γ controls steepness of the probability weighting function. The inverse S-shape is obtained with $\gamma < 1$ if δ is within a reasonable range $(0.5 < \delta < 1.5)$.

$$w(p) = \exp(-\delta(-\ln(p))^{\gamma}) \quad \delta > 0 \quad \gamma > 0 \tag{2.10}$$

2.2.4 Probabilistic version of CPT with logistic choice function (logistic-CPT)

The logistic function (firstly used to study population growth) has one parameter. φ controls the form of S-shape function. An increase in value of φ would imply a steeper logistic function.

$$f_B = \frac{1}{1 + e^{\varphi(U_A - U_B)}}, \tag{2.11}$$

What is really interesting in this function is that it could be linked to the partition function from statistical mechanics. Moreover, the probability of a state is ρ .

$$\rho_i = \frac{1}{\sum_j e^{-\beta E_j}} e^{-\beta E_i} = \frac{1}{1 + e^{-\beta(E_i - E_2)}} \tag{2.12}$$

For the first equality, we assume only two energy states. Contrary to decision theory where high utility lottery have a high probability to be chosen, high energy states have a low probability to appear.

2.2.5 Scale invariance: comparison of lotteries with different payoff magnitudes

Scale invariance means that the proportion of choice is identical for lotteries' pairs with different magnitudes of payoff. In other words, if the proportion of choice for lottery A is 80% within pair (A, B), then the proportion of choice for lottery A′ within pair (A′, B′) should be 80% too, where ′ means that all payoffs are multiplied by a constant "c".

For logistic-CPT to be scale invariant, if all payoffs are multiplied by "c", parameter φ from the logistic function should be multiplied by $c^{-\alpha}$ (with α the parameter from the value function (equation 2.8)).

Scale invariance could be test with ((Holt and Laury, 2002),figure 6). In this article, there is an example of data with different magnitudes of payoff. "c" takes as values : 20, 50 and 90. They are indications that the parameter φ should respects the rule "$c^{-\alpha}$" within a small margin of error.

2.3 Likelihood estimation

The answer to lotteries' pair j from decision maker i is denoted by Φ^i_j

$$\Phi_j^i = \begin{cases} 0 & \text{if subject i chooses A for the j}^{th}\text{ gamble} \\ 1 & \text{if subject i chooses Bs for the j}^{th}\text{ gamble} \end{cases} \tag{2.13}$$

The likelihood function to maximize at the aggregate level is

$$\Pi^{agg} = \prod_{i=1}^{142} \prod_{j=1}^{91} pBj^{\Phi_j^i} (1 - pBj)^{1-\Phi_j^i} \tag{2.14}$$

Where pBj is the probability from a model (logistic-CPT, QDT) to choose lottery Bj in lotteries' pair j. To optimize computation, negative loglikelihood (referred to as NLL) is minimized.

Dataset 1

Two datasets of binary choices are analyzed in this thesis. The first dataset (hereafter referred to as Dataset 1) originates from (Schulte-Mecklenbeck et al., 2016), and was previously analyzed in (Murphy and ten Brincke, 2017; Vincent et al., 2017), in their calibrations of stochastic version of cumulative prospect theory and quantum decision theory respectively. The second dataset (hereafter referred to as Dataset 2) was presented in (Loomes and Pogrebna, 2017) to investigate preference reversal pattern. In the following sections, the corresponding datasets are described and a further analysis within quantum decision theory approach is presented.

3.1 Description of the experiment

Dataset 1 includes 91 pairwise choices between two gambles - lotteries A and B. Each lottery consists of two outcomes (V_i, i=1,2) in a range from -100 to 100 monetary units (MU), which occur with probabilities that sum to one. Thus, a lottery is in the form of "(V_1,p) or (V_2,1-p)". A two-alternatives forced choice (2AFC) was implemented, as decision makers were not allowed to express indifference. The dataset comprises four types of lotteries: 35 pairs of lotteries with gains only; 25 pairs with losses only; 25 pairs of mixed lotteries with both gains and losses; and 6 pairs of mixed-zero lotteries with one gain and one loss and zero (status quo) as the alternative outcome. Choices of 142 participants from the subject pool at the Max Planck Institute for Human Development in Berlin were collected during two repetitions of the experiment (Schulte-Mecklenbeck et al., 2016). The two experimental sessions (hereafter referred to as time 1 and time 2) were conducted with the same decision makers at an approximately 2 weeks interval. To mitigate order and presentation effects, the sequence of lottery pairs and spacial representation within each pair was randomized at time 1, and was inverted at time 2. The bipartite incentive mechanism included a fixed remuneration and a varying payment based on one lottery from the participant's choice set, which was randomly selected and played out at the end of each experimental session.

Dataset 1, including complete list of lotteries, was previously presented in (Murphy and ten Brincke, 2017) and used to calibrate stochastic version of cumulative prospect theory. A hierarchical maximum likelihood parameter estimation (HML) method was implemented to improve reliability of individual parameters.

In (Vincent et al., 2017) the same dataset and HML method were used to calibrate quantum decision theory. Empirical observations support probabilistic approach to choice modelling, such as QDT. For example, the overall stability of choices at the aggregate level is accompanied by their variability at the individual level. Importantly, a simple probabilistic choice formulation without model assumption or adjustable parameters was proposed to quantitatively accounts for the fraction of choice reversals between two repetitions of the experiment. The prediction of choice reversal is based only on the observed frequency of the most common choice (i.e. majority choice) at time 1, and can be improved by introducing heterogeneity between decision makers. Thus, two groups of similar size were identified : "The decision makers referred to as "overconfident" tend to exhibit much less uncertainty towards the most common choice. In contrast, the decision makers that we call "contrarian" tend to weaken or even oppose the most common choice.". The next section gives some details on the model and we propose some modifications.

3.2 Probabilistic choice shift model

The model for probabilistic choice shift (without heterogeneity) is based on those assumptions:

- experiment's iterations are independent (people forget their choice and do not learn how to behave);
- for each pair of lotteries, probability p of choosing one of those is stable over time;
- all decision makers are identical.

A quick and informal computation of their model can be made to give some incentives to the reader. If p (resp. 1-p) is the probability of choosing the first (resp. second) lottery then $p(1-p)$ is the probability to shift from time 1 to 2 ((1-p)/p would be from time 2 to 1). All together this would lead to,

$$\mathbb{P}(shift) = 2p(1-p). \tag{3.1}$$

Probabilistic choice shift model with heterogeneity

Relaxing the assumption of all participants homogeneity and assuming two groups $i \in \{1,2\}$ of decision makers of size $N\,F$ respectively $N\,(1-F)$ (with $0 < F < 1$ and N subjects), for which p_1 and p_2 are the probability of the most frequent choice for each group.

The aggregate choice probability,

$$p = p_1 F + p_2(1-F) \tag{3.2}$$

(Vincent et al., 2017) proposed this ansatz for p_1 and p_2 :

$$\begin{cases} p_1 = p + \alpha p(1-p) & \alpha \in [0,1] \\ p_2 = p - \beta p(1-p) & \beta = \alpha F/(1-F) \in [0,1] \\ & \text{correction: } \beta \in [0,2] \end{cases} \tag{3.3}$$

where the value from β derives from equation 3.2.

The upper bound of constraint on β could be relaxed to 2 because p_2 is increasing and larger than 0 for $p \in [0.5,1]$ and $\beta \in [0,2]$ (proof : $p_2' = 1 - \beta + 2p\beta > 1$ since $p \in [0.5,1]$)

3.2.1 Heterogeneity based on majority choice

In this section, the hypothesis from (Vincent et al., 2017) on two groups of decision makers is inquired. Participants are divided into two groups according to the frequency that they follow the most common choice. The most common choice (i.e. majority choice) corresponds to the most frequently (among 142 subjects) chosen option (either A, or B) in each pair of lotteries. The most common choice is quite stable between time 1 and 2, as there are only 4 changes (see table 3.1)

Figure 1.1 presents histogram, over all 142 subjects, of proportion of the most common choices in the full choice set of a subject, observed at time 1. The histogram is bimodal and supports the proposed classification of decision makers in two groups according to their tendency to follow (or oppose) majority choice.

To test this proposition, the following two hypotheses are formulated. H_0: a single homogeneous group of decision makers, which can be modeled with a normal (Gaussian) distribution $\mathcal{N}(\mu_1, |\sigma_1^2)$. H_1: heterogeneous population with two groups, which can be modeled with two normal (Gaussian) distributions $\mathcal{N}(\mu_i, |sigma_i^2), i \in \{1,2\}$. For H_1, groups are of size 142 F and 142 (1-F), with $F \in [0,1]$

Lottery A	Lottery B
(72, 0.95) or (56, 0.05)	(95, 0.05) or (68, 0.95)
(88, 0.29) or (78, 0.71)	(91, 0.71) or (53, 0.29)
(-8, 0.66) or (-95, 0.34)	(-30, 0.07) or (-42, 0.93)
(96, 0.61) or (-67, 0.39)	**(71, 0.5) or (-26, 0.5)**

Table 3.1: The only 4 pairs of lotteries from Dataset 1 ((Schulte-Mecklenbeck et al., 2016)), for which the most common (i.e. majority) choice has shifted between two repetitions of the experiment (time 1 and time 2). The choice is between lottery A and B. The most common choice at time 1 is highlighted in bold.

Values of parameters are summarized in table 1.2, and corresponding probability density distributions are indicated in figure 1.1.

According to Wilks' likelihood ratio-test (Wilks, 1938), H_0 is rejected at both repetitions of the experiment: at time 1 with p value 0.005%, and time 2 with p value 0.02%. The threshold that separates the two groups - overconfident and contrarian - is close to 65% of the most common choices in the choice set of a subject.

Contrarian group (37 subjects) is estimated three times smaller, than overconfident group (105 subjects). This finding contrasts with the prediction of similar sized groups that was proposed in (Vincent et al., 2017).

		F	μ_1	σ_1	μ_2	σ_2	Threshold	p-value
Time 1	H_0	1	0.72	0.09	0	0	-	-
	H_1	0.75	0.59	0.06	0.76	0.04	0.65	0.005 %
Time 2	H_0	1	0.72	0.09	0	0	-	-
	H_1	0.78	0.58	0.04	0.75	0.06	0.64	0.02%

Table 3.2: Parametrization, for 142 subjects, of normal (Gaussian) distributions of proportion of the most common choices in a choice set of a subject, for time 1 and time 2 for Dataset 1. Two hypotheses are tested: a single homogeneous group (H_0), or heterogeneous population with two groups of size 142F and 142(1-F), with $F \in [0, 1]$ (H_1).

For time 2 the separation between subjects is the same for 138 out of 142. Such regularity is an argument to justify this decomposition.

Shift of the selected lottery

The upper histogram of figure 3.2 shows proportion for each subject of choosing the same gamble between time 1 and time 2. As overconfident de-

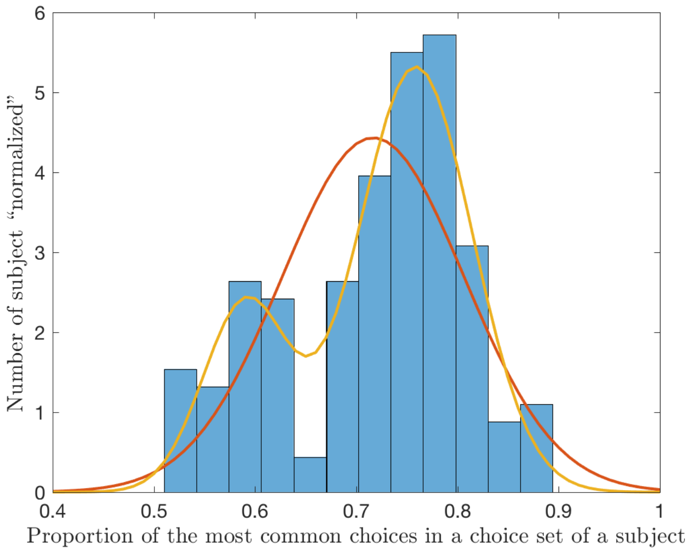


Figure 3.1: Histogram, over all 142 subjects from Dataset 1, of proportion of the most common (i.e. majority) choices in the full choice set of a subject, observed at time 1. The histogram is normalized to be compared with density functions. Solid lines indicate normal (Gaussian) probability density functions for the two hypotheses: a single homogeneous group (H_0), heterogeneous population with two groups (H_1). According to Wilks' likelihood ratio test, H_0 is rejected with p value 5×10^{-5}.

cision makers choose the most common lottery in a higher proportion, they should change their choice between times 1 and 2 less often than contrarian decision makers. This would lead to a decomposition of the histogram in two modes.

The lowest parts of 3.2 show the same histogram for the two groups of subjects and two mode can be distinguished. It seems to be in line with (Vincent et al., 2017): overconfident decision makers tend to exhibit much less uncertainty towards the most common choice. Contrarian decision makers have a broader distribution, which may indicate they are less determined.

Overconfident decision makers are more faithful to their choice. On average, they shifted their choice between time 1 and time 2 in 26% of lottery pairs (with standard deviation of 7%). Contrarian decision makers changed their answers in 37% of pairs (with standard deviation of 13%).

The separation of decision makers in two groups (overconfident: 105 subjects and contrariant: 37 subjects) gives good results: small p-value (5×10^{-5}), stability between time 1 and 2 and prediction of choice shift (choice of the other lottery within the pairs for time 2). We also learn that overcon-

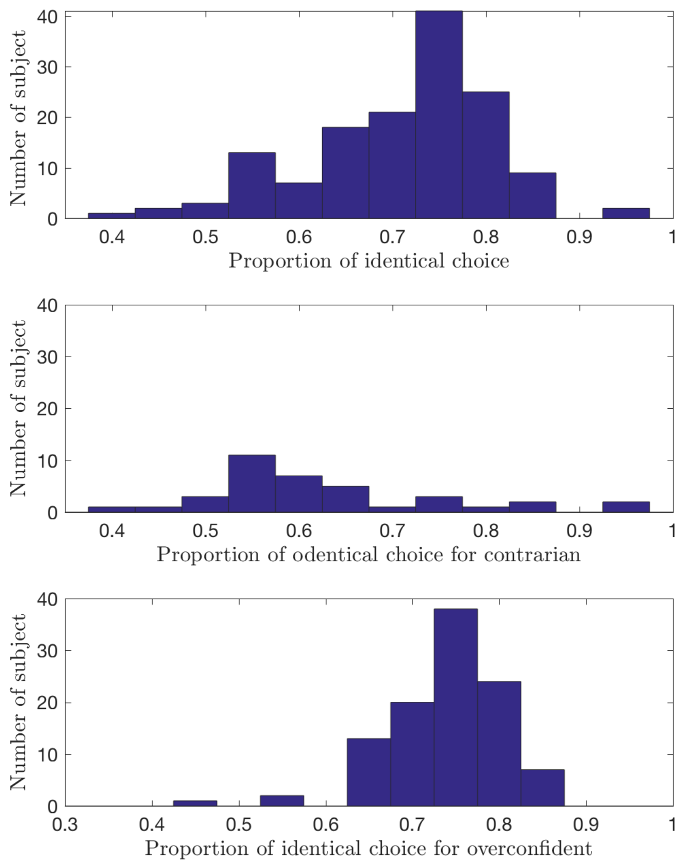


Figure 3.2: Each subject from Dataset 1 classified according to the proportion of taking the same lottery for time 1 and 2. The first histogram is about all subjects, the second one about contrarian decision makers and the final one overconfident decision makers. Groups are separated according to the two modes that we can see in figure 3.1, where contrarian (overconfident) decision makers chose less (more) often the most chosen gamble.

fident decision makers tend to exhibit much less uncertainty towards the most common choice than contrarian decision makers.

3.2.2 Optimisation of the choice shift model

Figure 3.3 shows the residual sum of square (RSS), as a function of F and β, which minimizes in order to find a solution of equation 3.3. The solution from (Vincent et al., 2017) ($F = 0.5$, $\alpha = \beta = 1$) is at the "old boundary" $\beta = 1$ and it is not the only minimum.

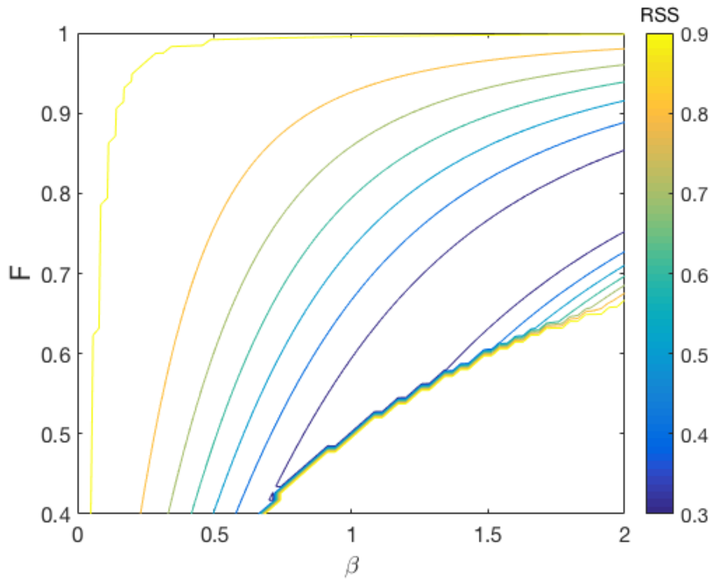


Figure 3.3: Contour plot of figure 3.3. Residual sum of square (RSS), as a function of F and β, minimizes in order to find a solution of the best values for equation 3.3.

In the next section we make an attempt to classify subjects in the two groups - contrarian and overconfident. A new dataset (Dataset 2) is analyzed with respect to this probabilistic choice shift in section 4.2.

Test of probabilistic choice shift model with $F = 0.75$

In section 3.2, the choice shift model with heterogeneity appears to have different minima. We will test if adding the constraint F=0.75 satisfies one of those minima.

Table 3.2 shows different sets of parameters that are at the minima. With $\beta < 2$ and no constraint on F, there are different minima (only a few of them are presented). Adding the constraint $F = 0.75$, it produces only one minimum at $\beta = 1.73$ and $\alpha = 0.58$.

Comparison with prior results

Figure 3.4 compares probabilities p_1 and p_2 from (Vincent et al., 2017) with our solution for $F = 0.75$. Values of our solution are between $[0, 1]$ and are

Contraints:	RSS	β	F
$\alpha < 1$ and $\beta < 1$	0.2331	1	0.5
$\alpha < 1$ and $\beta < 2$	0.2331	1.21	0.60
	0.2331	1.40	0.66
	0.2331	1.67	0.74
$\alpha < 1$, $\beta < 2$ and $F = 0.75$	0.2331	1.73	0.75

Table 3.3: The model calibration represented by equation 3.3 is depending on different constraints. The residual sum of square (RSS) is minimized to find a solution. We find many solutions for the same constraint and this shows there are different minima. The constraint $F = 0.75$ comes from 3.2.1.

smaller than their counterparts from (Vincent et al., 2017).

To verify our model, we reproduce a result from (Vincent et al., 2017) (see these articles for more information). It is shown in figure 3.5 the proportion of subjects having shifted their choice between time 1 and time 2 as a function of the proportion of the most chosen gamble . The blue line represents equation 3.2. The shaded area is the 5% and the 95% quantiles.

Proportion of the most common choice

Computing the proportion of the most common choice for contrarian ($\widehat{p_1}$) and overconfident ($\widehat{p_2}$) decision makers would allow to compare the probabilistic choice shift model with figure 3.1.

If the most common choice is available, ($\widehat{p_1}$) and ($\widehat{p_2}$) are the average of equations 3.3 over all lotteries' pairs. For contrarian (resp. overconfident) decision makers, $\widehat{p_1}$ = 0.40 (resp. $\widehat{p_2}$ = 0.82), while the mean of the Gaussian from figure 3.1 is $\mu_1 = 0.58$ (resp. $\mu_2 = 0.75$).

The difference may come from the probabilistic choice shift model (equation 3.1) or from the formulation of p_1 and p_2. To find the solution, we could test the probabilistic choice shift model, which can be used on data on both groups.

We have shown that the constraint $\beta < 1$ from expression 3.3 should be corrected by $\beta < 2$ and that there are different minima of the same amplitude for the objective function. One of these minima is satisfied and coincides with the constraint $F = 0.75$. This constraint comes from the separation (figure 3.1) of people in two groups (contrarian: 37 subjects, overconfident: 105 subjects and $F = 0.75$).

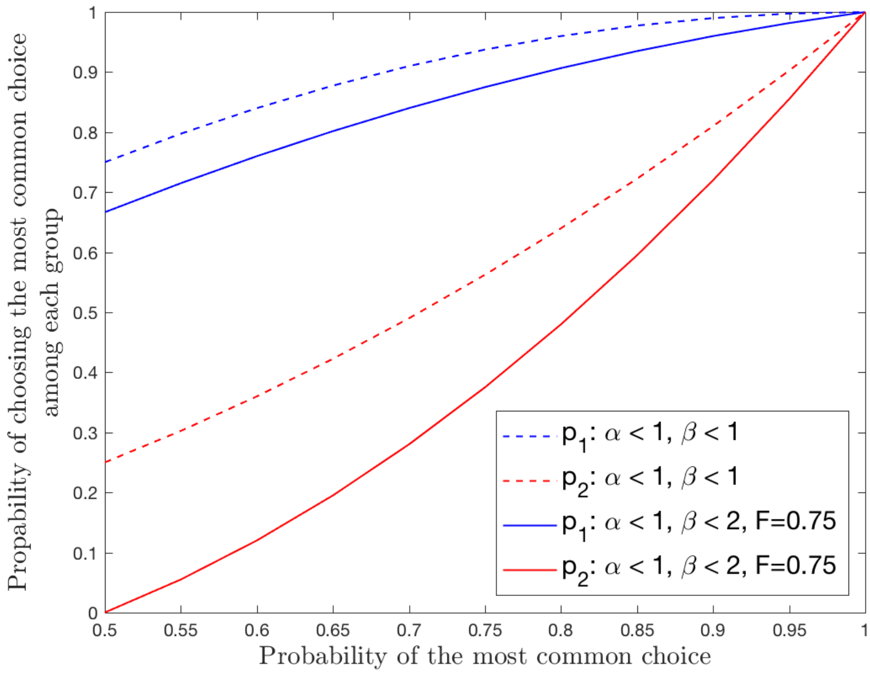


Figure 3.4: Comparison with results from (Vincent et al., 2017). "Probabilities p_1 and p_2 given by equation 3.3 with which the most common choice is chosen for each of the postulated two groups of decision makers as a function of the average choice probability p aggregated over the whole population. In other words, the top (resp. bottom) curve shows the decision probability p1 (resp. p2) of the "overconfident" (resp. "contrarian") decision makers as a function of the frequency p of the most common choice." The two sets of parameters are in table 3.3.

3.3 QDT parametrization for two groups

Two intuitions that someone may have while decomposing a population in two groups with distinct behaviours in order to model their choice are the following:

- It would be easy to model separately two groups because they should be more homogeneous.
- Particularities from one group may be counterbalanced by the ones from the other group so it would be harder.

If the criteria on which the decomposition has been made is correlated to some parameters of the model, it would make sense to do such a decompo-

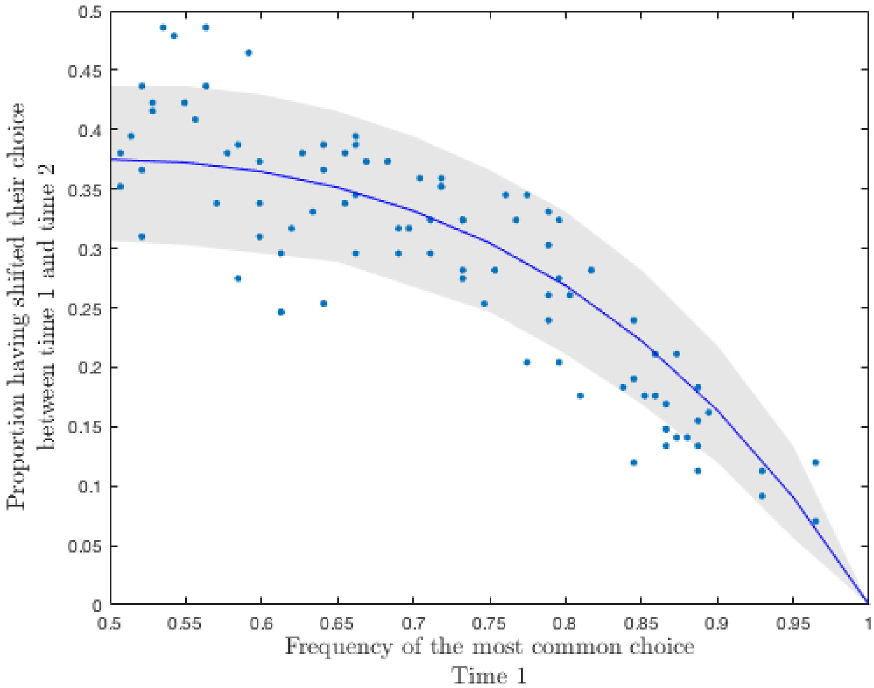


Figure 3.5: Same figure as in (Vincent et al., 2017) obtained with the new set of parameters $\beta = 1.73$, $\alpha = 0.58$ and $F = 0.75$. "Proportion of decision makers having shifted their choice between time 1 and time 2 as a function of the proportion choosing the most frequently chosen option at time 1 (there are 91 points, each one represents a pair of lotteries). The solid line represents the proportion of shifts predicted by expression 3.2 ($\beta = 1.73$, $\alpha = 0.58$ and $F = 0.75$) explained in the text. The shaded area represents the 5% and the 95% quantiles, ie. the area where 90% of the shifts should fall according to Monte Carlo simulations using the above model with two groups (3000 simulations per pairs of lotteries)."

sition and the fit should be better. (i.e. The distribution of height of subjects is unimodal if we look at men and woman separately). In counter part, if such link is not relevant, separating both populations may just increase the noise.

Introducing heterogeneity allows to reduce mean square error (MSE) of the logistic-CPT fit (at time 1) for contrarian group (table 3.4). However, the mean square error (MSE) of the logistic-CPT prediction (at time 2) is increased. At contrary, for overconfident, mean square error (MSE) of the logistic-CPT fit and prediction are increased.

Two parameters - absolute risk\loss aversion η and sensitivity a - are added

		logit-CPT	QDT
ALL	Fit (MSE)	0.0080	0.0058
	Prediction (MSE)	0.0083	0.0065
	Negative log-likelihood	51.91	52.33
Overconfident	Fit (MSE)	0.011	0.016
	Prediction (MSE)	0.012	0.016
	Negative log-likelihood	47.31	46.67
Contrarian	Fit (MSE)	0.0064	0.0087
	Prediction (MSE)	0.012	0.016
	Negative log-likelihood	59.68	59.57

Table 3.4: Mean square error (MSE) for the fits (time 1) and predictions (time 2) for both parametrizations, logistic-CPT and QDT, and with a separation of the population in two groups according to the best fit by two Gaussians of graph (figure 3.1). Overconfident decision makers choose most frequently the most common lottery than contrarian decision makers .

to logistic-CPT parameters' (α, γ, δ, λ and φ) to form the q-factor (equation 5.3) of QDT. Introduction of the attraction factor decrease kink parameter λ (this parameter increase the value of losses compared with gains) : "this means that, though loss might loom more than gain in general ($\lambda > 1$), this effect can be partly transferred to a risk aversion for big losses that was incorporated in the attraction factor ($q \neq 0$)." ((Vincent et al., 2017)).

Contrarian decision makers are less determined than overconfident decision makers (smaller logistic parameter φ). The impact of the q-factor is greater on overconfident decision makers , 25 pairs of lotteries have a q-factor greater than 0.01 while for contrarian decision makers, only 8 pairs are greater than 0.1. This effect can be confirmed by a smaller η and larger a.

λ should be greater than 1 (to give a bigger weight for gains than for losses), it is not the case for contrarian decision makers.

Introduction of heterogeneity allows to identify particularities of both groups. Contrarian decision makers are less determined (smaller φ parameter) and are less impacted by the q factor than overconfident decision makers.

Conclusion on section 3

Separation of participants in two groups (contrarian :37 subjects, overconfident :105 subjects and the proportion $F = 0.75$) is done according to the proportion of most common choices in the full choice set of a subject. This separation gives good results: small p-value (5×10^{-5}), stability between

Parameters		α	δ	γ	λ	φ	η	a
Logistic-CPT	ALL	0.73	0.88	0.65	1.11	0.30	-	-
	Overconfident	0.72	0.91	0.69	1.13	0.40	-	-
	Contrarian	0.87	0.78	0.38	1.002	0.076	-	-
QDT	ALL	0.69	0.89	0.63	1.02	0.37	0.05	1.5
	Overconfident	0.67	0.92	0.66	1.03	0.49	0.05	1.4
	Contrarian	0.81	0.76	0.38	0.95	0.104	0.14	0.76

Table 3.5: Comparison of the parameters with the decomposition of the population in two groups according to the best fit by two Gaussians of graph (figure 3.1). Overconfident decision makers choose most frequently the most common lottery than contrarian descision makers. All refer to all the subjects, it is the same result found by (Vincent et al., 2017). The f-factor of QDT is Logistic-CPT and high loss aversion is the q-factor.

time 1 and 2, prediction of choice shift (choice of the other lottery within the pairs for time 2) and information on each group (contrarian decision makers are less determined and q-factor impact them less). The only down side comes from the lake of improvement in the parametrization of QDT.

One of the minima of the probabilistic choice shift model corresponds to the constraint $F = 0.75$

It would be interesting to have additional information on participants (gender and field of education) in order to find what factors may influence this decomposition. In the next section, we present a new dataset to continue our researches.

Dataset 2

4.1 Description of the experiment

To get a better understanding on how to parametrize QDT, another dataset is analyzed. Lotteries within pairs of Dataset 2 have a gain V in £ with probability p and 0 with probability 1-p (lotteries describe as (V, p)). Due to the 0 payoff, only the probability "p" is important (we formulate the assumption that 1-p with gain 0 is neutral) and comparing "p" of both lotteries within a pair, it is possible to know which lottery is the riskiest (same classification used in (Holt and Laury, 2002)).

The empirical data were collected during two experiments with 101 and 184 participants at the University of Warwick for an article on preference reversal (Loomes and Pogrebna, 2017). They were provided at the individual and aggregate levels for analysis by Prof. G. Loomes.

In the first experiment, the 101 participants answered 144 binary choices (BC) questions (and 36 other questions). The 144 binary choices are four repetitions of:

- (40, 0.3) vs (M, 1), M$\in [4;12]$;
- (15, 0.7) vs (M, 1), M$\in [4;12]$;
- (40, 0.3) vs (15, p'), p'$\in [0.5;0.9]$;
- (40, p'), p'$\in [0.15;0.55]$ vs (15, 0.7).

For each participant, they use those binary choices to compute stochastic indifference (SI) value for two lotteries ($\$_1$-bet = (40, 0.3) and P_1-bet = (15, 0.7)). The SI value represents sure-sum that would be chosen 50% of the time. More information on the process can be found in their article (Loomes and Pogrebna, 2017).

For the second experiment, the 184 participants answered 120 binary choices (BC) questions and 20 direct valuation (DV) tasks. The BC questions:

- (50,0.25) vs (M,1), M$\in [4;11]$,
- (12, 0.8) vs (M, 1), M$\in [4;11]$,
- (50, 0.25) vs (12,p'), p'$\in [0.65;0.95]$,
- (50, p'), p'$\in [0.1;0.4]$ vs (12, 0.8)

were used to compute the SI value for the following lotteries $\$_2$-bet = (50, 0.25) and P_2-bet = (12, 0.8)). The 20 DV tasks concern five lotteries ($\$_1$-bet, $\$_2$-bet, P_1-bet, P_2-bet and $P\$$-bet = (40, 0.8)). For each lottery, the DV value is computed by taking the median of four tasks. The intention of the authors was to avoid any framing effect such as buying or selling.

The $P\$$-bet SI value was computed from another part of experiment 1 with 81 other individuals.

DV and SI values at the aggregate level are on table 4.1.

Lottery	Representation	sample mean DV [£]	sample mean SI [£]
$\$_1$-bet	(40, 0.3)	15.75	**7.22**
P_1-bet	(15, 0.7)	9.75	**8.64**
$\$_2$-bet	(50, 0.25)	18.3	6.55
P_2-bet	(12, 0.8)	8.87	8.18
$P\$$-bet	(40, 0.8)	29.02	**25.12**

Table 4.1: Data from (Loomes and Pogrebna, 2017) at the aggregate level. Direct valuation (DV) tasks are meant to get the value that subjects give to a lottery while avoiding any framing effect such as buying or selling. Stochastic indifference (SI) is the sure-sum that would be chosen 50% of the time against a lottery. Value in bolt are computed from question of the first experiment of Dataset 2.

A particularity of Dataset 2 is that lotteries pairs can be decomposed in what we call "type of lotteries' pairs" or "type of pairs". What characterize a type, is that pairs of lottery differs only in one point, either a probability of a payoff or a payoff of one of the lottery.

Type of pairs with varying payoff, contain a sure-amount lottery ((M,1) where M takes different values). As mentioned in (Allais, 1953) sure-amount lottery induce one of Allais's paradox. The common ratio effect is an example of violation of expected utility theory (a sure-amount lottery in a pair with two-outcomes lottery with higher expected value and one zero payoff, is chosen with a frequency to high compared to what expected utility theory predicts).

Summary of data used

Table 4.2 is a summary of data used in followings section. For DV tasks and BC questions, there is fours iterations.

#	Exp.	Description	# of points	# of participants
1	1	SI value	3	101
2	1	BC questions	36	101
3	2	SI value	2	184
4	2	BC questions	30	184
5	2	DV tasks	5	184
6	2	"Cleaned" SI value	2	132
7	2	"Cleaned" BC questions	30	132
8	2	"Cleaned" DV tasks	5	132

Table 4.2: Summary of data used in followings section. Row number allow a quick identification. "Exp." give the number of the experiement from Dataset 2. "Cleaned" means that subjects who gave extreme answers were excluded (more explanation in section 4.3.2)

4.2 Dynamic of choice shift

The probability of shifting was mentioned in section 3.1 when talking about contrarian and overconfident decision makers. Dataset 2 contains four repetitions of each question. Data used in this section are from experiment 2 of Dataset 2 to (row 4 of table 4.2). It gives an opportunity to compare with (Vincent et al., 2017) previous results.

Sequential iterations

Dataset 2 is composed of four repetitions, thus allowing three sequential comparisons (iteration i with i+1). Figure 4.1 shows that the probability of shifting is smaller than the prediction from the homogeneous choice shift model (equation 3.1). This difference is bigger for lottery with a low frequency of the most common choice (near 50%). This memory effect can be explained from the first assumption of the model for probabilistic choice shift (independence of experiment's iterations). This assumption is not respected because in Dataset 2, the four iterations were done in one session.

Probability of shift seams to be lower over time ($\mathbb{P}$(shift between i+1 and i+2) smaller than $\mathbb{P}$(shift between i and i+1) for $i = 1, 2, 3$). The mean of

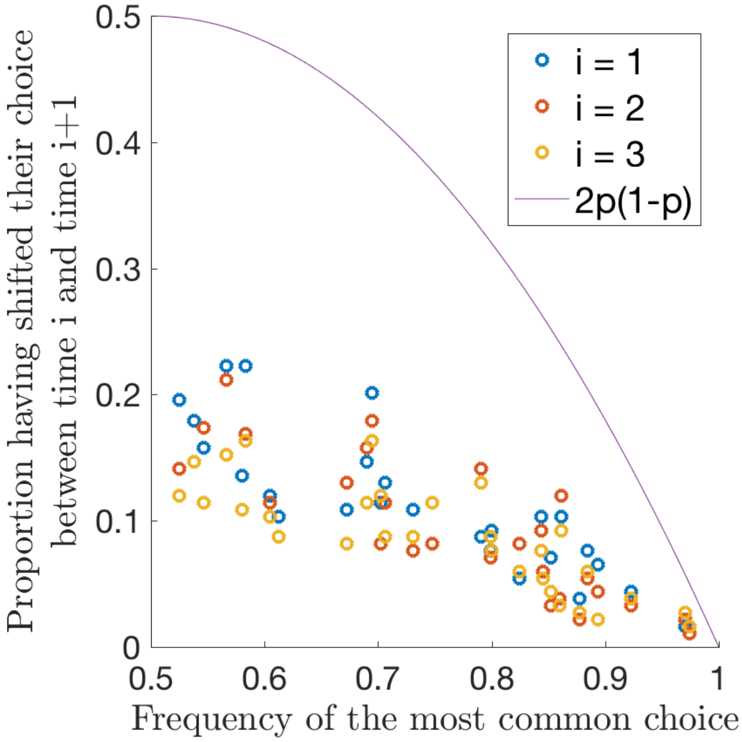


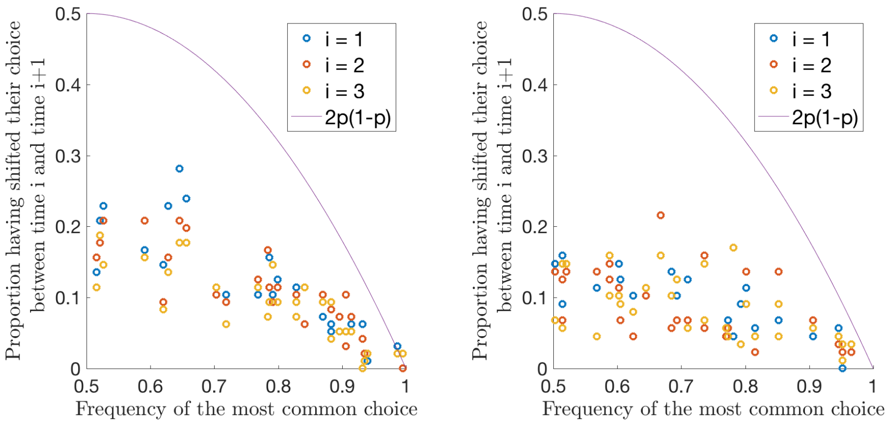
Figure 4.1: Proportion of decision makers having shifted their choice between time i and time i+1 as a function of the proportion choosing the most frequently chosen option at time i (i=1,2,3). For each of the three comparisons, there are 30 points representing each pair of lotteries. The solid line is the result that model 3.1 predicts. Data are from the second experiment of Dataset 2 (row 4 of table 4.2).

shifts between time i and i+1 confirm this impression (see table 4.3) but the standard deviation is too large for results to be significant.

Shift between :	time 1 and 2	time 2 and 3	time 3 and 4
All	0.11 (0.06)	0.10 (0.05)	0.09 (0.04)
Women	0.11 (0.07)	0.10 (0.06)	0.09 (0.05)
Men	0.10 (0.05)	0.09 (0.05)	0.08 (0.04)

Table 4.3: Mean and standard deviation on the 30 pairs of lotteries for the proportion of decision makers having shifted their choice between time i and time i+1. At first, decision makers are all subjects (figure 4.1), then women (figure 4.2a) and finally men (figure 4.2b). Data are from the second experiment of Dataset 2.

Figure 4.2 represents the same data than 4.1 but for women and men separately. As shown in table 4.3, the average proportion of shift is slightly larger for women than for men. The difference is associated with the most common lotteries are chosen with a smaller proportion (between 0.5 and 0.7), for which women shift more often than men. In a following section (4.3.1), we will continue analysis of the difference between genders and effectuate some statistical tests.



(a) Women (b) Men

Figure 4.2: Proportion of decision makers (women and then men) having shifted their choice between time i and time i+1 as a function of the proportion choosing the most frequently chosen option at time i (i=1,2,3). For each of the three comparisons, there are 30 points representing each pair of lotteries. The solid line is the result that model 3.1 predicts. Data are from the second experiment of Dataset 2 (row 4 of table 4.2).

Dynamic of absolute change

To analyse the dynamics of shifting choice (memory effect), figure 4.3 shows the difference of choice between time i and i+1 for $i = 1, 2, 3$ (x axis). Colors correspond to lottery pairs, and the line connecting them is here to improve visualization.

The main dynamics which could be observed is the one already mentioned earlier; the difference of choice between time i and i+1 gets in average smaller as i grows. It shows a memory effect.

A binomial test on each choice between two lotteries comparing the first frequency of choice with the three other repetitions would tell us if these changes are significant. To do that, we used the "myBinomTest" script from

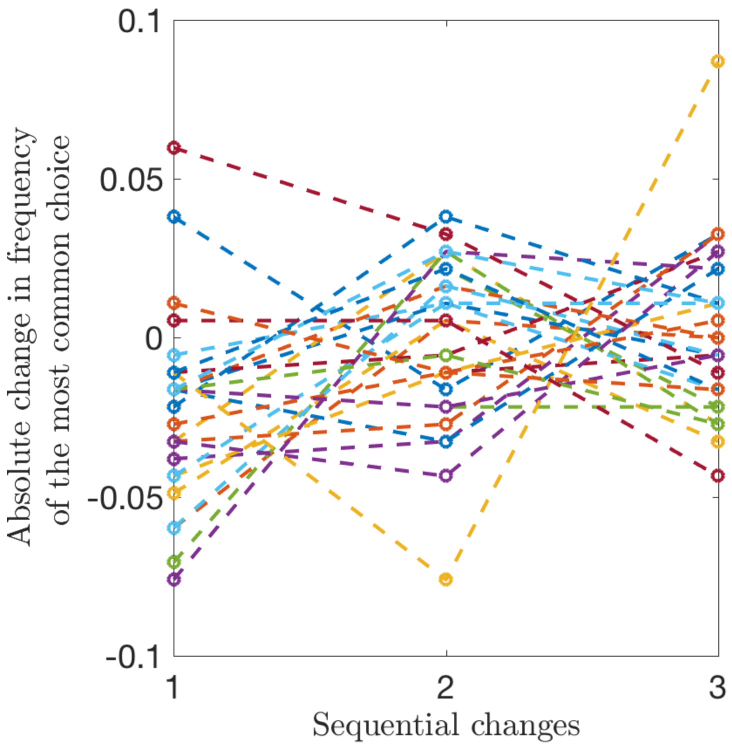


Figure 4.3: Dynamics of the absolute changes in frequency of the most common choice for each pair of lotteries (color-coded) between time i and (i+1), for i-1,2,3 (x axis) and data from the second experiment of Dataset 2 (row 4 of table 4.2). Frequency is discrete due to the limited number of subjects (an increment is 1 / 184 subjects = 0.0054).

the Mathworks' community File Exchange website. For each of the three comparisons, -2, 5 and 3- lotteries out of 30 have a p-value smaller than 0.05. Average p-values are in table 4.4.

	Freq. 2	Freq. 3	Freq. 4
Average p-value	0.42	0.44	0.36
Number significant	2	5	3

Table 4.4: Binomial test on choice frequency (of the most common choice) for each of 30 pairs of lotteries. The comparison is between the first frequency and the three other frequencies. The average p-value is over the 30 choices between two lotteries. Number of significant results represent the number of decisions between two lotteries with a p-value smaller than 0.05.

Conclusion on the dynamic of choice shift

As expected, a lack of independence between iterations induces a strong deviation from the probabilistic choice shift model (equation 3.1). In addition, the dynamic of choice shifting (choosing another lottery within a pair on the next iteration) shows significant sign of memory effect for 10 significant sign of memory effect.

4.3 Partitioning

Partitioning is an art in itself. It can confirm or reject the population's homogeneity and is a way to cluster comparable participants. Clustering subsets may be useful since it should simplify characterization of those subgroups and may help to predict their behavior more accurately.

Gender is a natural feature that could be used to break down a population in subgroups. Another possibility could be to divide subjects according to their behavior towards risk (based on probability of gains and not based on possible losses).

4.3.1 Gender effect

(Favre et al., 2016) found evidences that support larger risk-aversion of female decision makers but their sample was too small to draw any meaningful conclusions thereon. The probability of shifting could be a particularity different from men than women.

In the previous section (4.2), it was shown that on average women shift their choices more often, than men; and this gender effect is more profound for lotteries with lower probability of the most common choice (between 0.5 and 0.7). This section focuses on gender differences in attitude towards risk. The first lottery within a pair is the riskiest (smaller probability of the single payoff).

This section continues analysis of experiment 2 from Dataset 2 (row 4 of table 4.2). There are 184 subjects, including 96 women and 88 men. The mean choice frequency of the safest lottery within a pair, is larger for women than men ($p_{exp\,W} = 0.70$ and $p_{exp\,M} = 0.56$ see table 4.5).

We recall that, the full data set can be braked down in four type of lotteries' pairs :

- (50, 0.25) v.s. M$\in [4; 11]$;
- (12, 0.8) v.s. M$\in [4; 11]$;

Subjects:	Mean p_{exp}	σ
All	0.63	0.25
Women	0.70	0.25
Men	0.56	0.26

Table 4.5: Average choice frequency p_{exp} and standard deviation σ of the safest lottery within a pair for all subjects, women and men.

- (50, p') p'$\in$ [0.1; 0.4] v.s. (12, 0.8);
- (12, p') p'$\in$ [0.65; 0.95] v.s. (50, 0.25).

Figure 4.4 shows the choice frequency of women as a function of the choice frequency of men. All values are above bisectrix f(x)=x, indicating stronger risk-aversion of female group. Women in our sample always choose the safest gamble in a bigger proportion than men. The mean difference between choice frequency is : mean$(p_{exp,W} - p_{exp,M}) = 0.14\,(0.06)$

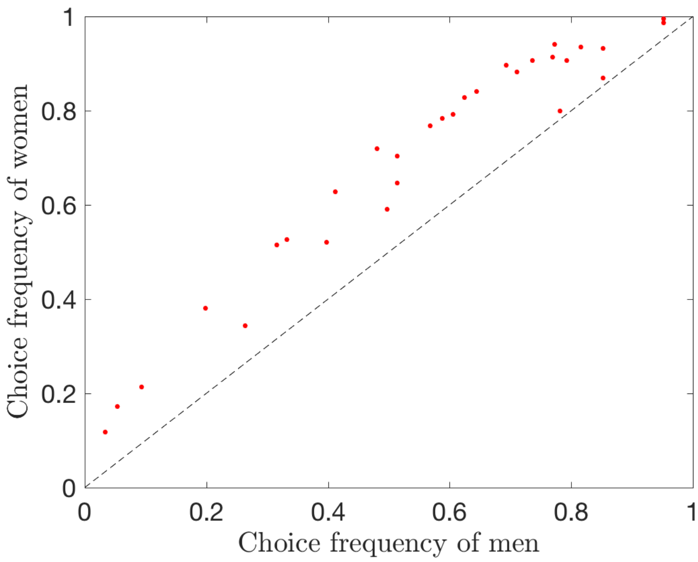


Figure 4.4: Choice frequency of the safest lottery for women as a function of the choice frequency of the safest lottery for men. Values above bisectrix f(x)=x, indicate a stronger risk-aversion of female group. Data are from the second experiment of Dataset 2 (row 4 of table 4.2)

Kolmogorov-Smirnov test

For Dataset 2 from (Loomes and Pogrebna, 2017), the quantitative gender comparison using a two-sample Kolmogorov-Smirnov test (Kolmogorov, 1933) on the two distributions of attraction factors (women and men), rejects the null hypothesis that both distributions are the same. The p-value is 0.06%.

Binomial test

At the aggregate level, choices from population of men (88 subjects) and women (96 subjects) would give the choice frequency $p_{exp\,W\,j}$ and $p_{exp\,M\,j}$ for the lotteries' pair j (j=1...30). If we make the assumption that this frequency is the probability of each participant to choose a given lottery (homogeneous groups), the choice frequency of the pair of lotteries j $p_{exp\,j}$ follows a binomial distribution. Since women always chose the safest lottery in a higher proportion (figure 4.4), we could perform a one-sided binomial test on each lotteries' pair comparing the choice frequency of women with men.

The hypothesis that the choice frequency is the same for men and women is rejected for 28 out of 30 pairs of lotteries at the 5% significance level.

Poisson binomial test

Removing the assumption that participants within a group are identical, the choice frequency follows a Poisson binomial distribution. This distribution can be approximated by using a discrete Fourier transform (Fernández and Williams, 2010).

$$\begin{cases} \Pr(x = k/N) = \frac{1}{N+1} \sum_{l=0}^{N} C^{-lk} \prod_{i=1}^{N} \left(1 + (C^l - 1)p_{ij}\right) \quad k \in \{0, ..., N\} \\ C = \exp\left(\frac{2i\pi}{n+1}\right) \end{cases}$$

Where p_{ij} is the probability of participant i who chooses the safest lottery within a pair j (j=1...30) and N the number of women (N =96). We could approximate this probability for each woman with the average over the four repetitions of each BC question.

Since women always choose the safest lottery in a higher proportion than men (figure 4.4), computing the p-value for men is just looking at the left tail: p-value $= Pr(x < p_{exp\,M\,j})$. Discreteness of probability distribution imposes to multiply $p_{exp\,M\,j}$ by the number of men (N_M) and then to round it to the upper integer:

$$Pr(x < p_{exp\,M\,j}) \approx CDF_j(\lceil N_M p_{exp\,M\,j} \rceil) \;, \tag{4.1}$$

where $\lceil x \rceil$ is the ceiling function and $CDF_j(x)$ is the cumulative distribution function for pair j. The hypothesis that the choice frequency for men is the same as for women is rejected for 28 out of 30 pairs of lotteries at the 5% significance level. It is in agreement with the binomial test.

It is important to emphasize some of our assumptions. Firstly, we suppose that four repetitions give a good estimate of the real probability for women. At the aggregate level (binomial test), this approximation makes sense since there are 384 (4 repetitions for 96 women) data points. In contrast, at the individual level (Poisson binomial test), there are only four data points, thus p_{ij} can only takes five values $\{0, 0.25, 0.5, 0.75, 1\}$.

Analyses of gender difference in 4 types of lotteries pairs

To compare interferences for women and men, the attraction factor is derived from equation 2.1, $q = p - f$. Using the ratio of expected values (equation 2.6) as f-factor (hereafter referred to as EV), we obtain q-factor for all subjects and each gender is $q_{All} = 0.2(0.2)$, $q_{Women} = 0.3(0.2)$ and $q_{Men} = 0.1(0.2)$. The average q-factor is close to the quarter law 2.5. Women show larger positive interferences towards safer option than men. The shape of the q-factor is too chaotic, so we try to understand the probability as a whole ($p = f + q$).

To reduce the problem, we can distinguish two kinds of type of pairs: with and without a sure-amount (M) lottery inside a pair. Frequency of choosing the safest lottery within a pair is displayed as a function of f-factor in figure 4.5 for the type of pairs without a sure-amount, i.e. where a choice is between two risky lotteries: ((50, p') p'∈ $[0.1; 0.4]$ v.s. (12, 0.8), and (12, p') p'∈ $[0.65; 0.95]$ v.s. (50, 0.25)). Type of pairs with sure-amount ((50, 0.25) v.s. M∈ $[4; 11]$, and (12, 0.8) v.s. M∈ $[4; 11]$) are on figure 4.6.

Figure 4.5 clearly demonstrates the gender effect (i.e. larger attraction of females to a safer prospect). At the same time, varying probability (p') has little impact on both groups, males and females. In contrast, on figure 4.6 gender effect is less strong than an influence of different lottery pairs. In this case, for both, men and women, their attraction towards a lottery with a sure-amount (M) grows substantially, when risk of an alternative lottery increases.

According to QDT, if attraction factor tends to 0 ($q \to 0$) and utility factor f is defined by the ratio of expected values of the lotteries in a pair, subjects are expected to choose in equal proportion lotteries with identical expected value (i.e. q=0,$f = 0.5$ and $p_{exp} = 0.5$). The positive difference on both figures (4.5 and 4.6) confirms a higher attraction towards the safest lottery, herewith for all types of lottery pairs risk-aversion of female group is more profound than male.

EV alone does not explain the data. Within QDT, if EV is used for f-factor, then the deviation from a straight line is q-factor. Thus, the observed deviation supports q and QDT.

In section 6, we will show how logistic-CPT fits those data and what problems occur.

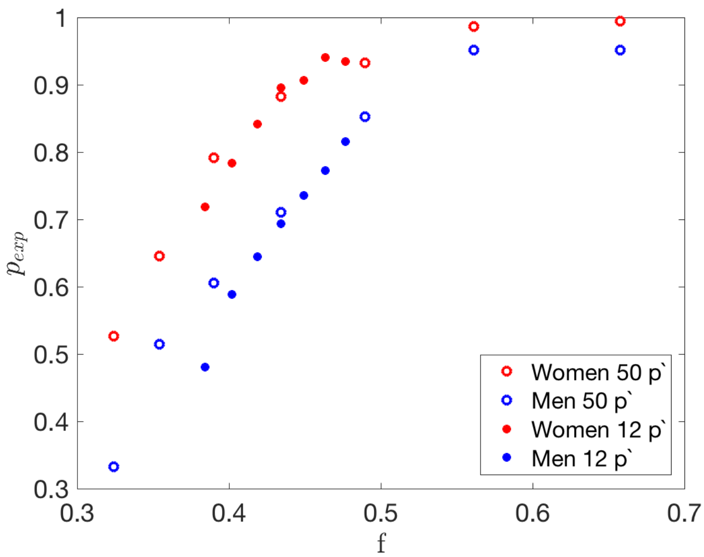


Figure 4.5: Observed frequency p_{exp} of choosing a safer lottery within a pair as a function fits utility factor f. The f-factor is a ratio of expected value (equation 2.6). Lotteries are from the second experiment of Dataset 2 (row 4 of table 4.2), and include the following types: “50 p′ ” stands for (50, p′) p′∈ [0.1; 0.4] v.s. (12, 0.8) and “12 p′ ” stands for (12, p′) p′∈ [0.65; 0.95] v.s. (50, 0.25)

Conclusion and comparison with a previous article on QDT and gender differences

Our findings concerning gender difference in decision makers’ attitude towards risk supports previous study presented in (Favre et al., 2016). The latter used similar QDT approach and type of data, i.e. binary choice composed of a sure-amount versus a single positive payoff lottery. Likewise,

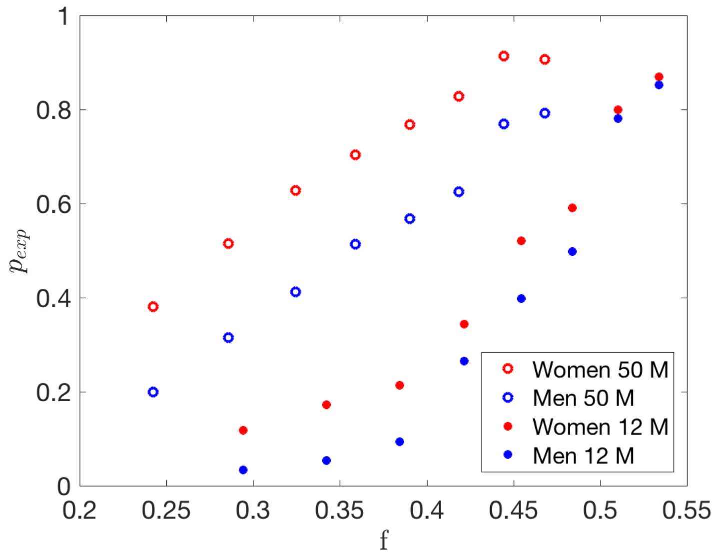


Figure 4.6: Observed frequency p_{exp} of choosing a safer lottery within a pair as a function of its utility factor f. The f-factor is a ratio of expected values of lotteries in a pair (equation 2.6). Lotteries are from the second experiment of Dataset 2 (row 4 of table 4.2), and include the following types: "50 M" stands for (50, 0.25) v.s. M$\in$ [4; 11]; and "12 M" for (12, 0.8) v.s. M$\in$ [4; 11]

(Favre et al., 2016) found evidences of larger risk-aversion of female decision makers, though the results were inconclusive due to the small sample size. The two studies together strengthen the argument.

4.3.2 Partitioning of risk attitude

In the last section, a separation of subjects according to their gender shows that women are more risk-averse than men. In this section, an alternative clustering of participants is based on their risk-attitude, which was revealed in 5 direct valuation (DV) tasks during experiment 2 (included in Dataset 2).

Expected value (EV) and expected utility (EU) theory with linear utility function, as well as QDT with $q \to 0$ and f defined as ratio of prospects' EV, predict the choice of a decision maker that maximizes his/her EV.

We recall that the DV questions were made in such a way to give a monetary value to a lottery while avoiding any reference towards buying or selling. For a single payoff lottery, DV/V has no unit like a probability (DV refers to the value of a DV question with a single payoff of value V). As in (Tversky and Kahneman, 1992) it can be interpreted as a subjective probability $p_{sub.\,DV}$

(i.e. $p_{sub.} \cdot V = DV$). The difference between $p_{sub.DV}$ and the probability "p" of "V" would give a measure of taste for risk.

Cleaning Dataset 2

The range of answers for DV tasks goes from 0 to almost the value of the single payoff. For example, for lottery (50,0.25), DV$\in$ $[0, 49.9]$, while its expected value is 12.5. Thus, values that are out of the range $[2, 40]$ could be qualified of extreme. Choosing one of the extreme values either 0 or 49 for lottery (50,0.25) would correspond to an unusually strong risk attitude (an extreme risk-aversion/risk-seeking). There are no reasons that would justify such a small or large value for that lottery.

Analysis of answers in DV tasks shows that some participants provided those extreme valuations. Table 4.6 shows number of participants with extreme answers in DV task, which was repeated four times.

Number of extreme answers	1	2	3	4
Number of participants	46	34	26	23

Table 4.6: Number of participants with extreme answers in DV tasks.

Theoretically, some decision makers could have abnormal risk attitudes. Such subjects could be separated in a distinct group and an adequate approach to analysis should be developed. On the other hand, for the present experiment (experiment 2 of Dataset 2), it is (more) plausible that reported extreme DV are not reliable. This type of answers may be associated with a misunderstanding of the experimental setup, lack of attention or diligence of some participants, which undermines validity of corresponding valuations. Thus, for the subsequent analysis of Dataset 2 (section 4.3.2), we choose to be the most conservative, participants with one extreme answer were excluded, reducing the number of subjects from 184 to 132 for the second experiment of dataset 2. Thus, data are from rows 6,7 and 8 of table 4.2.

K-means clustering

K-means clustering (Lloyd, 1982) is a technique that aims to partition data around k centroids in order to minimize the within-cluster sum of squares (variance in each cluster). It was first used for signal processing. It is still in use today, either as a full algorithm or as preprocessing algorithms for more complex algorithms of data mining.

Figure 4.7 represents the 132 selected subjects in a two dimensional space of abscissa $\sum \left(\frac{DV}{V} - p\right)$ (direction of deviation from risk-neutrality) and ordinates $\sum \|\frac{DV}{V} - p\|$ (amplitude of deviation from risk-neutrality). Data used are from the second experiment of dataset 2 (row 8 of table 4.2)

K-means clustering for three clusters is used on experiment 2 of Dataset 2 (figure 4.7). The lowest (center) cluster is formed with decision makers (43 of subjects) who are risk-neutral (their subjective probability is the closest to the real probability). The left group (32 of subjects) has a tendency for a subjective probability to be smaller than objective probability p, thus, they could be referred to as risk-averse. Finally, the group on the right (63 of subjects) has a tendency for a subjective probability to be bigger than p, and they could be referred to as risk-seekers.

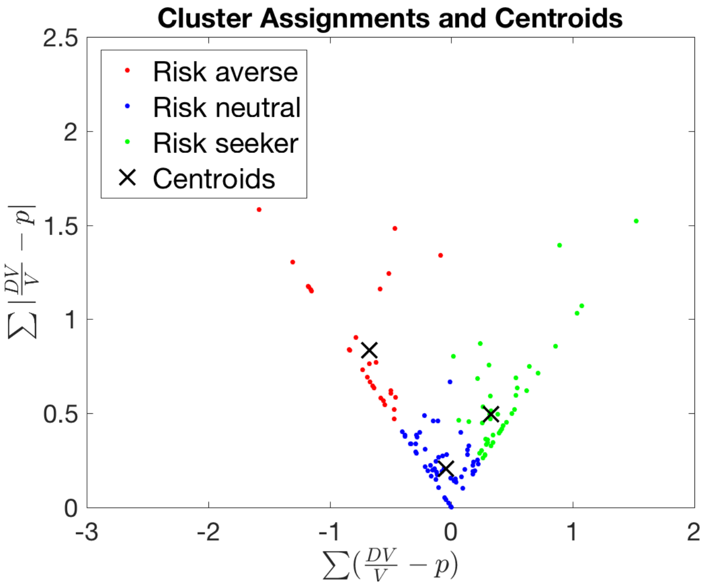


Figure 4.7: Partition of 132 subjects of the second experiment from Dataset 2 (row 8 of table 4.2) in three clusters with k-means clustering. $\frac{DV}{V}$ can be interpreted as a subjective probability.

Comparison and analyses of clusters

We look at different points to analyse the pertinence of this separation, for example: proportion of each gender, choice frequency, subjective probability. It is important to recall that we use experiment 2 from Dataset 2. Which contains 5 values for DV tasks, 2 values for SI questions (table 4.1) and 30 BC questions (section 4.1).

Proportion of women in each cluster

To link to a previous section 4.3.1, the proportion of women and men in each cluster is presented in figure 4.8. Risk-averse cluster consists mostly of women (75% of cluster members), while majority among risk-seekers are men (60%). Risk-neutral cluster is a mixture of women and men decision makers, almost at parity. This clustering emphasizes results from the previous section that women are in average more risk-averse. An evolutionary origin of male risk-seeking was investigated in (Favre and Sornette, 2012).

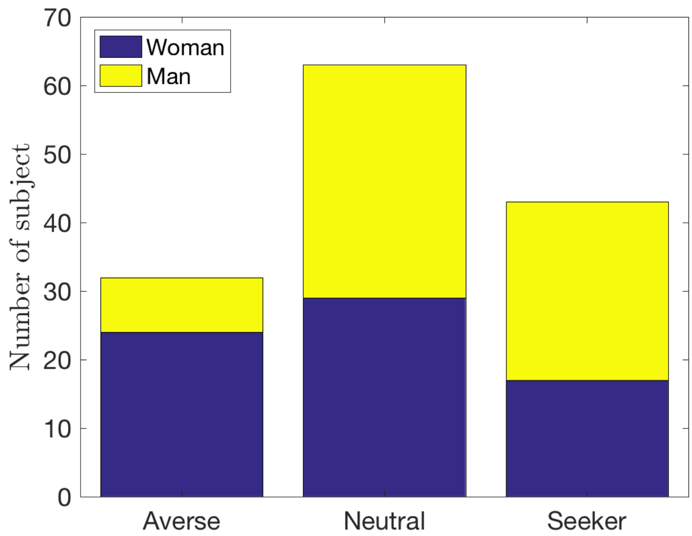


Figure 4.8: Gender composition of the three clusters: -riska-verse, risk-neutral and risk-seeker decision makers-. Based on k-means clustering (figure 4.7).

Choice frequency

The 30 binary choices (BC) questions from the same experiment 2 (row 7 of table 4.2) are ordered in a way that the choice frequency p_{exp} represents the safest lottery. If clustering according to DV tasks that was proposed in the previous section is correct, for a lotteries' pair, choice frequency of the safest lottery among risk-averse participants is expected to be greater than among risk-neutral or risk-seeking (and the frequency among risk-neutral is expected to be greater than the frequency among risk-seekers).

Figure 4.9 shows Δp_{exp} (the difference of choice frequency of a cluster with respect to the choice frequency of the same 132 participants, which were used for k-means clustering (4.7)) as a function of p_{exp} (the choice frequency of these 132 participants). Only risk-neutral and risk-seekers overlap and especially in a small proportion. It confirms the k-means clustering based on the 5 DV tasks and shows transitivity of risk behavior among different kinds of tasks (from DV tasks to BC questions).

From mean and standard deviation of Δp_{exp} over the 30 pairs of lotteries (BC questions) for each group, one can makes the following remarks:

- risk-averse decision makers with $\Delta p_{exp} = 0.16$ (0.08) confirms that they chose more often the safest option in a pair;
- in a prefect separation, risk-neutral decision makers should have a value of 0 (but with $\Delta p_{exp} = -0.03$ (0.03), it is within the range of error);
- risk-seeker decision makers have a negative value $\Delta p_{exp} = -0.07$ (0.04).

It is important to note that we compare deviation of a cluster choice frequency to a total choice frequency. Thus, the deviation of risk-neutral depends on number of participants in each cluster and severity of deviations in other clusters.

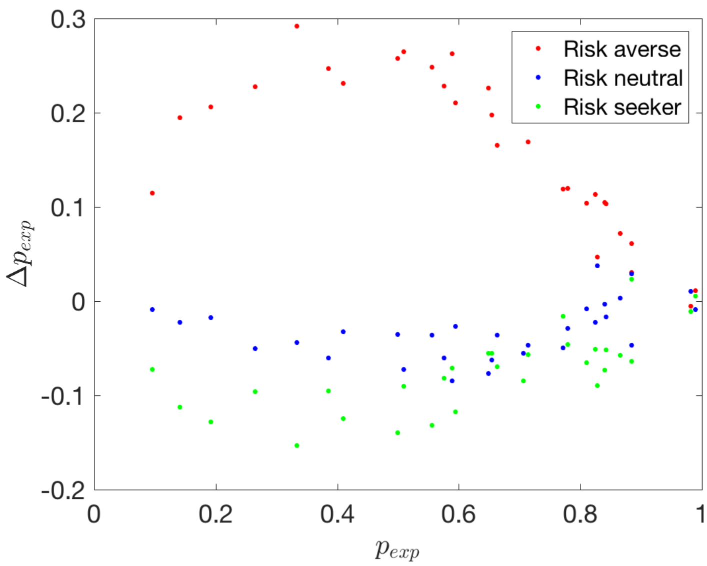


Figure 4.9: The difference of choice frequency of a cluster with respect to the choice frequency δp_{exp} as a function of the choice frequency of these 132 participants p_{exp}. Each point is one of the 30 lottery pairs that where presented in a binary choice task from experiment 2 of (Loomes and Pogrebna, 2017). Clusters are identified with k-means clustering (figure 4.7). Data are from row 7 of table 4.2.

Measure of inconsistency of choice

One measure available for all subjects would be the "inconsistency of choice". It is a measure of the variation in answer over the four iterations of DV tasks. The inconsistency for one choice is noted σ_{L1}). Because of the small sample size the L1-norm was chosen, $\sigma_{L1}(x) = \sum_{l=1}^{4} \|x_l - \mu\|$ where μ is the average of x over the four iterations (l=1..4) of the DV questions. To compute the estimate of inconsistency of choice (table 4.7), we first take for each lottery j (j=1...5) the median of $\frac{\sigma_{L1}(DV)}{V}$ among all subjects in the cluster. Then compute the mean over the five lotteries from Dataset 2 (table 4.1 or row 8 of table 4.2):

$$Precision = \frac{1}{5}\sum_{j=1}^{5} median_{subject\,i}(\frac{\sigma_{L1}(DV_{ij})}{V_{ij}}) \quad (4.2)$$

	Total	Risk-averse	Risk-neutral	Risk-seeker
Inconsistency of choice	0.045	0.062	0.039	0.047

Table 4.7: Inconsistency of choice for all 132 subjects together and for each cluster, mean, over the five DV tasks, of the median of $\frac{\sigma_{L1}(DV_{ij})}{V_{ij}}$. With $\sigma_{L1}(x) = \sum_{l=1}^{4} \|x_l - \mu\|$ and μ is the average of the four iterations (l=1..4) of one DV task. Clusters are identified with k-means clustering (figure 4.7). Data used are from row 8 of table 4.2.

Risk-averse decision makers have a inconsistency of choice that is 32% higher than inconsistency of choice of risk-seekers and 58% larger than risk-neutral. Thus risk-averse decision makers are less precise/more concentrated than risk-neutral and risk-seekers.

Subjective probability with the DV tasks

Section 2.2.3 presented how (Tversky and Kahneman, 1992) proceeded to compute the subjective probability from the "certainty equivalent of a lottery". Using the same process for the DV value (and then for the SI value), it is possible to compute subjective probability from other tasks ($p_{sub.DV} = \frac{DV}{V}$).

Figure 4.10 shows $p_{sub.DV}$ as a function of "p", the probability of the only non-zero payoff "V", for the five main lotteries from Dataset 2 (table 4.1 or row 8 of table 4.2), for all (132 subjects), risk-averse, risk-neutral and risk-seeking decision makers. This graph confirms names given to each group.

The line f(x)=x represents risk-neutrality, so upper and lower points correspond respectively to risk-seekers and risk-averse behavior. Circles correspond to median values, and error-bars are boarders of the first and third quartiles. Profound subjective deviations are observed :

- for risk-seekers as overestimation of small probabilities (i.e. $-bets);
- for risk-averse subjects as underestimation of large probabilities (i.e. P-bets).

Thus, the two effects, which are usually captured by an inverse S-shape, with this dataset were separated. We also observe that subjective probability, which is defined in the proposed way based on DV tasks, for risk-averse group (and only for this group) may be affected by payoff structure (risk-averse gives a lower subjective probability for $(40; 0.8)$ than for $(12; 0.8)$). For risk-averse, the highest payoff is associated with more profound decrease in subjective probability estimation, then risk-averse are "least greedy" (the largest decrease in utility of money).

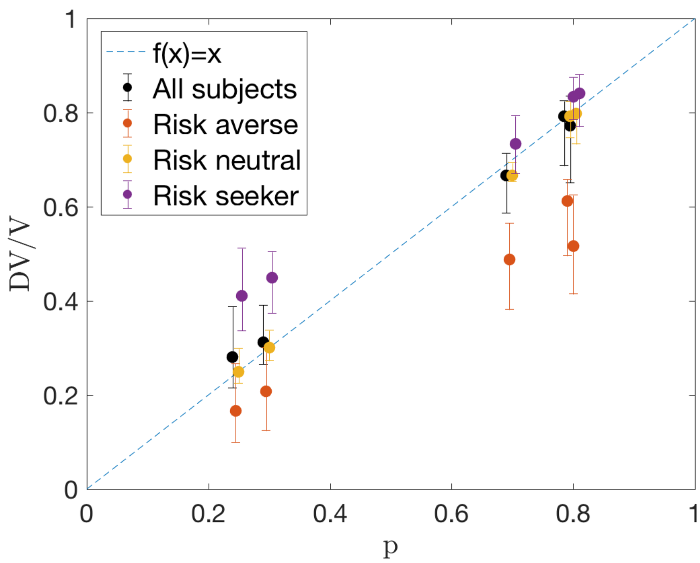


Figure 4.10: Quartiles of the subjective probability, defined based on DV tasks, $p_{sub.DV} = \frac{DV}{V}$ for all 132 subjects and each cluster from figure 4.7. "p" is the probability of the only non-zero payoff "V". Circles correspond to median values and error-bars are boarders of the first and third quartiles. To increase the visibility, the two points of abscissa 0.8 are shifted to the left for $(12; 0.8)$ and to the right for $(40; 0.8)$. Each cluster is a bit shifted too. Data are from row 8 of table 4.2.

Subjective probability with the SI value

In this section, the subjective probability obtained from the SI task, $p_{sub.SI} = \frac{SI}{V}$ is analyzed. Figure 4.11 shows $p_{sub.SI}$ as a function of "p", the probability of the only non-zero payoff "V", for the two main lotteries with SI values from experiment 2 of Dataset 2 (table 4.1 or row 6 of table 4.2), for all (132 subjects), risk-averse, risk-neutral and risk-seeker decision makers. Circles correspond to median values and error-bars are boarders of the first and third quartiles.

For all clusters, subjective probabilities which are based on SI values, are smaller than objective probabilities (below $f(x) = x$). One explanation could be that SI estimation is based on the lottery pairs, in which one of the choice options is a sure-amount (M). Thus, it results in a higher positive attraction towards the sure-amount option. This decreases the SI value and finally reduces $p_{sub.SI} = \frac{SI}{V}$.

The median of $p_{sub.SI}$ is in the order that confirms proposed k-means clustering: from bottom-up, risk-seeker, risk-neutral and risk-averse decision makers.

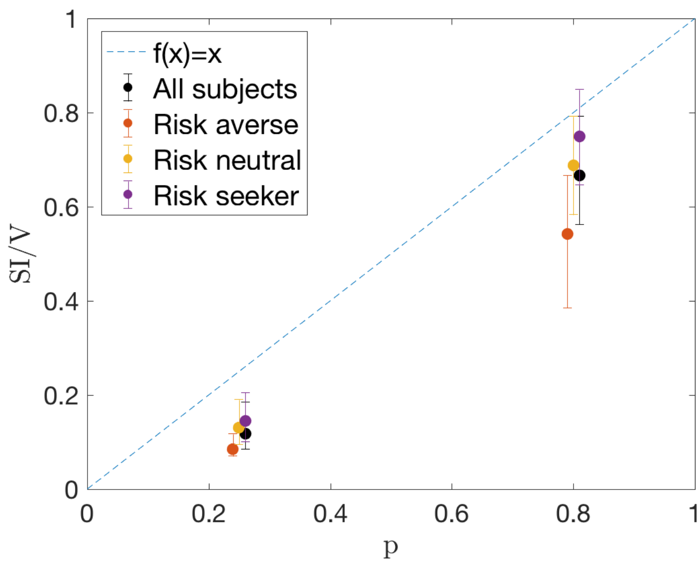


Figure 4.11: Quartiles of the subjective probability$p_{sub.SI} = \frac{SI}{V}$ for all 132 subjects and each cluster from figure 4.7. "p" is the probability of the only non-zero payoff "V". Circles corresponds to median values and error-bars are boarders of the first and third quartiles. To increase the visibility, each cluster is a bit shifted. Data are from row 6 of table 4.2.

Other clustering according to the SI value

Finally, if the process of k-means clustering (section 4.3.2) is applied on $\frac{SI}{V}$ (data from the second experiment of Dataset 2, row 6 of table 4.2), 62 out of 132 analyzed participants are classified in the right (risk-seeking) $\frac{DV}{V}$ group. In comparison, a random clustering would give 33% (46 subjects). It is important to recall that there are only two SI values per subject so this clustering may contain more errors.

Figure 4.12 compares k-means clustering based on DV value, with k-means clustering based on SI questions. Columns represent: DV k-means clustering and colors: SI k-means clustering. Two remarks, the extreme migration (averse to seeker and vice-versa) is relatively low. Secondly, DV k-means clusters are present in great quantity in their respective SI k-means clusters.

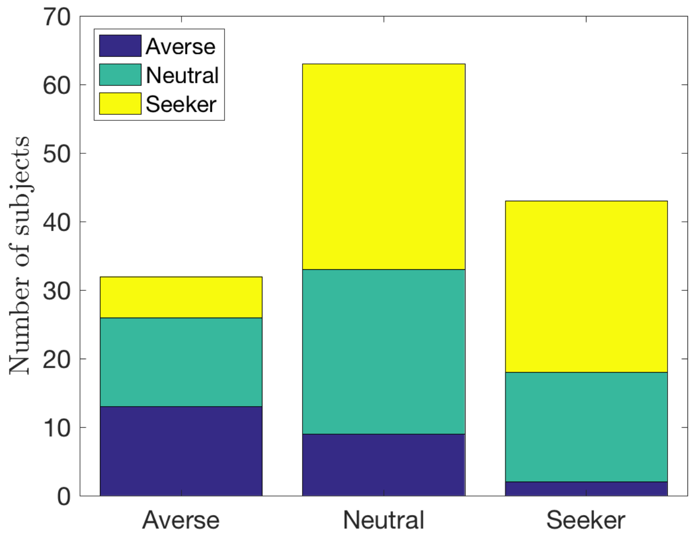


Figure 4.12: Comparison of DV k-means clustering from figure 4.7 -risk-averse, risk-neutral and risk-seeker decision makers- with k-means clustering on SI questions (represented by colors in the graph). SI k-means clustering is computed with the same process than DV k-means clustering but use SI values instead of DV values. Data used are from experiment 2 of Dataset 2 (rows 6 and 8 of table 4.2)

Discussion of the clustering

The goal of k-means clustering was to see what can be learnt from a few DV tasks and if risk-aversion is dependent of a type of questions. The difference in choice frequency between cluster confirms that the separation is meaningful because it separates well BC questions. K-means clustering is confirmed by gender as women are in a bigger proportion in the risk-averse group. Finally, the inverse S-shape from subjective probability is decomposed in behaviors that arises from two extreme groups -risk-seeker and risk-averse decision makers-.

QDT parametrization

5.1 QDT interpretation of preference reversal, with SI ratio for f-factor

As Dataset 2 is from an article on preference reversal, it is interesting to compare it with a previous article on QDT and preference reversal (Yukalov and Sornette, 2015a). Data used are from both experiments of Dataset 2 (rows 1 to 4 from of table 4.2)

Preference reversal is an effect, which was firstly expressed by (Lindman, 1971) and (Lichtenstein and Slovic, 1971) and (1973), where subject shift their preferences for a lottery after a pair is formed with lotteries. The decision is not the same when lotteries are valued separately or within a pair. In review articles such as (Slovic and Lichtenstein, 1983) , (Tversky et al., 1990) and (Tversky and Thaler, 1990) more quotes and information can be found.

In (Yukalov and Sornette, 2015a) the f-factor was estimated with the fraction (equation 2.6) that we recall here with expected values (EV).

$$f_{EU\,i} = \frac{EV_i}{EV_i + EV_j}. \tag{5.1}$$

Where i, j are names of lottery in a pair. To determine if there is some preference reversal they compare the expected value with the choice frequency on data from (Tversky and Thaler, 1990). Finally, they were able to explain those differences with the quarter law 2.5. It is important to mention that no parametrization is needed.

In (Loomes and Pogrebna, 2017) they compare the SI and DV value for both bets with the predominance of choice from ($-bet vs P-bet). Their conclusion is only a fraction (7%-10%) of participants have exhibited a preference reversal of small magnitude for SI value while for DV tasks, 117 subjects produce preference reversal.

From the description of preference reversal pattern and findings of (Loomes and Pogrebna, 2017), preference reversal can be explained with QDT. The SI values may be linked to f-factor, which is intrinsic for each individual lottery but when lotteries are compared to each other an additional q-factor arises, which affects/deviates this binary choice and may lead to preference reversal.

An important point to discuss is why the f-factor would come from the SI value. First, I would like to refresh what is the utility factor:

- diagonal terms of the norm of a scalar product between vectors in a Hilbert space;
- interaction of the same modes (not interferences between different modes);
- all characteristics a subject focuses their attention on.

SI values are determined by the choice between a lottery and a sure-amount. This task is simpler, then there are less interferences, which will finally gives a value closer to the intrinsic utility that someone may have for a lottery.

To compare the SI and the expected value, utility factor could be as well estimated from the SI value in the following way,

$$f_{SI\,i} = \frac{SI_i}{SI_i + SI_j}. \tag{5.2}$$

Where i, j are names of lottery in a pair. Values for pairs ($\$_1$,$P_1$) and ($\$_2$,$P_2$) are in table 5.1 as well as values for f-factor made with expected utility (equation 5.1).

Pair	Name	Values	f_{SI}	f_{EV}	p_{exp}
($\$_1$, P_1)	$\$_1$	(40, 0.3)	0.455	0.533	0.17
	P_1	(15, 0.7)	0.545	0.467	0.83
($\$_2$, P_2)	$\$_2$	(50, 0.25)	0.445	0.566	0.2
	P_2	(12, 0.8)	0.555	0.434	0.8

Table 5.1: F-factor computed with 5.1 for expected value and the SI value (monetary equivalent amount) from Dataset 2. p_{exp} is the frequency of choosing a lottery and it sums to one for a prospect.

Table 5.1 shows the utility factor from the SI and the EV. Interverting 2.1 Q-factor for lottery pairs $\$_i$ vs P_i i = 1,2 are in good agreement with the quarter law 2.5 $q_{\$_1,P_1} = 0.285$ and $q_{\$_2,P_2} = 0.245$. On the other side, if we use the f-factor computed with expected utility: $q_{EU\ \$_1,P_1} = 0.363$ and $q_{EU\ \$_2,p_1} = 0.366$, it is further away from 0.25. Better agreement with the quarter law indicates that SI values would be a better alternative than EV to calculate the f-factor.

Preference reversal is better explained by the quarter law if the f-factor is made with SI values. This confirms the intuition that SI value gives a value closer to the intrinsic value of a lottery.

5.2 Exploration of origins of q-factor with priority model, and novel “safe” q-factor

In article (Rieskamp, 2008), different theories are compared - a probabilistic version of CPT, priority model (hereafter referred to as PM) and decision field theory (hereafter referred to as DFT) - with their deterministic counterparts. This comparison has shown some results with for instance, the main one which is the supremacy of probabilistic theory over their deterministic counterparts. Another finding is that CPT and DFT outperform the priority model.

The choice between options is made based on sequential comparison of the following three aspects.

To understand which aspect of a lotteries’ pair influences the most decision makers, we have implemented the priority model. The choice between options is made based on sequential comparison of the following “three aspects”:

Aspect 1 “the minimum outcome of each gamble”;

Aspect 2 “the probability of each minimum outcome occurring (called minimum probability)”;

Aspect 3 “the maximum outcome of each gamble”.

To understand how the priority model works, it is advised (but not mandatory) to read appendix from (Rieskamp, 2008). To allow an understanding of our results, we briefly explain the model:

During the calibration of our model, decision weights and decision threshold for the “three aspects” are adjusted. If the first aspect is smaller than the threshold, the second aspect is considered and then the last aspect too. Decision weight influences the probability of selecting a specific aspect: “The probability of selecting a specific aspect is determined by the ratio of the specific weight to the sum of the importance weights of all aspects not considered so far. For instance, the probability of considering the minimum outcome first is determined by wmin/(wmin + wpr + wmax).”

The priority model has the particularity to work well for small differences in expected value. Dataset 1 was restricted to lotteries' pairs with a difference in expected value smaller than 20% of the largest payoff in absolute value. With this restriction, there are 18 and 24 lotteries for gain respectively loss and mixed domain. Parametrization of the priority model for these three domains can be seen on table 5.2. The impact of the"three aspects" can be deducted from decision weights. For gain domain, the most important aspect is the number 1 followed by 2 and finally by 3. For loss domain, the order changes : 1, 3 and 2 and finally for mixed domain, the order is also different : 2, 3 and 1.

Domain	w_{min}	w_{pr}	w_{max}
Gain	0.46	0.36	0.18
Loss	0.67	0.02	0.31
Mixed	0.26	0.47	0.27

Table 5.2: Decision weight of the priority model on restricted Dataset 1 (with this restriction, there are 18 and 24 lotteries for gain respectively loss and mixed domain). The model is the same as in appendix from (Rieskamp, 2008). The priority model has the particularity to work well for small differences in expected value. For this reason, Dataset 1 was restricted to lotteries' pairs with a difference in expected value smaller than 20% of the largest payoff in absolute value.

For different types of lotteries (gain/losses/mixed) the order of aspects that better explains the observed choices may vary, i.e. the main determining factor of the choice may depend on the setup/context/type of lotteries. This motivates the following proposition that QDT attraction factor q (and its analytical formulation) may also be context dependent, highlighting the most profound interfering factor (aspect) of a particular choice situation.

Big loss aversion attraction factor

(Vincent et al., 2017) created "big loss aversion" attraction factor:

$$q_{\text{Loss B}} = min(fA, fB)tanh(a(U_A - U_B)) \tag{5.3}$$

"Were U is the constant absolute risk-aversion (CARA) function for an initial wealth of 100 corresponding to the amount given to subjects at the beginning of the experiment:"

$$U(V) = 1 - e^{\eta(100+V)} \tag{5.4}$$

Novel “safe” q-factor

In finance, one of the mostly used measure of risk is standard deviation. It gives information on how far some numbers are spread around their mean. The formula is recalled,

$$\sigma^2 = \sum_i p_i (x_i - \mu)^2, \tag{5.5}$$

where p_i is the probability of x_i and $\mu = \sum_i (p_i x_i)$, as example, for a risky lottery $(40; 0.25)$ $\sigma = 15$ and for a less risky, $(12; 0.8)$, $\sigma = 2.1$. An attraction factor towards the safest lottery can be build:

$$q_{\text{Safe B}} = min(fA, fB)a(\sigma_A^\eta - \sigma_B^\eta) \tag{5.6}$$

where “a” is a scaling factor and “η” a power exponent.

5.3 QDT parametrization of Dataset 2 with EV for f-factor and novel “safe” q-factor

To test “safe” q-factor, QDT is calibrated on Dataset 2 (rows 2 and 4 of table 4.2) with expected value as utility factor. Expected utility with the identity function ($f(x) = x$), as a value function, gives expected value. Expected value of two lotteries from a pair can be composed with the equation 5.1 ($f_B = \frac{EV_B}{EV_A + EV_B}$) to form a f-factor. Two implementations of QDT are tested (called: one and two parameters “safe” q-factor). Only “safe” q-factor attraction factors change from one to another. The first one is the equation 5.6 with two parameters-power exponents η and scale a-. The equation 5.6 is the second.

Parameters are estimated with maximum likelihood. Wilks’ likelihood ratio test (Wilks, 1938) is used to compute p-values. Table 5.3 contains results of parametrizations of Dataset 2.

I would like to share you two main observations : First, the attraction factor with only one parameter works better owing to the fact that it has a smaller p-value than the attraction factor with two parameters. This improvement comes from the lower number of free parameters on Wilks’ likelihood ratio test. Secondly, “safe” q-factor with one and two parameters improves more the second than the first experiment where the difference in risk between two lotteries is larger than in the first experiment.

Figure 5.1 compares the frequency of choosing the safest lottery with results from models -only f-factor, one parameter “safe” q-factor and two parameters “safe” q-factor-. QDT improves mostly the fit for pairs of lotteries

Parametrization	η	a	NLL	MSE	E. frac.	p-value	mean $\lvert q \rvert$
Exp 1	-	-	23.89	0.079	0.54	-	-
"Safe" q-factor, $\eta = 1$	1	0.043	20.33	0.033	0.69	0.0076	0.21 (0.08)
"Safe" q-factor	1.81	0.0038	20.03	0.030	0.72	0.0210	0.22 (0.11)
Exp 2	-	-	21.02	0.084	0.47	-	-
"Safe" q-factor, $\eta = 1$	1	0.041	16.84	0.019	0.72	0.0038	0.24 (0.11)
"Safe" q-factor	1.50	0.0084	16.73	0.018	0.74	0.0138	0.23 (0.13)

Table 5.3: Calibration of QDT -expected value as f-factor (equation 5.1), "safe" q-factor (equation 5.6) and maximum likelihood estimation (minimizing negative loglikelihood NLL)- on Dataset 2 is at the aggregate level. The mean squared error (MSE) and the explained fraction (E. frac.) give information on the precision of the fit. Wilk's likelihood ratio test gives the p-value for H_0 (no q-factor) v.s. H_1 (existence of q-factor).

with a high choice frequency of the safest lottery. Pairs of lotteries with a small/large choice frequency are overestimated/underestimated by QDT.

"safe" q-factor improves the simple expected value model. This improvement is more meaningful for "safe" q-factor with one parameter than two (p-value three times smaller). Pairs of lotteries with a high choice frequency of the safest lottery show the best improvement.

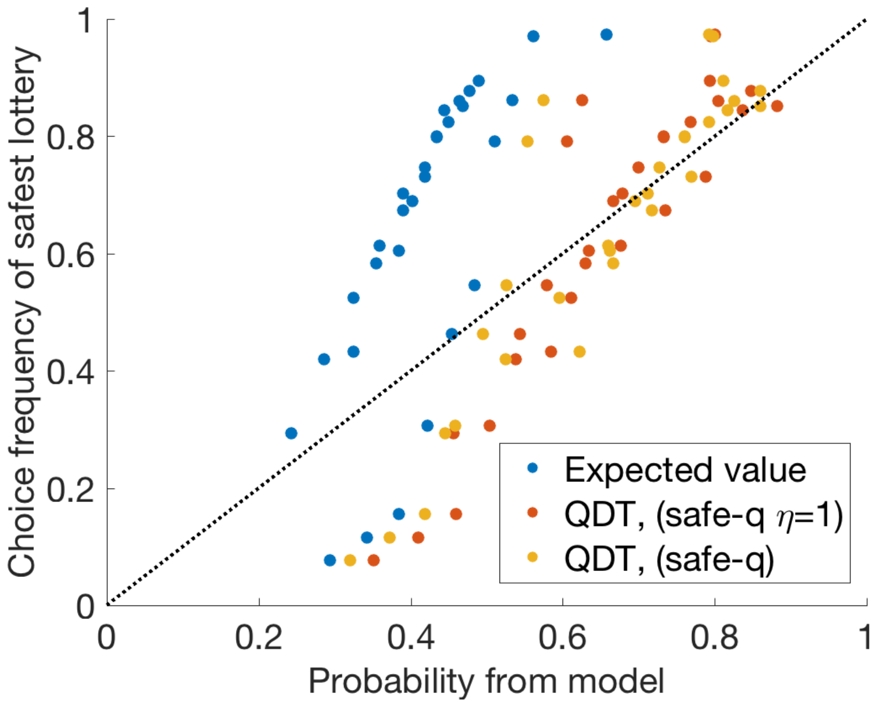


Figure 5.1: Comparison between choice frequency p_{exp} and QDT parametrization p -expected value as f-factor (equation 5.1), "safe" q-factor (equation 5.6) and maximum likelihood estimation (minimizing negative log-likelihood NLL)- on experiment 2 of Dataset 2 (row 4 of table 4.2) is at the aggregate level.

Improving logistic-CPT parametrization for Dataset 2

Calibration of logistic-CPT on Dataset 2 produces parameters out of a normal range. In the meanwhile, weighting, value and logistic functions have an unexpected shape. We try to understand what is happening in order to gain information on the model. Another goal is to see if there is a limit on predictability.

Parametrizing logistic-CPT on Dataset 2 is done with maximum likelihood estimation 2.3. Figure 6.1 compares probability from the model (logistic-CPT) with choice frequency. It is shown that the fit is not perfect (points away from the line $f(x) = x$).

Bootstrapping (Efron and Tibshirani, 1994) is used to compute parameters estimate and standard deviation (in parenthesis). For convergence towards true parameters, data (lotteries' pairs) should be independent of each other. Here it is not the case, but it will give some information on the stability of the parametrization. A thousand dataset of equal size are sampled from the original dataset with repetitions. After that, it is then possible to compute mean and standard deviation on parameters.

Parameters obtained can be qualified of "weird" for two reasons: in comparison with parameters from others studies (table 6.1) and from the shape of constituent functions of logistic-CPT (section 6.2).

6.1 Comparison of logisitc-CPT's parameters with other studies

To confirm our impression that logistic-CPT's parameters (table 6.1) for Dataset 2 are unusual, we first compare them with other studies: (Murphy and ten Brincke, 2017; Tversky and Kahneman, 1992; Nilsson et al., 2011).

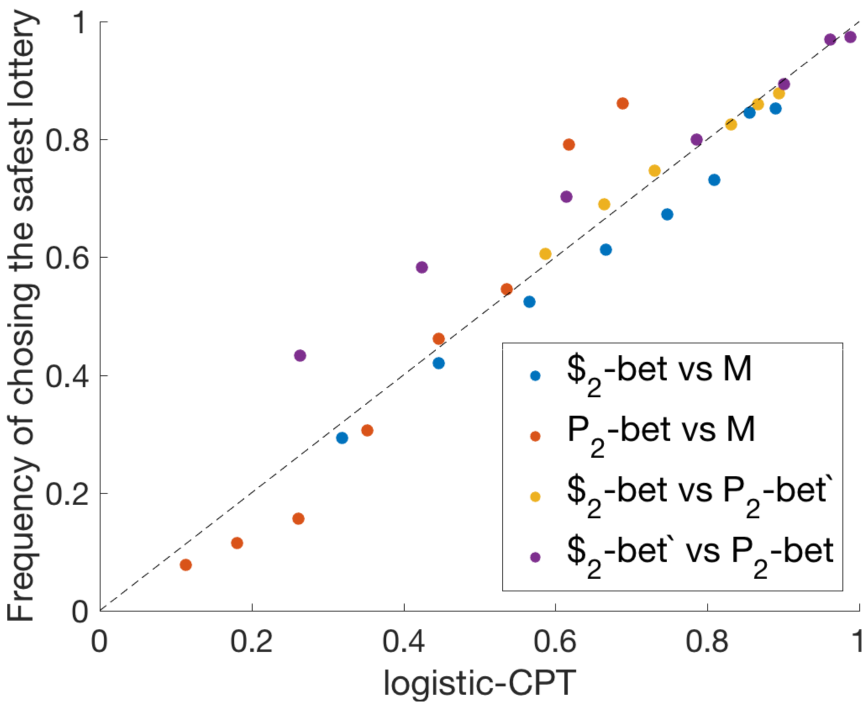


Figure 6.1: Frequency of choosing the safest gamble p_{exp} as a function of logistic-CPT. The four types of lotteries' pairs are in this order: (50, 0.25) vs M'∈ [4; 11]; (12, 0.8) vs M'∈ [4; 11]; (50, 0.25) vs (12, p') p'∈ [0.65; 0.95] and (50, p') p'∈ [0.1; 0.4] vs (12,0.8).

Some studies use "two-part power function" as weighting function (equation 2.9). To convert those parameters in "Prelec II" function's parameters (equation 2.10), we sample data-points from the "two-part power function" with parameters from a study. Then, data-points that are fitted using Non-linear least squares with the Prelec II function (e.g. for (Tversky and Kahneman, 1992): $\gamma = 0.61$ becomes $\gamma_{PrelecII} = 1.07$ $\delta_{PrelecII} = 0.52$ and $\delta = 0.69$ becomes $\gamma_{PrelecII} = 1.01$ $\delta_{PrelecII} = 0.63$ but δ the parameters are for losses lottery so, useless for our instance. After this example, δ (reps. γ) refers to parameters of the Prelec II function for general elevation of the curve respectively curvature parameters.

Comparison of parameters from Dataset 2 with another set of table 6.1 gives the largest difference. In the other hand, comparison of two sets of parameters not from Dataset 2 has a less significant difference. Power exponent (α) and δ are smaller, while γ and curvature of logistic function (φ) are larger, respectively much larger for curvature.

Standard deviations of parameters for Dataset 2 are larger than for other

Parametrization	α	δ	γ	λ	φ
Dataset 2, exp. 1	.20(.22)	.31(.34)	1.15(.08)	-	44(42)
Dataset 2, exp. 2	.44(.31)	.70(.49)	1.05(.07)	-	19(34)
(Murphy and ten Brincke, 2017), 1	.73	.88	.65	1.11	.30
(Murphy and ten Brincke, 2017), 2	.73	.84	.68	1.18	.29
(Tversky and Kahneman, 1992)	.88	1.07	.52	2.25	-
(Nilsson et al., 2011)	.91(.16)	1.02	.62	1.02(.26)	.18(.09)

Table 6.1: Logistic-CPT parameters for different studies. Parameters for (Tversky and Kahneman, 1992; Nilsson et al., 2011) are the median from "fit" at the individual level. Others parameters come from "fit" at the aggregate level. For Dataset 2, parameters and standard deviation (in parenthesis) are obtained with bootstrapping (1000 re-sampling with repetition).

studies. One explanation comes from the "four types of lotteries' pairs". Parametrization of each type separately gives different parameters (section 6.3). While re-sampling, some data of a type of lotteries' pair may be in a bigger proportion and thus create those differences. Another cause is maybe due to over-fitting. Convergence at the aggregate level is stable then it is not a cause for a large variance.

As expected, calibration of logistic-CPT's on Dataset 2 parameters are different than in other studies.

6.2 Constituent functions of logistic-CPT

To understand what those "weird" parameters may imply, we can look at functions constituting logistic-CPT.

Prospect theory and cumulative prospect theory were designed with a strong empirical background (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Gonzalez and Wu, 1999; Neilson and Stowe, 2002) which are based on some of those points:

- small (resp. large) probability are over (resp. under)-estimated. $\approx$ Diminishing sensitivity away from $p = 0$ or 1;
- the marginal value of money is decreasing;
- losses matter more than gains .

Figure 6.2 shows subjective probability with the Prelec II function with different values of δ and γ. Subjective probability from parametrization of Dataset

2 does not respect the first point as subjective probabilities of Dataset 2 are only concave and they over-estimate large probabilities.

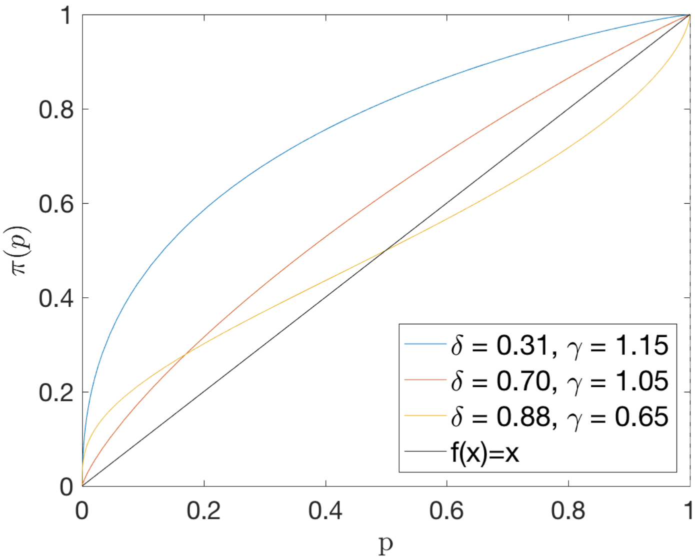


Figure 6.2: Prelec II (subjective probability) function 2.10 that modifies the value of probability. The two first sets of parameters are from parametrization of Dataset 2 and the last for the first experiment of (Murphy and ten Brincke, 2017).

The power parameter α controls the shape of the value function 2.8. This function respects the second point (decreasing marginal value of money) but with a too strong effect. If α is small (e.g. 0.2), payoff weights are almost constant for values in a range from 4 to 50 monetary units (figure6.3).

The logistic function (equation 2.11) transforms a difference in utility between two lotteries into a probability. A larger parameter φ implies a fewer difference in utility that is needed to change the probability. Utility for lotteries are reduced by the value function, so the difference in utility get smaller too. This would explain why φ gets small but it does not justify such a modification of p and V with the weighting and value function.

(Gonzalez and Wu, 1999) discusses psychological implication of the weighting function. They concluded the weighting function is characterized by two psychological properties that are independent: “Discriminability refers to how people discriminate probabilities in an interval bounded away from 0 and 1. Attractiveness refers to the degree of over/under weighting”. In their articles, they cite (Lopes, 1987, 1990) for their characterisation of the weighting function (“Security-minded” for convex and below the iden-

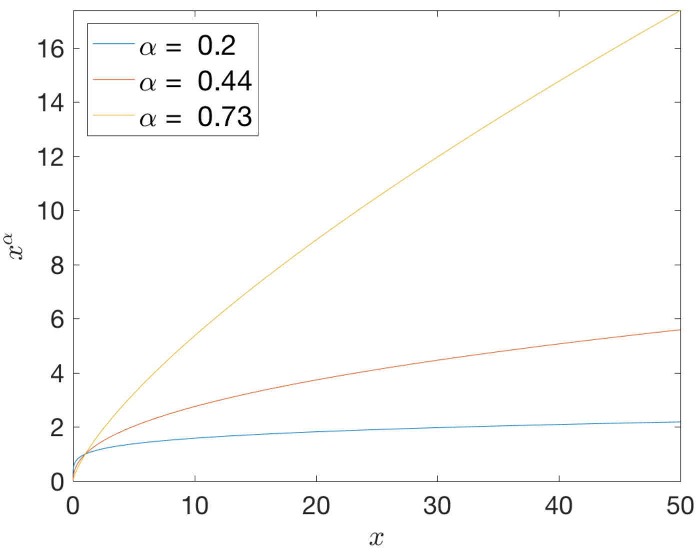


Figure 6.3: Value function (2.8) modifies the value of money. The two first sets of parameters are from parametrization of Dataset 2 and the last set is from the first experiment of (Murphy and ten Brincke, 2017).

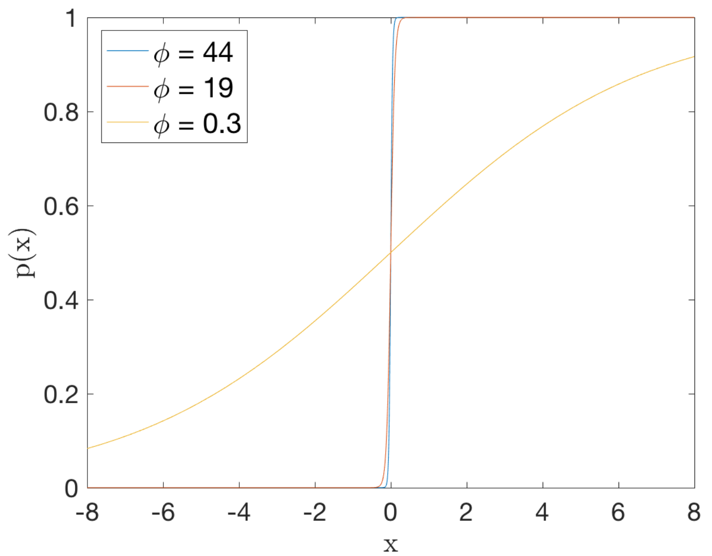


Figure 6.4: Logistic function 2.11 that modifies a difference of utility between two lotteries into a probability. The two first sets of parameters are from parametrization of Dataset 2 and the last set from the first experiment of (Murphy and ten Brincke, 2017).

tity line, “potential-minded” for concave and above the identity line and “cautiously-hopeful” for the inverse S-shape form). None of the examples of weighting functions at the individual form (Gonzalez and Wu, 1999) features a shape as convex as the one for Dataset 2. It is important to notice that at the aggregate level they have a nice inverse S-shape.

We don’t advocate parameters should be universal and of course they are not, but parameters from the calibration of Dataset 2 with logistic-CPT give strange result: their values; standard deviation and the shape of the functions that they produce.

6.3 Separation in type of lotteries’ pairs

To understand why parameters from the parametrization of Dataset 2 with logistic-CPT give “strange” parameters and functions, we try to understand the structure of the dataset. The four types of data from (Loomes and Pogrebna, 2017) were already plotted with the choice frequency p_{Exp} as a function of the f-factor (figure 4.5 and 4.6). An alternative would be to plot them as a function of the varying parameter for each type (either p or M). To have the same abscissa for all of them, we choose to plot the frequency as a function of the difference in expected value ($EV_{safe} - EV_{risky}$).

In figure 6.5, the four types of lotteries’ pairs are ordered and cleaned (points are in an increasing order and are not far away from a smooth curve.) Deviations from the curve are mostly due to “rounded price” ($M = 10$, next to last points for blue and orange) (Ball, 1988) or “rounded probability” ($p = 0.1$ last point for purple dots).

They all show a transition from low frequency to high frequency but there are two main differences between types of lotteries’ pairs: the slope and intercept. A different slope expresses either more errors or a larger probabilist component. A higher intercept than 0.5 shows attraction towards the safest lottery.

A model for the four types of lotteries’ pairs should either have for each type of lotteries’ pair, a point where $U_{safe} - U_{risk} = 0$ and p_{Exp} = 0.5 or should be corrected afterwards, with for instance a q-factor. For logistic-CPT, there is no q-factor. Thus, the model takes weird parameters for the subjective probability and the weighting function in order to obtain this point on the four types of lotteries’ pairs. Another requirement is: all types of lotteries’ pairs need to have the same slope (slope mentioned in the previous paragraph)

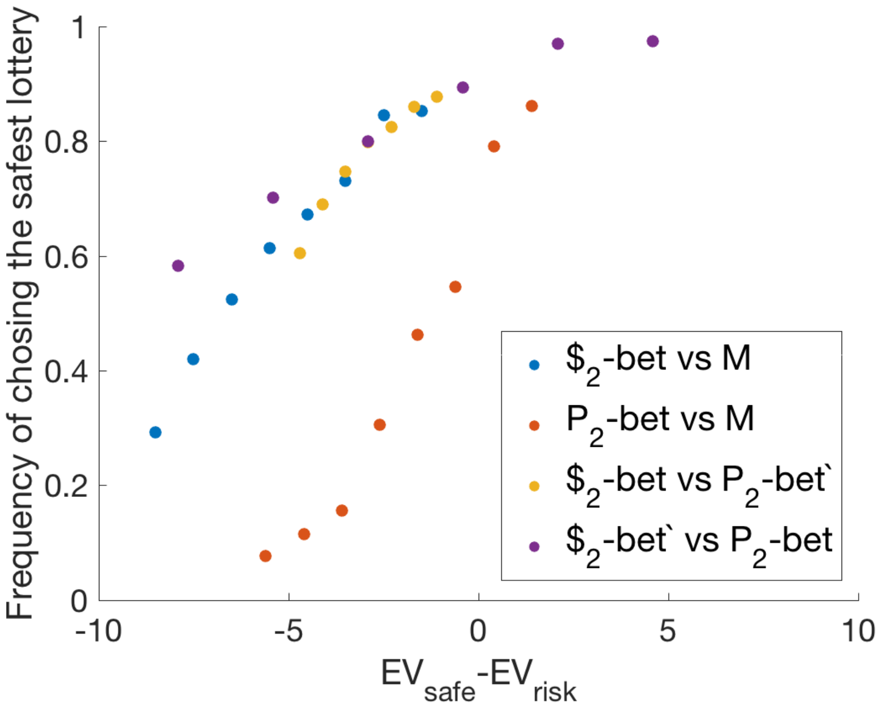


Figure 6.5: The frequency of choosing the safest gamble p_{exp} as a function of the difference between lotteries' expected value for each lotteries' pair. The four types of lotteries' pairs are in the same order: (50, 0.25) vs M'$\in$ $[4; 11]$; (12, 0.8) vs M'$\in$ $[4; 11]$; (50, 0.25) vs (12, p') p'$\in$ $[0.65; 0.95]$ and (50, p') p'$\in$ $[0.1; 0.4]$ vs (12,0.8).

Logistic-CPT on each type of lotteries' pairs

In section 6, parametrization of the logistic-CPT was on the second experiment of Dataset 2. Table 6.2 shows a comparison with parametrization on each type of lotteries' pair (from the second experiment of Dataset 2). Due to a few number of data per type of lotteries' pairs (7 or 8), errors on parameters are not computed with bootstrapping. Instead of that, Jackknife re-sampling is used (removing each data point once) to give some information on the stability of the parametrization.

As we can see on the figure 6.5, data distribution within a type of lotteries' pairs is uniform, so its parametrization should be feasible with accuracy. The mean square error is one order of magnitude lower when each type is parametrized separately (MSE experiment 2 = 0.0045), this confirms the previous statement.

Parameters are quite different across types of lotteries's pairs, such differ-

Parametrization	α	δ	γ	φ	NLL	MSE
$\$_2$-bet vs M	.54(.18)	.48 (.18)	2.7 (.11)	1.7(.93)	4.716	0.0003
P_2-bet vs M	1.0 (.0)	5.6 (.03)	1.8 (.01)	0.64 (0.02)	3.992	0.0009
$\$_2$-bet vs P_2-bet′	.04 (0.3)	.05 (.04)	1.3 (.04)	79 (28)	3.597	0.00001
$\$_2$-bet′ vs P_2-bet	.24 (.10)	.38 (.14)	2.1 (1.3)	2.5 (41)	3.072	0.0002

Table 6.2: Calibration of logistic-CPT on type of lotteries' pairs, at the aggregate level with maximum likelihood estimation (minimizing negative loglikelihood NLL). Jackknife re-sampling gives the standard deviation (in parenthesis). The four types of lotteries' pairs from the second experiment of Dataset 2 are in the following order: (50, 0.25) vs M′∈ [4; 11]; (12, 0.8) vs M′∈ [4; 11]; (50, 0.25) vs (12, p′) p′∈ [0.65; 0.95] and (50, p′) p′∈ [0.1; 0.4] vs (12,0.8). Mean squared error (MSE) shows how well it fits.

ence of parameters induces a large change on the utility of a lottery. Table 6.3 shows utilities of the $\$_2$-bet (50,0.25) and P_2-bet (12,0.8). If it was just a matter of scale, it wouldn't be a problem as φ could encompass for that, but the relative difference between two lotteries changes a lot (i.e. 0.02 for the pair -$\$_2$-bet vs P_2-bet′- and 8.3 for the pair -$\$_2$-bet vs M-).

Utility of lottery:	(50,0.25)	(12,0.8)
Exp. 2 composed of :	1.8	2.1
$\$_2$-bet vs M	2.6	3.8
P_2-bet vs M	0.0017	8.3
$\$_2$-bet vs P_2-bet′	1.07	1.09
$\$_2$-bet′ vs P_2-bet	1.2	1.8

Table 6.3: "Utility" of two lotteries for parametrization of logistic-CPT on the second experiment of Dataset 2 and its different types of lotteries' pairs. The four types of lotteries' pairs from the second experiment of Dataset 2 are in same order: (50, 0.25) vs M′∈ [4; 11]; (12, 0.8) vs M′∈ [4; 11]; (50, 0.25) vs (12, p′) p′∈ [0.65; 0.95] and (50, p′) p′∈ [0.1; 0.4] vs (12,0.8)

A simpler model

Another way to convince ourselves that logistic-CPT chose weird parameters in order to shift $U_{safe} - U_{risk}$ towardss 0 at the ordinate p_{Exp} = 0.5, is to make a simpler model (called shift model) with those assumptions and to compare the result with logistic-CPT.

The shift model is made up of the logistic function 2.11 with parameter φ and a shifting parameter "shift":

$$f = \frac{1}{1 + exp{-}\varphi(UB - UA + shift)} \tag{6.1}$$

Shift parameters express preference for the safest lottery. φ gives information on the difficulty of a task, which could be compared across data types as utilities are not modified. A good illustration of that is when φ is large, the logistic function is steeper and thus, the choice is simpler because the transition of the probability from a lottery to another is over a smaller difference in utility.

Parametrization	shift	φ	NLL	MSE
$\$_2$-bet vs M	6.5	.36	4.719	0.0004
P_2-bet vs M	1.3	.65	3.992	0.0009
$\$_2$-bet vs P_2-bet′	5.9	.44	3.599	0.0001
$\$_2$-bet‘ vs P_2-bet	9.1	.26	3.078	0.0003

Table 6.4: Calibration of shift model represented by equation 6.1 on the four types of lotteries' pairs from the second experiment of Dataset 2. The four type of lotteries' pairs are in the same order: (50, 0.25) vs M′$\in$ $[4; 11]$; (12, 0.8) vs M′$\in$ $[4; 11]$; (50, 0.25) vs (12, p′) p′$\in$ $[0.65; 0.95]$ and (50, p′) p′$\in$ $[0.1; 0.4]$ vs (12,0.8)

Compared to logistic-CPT, this parametrization with only two parameters (table 6.4) works well. Mean square error is within a negligible range (0.0001). Akaike information criterion (AIC) (Akaike, 1974) is smaller for the shift model than for logistic-CPT (i.e. the worst case is "$\$_2$-bet‘ vs P_2-bet" and $AIC_{shift} = 10.2$ while $AIC_{CPT} = 14.1$).

If we compare the two first types of lotteries' pairs (pairs with sure-amount), shift parameter is five time bigger for the type "$\$_2$-bet vs M" than "$P_2$-bet vs M". φ is for the type "$\$_2$-bet vs M" the half of the value of φ for the type "P_2-bet vs M" (table 6.4). Comparison of the two other types of lotteries' pairs (pairs with varying probability), the type with varying probability of the $\$_2$-bet (last line) has a bigger shift and smaller φ than the type with varying probability of the P_2-bet.

Conclusion

The division in four types of lotteries' pairs shows that choice frequency at the aggregate level is quite homogeneous within a type. However, there are differences between types. Owing to the homogeneity, it would be possible to make a model that perfectly fits all data together. This model should take into account the decreasing utility of money, the inverse S-shape of probability, the shift for safer choice and the difference in difficulty between type of lotteries' pairs. Without that, it may lead to a limit on the goodness fit. Even logistic-CPT with weird parameters is not able to perfectly fits the 30 lotteries' pairs from the second experiment of Dataset 2 (figure 6.1).

"Shift for safer choice" and the "difference in difficulty" are partially taken into account by modification of value and probability (with value and weighting function). If parameters stay within a meaningful range, this modification is too small.

Conclusion

Decision theory is at the center of main economical theories. Its different implementations lead to paradoxes, when compared to actual behavior of people, or apply only to a special type of decision. Quantum decision theory (QDT) offers, a new way (quarter law, decomposition of probability in utility and attraction factor) to look at data in order to reach a better understanding of decision makers (gender difference, risk behaviour, choice shifting).

Our work is a continuation of (Vincent et al., 2017). At first, we based our analyses on the same dataset (Dataset 1: Two repetitions with two week intervals of 91 choices between two lotteries made by 142 subjects). Then, we corrected their probabilistic choice shift model by changing a constant. And finally, we found a way of implementing their theoretical separation of subjects in two groups (overconfident: 105 subjects, and contrarian: 37 subjects). This separation, which is based on the majority choice, is another solution of the probabilistic choice shift model. Thus, it gives evidences of heterogeneity of participants and intrinsic probabilistic nature of choice.

Parametrization of the priority model 5.2 on Dataset 1 showed variations of aspects (i.e. choice criteria) order, which better explained the observed choice. The best fitting order of aspects depends on type of lotteries. This process motivates the following proposition: QDT attraction factor q (and its analytical formulation) may also be context-dependent, highlighting the most profound interfering factor (aspect) of a particular choice situation. Therefore, we created a novel "safe" q-factor, which is an attraction factor towards the safest lottery. This attraction factor is based on standard deviation measure of risk.

To continue our inquiry on heterogeneity, on shifting probability and on the novel "safe" q-factor, other data were needed. Dataset 2 is composed of two experiments with 4 repetitions and 3 types of data: 66 binary choice (BC) questions, 5 direct valuation (DV) tasks and 5 stochastic indifference (SI) values. Given different experimental setup, a lack of independence between iterations induces a strong deviation from the probabilistic choice shift model and indicates a memory effect.

Gender is a characteristic used to search for heterogeneity. Our findings, concerning gender difference in decision makers' attitude towards risk, support previous studies presented in (Favre et al., 2016). They found evidences of larger risk-aversion of female decision makers. However, the results were inconclusive due to the small sample size. The two studies together strengthen this argument.

Clustering of participants (with the k-means algorithm) in three groups - risk-averse, risk-neutral and risk-seeker- shows a good sign of transitivity across the type of questions/values (BC, DV and SI). K-means clustering confirms identified gender effects, as women are in a bigger proportion in the risk-averse group, while men constitute the majority among risk-seekers. Finally, the inverse S-shape of subjective probability function is decomposed in behaviors that arise from two extreme groups: risk-seeking and risk-averse decision makers.

QDT with the quarter law is an alternative interpretation of preference reversal. The quarter better explained preference reversal with a novel f-factor based on SI values, than the f-factor based on expected values. It confirms the intuition that SI values give a value closer to the intrinsic value of a lottery. Next, QDT parametrization with our new "safe" q-factor improved the simpler expected value model.

In the last section, Dataset 2 was divided in four types of lotteries' pairs with the aim of examining "weird" parameters obtained with logisitc-CPT. Plot of types of lotteries' pairs encourage the use of QDT as they exhibit requirements for an attraction factor and for probabilistic behavior (in the form of the logistic function). Within a type of lotteries' pairs (its plot), the choice frequency is homogeneous. This, would encourage a possibility of a perfect parametrization, but plot of types of lotteries' pairs differ from themselves (by their slope and ordinate). Those differences raise concerns on the limit of predictability.

To conclude, the separation of lotteries' pairs in different types improved the goodness of fit, while the separation of subjects showed no real improvement. To increase the predictability, it is important to classify choice tasks (e.g. pairs of lotteries) according to their characteristics (e.g. type of outcomes: gain/loss; $-bet/P-bet type). It would be useful to try to combine different theories or sets of parameters within QDT to have for example, an attraction factor that would be gender dependant.

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Appendix: Quantum decision theory (QDT)

This appendix is borrowed from article (Vincent et al., 2017). Developed by Yukalov and Sornette in a series of articles (Yukalov and Sornette, 2008, 2009, 2010, 2015b). quantum decision theory (QDT) has recently been introduced as an alternative formulation to existing theories. It is based on two essential ideas: (i) an intrinsic probabilistic nature of decision making and (ii) a generalisation of probabilities using the mathematics of Hilbert spaces that naturally accounts for entanglement between choices.

Mathematical structure of QDT

Let us recall briefly the mathematical construction of quantum decision theory (which can be found in more details in (Yukalov and Sornette, 2010)).

- **Definitions: actions, prospects and state of mind**

Definition .1 (Action ring) *The action ring* $\mathcal{A} = \{A_n : n = 1, 2, \ldots, N\}$ *is the set of intended actions, endowed with two binary operations:*

- *The reversible and associative addition.*
- *The non-distributive and non-commutative multiplication, which possesses a zero element called empty action.*

The interpretation of the sum $A + B$ is that A or B is intended to occur. The product AB means that A and B will both occur. The zero element is the impossible action, so $AB = BA = 0$ means that the actions A and B cannot occur together: they are disjoint.

Definition .2 (Composite action and action modes) *When an action* A_n *can be represented as an union (i.e. is the sum of several actions), it is referred to as composite. Otherwise it is simple.*

The particular ways A_{jn} of realizing a composite action A_n are called the action modes and are disjoint simple elements:

$$A_n = \bigcup_{j}^{M_n} A_{jn} \quad M_n > 1 \tag{.1}$$

Definition .3 (Elementary prospects) *An elementary prospect e_α is an intersection of separate action modes,*

$$e_\alpha = \bigcap_n A_{\alpha n} \tag{.2}$$

where the $A_{\alpha n}$ are action modes such that $e_\alpha e_\beta = 0$ if $\alpha \neq \beta$.

Definition .4 (Action prospect) *A prospect π_n is an intersection of intended actions, each of which can be simple (represented by a single action mode) or composite*

$$\pi_n = \bigcap_j A_{n_j} \tag{.3}$$

To each action mode, we associate a mode state $|A_{jn}\rangle$ and its hermitian conjugate $\langle A_{jn} \mid$. Action modes are assumed to be orthogonal and normalized to one, so that $\langle A_{jn} \mid A_{kn} \rangle = \delta_{jk}$. This allows us to define orthonornal basic states for the elementary prospects:

$$|e_\alpha\rangle = |A_{\alpha 1} \ldots A_{\alpha N}\rangle \quad \text{and} \quad \langle e_\alpha \mid e_\beta \rangle = \prod_n \delta_{\alpha_n} \delta_{\beta_n} = \delta_{\alpha\beta} \tag{.4}$$

Definition .5 (Mind space and prospect state) *The mind space is the Hilbert space*

$$\mathcal{M} = \text{Span}\,\{|e_\alpha\rangle\} \ . \tag{.5}$$

For each prospect π_n, there corresponds a prospect state $|\pi_n\rangle \in \mathcal{M}$

$$|\pi_n\rangle = \sum_\alpha a_\alpha \, |e_\alpha\rangle \ . \tag{.6}$$

Definition .6 (Strategic state of mind) *The strategic state is a normalized fixed state of the mind space $\mathcal{M}$ describing a decision maker at a given time:*

$$|\psi\rangle = \sum_\alpha c_\alpha \, |e_\alpha\rangle \ . \tag{.7}$$

The strategic state characterizes a particular decision maker at a given time, it includes his/her personal attributes and is related to the information available to the decision maker.

• **Prospect probabilities**
In the context of quantum decision theory, the preferences of a decision maker depend on his/her state of mind and on the available prospects. Those preferences are expressed through prospect operators.

Definition .7 (Prospect operator) *For each prospect π_n, we define the prospect operator*

$$\hat{P}(\pi_n) = |\pi_n\rangle \langle \pi_n| \ . \tag{.8}$$

By this definition, the prospect operator is self-adjoint. Its average over the state of mind defines the prospect probability $p(\pi_n)$:

$$p(\pi_n) = \langle \psi \mid \hat{P}(\pi_n) \mid \psi \rangle \ . \tag{.9}$$

The decision maker is more likely to choose the prospect with the highest prospect probability. The probabilities should correspond to the frequency with which the prospect would be chosen if the choice could be made several times in a same state of mind.

By definition .5 and .6, we can distinguish two terms in the expression of $p(\pi_n)$: a utility factor $f(\pi_n)$ and an attraction factor $q(\pi_n)$:

$$p(\pi_n) = f(\pi_n) + q(\pi_n) \tag{.10}$$

$$f(\pi_n) = \sum_{\alpha} \mid c_\alpha^* a_\alpha \mid^2 \tag{.11}$$

$$q(\pi_n) = \sum_{\alpha \neq \beta} c_\alpha^* a_\alpha a_\beta^* c_\beta \tag{.12}$$

Within the framework of quantum decision theory, the utility and attraction terms are subjected to additional constraints:

- $f(\pi_n) \in [0,1]$ and $\sum f(\pi_n) = 1$ (normalization of the utility factor) ,
- $q(\pi_n) \in [-1,1]$ and $\sum q(\pi_n) = 0$ (alternation property of the quantum factor) .

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