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Master Thesis

Physics MSc.

topic:	Extension of the Markowitz Portfolio Optimization to include diversification measures such as the Herfindahl Index and other means
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Abstract

The paper aims at adding different measures of diversification to the Markowitz portfolio, since solely focussing on return and risk maximization might lead to a limited portfolio size of only a few securities. Throughout the paper we discuss the performance of the naive $\frac{1}{N}$-rule compared to three distinct portfolio concepts - namely the Generalized Herfindahl Index, state-dependent allocation and the Kelly Criterion - by deploying several relevant performance metrics such as Up / Down Ratios, Sharpe Ratio-like measures and cumulative returns. The Generalized Herfindahl Index is defined in an analogous manner to the Herfindahl Index, which results in a performance similar to the $\frac{1}{N}$-benchmark. In contrast to this observation, the state-dependent allocation strategy, which switches from the naive $\frac{1}{N}$-rule to the minimum variance portfolio depending on the value of the DS LPPLS^{TM} confidence indicator, is able to generate consistently superior returns if indicator thresholds below 0.8 are applied. The last strategy deploys the Kelly criterion which results in portfolios that are not only well diversified but also heavily dependent on the risk-free asset. Understandably, the Kelly portfolio behaves very differently compared to the naive $\frac{1}{N}$-rule, but fails to outperform it. The above-mentioned strategies were tested using three data sets that vary with respect to asset number and estimation window size. The first data set includes nine S&P500 sectors from 2000 to 2009, whereas the second one relies on 39 S&P500 assets from 1990 to 2009 and finally the third data set utilizes 50 S&P500 assets from 1974 to 2011.

Contents

1. Introduction

"One should always divide his wealth into three parts: [investing] a third in land, a third in merchandise, and [keeping] a third ready to hand." - This ancient quote by rabbi Isaac bar Aha dates back to the 4th century and is to be found in the Babylonian Talmud. The rabbi's simplistic guide to diversification marks the starting point of a longsome quest in which businessmen have been after the optimal investment strategy. Even Shakespeare's fictitious venetian merchant Antonio proudly reveals his very own diversification approach: "My ventures are not in one bottom trusted / Nor to one place, nor is my whole estate / Upon the fortune of this present year." In fact, diversification has been mentioned already much earlier in the book of Ecclesiastes (935 BC), where it reads: "But divide your investments among many places, for you do not know what risks might lie ahead", which to a large extent relates to today's common proverb "don't put all your eggs into one basket".

Speaking of today: The manifestations of modern portfolio theory were established by Harry Markowitz in 1952. Over the years various amendments to the Markowitz portfolio have been proposed for the purpose of overcoming several weaknesses that come with some of the assumptions made by Markowitz. For instance, Fama and French coined a three-factor model, which itself can be interpreted as an advanced version of the capital asset pricing model (CAPM). The model is based on their observation that stocks with small market capitalization or small price-to-book ratio have a tendency to outperform the market. An alternative to the CAPM is presented by the widely-used arbitrage pricing model, which does not require a market equilibrium, but only an arbitrage-free stock market. It turned out that any of these developments were helpful inasmuch as they successfully model the market more realistically than the original Markowitz model in many aspects. In our paper we will extend the Markowitz portfolio theory to include diversification contraints such as Herfindahl Index (HHI) and other related means in order to avoid that our Markowitz portfolio condensates into a few assets only.

In the theory part (Section 2) we will describe the general Markowitz portfolio itself before introducing the naive $\frac{1}{N}$-rule, which will serve as a performance benchmark throughout the entire work. The benchmark will be compared to three distinct investment strategies: At first, we will examine the peculiarities of a generalized version of the Herfindahl Index, which does not measure the concentration among individual assets, but whole sectors. As the $\frac{1}{N}$-rule and the Herfindahl Index are formulated analogically we expect very similar results. After that, we will present a state-dependent allocation strategy that switches between the naive $\frac{1}{N}$-rule and a minimum variance portfolio depending on the value of the LPPLS bubble indicator. Hence, whenever a crisis is imminent and correlations between assets increase, we primarily depend on risk minimization instead of return maximization. We hope to outperform the naive $\frac{1}{N}$-portfolio provided that the bubble indicator sufficiently predicts crises. Finally, we will introduce the Kelly criterion and build a portfolio based on it. The unconstrained implementation of such a Kelly portfolio leads to an investment into all available assets and is therefore considered well diversified.

In the data part (section 3), we will show three data sets that comprise a range of securities over different observation periods (2000 - 2009, 1990 - 2009 and 1974 - 2011, respectively). Furthermore, we will disprove Markowitz' assumption of normality by applying a plethora of evaluation and performance metrics which illustrate that stock returns do not follow a Gaussian distribution. On top of that, we will justify the choice of the data sets with respect to the phenomenon of regime changing. The implementation part (section 4) will demonstrate how to carry out each of the three above-mentioned portfolio concepts, namely Generalized Herfindahl Index, state-dependent allocation as well as Kelly criterion. Moreover, we will explain how to deal with a rolling window approach as opposed to a "buy-and-hold" strategy. In addition, the result part (section 5) will show the respective weight distributions for any of the strategies by visualizing them in risk-return diagrams. Besides, it will draw an empirical connection between the Herfindahl Index and entropy-like diversification.

The Analysis part (section 6) will take the performance evaluation tools defined in the data part and apply them to our three strategies for the sake of comparison with the performance of the naive $\frac{1}{N}$-rule. It will become clear that the Generalized Herfindahl Index (GHHI) and the usual Herfindahl Index work in a very similar manner so that optimizing them will lead to a portfolio that performs more or less like the $\frac{1}{N}$-portfolio. On the contrary, the state-dependent allocation strategy is actually able to outperform the $\frac{1}{N}$-portfolio, if the strategies are switched whenever the bubble indicator surpasses a small threshold. At last, the Kelly portfolio will be assessed. Since the Kelly portfolio heavily relies on the risk-free asset it behaves totally different than the previously evaluated strategies as well as the $\frac{1}{N}$-strategy, but just like the GHHI portfolio it does not consistently outfperform the $\frac{1}{N}$-benchmark.

In the conclusion part (section 8) we will suggest numerous possible improvements to our approach. For example it is advisable to spend more time on properly selecting the data set, since the results are very sensitive to the length of the observation window as well as the choice of securities. Aside from this technical issue, we will propose a scheme on how to combine our three strategies in a meaningful way based on the results of this paper.

2. Theory

2.1 Description of the Markowitz Problem

Instead of solely evaluating the risk of individual assets separately, Markowitz Portfolio Theory takes into account their effect on the joint risk of the entire portfolio. In other words, the total risk of a portfolio is constituted not only by the mere sum of the risks of the single assets, but also by a correlational contribution that incorporates the dependencies between different assets. Accordingly, the squared risk of a portfolio X with assets x_i is given by the variance:

$$\sigma^2(X) = var[X] = \sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j \tag{1}$$

where:

w_i: weight of asset x_i

Σ_{ij}: covariance matrix

The corresponding covariance matrix is defined as follows:

$$\Sigma_{i,j} = \Sigma(x_i, x_j) = var(x_i, x_j) = E\left[\left(X_i - E\left[X_i\right]\right)\left(X_j - E\left[X_j\right]\right)\right] \tag{2}$$

For n assets the covariance matrix takes the following form:

$$\Sigma_{i,j} = \begin{bmatrix} var\left(x_1, x_1\right) & cov\left(x_2, x_1\right) & \cdots & cov\left(x_{n-1}, x_1\right) & cov\left(x_n, x_1\right) \\ cov\left(x_1, x_2\right) & var\left(x_2, x_2\right) & \ddots & \ddots & cov\left(x_n, x_2\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ cov\left(x_1, x_{n-1}\right) & \ddots & \ddots & var\left(x_{n-1}, x_{n-1}\right) & cov\left(x_n, x_{n-1}\right) \\ cov\left(x_1, x_n\right) & cov\left(x_2, x_n\right) & \cdots & cov\left(x_{n-1}, x_n\right) & var\left(x_n\right) \end{bmatrix} \tag{3}$$

Taking the square root of the variance leads to the standard deviation σ, which is represented by price variations of an asset or a portfolio and poses as a measure of risk. The total return of the portfolio is given by the sum of the expected returns of assets x_i:

$$E\left[R(X)\right] = \sum_{i=1}^{n} E\left[R\left(x_i\right)\right] = \sum_{i=1}^{n} \mu_i w_i \tag{4}$$

where:

μ_i: return associated with asset x_i

Markowitz assumes that on the one hand investors are fond of large profits, but on the other hand they perceive price fluctuations as risky. By combining several assets in a certain way it is possible to optimize the total output of an investment resulting in higher return at the same risk or lower risk at the same return. It follows that investors consider their portfolio optimal if they are able to maximize the return of their portfolio at the risk that they are willing to take (or vice versa). To sum up, the optimal size of w_i - the relative contribution of asset x_i to the whole portfolio - is determined by three parameters, namely the future return of x_i, the fluctuation range of the profit of x_i and the development of the profit of x_i with respect to the other assets x_j (measured by the correlation value ρ_{ij}). Furthermore, Markowitz defined the profit of asset x_i as follows:

$$\mu_{t-1,i} = \frac{P_{t,i}}{P_{t-1,i}} - 1 \tag{5}$$

where:

$P_{t,i}$: price of asset x_i at time t

$P_{t-1,i}$: price of asset x_i at time $t-1$

Since the above description of the return does not allow a simple addition of the returns we will use logarithmic definition of returns throughout this paper which is also known as the continuously compounded return:

$$\mu_{t-1,i,\log} = \log\left(\frac{P_{t,i}}{P_{t-1,i}}\right) \tag{6}$$

It can be shown that these two definitions are equivalent up to quadratic terms. Both definitions also illustrate one of several weaknesses of Markowitz Portfolio Theory, as the estimates of both return and risk are based on past price fluctuations. In general, this contradicts the weak form of the Efficient-market Hypothesis, which states that past price developments have no influence on future prices. Hence, technical analysis techniques can not be used to generate persistent excess returns. In addition, we impose two linear constraints on the Markowitz Problem:

$$\sum_{i=1}^{n} w_i = 1 \tag{7}$$

$$w_i \geq 0 \quad \forall i \tag{8}$$

The first constraint ensures that the investors invest all their money. The second one excludes the possibility of short selling, which yields more consistent optimization results. Plotting the sum of all possible portfolios into a return-risk diagram we obtain a feasible set:

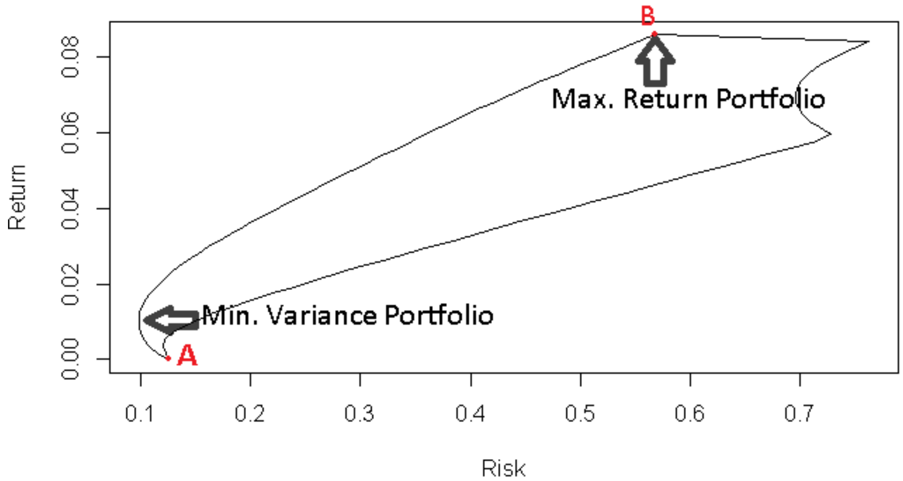


Figure 1: risk-return diagram showing the feasible set of six assets / indices from the LPP2005 data set ("SBI" (swiss bonds), "SPI" (swiss equities), "SII" (real estate), "LMI" (american bonds), "MPI" (international equities), "ALT" (alternative investments)) including indication of minimum variance and maximum return portfolios; points A and B indicate the locations of two single assets within the diagram.

The feasible set represents all possible combinations of assets. The efficient frontier, which comprises all portfolios that are pareto-optimal in terms of risk and return, is defined by the outer left line of the feasible set reaching from the maximum return portfolio to the minimum variance portfolio. Considering only two assets (with: $r_A < r_B$ and $\sigma_A < \sigma_B$) yields a single line that starts at a portfolio purely consisting of asset A and ends at a portfolio that only contains asset B. While altering the concentration of asset A and B within our portfolio we first move along the inefficient frontier, then pass the minimum variance portfolio, before we move along the efficient frontier and end up at the maximum return portfolio. Efficient portfolios are those portfolios that lie on the efficient frontier of the risk-return diagram. A portfolio dominates a second portfolio if the return of the first portfolio is $r_1 \geq r_2$ and its standard deviation is $\sigma_1 \leq \sigma_2$. A particular portfolio is pareto-optimal if it is not dominated by another portfolio within the feasible set.

2.2 Criticism of Markowitz Approach

Many financial scientists have criticized Markowitz Portfolio Theory in different ways. First of all, Markowitz assumes the market to be efficient which in general leads to high-risk assets yielding more return than low-risk assets. But, at the same time Markowitz also neglects the Efficient-market hypothesis. Its weak form states that any information of past price variations offer no valuable clues to future price developments. The semi-strong form suggests that it is not possible to deduce abitrage opportunities from any publicly available information. Finally, the strong form claims that there are no arbitrage opportunities at all no matter which information we are looking at. This form is backed up by statistics showing that only very few pension fund managers are able to generate sustainable profits although they have access to almost any information. However, Markowitz portfolio theory seems to even disregard the manifestations of the weakest form by making up a strategy based only on the consideration of risk and return values that were obtained in the past through technical analysis.

Another school of thought criticizing Markowitz portfolio theory was established by Robert A. Haugen. He opposed the concept of efficient markets after having examined the price fluctuations of American assets over long periods of time. His results showed that more risky assets such as equities do not yield significantly higher returns than less risky assets such as bonds, which fundamentally contradicts Markowitz portfolio theory, since Markowitz assumes that high-risk assets need to yield disproportionately large returns in order to be favored over low-risk assets. Haugen's proposal is supported by the existence of over- or underpriced assets that can easily be identified in any kind of market, which indicates that markets might not be as efficient as Markowitz suggests. Furthermore, Haugen suspects that asset prices are not mainly driven by news, but rather by price variations of other assets. This idea would explain why markets sometimes do not react to relevant events such as the US entry into World War Two in 1941, but sometimes dramatically depreciate without any observable inducement from the news.

2.3 Measures of Diversification

Thirdly, Markowitz portfolio theory fails to take into account diversification. For instance, optimizing with respect to covariance risk and average return might lead to a portfolio that relies on bonds and equities alone. But it might actually be advantageous to invest in a broader selection of asset classes, since bonds and equities display some slight correlations which might lead to a complete loss of the whole portfolio under certain circumstances. Therefore, one might give consideration to investing into other asset types such as commodities, currencies, real estate or even gold.

Defining the market portfolio as synonymous to complete diversification the literature seeks to find the portfolio volume required to be adequately diversified. Adequate diversification can be interpreted as a level of diversification at which the volatility of the portfolio does not deviate considerably from the market volatility. The number of assets associated with this level of diversification marks a threshold above which the marginal costs of investing in an additional asset outweigh the diversification gain. Unfortunately, most papers only deal with portfolios where investors portion their money equally among all available securities. Two central pieces of literature on diversification are presented by the works of Evans & Archer (1968) and Fisher & Lorie (1970). Evans and Archer measure the volatility of evenly distributed portfolios with different asset numbers and regress the volatility against the number of assets in the portfolio, while the latter study states that the lion's share of risk mitigation is already achieved when simply possessing just 8 assets. More recent contributions suggest that more securities are necessary to diversify adequately. Some of the approaches deployed in these papers involve the computation of the volatility of the volatility (Upson et. al., 1975) or no-load index funds (Statman, 1987). On the contrary, adding transaction fees actually reduces the number of assets needed for adequate diversification (Goldsmith, 1976 / Shanker, 1989). Other works recommend hedging for exchange rate risk (Solnik, 1974), primarily investing in assets from the S&P Stock Guide (Wagner & Lau, 1971) or in low-beta securities (Martin & Klemkosky, 1975) and incorporating stock classifications (Martin & Klemkosky, 1976).

In the following we will shed light on a few different measures of diversification including Herfindahl- and entropy-like approaches, which are adopted from the works of of Woerheide & Persson (1993) and Anand & Ramasubramanian (2015). As opposed to the number of assets which only works for equally weighted investments, the diversification metrics below can handle any contribution of asset weights. First of all, the Herfindahl-Hirschman Index (from now on referred to as Herfindahl Index) is defined via the weights w_i of the respective assets $i \in [1, N]$:

$$HHI = \frac{\sum_{i=1}^{N} w_i^2}{\left(\sum_{i=1}^{N} w_i\right)^2} \tag{9}$$

In our case the denominator can be neglected since the sum of all weights w_i is always one. Herfindahl Index can take values between $\frac{1}{N}$ and 1, while the bottom limit corresponds to an equal weights distribution and the upper limit represents the investment in a single asset only. Besides, the inverse of the Herfindahl Index corresponds to the effective number of securities among which the risk is mitigated within a portfolio. Secondly, it exists a measure that orders the weights by size and multiplies the weights with their respective rank, namely the Rosenbluth Index (Woerheide & Persson, 1993):

$$RI = \frac{1}{2\sum_{i=1}^{N} i \cdot w_i - 1} \tag{10}$$

Similar to the Herfindahl Index, values for the Rosenbluth Index range between $\frac{1}{N}$ and 1. A third measure introduced by Christian Marfels in 1971 takes the exponential of the entropy measure, which is defined later on (Woerheide & Persson, 1993):

$$EI = \prod_{i=1}^{N} w_i^{w_i} \tag{11}$$

Again, the Exponential Index (EI) is as well limited by the values $\frac{1}{N}$ and 1. All of the above three metrics interpret diversification for unevenly weighted investments analogous to how the number of securities does it for equally distributed portfolios. Hence, the portfolio size is a discretized version of the three before-named indices. Since the first index is the most popular among these diversification measures, we will mainly focus on testing the Herfindahl index and its extensions throughout this paper, although there is no apparent reason to prefer one of these metrics over another. One such possible extension is presentend by Anand and Ramasubramanian (2015), namely the Generalized Herfindahl-Hirschman Index (GHHI, from now on referred to as Generalized Herfindahl Index). In contrast to the Herfindahl Index, which does not take into account correlations between the different securities, the Generalized Herfindahl Index assumes correlations between assets that are part of the same sector. Accordingly, both indices are equivalent if all assets originate from different sectors. The effect of subsectors can be incoroporated by considering full correlation between assets from the same subsector, finite, but smaller correlation between securities from the same main sector, yet different subsectors and finally zero correlation between any stocks from different main sectors. In order to derive the Generalized Herfindahl Index we first have a look at the return variance σ_p^2 of a Markowitz portfolio for two assets (Anand & Ramasubramanian, 2015):

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\varrho_{12} \tag{12}$$

Ignoring the riskiness of the single assets allows to set $\sigma_1 = \sigma_2 = \sigma$ and divide the return variance by σ:

$$\frac{\sigma_p^2}{\sigma^2} = w_1^2 + w_2^2 + 2w_1w_2\varrho_{12} \tag{13}$$

Assuming zero correlation $\varrho = 0$ yields the conventional Herfindahl Index introduced above:

$$\frac{\sigma_p^2}{\sigma^2} = w_1^2 + w_2^2 = HHI = \sum_{i=1}^{N=2} w_i^2 \tag{14}$$

Taking a third asset into account it becomes obvious that the previously established principle proves true even if we assume securities from the same sector to be fully correlated ($\varrho = 1$):

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2\left(w_1w_2\sigma_1\sigma_2\varrho_{12} + w_2w_3\sigma_2\sigma_3\varrho_{23} + w_3w_1\sigma_3\sigma_1\varrho_{31}\right) \tag{15}$$

Subject to the condition that securities W_1 and W_2 are part of the same sector S_1, while W_3 belongs to sector S_2, we can ignore the correlation values $\varrho_{23} = \varrho_{31} = 0$ and hence only stay with $\varrho_{12} = 1$:

$$\frac{\sigma_p^2}{\sigma^2} = w_1^2 + w_2^2 + w_3^2 + 2w_1w_2 = (w_1 + w_2)^2 + w_3^2 = s_1^2 + s_2^2 \tag{16}$$

Correspondingly, the Generalized Herfindahl Index computes as follows:

$$GHHI = \sum_{i}^{N} w_i^2 + \sum_{i}^{N} \sum_{i \neq j} 2 w_i w_j \varrho_{ij} = \frac{\sigma_p^2}{\sigma^2} \tag{17}$$

where:

σ_p^2: portfolio risk

σ^2: average risk in underlying assets

Obviously, the Generalized Herfindahl Index rather sums over the squared holdings invested into the same sector or asset class instead of adding up the squared weights of each single security. Unlike the first three measures the subsequent index does not shift as rapidly from 1 to $\frac{1}{N}$ when raising the portfolio size. The Comprehensive Concentration Index (CCI) attributes extra emphasis on the largest portfolio weight w_1 (Woerheide & Persson, 1993):

$$CCI = w_1 + \sum_{i=2}^{N} w_i^2 \left[1 + (1 - w_i)\right] \tag{18}$$

As mentioned before the Comprehensive Concentration Index scales more slowly between 0 and 1 than the other metrics above, which makes it worth to consider in the following. The concept of entropy concentration presents a diversification measure, which can take values starting from 0 to $\ln(N)$, where $\ln(N)$ matches the equal weights portfolio (Woerheide & Persson, 1993):

$$S = -\sum_{i=1}^{N} w_i \ln(w_i) \tag{19}$$

In the course of our work it will be shown that optimizing a Markowitz portfolio with respect to Herfindahl Index or entropy concentration leads to an equal weights portfolio (EWP). Following the works of deMiguel et. al. (2006) we use the EWP or $\frac{1}{N}$-strategy as a benchmark strategy to evaluate the performance of three distinct portfolio approaches: One that optimizes with respect to the Generalized Herfindahl Index, another one that pursues a state-dependent allocation strategy switching between the equal weights portfolio and a minimum-variance portfolio depending on the value of a crisis index and a third one that is based on the so-called Kelly Criterion. In the following we will lose a few words about the $\frac{1}{N}$-portfolio as well as the bubble indicator that will be applied in the case of the the state-dependent allocation strategy and finally also the Kelly Criterion.

2.4 $\frac{1}{N}$-Portfolio

In addition to being easier to implement, the $\frac{1}{N}$-strategy does not estimate any parameters like mean return and covariance risk and consequently is not affected by inaccurate data assessment, which poses a crucial advantage over other portfolio practices. Comparing the out-of-sample performance of optimal portfolio strategies and the equal weights portfolio provides information about how much the benefits from optimization are mitigated by estimation errors. Followingly, deMiguel et. al. (2006) were not able to identify an investment strategy that significantly outperforms the $\frac{1}{N}$-Strategy with regard to several metrics such as Sharpe-ratio, certainty-equivalent return (CEQ) or turnover. Only after having trailed stock returns for a couple of centuries for the purpose of parameter estimation optimizing portfolios would actually yield statistically stable surpluses. Suprisingly, even strategies that aim at avoiding estimation problems could barely scale down the necessary estimation window whose length also largely depends on the number of securities available. Besides, deMiguel et. al. (2006) concluded that the error inherent in computing the expected return deteriorates the out-of sample performance to a much larger extent than the error associated with covariance risk (see also: Merton, 1980). Thus, yet small mistakes originating from moment parameter estimation can tremendously change the asset distribution, since return-maximizing models tend to appreciate very quickly to performance imbalances between different assets (Michaud, 1989 and Best &Grauer, 1991). Constraining the portfolio weights to $[0, 1]$ and hereby forbidding short-selling and borrowing measurably diminishes the influence of parameter estimation errors. Moreover, deMiguel et. al. (2006) advocate to handle moment assessment much more circumspectly. Prior to deMiguel et. al. (2006) possible approaches to tackle this issue were for instance already presented by Jorion (1985) and Philippatos & Wilson (1972), who recommend deploying shrinkage estimators for expected return valuation and substituting covariance risk by entropy or expected information, respectively. More recently, Escobar et. al. (2013) suggested using the aforementioned state-dependent switching strategy. They claim that choosing the $\frac{1}{N}$-strategy in case of a bull market and crossing over to a minimum-variance strategy when the market becomes more volatile effectively yields superior gains compared to solely relying on an equal weights portfolio. One the one hand, Escobar et. al. (2013) argue for a choice of assets that is as wide as possible in order to profit from the upswing of a “positive” market , while on the other hand when assuming a “negative” market, they prefer a Markowitz portfolio that minimizes risk, since strong return correlations arise between any kind of stock during crises. Hence, in times of very volatile markets having to deal with covariance estimation errors might be the lesser evil, because investing in the whole market and thereby totally neglecting optimization would expose your portfolio to heavy loss risks. Below we will devote a few remarks to the crisis indicator deployed.

2.5 DS LPPLS Confidence^{TM} indicator

The crisis indicator used throughout this paper identifies feedback reactions resulting from the interaction of value and noise traders as the origin of super-exponential price growth. Deviations from this price growth appear as oscillations whose periodicity is related to the logarithm of the time to the burst of the bubble. After having reached a certain threshold the increasing positive feedback forces the price to move away tremendously from the actual value. At a critical time t_c the mispricing associated with this phenomena either smoothly evolves into a different regime or the bubble bursts. The logarithm of the observable asset price is given as follows (Johansen et. al., 2000):

$$\frac{d\left(p\right)}{p} = \mu\left(t\right)dt + \sigma\left(t\right)dW - kdj \tag{20}$$

where:

dW: infinitesimal increment of a standard Wiener process

dj: represents a discontinuous jump such that j = n before and j = n + 1 after a crash

As mentioned before, we are dealing with two different types of agents: Firstly value investors that act rationally and secondly noise traders that trigger the mispricing through their collective herding behavior. The crash hazard rate attempts to mathematically capture the behavior of the latter kind of traders (Johansen et. al., 2000):

$$h\left(t\right) = \alpha\left(t - t_c\right)^{m-1}\left\{1 + \beta\cos\left[\omega\ln\left(t - t_c\right) - \phi'\right]\right\} \tag{21}$$

where:

$\alpha, \beta, \omega, t_c$: parameters

Accordingly, the crash risk associated with the actions of noise traders is related to the power law singularity $\alpha\left(t - t_c\right)^{m-1}$. The excess return $\mu(t)$ is proportional to the above-introduced crash hazard rate due to the no-arbitrage setting that we are dealing with. If we set the expected value of dp equal to zero before the crash (ergo $dj = 0$) we obtain $\mu = kh\left(t\right)$. Now, integrating results in the expected trajectory of the price logarithm during a bubbly trajectory provided that the crash has yet to happen (Johansen et. al., 2000):

$$E\left[\ln p\left(t\right)\right] = A + B_c \mid t_c - t \mid^m + C \mid t_c - t \mid^m \cos\left(w\ln \mid t_c - t \mid - \phi\right) \tag{22}$$

where:

$B = \frac{-k\alpha}{m}$

$C = \frac{-k\alpha\beta}{\sqrt{m^2+w^2}}$

The calibration of the model is conducted using Least Squares. For every point in time t_2 the price time series is fitted to shrinking estimation windows whose lengths vary between 30 and 750 trading days. Now the DS LPPLS Confidence^{TM} indicator is defined as the fraction of fitting windows that satisfy the conditions in the following table:

Item	Notation	Search Space	Filtering Condition 1	Filtering Condition 2
3 nonlinear parameters	m	$[0, 2]$	$[0.01, 1.2]$	$[0.01, 0.99]$
	ω	$[1, 50]$	$[6, 13]$	$[6, 13]$
	t	$[t_2 - 0.2dt, t_2 + 0.2dt]$	$[t_2 - 0.05dt, t_2 + 0.1dt]$	$[t_2 - 0.05dt, t_2 + 0.1dt]$
Number of oscillations	$\frac{\omega}{2} \ln \left[\frac{t_c - t_1}{t_2 - t_1}\right]$	-	$[2.5, +\infty)$	$[2.5, +\infty)$
Damping	$\frac{m\|B\|}{\omega\|C\|}$	-	$[0.8, +\infty)$	$[1, +\infty)$
Relative Error	$\frac{p_t - \widehat{p}_t}{\widehat{p}_t}$	-	$[0, 0.05]$	$[0, 0.2]$

Table 1: search space and filter conditions for the qualification of valid LPPLS fits

Correspondingly, a large indicator value strongly hints at a bubble, while a small value indicates a slight fragility, as the LPPLS pattern is observed only in some windows. Applying different estimation windows allows us to distinguish between long-, medium- and short-term bubbles. Furthermore, it is possible to separate positive from negative bubbles. In addition to the above-presented mean-variance approaches, we will now focus on a concept known as Kelly Criterion, which can be adopted to generate portfolios that attempt to maximize the exponential growth rate of the investor's capital.

2.6 Kelly Portfolio

As the Markowitz Problem is only applicable to rather unlikely scenarios such as a quadratic utility function, it was suggested to utilize a criterion proposed by J. L. Kelly in 1956 to create portfolios that do not rely on utility functions. The Kelly Criterion tries to optimize the investor's capital by maximizing the expected value of the logarithm of the wealth per each time period. It can therefore be exemplified using the simple model of a gambler that learns about the outcomes of a bet, before the respective activity of interest is even finished and the results are published. Being able to bet at the original publicly available odds the betor seeks to maximize their total capital in the long run. In the example below we will consider a chance event with two equally probable outcomes. The corresponding exponential rate of growth is defined as follows (Kelly, 1956):

$$G = \lim_{N \to \infty} \frac{1}{N} \log_2 \left(\frac{V_N}{V_0} \right) \tag{23}$$

where:

N: number of bets

V_N, V_0: capital after N bets and initial capital, respectively

Provided that the binary channel is noiseless the exponential rate of growth is equal to one, since the gambler will always be right about the results of the bet. Therefore, the betor pursues the optimal strategy by investing their entire capital at each round. On the contrary, a noisy binary channel will cause the transmission of the results to be faulty with a probability p (thus correct with probability $q = 1 - p$). Accordingly, the bettor will end up loosing everything if they invest their entire capital at each instance for an infinite number of bets. Therefore, the gambler should only invest a fraction l of their capital leading to the following final capital V_N (Kelly, 1956):

$$V_N = (1 + l)^W (1 - l)^L \tag{24}$$

where:

W, L: number of wins and losses, respectively

Subsituting this expression into the exponential rate of growth established before yields the formula below (Kelly, 1956):

$$G = \lim_{N \to \infty} \left[\frac{W}{N} \log_2 (1 + l) + \frac{L}{N} \log_2 (1 - l) \right] = q \log (1 + l) + p \log (1 - l) \tag{25}$$

Finally maximizing the exponential rate of growth G with respect to the investment fraction l results in the following equation (Kelly, 1956):

$$G_{\max} = 1 + p \log p + q \log q \tag{26}$$

In order to implement the Kelly Criterion as a portfolio strategy we need to establish an authentic scenario which fits the assumptions made by Kelly. We start with lognormally distributed prices $p_i(t)$ that experience an uncorrelated random walk and can thus be written as follows (Laureti et. al., 2009):

$$p_i(t) = p_i(t - 1) \exp [\eta_i(t)] \tag{27}$$

where:

$\eta_i(t)$: randomly generated Gaussian numbers with homoskedastic mean m_i and variance D_i

In this respect, we consider the random walk parameters m_i and D_i to be well known to any portfolio holder. Additionally, a risk-free asset is introduced, while dividends, transaction costs and taxes are neglected. The previously established prices result in lognormally returns $R_i(t)$ (Laureti et. al., 2009):

$$R_i(t) = \frac{[p_i(t) - p_i(t-1)]}{p_i(t-1)} = \exp[\eta_i(t)] - 1 \tag{28}$$

The mean and variances of the returns are defined by the formulas below (Laureti et. al., 2009):

$$\mu_i = E(R_i) = \exp\left[m_i + \frac{D}{2}\right] - 1 \tag{29}$$

$$\sigma_i^2 = E[(R_i - \mu_i)] = [\exp(D_i) - 1][\exp(2m_i + D_i)] \tag{30}$$

The generated portfolio is constituted by the portfolio weights w_i, where w_0 describes the percentage invested in cash. As m_i and D_i are supposed to be fixed in our homoskedastic set-up, the Kelly portfolio is a discrete optimization scheme. If the initial wealth of the investor is equal to one, the wealth after one optimizing step is given as (Laureti et. al., 2009):

$$W_1 = 1 + \sum_{i=1}^{N} w_i R_i = 1 + R_p \tag{31}$$

where:

w_i: portfolio weight of asset i

R_i: return of asset i

R_p: portfolio return

Similar to the mean-variance portfolio, it is possible to forbid short-selling and borrowing by imposing the constraints $w_i \geq 0$ and $\sum_{i=1}^{N} w_i \leq 1$. The example of the noisy binary channel depicted that it is not advisable to optimize the expected value of the final capital, because in the long run the final capital can be heavily affected by extreme tail events due to the multiplicative nature of the random walk. Instead we hope to maximize the the logarithm of the capital after one optimization period by optimizing the average exponential rate of growth of the wealth:

$$\nu = E(\ln W_1) \tag{32}$$

Differentiating this quantity with respect to the individual weights w_i leads to the following set of equations (Laureti et. al., 2009):

$$E\left(\frac{R_i}{1 + \sum w_i R_i}\right) = 0 \tag{33}$$

Here, the before-mentioned constraints, namely forbidded short-selling and borrowing come into play, as they ensure that W_1 - the capital after one time step - stays positive and hence well defined. Not including these constraints would imply a finite chance of loosing the whole investment leading to an almost certain loss of the entire capital for large number of optimization steps.

In the case of one risky asset we screen the random walk parameter m for a starting value, for which it begins to be reasonable to invest into the respective asset, as well as an end value, for which it makes sense to put all money into this particular security. With the help of these two values ($m_{\pm} = \pm^{D}/_{2}$) and under the assumption of small m and D we receive the weight w allocated to the risky asset (Laureti et. al., 2009):

$$w = \frac{1}{2} + \frac{m}{D} \tag{34}$$

The remaining capital is thus kept as cash. Since m and D grow linearly as time passes, the weight w invested into the risky asset is independent of the length of the optimization step. In the case of more than one risky asset we can introduce a Lagrange multiplier γ which constrains the sum of all weights w_i to one $\left(\sum_{j=1}^{N} w_i = 1\right)$ yielding the following weight expression (Laureti et. al., 2009):

$$w = \frac{1}{2} + \frac{m + \gamma}{D} \tag{35}$$

Be aware of the fact that the above-established framework only deals with uncorrelated securities. To this effect, it is necessary to utilize the covariance matrix $\Sigma_{i,j}$ leading to a different formulation of the afore-defined expected value (Laureti et. al., 2009):

$$E\left[g\left(\eta\right)\right] = g\left(m\right) + \frac{1}{2} Tr\left(\Sigma_{i,j} V\right) \tag{36}$$

where:

$g(m)$: function $g\left(\eta\right) = \frac{R_i}{1+\sum w_i R_i}$ evaluated for $\eta = m$

V: matrix of second derivatives of $g\left(\eta\right)$

If $g\left(\eta\right)$, $\Sigma_{i,j}$ and m are known, it is possible to treat the equation above as in the uncorrelated scenario. In the "Implementation / Results" section we will create three different Kelly portfolios, while each originates from a different data set. The corresponding data sets will be presented in the next section and we will come up with a justification regarding the length of the data windows.

3. Data

The following section will elaborate on the data sets used and it will introduce some interesting evaluation metrics that show that financial data is not normally distributed contrary to Markowitz' assumption. We will use three different data sets which are downloaded from *finance.yahoo.com* using the R-package "fImport". First of all, we implement nine out of eleven Standard and Poor's 500 sectors, namely Consumer Staples (XLP), Financials (XLF), Health Care (XLV), Industrials (XLI), Materials (XLB), Consumer Discretionary (XLY), Energy (XLE), Utilities (XLU) and Technology (XLK). The sectors Real Estate and Financial Services were omitted because these two sectors were introduced some time after the perviously mentioned nine sectors and since we also want to include longer observation windows later on, we decided to leave them out in order to keep the selection of sectors / assets consistent over the course of our work. In order to examine the Generalized Herfindahl Index it was necessary to reproduce approximate copies of these S&P500 sectors by utilizing the return series of the 4-5 largest and most well-known companies from each sector resulting in a total number of 39 securities. For the nine S&P500 sectors we take an estimation window with daily returns of five years (01/01/2000 to 31/12/2004) and measure the out-of-sample for another five years (01/01/2005 to 31/12/2009), while for the 39 S&P500 assets we look at daily returns of a 10-years estimation window (01/01/1990 to 31/12/1999) and an out-of-sample period of yet 10 years again (01/01/2000 to 31/12/2009). Finally, the third data set comprises the prices of 50 companies from 1974 to 2011, from which the first 26 years are used for parameter estimation, whereas the remaining twelve years serve as out-of sample window. A table of all sectors and companies with their respective stock symbols can be found in the appendix (see Table 32 and 33). Plotting the returns of the first data set yields the following figure:

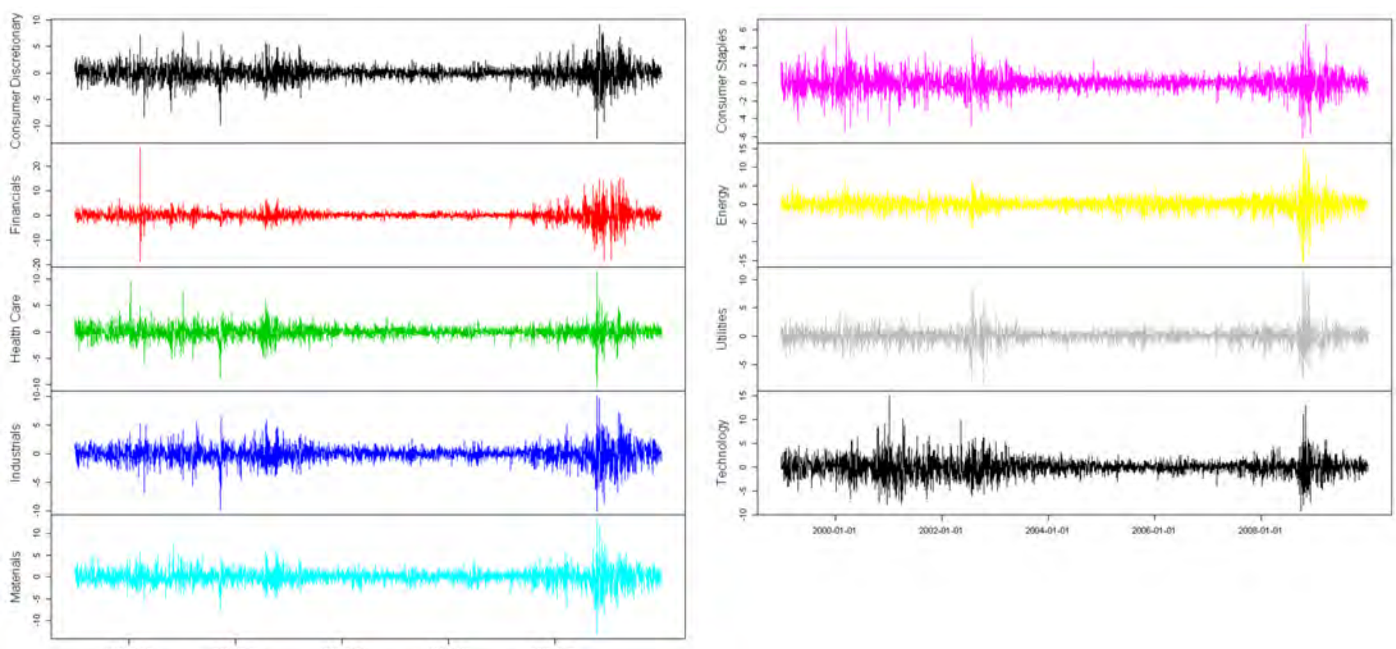


Figure 2: return plots for nine out of eleven S&P500 sectors ("XLP" (Consumer Staples), "XLF" (Financials), "XLV" (Health Care), "XLI" (Industrials), "XLB" (Materials), "XLY" (Consumer Discretionary), "XLE" (Energy), "XLU" (Utlities), "XLK" (Technology)) over a period of ten years (01/01/2000 - 31/12/2009); the plots hypothesize two rather turbulent periods (one approx. from 2000 to 2003 and the other starting in 2008).

Apparently, the different sectors show roughly the same price development meaning that the occurence of major peaks is well synchronized in time. Still, the different sectors exhibit different statistics such as mean or standard deviation. In order to better understand the diversity among the sectors we generated the following box plots:

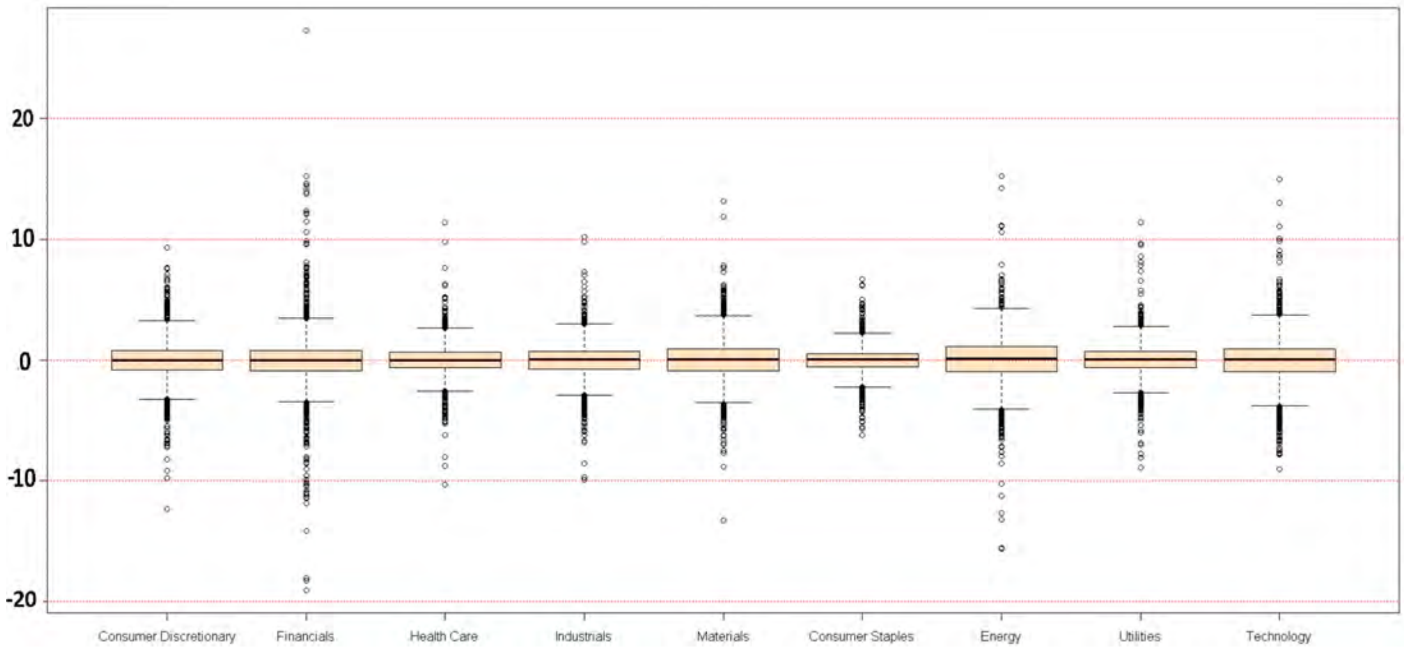


Figure 3: box plots for nine S&P500 sectors over a period of ten years (2000 - 2009); the skin colored boxes indicate the upper / lower quartiles, while the black line in the middle of each box represents the median; the black lines above and below the skin-colored boxes called antennae indicate the area in which 95% of the data points lie, while the points exceeding the antennae are outliers; expectedly, the financial sector exhibits larger fluctuations than the other sectors.

A box plot is a diagram that graphically displays the distribution of data points by showing in which region the points lie and how they are distributed over a particular area. In this regard, box plots make use of five key statistics, namely median, upper / lower quartile and upper / lower extreme values. They always consist of a rectangle / box and two lines that prolong the rectangle. The box corresponds to the area in which 50 percent of the data lies, while the lines represent the values that lie outside the box. Furthermore, the median is drawn in the middle of the box. We can see that over the course of ten years the financial sector experienced tremendously high fluctuations compared to the other sectors. Furthermore, the box plot suggests that contrary to Markowitz' assumption the assets are not normally distributed. If interested in the statistical similarities between the different sectors one might also look at the following dendrogram:

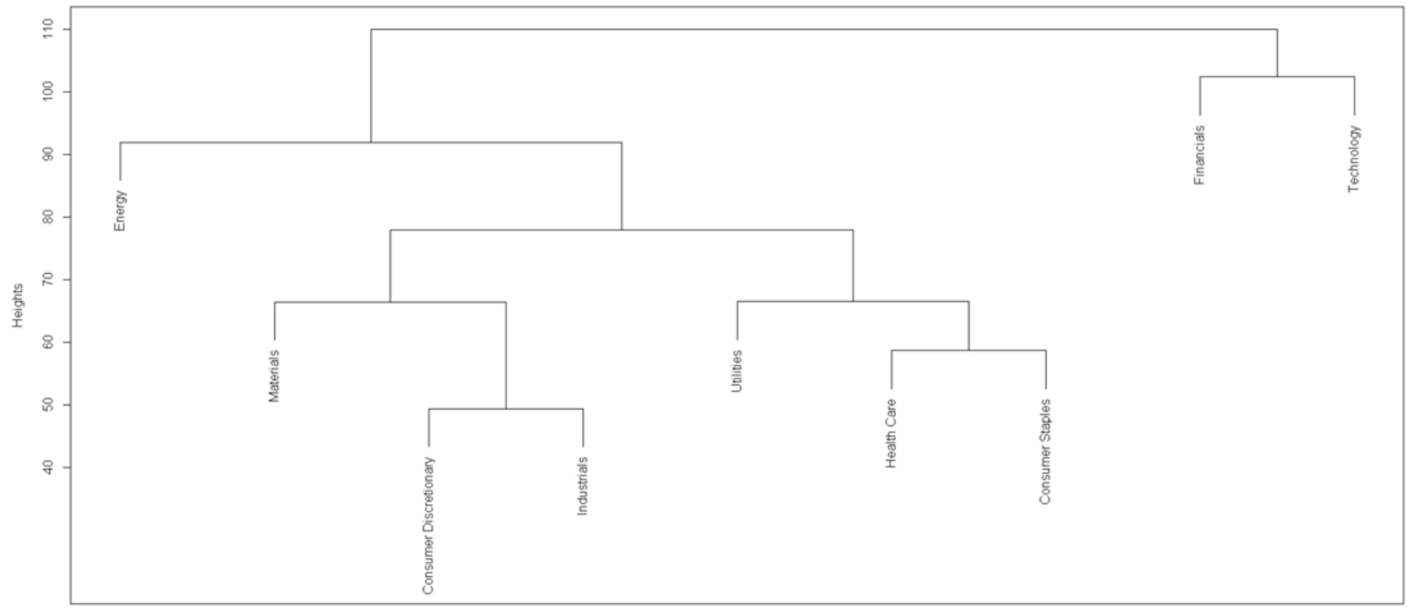


Figure 4: dendrogram for nine S&P500 sectors over a period of ten years (2000-2009) illustrating the sameness of the different sectors; the correpsonding euclidean distance measure values can be found in the appendix (see Table 34); the figure suggests two main clusters (one consisting of the sectors “Financials” and “Technology”, the other one comprising all the other sectors)

A dendrogram is a tree diagram which illustrates the hierarchical decomposition of a data set into partial data sets. The origin is represented by a single cluster which contains the whole data set O. The tree branches represent clusters that contain each individual partial data set. The first axis of the dendrogram labels the data objects, while the other one shows the distance / similarity between the data objects. It is possible to observe that some groups of sectors behave more alike than others. The Euclidean distance measure values calculated to generate this plot can be found in the appendix (see Table 34). Interestingly, “Technology” and “Financials” behave more alike than for example “Technology” and “Industrials”, although one might suspect a natural connection between these two sectors. Correspondingly, changes in the tech sector are appreciated far more rapidly by the financial industry than by any other branch, which might be the case because banks see more potential for quick returns in the tech sector than in other sectors and accordingly invest more heavily into tech-related enterprises. In addition, the dendrogram implies that some sectors come closer to an ordinary normal distribution than others.

3.1 Performance Measures

In the following we will introduce some important statistics such as performance and drawdown measures in order to further evaluate the different sectors. Note that these metrics will also be used throught the Analysis section, where we compare the performance of our three distinctive portfolio strategies with the naive $\frac{1}{N}$-rule. The figure below shows some basic statistics for the nine S&P500 sectors over the period from 01/01/2000 to 31/12/2009.

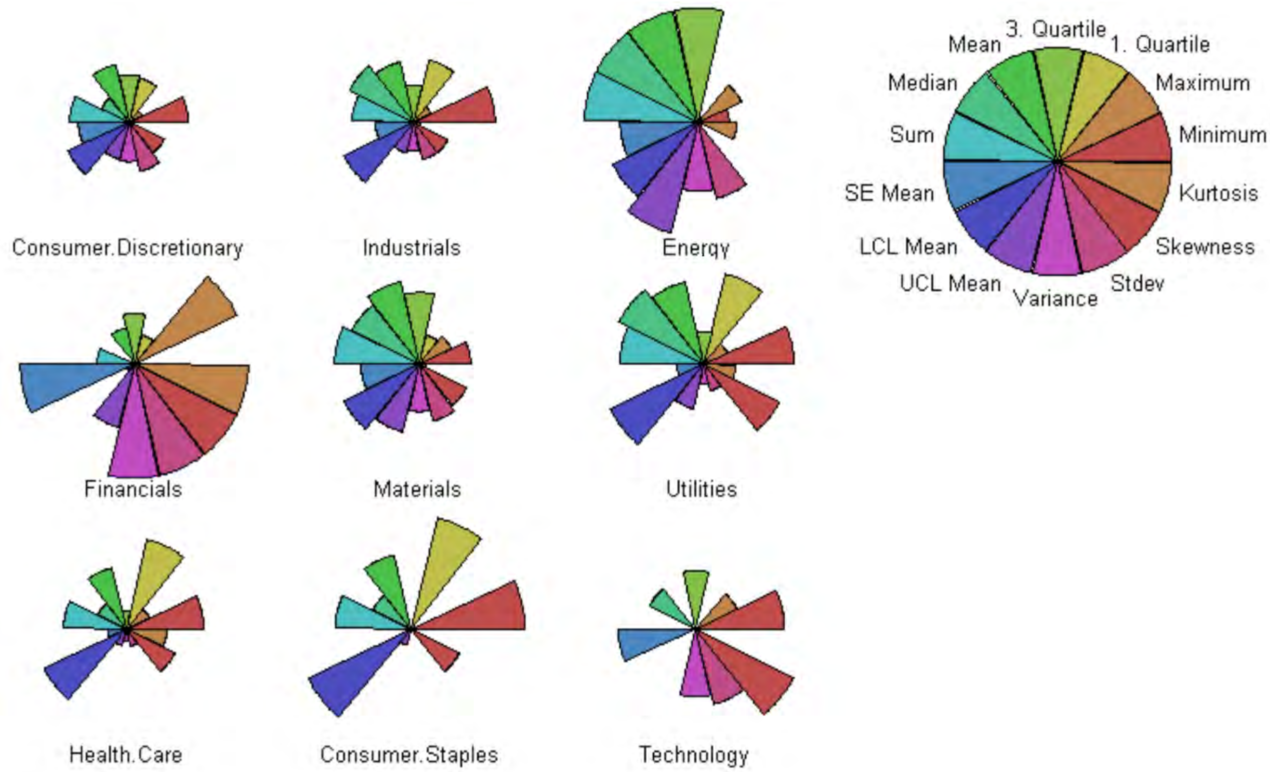


Figure 5: asset statistics for nine S&P500 sectors over a period of ten years (2000-2009); the pie plots are generated such that each color correpsonds to a single metric; the size of each pie piece is related to the largest value from all sectors for each metric; judging from the values for standard deviation, skewness and kurtosis investments to the financial sector embody the largest risks;

Again, the vast diversity among the sectors is illustrated and the emergence of skewness as well as kurtosis provide further evidence for non-normality. The most important key statistics from the figure above are summarized for each individual sector in the following table. Note that the most favorable value for each metric is colored green, while the least advantageous value for each metric is colored red.

Sector	Cons. Discret.	Financials	Health Care	Industrials	Materials	Cons. Staples	Energy	Utlities	Technology
Median	0.030	**0.000**	0.031	0.071	0.074	0.042	**0.112**	0.094	0.051
Mean	0.004	-0.009	0.006	0.005	0.019	0.014	**0.037**	0.019	**-0.032**
St. Dev.	1.646	**2.392**	1.269	1.504	1.739	**1.054**	1.975	1.375	1.964
Skewness	-0.165	**0.385**	-0.051	-0.171	-0.065	-0.055	**-0.454**	0.168	0.350
Kurtosis	4.867	**17.650**	9.256	5.323	5.380	**4.534**	8.969	8.189	4.664

Table 2: key statistics for nine S&P500 sectors over a period of ten years (2000-2009); the weakest value for each metric is colored red, while the strongest value is branded green; it is revealed that investments to the financial sector incorporate higher volatility than for instance holdings in the "Consumer Staples" sector.

Obviously, investments towards the financial sector are afflicted with large risk, whereas the sector "Consumer Staples" yields the lowest standard deviation and kurtosis which is generally associated with small tail behavior. To that effect, the plotting the quantiles of a normal distribution against the quantiles of our empirical data reveals that the financial sector only offers a sorry representation of a normal distribution, while the sector "Consumer Staples" mimics Gaussian behavior far better. However, one should be aware of the fact that none of the above introduced return distributions comes close to a normal distribution, as depicted by the figure below. The blue line in the center corresponds to the theoretical quantiles of a normal distribution, while the black dots are associated with the empirical distributions of the individual sectors. Note that each of the following quantile-quantile plots is scaled differently in order to fit the distinct plots into one figure:

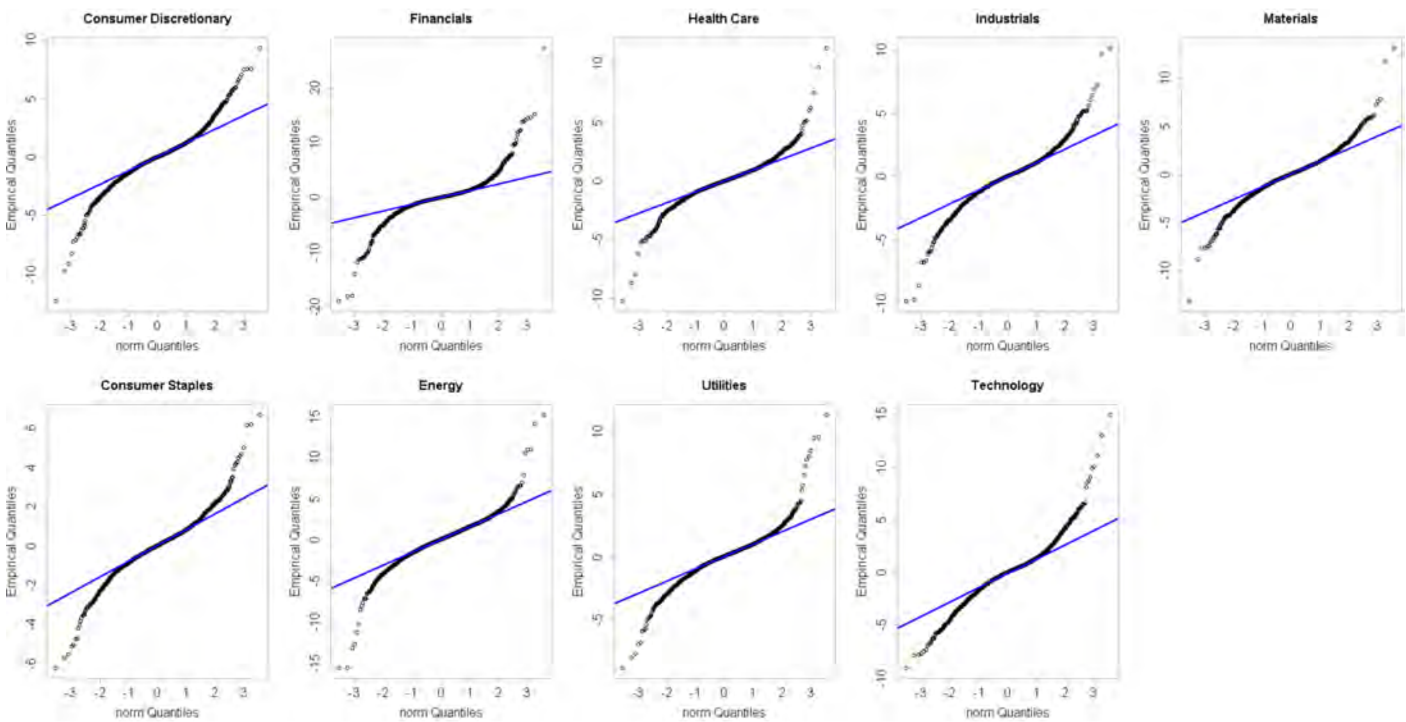


Figure 6: norm QQPlots for nine S&P500 sectors over a period of ten years (2000-2009); the blue line in the center of each plot indicates the theoretical quantiles of a normal distribution, whereas the black dots represent the empirical distribution; each plot is scaled differently; some sectors deviated largely from the normal distribution (e.g. "Financials", "Energy", "Technology"), while the sector "Consumer Staples" is much more suitable to such a distribution.

The whole summary of the basic statistics can be found in the appendix (see Table 35). Also the appendix contains further charts that display the depth and length of the drawdowns of the different sectors (see Figure 28) and two tables with all important drawdown statistics including explanations for each metric (see Table 36 and 37). Some key drawdown statistics are given below. Again note that the most favorable value for each metric is colored green, while the least advantageous value for each metric is colored red.

Sector	Cons. Discret.	Financials	Health Care	Industrials	Materials	Cons. Staples	Energy	Utilities	Technology
Avg. Drawdown Depth	26.087	81.262	**3.888**	12.328	19.074	27.599	**181.618**	12.237	20.318
Avg. Drawdown Length	73.029	38.453	278.889	249.900	56.023	**625.500**	19.427	313.625	**19.040**
Avg. Recovery Length	59.912	**3.109**	272.889	229.500	49.614	**618.750**	6.452	294.750	14.397
Cond. Drawdown at Risk	25.783	**81.046**	**6.671**	19.407	54.455	74.373	34.108	27.681	37.638
Pain Index	18.120	**21.247**	16.248	19.313	19.076	14.752	18.624	17.733	**7.005**
Ulcer Index	23.117	**31.789**	18.342	24.913	24.232	17.471	25.663	23.363	**7.226**

Table 3: key drawdown statistics for nine S&P500 sectors over a period of ten years (2000-2009); once again, it becomes apparent that the financial sector is more risky than other sectors.

Clearly, the previously established conjecture that the financial sector is extremely unstable is proven by many drawdown statistics. Looking at some of the tremendous drawdowns revealed above we can conclude that the different sectors only vaguely follow a Gaussian distribution. The first two metrics describe the average depth and length of all drawdowns within one time series. The third statistic measures the average time needed to recover from a drawdown. For a confidence level p the fourth metric takes the mean of the worst p percent of all drawdowns within one time series (Chekhlov & Uryasev & Zabarankin, 2003). The last two indices are formulated as follows:

$$PI = \sum_{i=1}^{d} \frac{D_i'}{n} \tag{37}$$

$$UI = \sqrt{\sum_{i=1}^{d} \frac{D_i'^2}{n}} \tag{38}$$

where:

d: number of all drawdowns

D_i': ith drawdown since pervious peak in period i

n: number of observations

The Ulcer Index strongly punishes underperformance by squaring the drawdowns since the last peak, so that a large index value is associated with portfolios that only reluctantly recover from crises. Attention has to be paid when selecting the frequency at which the return is assessed (hourly, daily, weekly etc.) since the Ulcer as well as the Pain Index considerably depend on the scale of the input variable. The Pain Index simply takes the average of all drawdowns over the full observation period. Unlike Ulcer Index it does not square the depth of the drawdowns and furthermore it has to be distinguished from the average drawdown depth since it does not divide by the total number of drawdowns but the total number of observations. As before, the drawdown statistics show that the sectors are very diverse and some sectors are distributed more Gaussian-like than others, but all sectors exhibit strong tail behavior.

Finally, we computed a few well-known performance measures and their extensions such as Sharpe Ratio, Omega-Sharpe Ratio or Sortino Ratio for each of the before-named sectors:

Sector	Cons. Discret.	Financials	Health Care	Industrials	Materials	Cons. Staples	Energy	Utitlities	Technology
Sharpe Ratio	0.0022	-0.0039	0.0045	0.0034	0.0109	0.0130	**0.0186**	0.0138	**-0.0163**
Adjusted SR	-0.0949	-0.2265	-0.0297	-0.0657	0.0354	0.1231	**0.1375**	0.1106	**-0.3680**
Omega SR	0.0063	-0.0132	0.0131	0.0098	0.0313	0.0373	**0.0541**	0.0408	**-0.0458**
Sortino Ratio	0.0031	-0.0056	0.0062	0.0047	0.0154	0.0183	**0.0255**	0.0193	**-0.0230**
Upside Potential Ratio	0.4910	**0.4202**	0.4835	0.4832	0.5064	**0.5095**	0.4963	0.4928	0.4784

Table 4: key performance measures for nine S&P500 sectors over a period of ten years (2000-2009); in contrast to the "Technology" sector, the "Energy" sector consistently outperforms the other sectors over the given observation period.

These and further downside performance measures including explanations can also be found in the appendix (see Table 38 and 39). Evidently, the "Energy" sector outperforms the remaining sectors over the estimation period, while the "Technology" sector performs poorly in terms of the below-defined performance measures. Sharpe Ratio is computed as mean return minus the risk free rate divided by standard deviation. Note that the risk free rate is chosen to be zero in our case for simplicity reasons.

$$SR = \frac{r_p - r_f}{\sigma_p} \tag{39}$$

The Adjusted Sharpe Ratio also takes skewness and kurtosis into account (Bacon, 2008):

$$ASR = SR\left[1 + \left(\frac{S}{6}\right)SR - \left(\frac{K-3}{24}\right)SR^2\right] \tag{40}$$

where:

S: Skewness

K: Kurtosis

Starting from the definition of the Omega Ratio it was possible to develop a Sharpe-like measure, namely the Omega-Sharpe Ratio. Therefore we take the difference between the portfolio return and a specified Minimum Acceptable Return (MAR) as numerator and the opposite of the Downside Deviation as denominator (Bacon, 2008):

$$OSR = \frac{r_p - MAR}{\sum_{i=1}^{n} \frac{\max(MAR - r_i, 0)}{n}} \tag{41}$$

Later on it was proposed to measure the performance of an investment by how much it fails to achieve a return goal, which yields a revised version of the Sharpe ratio, namely the Sortino Ratio. Sortino defines this return goal as the before-mentioned Minimum Acceptable Return. The proper choice of this return is crucial, since extreme values can alter the results significantly. Various papers recommend to choose the riskfree rate as MAR, which is zero in our case. The authors of this paper are aware of the fact that choosing the riskfree rate to be zero is not the proper way to incorporate the riskfree asset, but since this section only aims introducing the performance measures and showing that there is in general a behavior as well as a performance difference between the various sectors, we concluded that selecting the riskfree rate as such satisfies our needs for the time being. Omega-Sharpe Ratio and Sortino Ratio can also be computed via the Kappa (Sortino & Price, 1994). The Kappa is defined as the difference between mean and MAR over lth root of the sum of the lth lower partial moments (Bacon, 2008):

$$\kappa = \frac{r_p - MAR}{\sqrt[l]{\sum_{i=1}^{n} \frac{\max(MAR - r_i, 0)^l}{n}}} \tag{42}$$

Taking $l = 1$ results in the Omega-Sharpe Ratio, while $l = 2$ leads to the Sortino ratio. Further extending the Sortino Ratio yields the Upside Potential Ratio, which takes returns that exceed the MAR as numerator and returns that fall short of the MAR as denominator. Unlike Sortino Ratio, the UPR does not only consider the investment's performance during bull market, but also during bear markets (Sortino & Price, 1994):

$$UPR = \frac{\sum_{i=1}^{n} \max\left(r_i - MAR, 0\right)}{\sum_{i=1}^{n} \max\left(MAR - r_i, 0\right)} \tag{43}$$

The sum of all performance measures again underlines the strong diversity among the sectors. Note that the above-introduced drawdown and performance measures are also applied to compare the performances of the different portfolios in the Analysis section down below.

Similar statistics obtained from the second (1990 - 2009) and third data set (1974 - 2011) can be found in the appendix (see Tables 40 - 48 and 49 - 57). These two data sets need to be considered as well throughout the paper, since it is possible that a certain strategy might work when evaluating an individual data set, but not when looking at the other two data sets due to regime changing. To put it differently, the period assessed in particular data set might be dominated by some market behavior which opresses the benefits of the respective strategy. Hence, special emphasis must be attributed to choosing the size of the data set and to selecting the types of assets utilized in the respective data set. In our paper the out-of sample period was chosen to be within the first ten to twelve years of this millenium, simply because a plethora of economically interesting events happened during this time. Furthermore, we made sure to fulfill the condition that the assessed period is much longer than the number of securities considered: $T \gg N$, where T represents the number of price observations. If this condition is not met, a phase transition occurs that brings an immense estimation error with it due to diverging noise sensitivity. As most optimization attempts deploy 100 to 1000 securities as well as estimation periods of only a few years, they disregard this consideration and thereby the control parameter $\frac{N}{T}$ often exceeds a critical threshold (Ciliberti & Mézard, 2007). The three data sets utilized in our paper yield ratios of 0.0072, 0.0156 and 0.0077, respectively. In the following we will explain the implementation and present the results of the three concepts of interest, namely Generalized Herfindahl Index, state-dependent allocation and Kelly Criterion.

4. Implementation

As mentioned before we intend to solve the Markowitz problem by utilizing the statistical programming language R as well as AMPL ("A mathematical programming language"), which enables us to algebraically formulate optimization problems within the R environment. In addition, imposing the covariance budget constraint (QP2) or minimizing the the covariance risk (QP1) necessitates the use of nonlinear solvers such as "CPLEX" or "MINOS". The R-package "appRmetricsDashboard" allows us to implement AMPL code into R by generating three specific files, namely the model, the data and the run file. The respective code that was used to obtain the results below will be shown in the appendix. The following section will first deal with the general Markowitz problem before adding two different diversification constraints, which are the Herfindahl Index and the Generalized Herfindahl Index. Finally we also cover the implementation concerning the state-dependent allocation strategy and the Kelly portfolio.

4.1 QP3 Markowitz Problem

As explained in the theory section on the Markowitz problem we are dealing with an optimization problem: The QP1 Markowitz problem minimizes the covariance risk when the return is fixed, while the QP2 Markowitz problem maximizes the return while the covariance risk is fixed. Further generalizing these problems is done by weighting both objectives (return, risk) with factors λ_1 and λ_2, respectively. The ratio of these factors defines the investor's willingness to take risks. That is to say, large λ_1 would mean that the investor is rather keen on profit maximization than on risk minimization and vice versa. Each choice of λ_i yields a different portfolio within the feasible set. Hence, we aim at optimizing the following objective:

$$\max_w \left\{ \lambda_1 \left(\sum_{i=1}^{n} \mu_i w_i \right) \right\} + \min_w \left\{ \lambda_2 \left(\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j \right) \right\} \tag{44}$$

This equation can also be interpreted as $\max\{\lambda_1 \cdot return\} + \min\{\lambda_2 \cdot risk\}$. We further simplify this result after having realized that maximizing return is just the same as minimizing negative return, so that:

$$\min_w \left\{ -\lambda_1 \left(\sum_{i=1}^{n} \mu_i w_i \right) \right\} + \min_w \left\{ \lambda_2 \left(\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j \right) \right\} \tag{45}$$

Applying the constraint condition $\lambda_1 + \lambda_2 = 1$ we eliminate λ_1 by setting it equal to $1 - \lambda_2$:

$$\min_w \left\{ (\lambda_2 - 1) \left(\sum_{i=1}^{n} \mu_i w_i \right) \right\} + \min_w \left\{ \lambda_2 \left(\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j \right) \right\} \tag{46}$$

In order to make sure we obtain reasonable results, we have to normalize both return data and risk data. This can be done by dividing by $\Sigma_{\max}$ - the largest diagonal element of the covariance matrix - and by μ_{max} - the largest return value:

$$\min_w \left\{ (\lambda_2 - 1) \left(\frac{\sum_{i=1}^{n} \mu_i w_i}{\mu_{max}} \right) \right\} + \min_w \left\{ \lambda_2 \left(\frac{\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j}{\Sigma_{max}} \right) \right\} \tag{47}$$

Followingly, both return objective and risk objective can only take values between 0 and 1, which leads to a more consistent distribution of portfolios along the efficient frontier, since this kind of normalization shifts the distribution of portfolios toward the high-risk end of the efficient frontier. Another approach is presented by dividing the objective terms by the difference between the maximum and the minimum return / risk values, which yields a normalization with respect to the equal weights portfolio. Neglecting the shift due to the minimal risk or return value, the previous equation is therefore given as:

$$\min_w \left\{ (\lambda_2 - 1) \left(\frac{\sum_{i=1}^{n} \mu_i w_i}{\mu_{max} - \mu_{max}} \right) \right\} + \min_w \left\{ \lambda_2 \left(\frac{\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j}{\Sigma_{max} - \Sigma_{min}} \right) \right\} \tag{48}$$

At this point we will not go into further detail regarding the peculiarities of individual normalization measures. In order to explain the particular implementation of the QP3 problem into R and AMPL the corresponding model file is depicted in the appendix (see Box 1). Including a risk-free asset yields an objective function whose covariance risk term takes into account only those weights attributed to risky assets. This is equivalent to considering an asset with no correlation to the other assets resulting in a correlation matrix that only displays zeros in the row and column associated with the risk-free asset. Furthermore, the return μ_0 of the risk free asset - multiplied with the respective weight w_0 - is added to the return term of the objective function.

4.2 QP3 Markowitz Problem including EWP Diversification

In addition to the optimization with respect to return and risk we can also introduce a third goal, namely diversification. Diversification can be carried out with regards to different diversification measures. For example, we could aim at minimizing tail risks or risk budgets. However, the easiest approach is to diversify with respect to the equal weights portfolio by using the Herfindahl Index or entropy concentration. As other diversification optimization strategies build on the same technical structure as the EWP approach, it serves as a starting point for later sections of this project. In order to implement the EWP approach into R we have to introduce a third lambda value λ_3 that determines the investor's demand for diversification. Adopting the new normalization condition $\lambda_1 + \lambda_2 + \lambda_3 = 1$ yields the following objective:

$$\min_{w} \{-\lambda_1 \cdot return + \lambda_2 \cdot risk + \lambda_3 \cdot diversification\} \tag{49}$$

As before, return and risk are defined as $\frac{\sum_{i=1}^{n} \mu_i w_i}{\mu_{max}}$ and $\frac{\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j}{\Sigma_{max}}$, respectively. Analogous to the risk definition, the previously established Herfindahl Index constitutes the diversification term:

$$\sum_{i,j=1}^{n} w_i \delta_{ij} w_j \tag{50}$$

where:

δ_{ij}: Kronecker Delta

Accordingly, the objective function is written as follows:

$$\min_{w} \left\{ -\lambda_1 \left(\frac{\sum_{i=1}^{n} \mu_i w_i}{\mu_{max}} \right) + \lambda_2 \left(\frac{\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j}{\Sigma_{max}} \right) + \lambda_3 \left(\sum_{i,j=1}^{n} w_i \delta_{ij} w_j \right) \right\} \tag{51}$$

Again, the model file is shown in the appendix (see Box 3) in order to explain the particular implementation of the QP3 Markowitz Problem including the Herfindahl Index diversification. Instead of applying the Herfindahl Index as diversification measure we can also utilize the concept of entropy concentration, which yields the objective function below:

$$\min_{w} \left\{ -\lambda_1 \left(\frac{\sum_{i=1}^{n} \mu_i w_i}{\mu_{max}} \right) + \lambda_2 \left(\frac{\sum_{i,j=1}^{n} w_i \Sigma_{i,j} w_j}{\Sigma_{max}} \right) - \lambda_3 \left(\sum_{i,j=1}^{n} w_i \cdot \ln w_i \right) \right\} \tag{52}$$

The corresponding model file can be found in the appendix as usual (see Box 4). Both techniques yield very similar risk-return diagrams and can therefore be exchanged without loss of generality.

4.3 QP3 Markowitz Problem including GHHI Diversification

As opposed to the Herfindahl Index, the Generalized Herfindahl Index also takes into account correlations between assets from the same sector and not just between the assets itselves. So, replacing the diversification term in the objective function by the definition of the Generalized Herfindahl Index results in an alternative objective function which incoroporates a matrix that provides information about the sector affiliation of the individual securities:

$$\min_{w}\left\{-\lambda_1\left(\frac{\sum_{i=1}^{n}\mu_i w_i}{\mu_{max}}\right)+\lambda_2\left(\frac{\sum_{i,j=1}^{n} w_i\Sigma_{i,j}w_j}{\Sigma_{max}}\right)+\lambda_3\left(\sum_{i,j=1}^{n} w_i\varrho_{ij}w_j\right)\right\} \tag{53}$$

where:

ϱ_{ij}: GHHI correlation matrix

In our case the GHHI correlation matrix is defined through the 39 assets that are part of the second data set which is built up of individual securities from the different S&P500 sectors. The stock symbols can be looked up in the appendix (see Table 32). Note that each entry on the diagonal of following 9×9 matrix is in fact a matrix full of ones itself:

$$\varrho_{ij}=\begin{bmatrix} XLY & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & XLF & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & XLV & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & XLI & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & XLB & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & XLP & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & XLE & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & XLU & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & XLK \end{bmatrix} \tag{54}$$

Depending on how many securities we have chosen for each sector each block full of ones on the diagonal is a 4×4 or 5×5 matrix. The exact sector division can once again be looked up in the appendix. The model file is obtained by substituting the identity matrix in the model file for the EWP diversification with respect to the Herfindahl Index by the above-introduced 39×39 matrix. For the third data set we are dealing with a 50×50 matrix. Now we will demonstrate to implement the second strategy, namely the state-dependent allocation strategy.

4.4 State-Dependent Allocation Strategy

As mentioned before Escobar et. al. (2013) propose a portfolio strategy that switches between the naive $\frac{1}{N}$-strategy and a minimum-variance approach depending on whether the market experiences "normal" or "turbulent" times. We will test this strategy using all three data sets. Note, that we did not consider the risk-free asset in this case, because adding such an asset to a minimum-variance approach would mean to only invest into this particular asset, since it comes with the lowest risk. The portfolio is now implemented by using a simple if-statement such that our portfolio assumes the weights of the minimum variance portfolio whenever the crisis indicator exceeds a certain threshold. The model file corresponding to the minimum-variance portfolio can be found in the Appendix (see Box 5).

We estimate the minimum variance portfolio for various estimation periods within each data set aiming at introducing a rolling window approach. In order to obtain larger rebalancing frequency, we just omit some periods and substitue their weight distribution by the previous period's distribution. In the case of the second data set (1990 - 2010) we thereby obtain three different strategies, which rebalance the min-var weights every six, twelve and sixty months (ergo "buy-and-hold" strategy), respectively. After having generated time series for the min-var approach and the $\frac{1}{N}$-strategy we decide which portfolio to choose on a certain day depending on the indicator value given on the previous day. The allocation decision is conducted applying different indicator thresholds varying between 0.0 and 0.8. Instead of comparing different thresholds one could also relate the indicator value with a switching probability, so that whenever the indicator value is larger than a random number that is newly generated each day, the $\frac{1}{N}$-strategy is replaced by the minimum varriance approach. In the next subsection we will explain how to implement a Kelly portfolio.

4.5 Kelly Portfolio

As explained in the theory, the Kelly Portfolio will optimize the investor's wealth by maximizing its average exponential rate of growth (Medo et. al., 2009):

$$G = E\left[\ln\left(1 + \sum_{i=1}^{N} w_i R_i\right)\right] \tag{55}$$

Differentiating with respect to the individual w_i yields the following set of equations (Medo et. al., 2009):

$$\sum_{R} \frac{P\left(\overrightarrow{R}\right) R_i}{1 + \sum_{j=1}^{N} w_j R_j} = 0 \tag{56}$$

where:
$P\left(\overrightarrow{R}\right)$: probability of given vector of returns $\overrightarrow{R} = (R_1 ... R_N)$

Provided that the portfolio return $R_p = \sum_{i=1}^{N} w_i R_i$ is very small we can approximate $\frac{1}{1+\sum_{j=1}^{N} w_j R_j}$ as $1 - \sum_{j=1}^{N} w_j R_j$ leading to the set of equations below (Medo et. al., 2009):

$$< R_i >= \sum_{j=1}^{N} w_j < R_i R_j > \tag{57}$$

Higher order approximations of course imply tedious higher order cross terms, which is why we shall rely on expanding $\frac{1}{1+x}$ as $1 - x$ for the time being. Note that all negative percentages / weights are converted to zero due to short-selling contraints.

5. Results

In the following we will present the results (i.e. risk-return diagrams and weight distributions) for the general QP3 Markowitz Problem as well as the QP3 Markowitz Problem including the (Generalized) Herfindahl Index, before shifting the focus to the state-dependent allocation strategy and finally the Kelly Portfolio.

5.1 QP3 Markowitz Problem

Implementing the QP3 Markowitz Problem into R results in a graph of the feasible set, in which the efficient frontier is indicated by a red doted line. For the first data set we acquire the following risk-return graph. Note that the risk proxy is the portfolio's standard deviation.

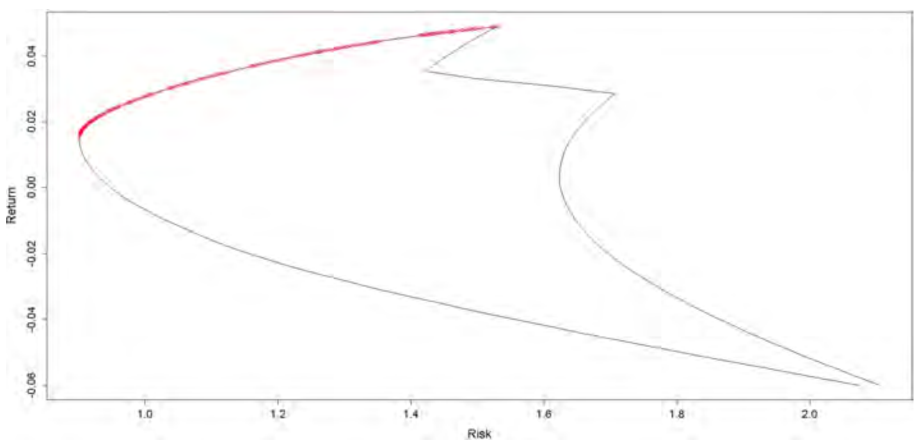


Figure 7: risk-return diagram showing the feasible set including red dots on the efficient frontier for nine S&P500 sectors over a period of five years (01/01/2000 - 31/12/2004); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return maximization and risk minimization; the striking outlier in the in the lower right corner of the figure is related to the poor performance of the "Technology" sector over the period of estimation.

Clearly, the red dots accumulate along the efficient frontier of the feasible set. The eye-catching "spike" in the far right / bottom corner is actually due to the "Technology" sector, which performed poorly over the estimation period in terms of mean return and covariance risk. Incorporating the concept of a risk-free asset into our optimization scheme lets us live outside the feasible set, which means that we can draw a straight line from any of our red dots to the risk-free rate on the y-axis of the above-shown risk-return diagram.

The attentive reader might wonder why we did not choose to implement the λ-parameters by minimizing the objective function $\left[\lambda_1 \left(-return + \frac{\lambda_2}{\lambda_1} \cdot risk\right)\right]$. The simple answer is that omitting the constraint condition $\lambda_1 + \lambda_2 = 1$ would fail to adequately project the efficient frontier, since the variation of red dots in the following risk-return diagram slightly deviates from the outer left line of the feasible set:

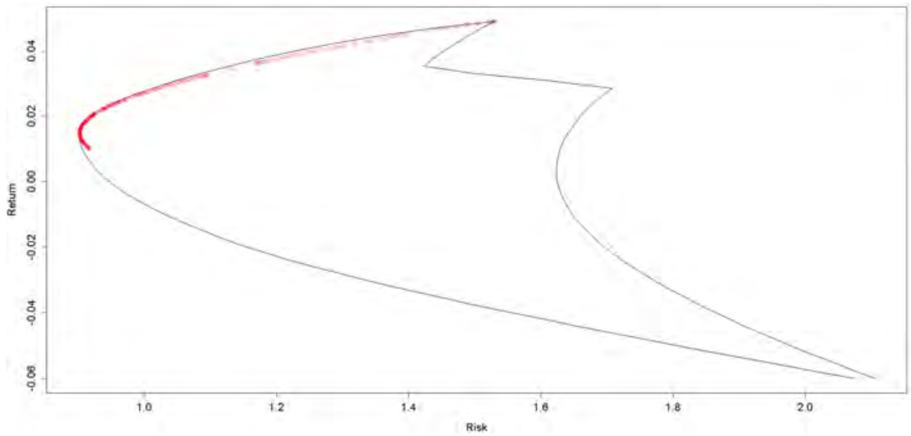


Figure 8: risk-return diagram showing the feasible set including red dots that slightly deviate from the efficient frontier for nine S&P500 sectors over a period of five years (01/01/2000 - 31/12/2004); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return maximization and risk minimization; the slight deviation proves that the constraint condition $\lambda_1 + \lambda_2 = 1$ has to be applied in order to obtain reasonable results

The code written to generate the diagram above can be found in the appendix (see Box 2). Risk-return diagrams for the second and third data set similar to Figure 7 are located in the appendix (Figure 29 and 30). In the next section we are going to extend the previous approach to also include a third goal named diversification. This will result in a cluster of red dots that is confined by the efficient as well as the sub-efficient frontier.

5.2 QP3 Markowitz Problem including EWP Diversification

Below two risk-return diagrams (Figures 9 and 10) are presented that illustrate the optimization competition between the three goals for the first data set. For the sake of convenience we will only show the plots for the Herfindahl Index. The diagrams once more show the feasible set but now including a distribution of red dots that ranges from the efficient frontier to the EWP portfolio in the center:

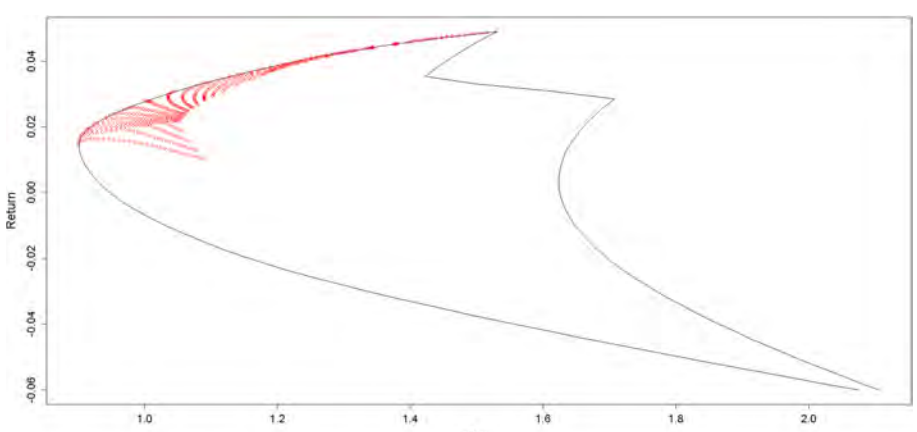


Figure 9: risk-return diagram showing the feasible set including red dots indicating the optimization competition with respect to return, risk and Herfindahl index for nine S&P500 sectors over a period of five years (2000-2004); each red dot

represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return / diversification maximization and risk minimization; the variation of red dots ranges from the minimum variance portfolio to the equal weights portfolio and the maximum return portfolio; taking into account the large size of the feasible set the equal weights portfolio performes rather well in terms of risk and return optimization.

Surprisingly, adding the diversification objective does not significantly diminish the return and risk goals considering how large the whole feasible set actually is, as the sub-efficient frontier lies very close to the efficient frontier. One could have guessed that investing into the "Technology" sector would mitigate the performance in terms of risk and return, which is not the case to a large extent. Apparently, holding the other sectors makes up for the drawbacks associated with the "Technology" sector. Zooming in and relating the degree of diversification with the color of the dots yields a more detailed plot, that is presented below. Note that blue color implies a low level diversification, whereas skin color stands for a high degree of diversification. It is important to realize that the risk and return proxies are the average return and the standard deviation of the equal weights portfolio:

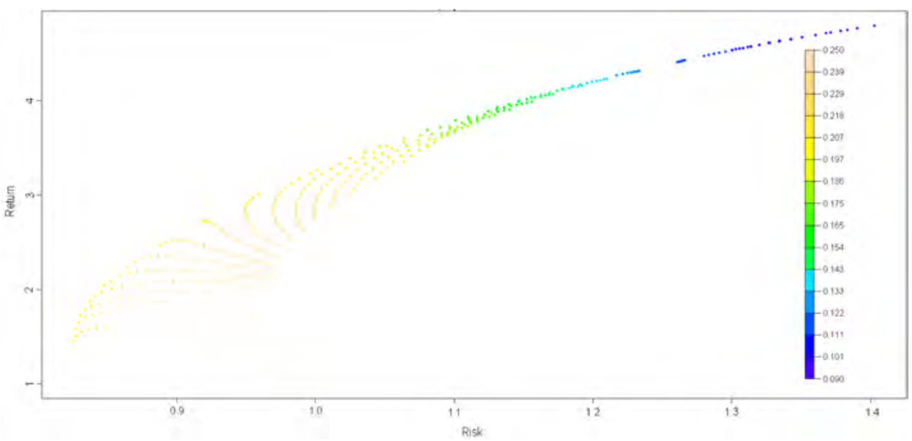


Figure 10: risk-return diagram showing the afore-exemplified variation of dots indicating the optimization competition with respect to return, risk and Herfindahl index for nine S&P500 sectors over a period of five years (2000-2004); as opposed to before, the picture is zoomed in and the level of diversification for each single portfolio is branded by a color (skin-color: well diversified, blue color: poorly diversified); as expected, well diversified portfolios can be found in the proximity of the equal weights portfolio, while poorly diversified portfolios are located near the maximum return portfolio.

Interestingly, there is a large area of relatively strong diversification between the EWP portfolio and the minimum variance portfolio, where the level of diversification changes only slowly. Including the risk-free asset changes the risk and return terms in the same way as for the general QP3 Markowitz problem, but in this case one also has to divide the diversification term by $(1 - w_0)^2$, where w_0 is the weight allocated to the risk-free asset. As before, we can draw a line from any colored dot to the respective risk-free rate on the y-axis of the above-presented risk-return diagrams and invest along the line. As before, the risk-return diagrams for the second and third data set are presented in the appendix (Figures 31 - 34).

In order to show that the two concepts of Herfindahl Index and entropy concentration are empirically more or less equivalent we computed the optimal weight distributions for the first data set for a fixed target return of 0.66 percent - the mean return of all sectors - but using different target covariance risk values. Note that we excluded the risk-free asset in this example, because we wanted to single out the effect of both optimization concepts. In this respect, taking into account a risk-free asset implies an additional estimation uncertainty, since the allocation decision would also be accompanied by a date decision, as investing at the wrong point in time implicates an unfavorable risk-free rate:

Target Risk	Diversification Measure	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU	XLK
1.0	Herfindahl	0.0447	0.0000	0.1559	0.0552	0.0683	0.2752	0.1390	0.1848	0.0769
	Entropy	0.0469	0.0207	0.1376	0.0534	0.0605	0.3093	0.1224	0.1729	0.0764
1.1	Herfindahl	0.0721	0.0325	0.1354	0.0814	0.0870	0.2067	0.1340	0.1607	0.0903
	Entropy	0.0710	0.0476	0.1266	0.0778	0.0816	0.2244	0.1246	0.1557	0.0906
1.2	Herfindahl	0.0900	0.0686	0.1227	0.0958	0.0983	0.1605	0.1250	0.1386	0.1005
	Entropy	0.0896	0.0735	0.1200	0.0945	0.0965	0.1659	0.1219	0.1374	0.1006
1.3	Herfindahl	0.1056	0.1001	0.1138	0.1073	0.1078	0.1235	0.1150	0.1184	0.1085
	Entropy	0.1056	0.1004	0.1137	0.1072	0.1077	0.1238	0.1148	0.1183	0.1085
1.4	Herfindahl	0.1198	0.1284	0.1072	0.1170	0.1161	0.0923	0.1047	0.0996	0.1150
	Entropy	0.1196	0.1293	0.1067	0.1167	0.1157	0.0932	0.1041	0.0996	0.1150
1.5	Herfindahl	0.1328	0.1542	0.1019	0.1255	0.1235	0.0653	0.0942	0.0820	0.1206
	Entropy	0.1315	0.1607	0.0991	0.1231	0.1203	0.0708	0.0912	0.0823	0.1209

Table 5: weight distributions for nine S&P500 sectors at fixed target return and varying target risk resulting from an estimation window of five years (2000-2004); for each target risk value two rows are presented showing the weight distributions for Herfindahl and entropy optimization, respectively; by comparing the rows pairwise it turns out that both optimization approaches are virtually equivalent from a empirical viewpoint.

Obviously, both optimization approaches lead to very similar portfolio weights under the same starting conditions (i.e. target return / risk). The largest deviations can be observed for the lowest target covariance risk value (~2 percent). Finally, we computed the different diversification measures for the above-introduced weight distributions:

Target Risk	Diversification Measure	Herfindahl	Rosenbluth	Exp. Entropy	CCI	Entropy
1.0	Herfindahl	0.1691	0.1847	0.1480	0.4482	1.9105
	Entropy	0.1744	0.1826	0.1433	0.4563	1.9428
1.1	Herfindahl	0.1334	0.1478	0.1231	0.3768	2.0944
	Entropy	0.1344	0.1472	0.1225	0.3825	2.0995
1.2	Herfindahl	0.1174	0.1281	0.1143	0.3334	2.1690
	Entropy	0.1175	0.1279	0.1143	0.3358	2.1694
1.3	Herfindahl	0.1115	0.1149	0.1113	0.1819	2.1954
	Entropy	0.1115	0.1150	0.1113	0.1818	2.1954
1.4	Herfindahl	0.1121	0.1173	0.1115	0.1807	2.1928
	Entropy	0.1121	0.1173	0.1116	0.1802	2.1926
1.5	Herfindahl	0.1172	0.1279	0.1143	0.1762	2.1688
	Entropy	0.1173	0.1280	0.1143	0.1726	2.1691

Table 6: diversification measure for nine S&P500 sectors at fixed target return and varying target risk resulting from an estimation window of five years (2000-2004); as anticipated, the Comprehensive Concentration Index moves more slowly within the range $[0, 1]$ than the first three indices, whereas the entropy is not bound to an upper limit.

As already explained before the first three diversification measures, namely the Herfindahl Index, the Rosenbluth Index and the exponential of the entropy shift more rapidly from 1 to $\frac{1}{N}$ with increasing degree of diversification than the Comprehensive Concentration Index. Only the last metric - entropy - is not bound to the range $[0, 1]$. Effectively, the first three measures become equivalent if the number of securities exceeds a certain threshold. Also Bouchaud et. al. (1997) assert that the entropy concentration S becomes equivalent to $\frac{Yq-1}{q-1}$ in the limit of q going to 1, where Y_q is the q-norm of the portfolio weights ($Y_q = \sum_i p_i^q$), which is equal to the Herfindahl Index for $q = 2$. These theoretical considerations and the previously presented empirical results show that Herfindahl Index and entropy diversification are very closely related. In the following we will extend the previous approach to include the Generalized Herfindahl Index.

5.3 QP3 Markowitz Problem including GHHI Diversification

The following subsection contains four risk-return diagrams (Figures 11 - 14) that illustrate the optimization competition between return, risk and the Generalized Herfindahl Index.

Second Data Set: 1990 - 2009

As the first data set only consisted of the distinguished sectors, we introduce two more data sets that comprise individual assets from the different sectors. The following risk-return diagram is generated by optimizing the objective function above for different choices of λ_i:

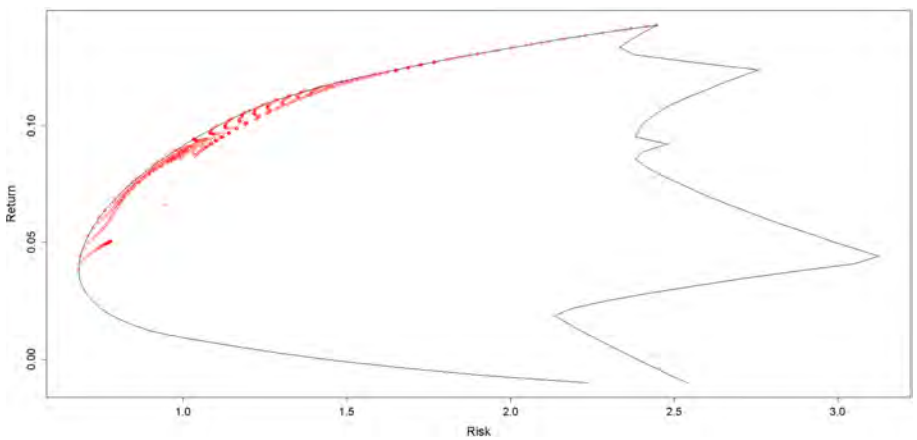


Figure 11: risk-return showing the feasible set including red dots indicating the optimization competition with respect to return, risk and Generalized Herfindahl index for 39 S&P500 assets over a period of ten years (1990-1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return / diversification maximization and risk minimization.

Compared to the previous risk-return diagrams, the dot distribution sticks much closer to the efficient frontier. As before, we zoom into the picture and relate the color of the dots with the degree of diversification in order to see more details. Note that the risk proxy is given by the standard deviation of the equal weights portfolio.

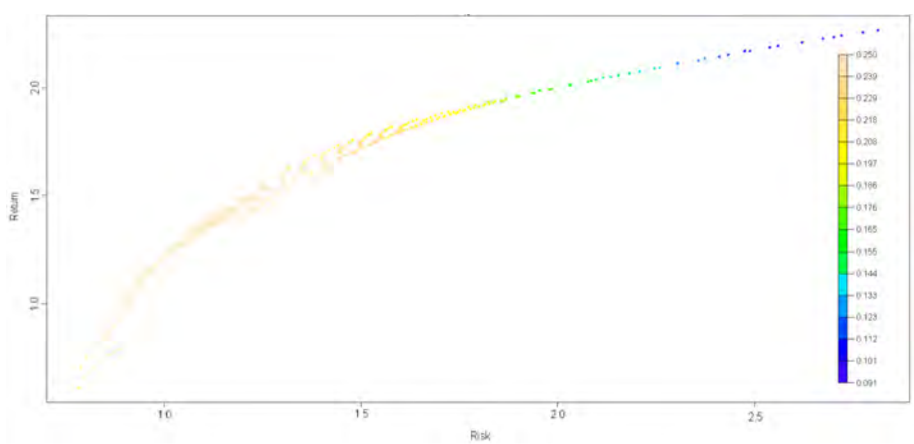


Figure 12: risk-return diagram showing the afore-exemplified variation of dots indicating the optimization competition with respect to return, risk and Generalized Herfindahl index for for 39 S&P500 assets over a period of ten years (1990-1999); as opposed to before, the picture is zoomed in and the level of diversification for each single portfolio is branded by a color (skin-color: well diversified, blue color: poorly diversified).

Apparently, the area of strongest diversification is slightly extended from the equal weights portfolio towards the maximum return portfolio. As explained before the Generalized Herfindahl Index tends to keep the concentration low within a certain sector and therefore a GHHI portfolio usually only invests in a few assets from each sector provided that these particular assets outperform the remaining securities from the same sector in terms of expected return and risk.

Third Data Set: 1974 - 2009

The third data set containing 50 assets from the nine S&P sectors over a period of 26 years (01/01/1974 - 31/12/1999) can be implemented using a 50×50 matrix with blocks of ones whose sizes vary between 3×3 and 9×9. The size of the blocks depends on how many companies could be identified for this particular period from the respective sectors. Note that the respective companies and their stock symbols can be found in the appendix (see Table 33). Optimizing the portfolio weights for different specific choices of λ_i results in the following feasible set:

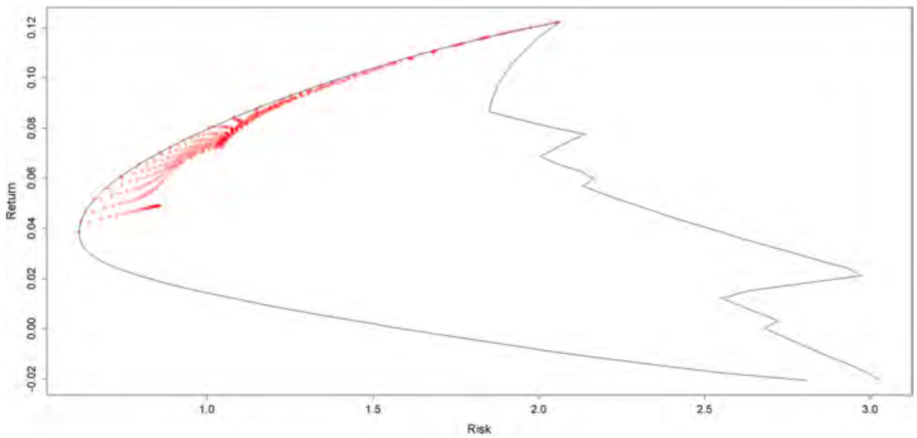


Figure 13: risk-return showing the feasible set including red dots indicating the optimization competition with respect to return, risk and Generalized Herfindahl index for 50 S&P500 assets over a period of 26 years (1974-1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return / diversification maximization and risk minimization.

Once more it can be observed that the sub-efficient frontier associated with the GHHI diversification is located in the proximity of the efficient frontier, altough the entire feasible set is a lot larger. As opposed to the equal weights portfolio, the "perfect" GHHI portfolio can be achieved in several ways, since merely the sums of the weights allocated to each sector have to be the same and not the weights allocated to each asset. As usual, zooming in to the cluster and giving the dots colors according to the level of diversification yields a different picture. Again, the standard deviation of the equal weight portfolio serves as risk proxy.

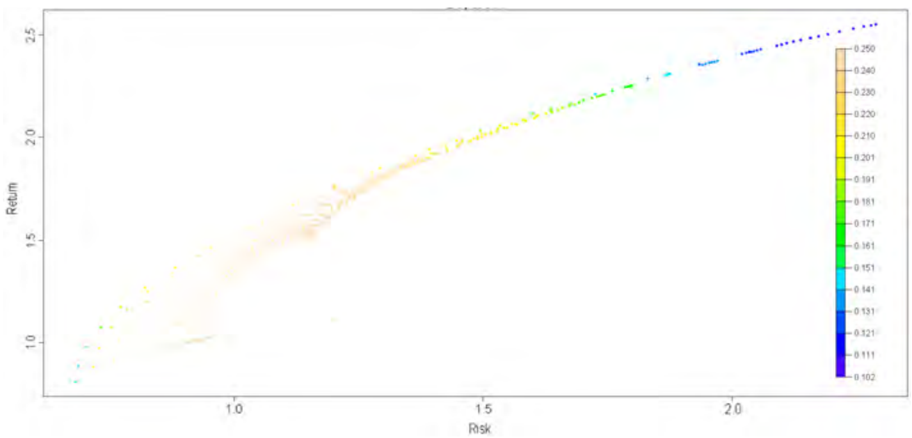


Figure 14: risk-return diagram showing the afore-exemplified variation of dots indicating the optimization competition with respect to return, risk and Generalized Herfindahl index for for 50 S&P500 assets over a period of 26 years (1974-1999); as opposed to before, the picture is zoomed in and the level of diversification for each single portfolio is branded by a color (skin-color: well diversified, blue color: poorly diversified).

Like before, those portfolios that are well diversified with respect to the Generalized Herfindahl Index lie more or less in the center of the cluster, while the poorly diversified portfolios are closer to the maximum return portfolio. Again, adding a risk-free asset necessitates altering the risk, return and diversification terms in the same way as for the previous objective functions. Because of analysis purposes we required the GHHI portfolio to match the mean and standard deviation of the respective $\frac{1}{N}$-portfolio.

All four plots (Figures 11 - 14) have demonstrated that both indices, namely the Herfindahl and the Generalized Herfindahl Index, work in a similar manner as the courses of the corresponding sub-efficient frontiers a very much alike. The exact location of the sub-efficient frontier depends on the number of securities selected from each sector. In general, it is advisable to choose larger stock numbers and to keep the number of assets from each sector the same if possible, since otherwise the GHHI approach might lead to a condensation of only a few assets. On top of that, some of the assets chosen to be part of the portfolio might even strongly underperform others in terms of expected risk and return, if some sectors offer only a small number of poorly performing assets. In the analysis section we will compare the performance of some GHHI portfolios to the $\frac{1}{N}$-strategy, but first we will have a look at the results regarding the state-dependent allocation strategy and the Kelly portfolio.

5.3 State-Dependent Allocation Strategy

First Data Set: 2000 - 2009

As before, the first data set incorporates nine out of eleven S&P500 sectors. Applying a min-var strategy in combination with a rolling window approach (period: six months) yields the following weight distribution:

Sector	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU	XLK	**Herfindahl**
01/2005 - 06/2005	0	0	0.2184	0	0.0001	0.4431	0.1353	0.2031	0	**0.3036**
07/2005 - 12/2005	0	0	0.1886	0	0	0.5338	0.0932	0.1844	0	**0.3632**
01/2006 - 06/2006	0	0	0.1729	0	0	0.5982	0.0648	0.1641	0	**0.4189**
07/2006 - 12/2006	0	0	0.1278	0	0	0.6752	0.0465	0.1504	0	**0.4970**
01/2007 - 06/2007	0	0	0.1144	0	0	0.7128	0.0527	0.1201	0	**0.5384**
07/2007 - 12/2007	0	0	0.1007	0	0	0.7538	0.0439	0.1017	0	**0.5906**
01/2008 - 06/2008	0	0	0.1828	0	0	0.6579	0.0234	0.1359	0	**0.4853**
07/2008 - 12/2008	0	0	0.2197	0	0	0.6658	0.0070	0.1075	0	**0.5032**
01/2009 - 06/2009	0	0	0.1899	0	0	0.8101	0	0	0	**0.6923**
07/2009 - 12/2009	0	0	0.1960	0	0	0.8040	0	0	0	**0.6750**

Table 7: weight distribution for a minimum variance portfolio that rebalances every six months using a rolling window approach over period of five years (2005 - 2009); the last column contains the Herfindahl Index for the respective weight distribution; accordingly, the state-dependent allocation strategy permantently revolves around effective numbers of 9 assets ($\frac{1}{N}$-rule) and 1.5 to 3 assets (min-var portfolio)

Note that we decided to not further increase the rebalancing frequency, since computational effort grows linearly with it and as the analysis section will show later on, there comes little to no benefit at all with additional increment. The minimum variance weight distribution changes slightly over the years. While we invest into five assets in the beginning, we end up with only two sectors remaining, namely "Health Care" and "Consumer Staples". Interestingly, the percentage invested into "Health Care" fluctuates within a small range of approximately ten percent, whereas the relative amount placed into "Consumer Staples" almost continously increases over time from 44 percent to over 80 percent. The last column containing the Herfindahl Index values documents how the weight distribution in general changes. As the inversed Herfindahl Index indicates the effective number of assets among which the portfolio is diversified, we see that the state-dependent allocation strategy switches between an effective number of 9 assets ($\frac{1}{N}$-strategy) and 1.5 to 3 assets (min-var portfolio). Switching once from the $\frac{1}{N}$-rule to the min-var portfolio generates a turnover that ranges from 1.1109 to 1.5555 depending on the investment date. In the following the second and the third data set are utilized to generate two distinct state-dependent portfolios that once again switch between minimum variance strategy and $\frac{1}{N}$-rule depending on the value of the crisis indicator.

Second Data Set: 1990 - 2009

In the case of the second data set the portfolio weights are computed based on the asset's performance from 01/01/1990 to 31/12/1999, while the out-of sample performance is evalauted from 01/01/2000 to 31/12/2009. The respective minimum variance weights for the period 2000 - 2009 are shown in the tables below:

Walt Disney	McDonalds	Goodyear	Nike	AON	Johnson&Johnson
0.0188	0.0250	0.0357	0.0049	0.0728	0.0092
Boeing	FedEx	Dow Chemical	Newmont Mining	Pfizer	Bristol-Myers Suibb
0.0042	0.0076	0.0447	0.0689	0.0025	0.0203
International Paper	Colgate	Exxon-Mobile	Chevron	Occidential Petroleum	American Electric Power
0.0245	0.0109	0.0563	0.0517	0.0414	0.2092
Duke Energy	Southern Company	Exelon	IBM	Apple	**Herfindahl**
0.1183	0.0663	0.0599	0.0414	0.0055	**0.0907**

Table 8: weight distribution for a minimum variance portfolio that is hold for ten years (2000 - 2009); the last column contains the Herfindahl Index for the respective weight distribution; therefore, the state-dependent allocation strategy switches between effective numbers of 39 assets ($\frac{1}{N}$-rule) and 11 assets (min-var portfolio)

As the table above shows, the main contributors to the minimum variance portfolio are “American Electric Power” ($\sim$ 20.1 percent) and “Duke Energy” ($\sim$ 11.9 percent). A Herfindahl Index of 0.0907 corresponds to an effective number of 11 assets. A turnover of 0.8545 is created whenever our strategy switches between the $\frac{1}{N}$rule and the minimum variance portfolio.

Third Data Set: 1974 - 2011

For the third data set the parameters are estimated over a period of 26 years (01/01/1974 - 31/12/1999) in order to check whether a longer estimation window has positive effects on the portfolio performance. Later on in the Analysis section the performance of this portfolio is matched against the $\frac{1}{N}$-strategy for another twelve years (01/01/2000 - 31/12/2011). The length of the data window is chosen according to the maximum period for which the bubble indicator is available. The minimum variance portfolio weights associated with the period 2000 - 2011 set are presented in the following table:

Footlocker	Bank of New York Mellon	Merck	Eli-Lilly	Sparton	Alcoa	John Deere
0.0018	0.0060	0.0084	0.0059	0.0187	0.0108	0.0059
Eaton	Ducommun	Procter and Gamble	Sysco	Exxon-Mobile	Chevron	American Electric Power
0.0215	0.0244	0.0046	0.0052	0.0375	0.0205	0.0742
Consolidated Edison	DTE Energy	Entergy	PC&E	Center Point Energy	Allete	Empire District Energy
0.0072	0.0817	0.0249	0.0775	0.0362	0.1183	0.2071
SJW	IBM	3M	United Technologies	**Herfindahl**		
0.1596	0.0189	0.0174	0.0058	**0.1068**		

Table 9: weight distribution for a minimum variance portfolio that is hold for twelve years (2000 - 2011); the last column contains the Herfindahl Index for the respective weight distribution; therefore, the state-dependent allocation strategy switches between effective numbers of 50 assets ($\frac{1}{N}$-rule) and 9 - 10 assets (min-var portfolio)

In this case the largest holdings come with three companies: “Empire District Energy” ($\sim$ 20.7 percent), “SJW Corporation” ($\sim$ 16.0 percent) and “Allete” ($\sim$ 11.8 percent). Similar to the second data set the Herfindahl Index implies an effective number of approximately 10 assets. A single re-allocation of our assets generates a turnover of 1.268. In the Analysis section we will compare the performance of these three portfolios to the respective $\frac{1}{N}$-benchmark. But first, we will present the results regarding the Kelly portfolio for all three data sets.

5.4 Kelly Portfolio

Applying the Kelly Cirterion to all three data sets leads to the following weight distributions:

Consumer Discretionary	Financials	Health Care	Industrials	Energy	Utilities	**Herfindahl**
0.0016	0.0151	0.0066	0.0235	0.0188	0.0075	**0.2320**

Table 10: weight distribution for a Kelly portfolio that is hold for five years (2005 - 2009); the last column contains the Herfindahl Index for the respective weight distribution; the percentage hold as cash is not included in the Herfindahl index

The Kelly portfolio generated from the first data contains 7.3 percent stocks and 92.7 percent cash, which results in a Herfindahl Index of 0.2320. The implementation of the risk-free rate by investing into a 5-years treasury bill on the January 1st 2005. The observation that the Kelly portfolios invests a relatively small amount in stocks remains prevalent in the second data set as well:

McDonalds	Ford	Nike	AON	Wells-Fargo
0.0007	0.0011	0.0030	0.0056	0.0063
American-Intl-Corp	Johnson&Johnson	Pfizer	General Electrics	Southwest Airlines
0.0066	0.0027	0.0119	0.0211	0.0051
E-I-du-Pont-de-Nemours	Dow-Chemical	Procter-and-Gamble	Wal-Mart	Colgate
0.0019	0.0008	0.0037	0.0080	0.0162
Exxon-Mobile	Chevron	Schlumberger	Duke-Energy	Southern-Company
0.0187	0.0046	0.0002	0.0119	0.0115
Exelon	IBM	Texas Instruments	Intel	**Herfindahl**
0.0009	0.0012	0.0045	0.0136	**0.0740**

Table 11: weight distribution for a Kelly portfolio that is hold for ten years (2000 - 2009); the last column contains the Herfindahl Index for the respective weight distribution; the percentage hold as cash is not included in the Herfindahl index

Again, the Kelly portfolio comprises a very small-sized stock portion of approximately only 16.2 percent, while the rest ($\sim$ 83.8 percent) is invested in the risk-free asset leading to a Herfindahl Index of 0.0740. The risk-free rate is achieved using a 10-years treasury bills. At last, the Kelly portfolio weights associated with the third data set are presentend below:

McDonalds	Ford	Altria-Group	Wells-Fargo	Bank-of-New-York-Mellon	Pfizer	Merck
0.0008	0.0076	0.0044	0.0031	0.0035	0.0022	0.0054
Bristol-Myers-Squibb	General-Electrics	Boeing	Alcoa	Eaton	Procter-and-Gamble	Wal-Mart
0.0090	0.0074	0.0105	0.0069	0.0081	0.0057	0.0200
Pepsico	Sysco	Exxon-Mobile	Chevron	Consilidated-Edison	DTE-Energy	PC&E
0.0011	0.0074	0.0238	0.0014	0.0104	0.0103	0.0027
Center-Point-Energy	Allete	Empire-District-Electric	SJW-Corp.	United-Technologies	HP-Inc	**Herfindahl**
0.0032	0.0047	0.0122	0.0165	0.0125	0.0009	**0.0584**

Table 12: weight distribution for a Kelly portfolio that is hold for twelve years (2000 - 2011); the last column contains the Herfindahl Index for the respective weight distribution; the percentage hold as cash is not included in the Herfindahl index

Still, the percentage allocated to stock ($\sim$ 20.2 percent) appears comparatively small, while the lion's share ($\sim$ 79.8 percent) is kept as cash. The corresponding Herfindahl Index is 0.0584. In the last part of the Analysis section we will match the performance of the three before-created portfolios against the $\frac{1}{N}$-benchmark.

6. Analysis

The following section is devoted to comparing the performances of our three optimizing concepts, namely the Generalized Herfindahl Index, the state-dependent allocation and the Kelly Criterion, with the $\frac{1}{N}$-benchmark. Therefore, we will utilize basic statistics such as mean and standard deviation, performance measures such as Sharpe Ratio and Sortino Ratio, drawdown metrics such as average recovery time and Pain Index as well as rather rarely used Up / Down Ratios such as Capture and Percentage Ratio. As ultimate performance comparison we will finally consult cumulative returns.

6.1 GHHI Diversification versus 1/N-Strategy

First Data Set: 2000 - 2009

In the following we used the GHHI diversification approach to generate two distinct portfolios and compare them to the $\frac{1}{N}$-strategy: The first portfolio is generated by optimizing with respect to the Generalized Herfindahl Index based on risk and return estimates obtained from a 10-years observation window (01/01/1990 to 31/12/1999) and then holding the respective portfolio for another ten years (01/01/2000 to 31/12/2009). On the contrary, the second portfolio does not adopt a "buy-and-hold" strategy, but rebalances the weight distribution every year based on the performance of the last ten years. Here the size of the rolling estimation window was chosen to be as large as possible whilst staying constant over the entire process in order to preserve the amount of information used for parameter estimation. The target risk and return values were set such that they correspond to the covariance risk and average return values of $\frac{1}{N}$-strategy for each respective year. The table below displays some key statistics regarding the respective portfolios. Note again that the weakest value per each metric is colored red, while the strongest branded green. Also be aware of the fact that a 10-years US treasury bill has been used to mimic the risk-free rate of 6.58 percent:

Company	$\frac{1}{N}$-strategy	GHHI "buy-and-hold"	GHHI "rolling window"
Median	**0.0512**	**0.0568**	0.0625
Mean	0.0095	**0.0083**	**0.0125**
St. Dev.	**1.3600**	**1.3440**	1.3564
Skewness	**-0.2073**	**-0.0315**	-0.0547
Kurtosis	**8.4689**	**7.7212**	7.9830
Pain Index	**14.2913**	**15.5579**	14.6864
Ulcer Index	20.4405	**20.8043**	**19.5927**
Sharpe Ratio	-0.0414	**-0.0427**	**-0.0399**
Adjusted SR	-0.0002	**-0.0112**	**0.0262**
Omega SR	-0.1175	**-0.1201**	**-0.1126**
Sortino Ratio	-0.0556	**-0.0579**	**-0.0541**
Upside Potential Ratio	**0.4177**	0.4242	**0.4261**

Table 13: key statistics for two distinct GHHI portfolios compared to the naive $\frac{1}{N}$-rule; the second column is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of each ten years (1990 - 1999 and 2000 - 2009, respectively), whereas the porftolio from the third column rebalances the weight distribution every year using a rolling window approach; as expected, the rolling window approach outperforms the "buy-and-hold" strategy.

First of all, we have to realize that the Sharpe Ratio as well as the remaining Sharpe Ratio-like measures are negative if we include the risk-free asset. One possible reason for this problem might be the fact that we limited the choice of securities to the S&P500 resulting in an "imperfect" $\frac{1}{N}$-portfolio. The phenomenon of negative Sharpe Ratios will continue throughout the other data sets. A potential remedy for this issue will be given in the conclusion at the end of this paper.

Anyways, we will continue to compare our optimization strategies with the $\frac{1}{N}$-benchmark. Obviously, the "rolling window" approach yields more favorable values than the "buy-and-hold" strategy regarding every statistic apart from standard deviation, skewness and kurtosis. Apart from Pain Index, the "rolling window" approach offers better performance than the $\frac{1}{N}$-strategy, but the differences are actually not significant except for the Adjusted Sharpe Ratio.

Up/Down Ratios open up another way to relative performance assessement. For instance, the Up (Down) Capture Ratio measures how well the investment performs during up (down) markets. The Up (Down) Capture Ratio is computed by dividing the returns of the optimizing portfolios by the returns of the $\frac{1}{N}$-strategy when the benchmark crosses an up (down) market. An Up (Down) Capture Ratio of 0.9 means that the investment gains (loses) 90 percent of its value, when the value of corresponding benchmark asset / portfolio increases (decreases) by 100 percent, such that a larger (smaller) value indicates superior performance (Bacon, 2008). The Up (Down) Number Ratio works similar, but takes the number of periods when the optimizing portfolio experiences an up (down) swing and divides it by the number of periods in which the benchmark is going through an up (down) market (Bacon, 2008).

The Up (Down) Percentage Ratio puts the number of days when the GHHI portfolio outperforms the $\frac{1}{N}$-strategy at times when the $\frac{1}{N}$-strategy was up (down) into the nominator, while taking the number of periods when the $\frac{1}{N}$-strategy was up (down) as denominator (Bacon, 2008). The following table demonstrates that the Up/Down Capture Ratios of both GHHI approaches are close to 1 which implies that they behave approximately like the $\frac{1}{N}$-strategy.

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
"buy-and-hold"	0.9575	0.9592	0.8989521	0.8989813	0.4438623	0.5551783
"rolling window"	0.9829	0.9809	0.9146707	0.9117148	0.4528443	0.5390492

Table 14: Up/Down Ratios for two distinct GHHI portfolios; the first row is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of each ten years (1990 - 1999 and 2000 - 2009, respectively), whereas the porftolio from the second row rebalances the weight distribution every year using a rolling window approach; again, the "buy-and-hold" strategy provides inferior results compared to the rolling window approach.

Plotting the Up Capture Ratio versus the Down Capture Ratio illustrates whether an investment outperforms the benchmark strategy. In the plot below the $\frac{1}{N}$-strategy is located in the center where Up and Down Capture Ratios are both equal to one. The Capture Ratios of both the buy-and-hold strategy and the rolling window approach are marked by red dots:

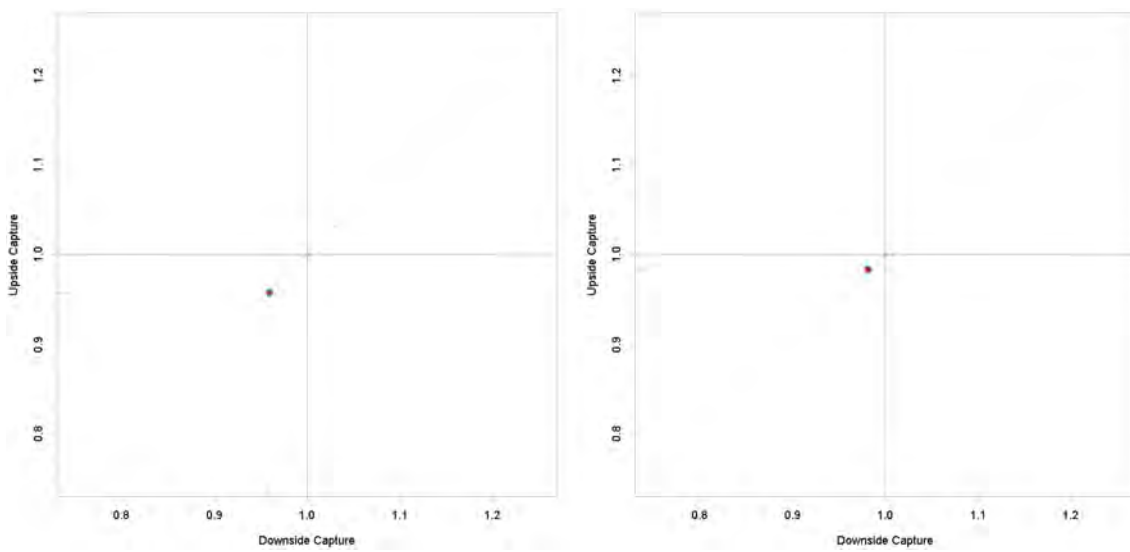


Figure 15: diagram showing the ratio of Up Capture Ratio (y-axis) and Down Capture Ratio (x-axis) for two distinct GHHI portfolios compared to the naive $\frac{1}{N}$-rule; the Up (Down) Capture Ratio is computed by dividing the returns of the optimizing portfolios by the returns of the $\frac{1}{N}$-rule when the benchmark experiences an up (down) market; the left panel is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of each ten years (1990 - 1999 and 2000 - 2009, respectively), whereas the porftolio in the right panel rebalances the weight distribution every year using a rolling window approach; obviously, the rolling window portfolio is slightly superior to the "buy-and-hold" strategy, but both optimizing portfolios fail to consistently outperform the $\frac{1}{N}$-rule.

As both red dots lie approximately on the gray linear line intersecting the origin and the center of the plot, it becomes obvious that none of the approaches performs significantly better than the $\frac{1}{N}$-strategy.

In fact, the "buy-and-hold" strategy is slightly inferior to the $\frac{1}{N}$-strategy, while the opposite applies to the "rolling window" approach. Another possibility of performance evaluation is provided by the Capital Asset Pricing Model. The CAPM is a linear factor equilibrium model that enquires which part the total risk of a portfolio is diversifiable, which yields further possibilites to value risky investments. The alpha intersect quantifies the amount of return that is not related to the benchmark return, while the beta slope embodies the returns that are directly induced by the benchmark performance. Thus, it appears that the alpha intersect effectively reveals to what extent the portfolio has the ability to generate superior returns compared to the benchmark investment. As an extension to the CAPM Fama and French proposed a three-factor model that can be used to assess the loss of diversification. Fama beta is defined as $\beta_F = \frac{\sigma_p}{\sigma_m}$, where σ_p and σ_m are the portfolio's standard deviation and the market risk, respectively. In addition to the before-mentioned CAPM measures the following table comprises the specific, the systematic and the total risk values for both GHHI strategies as well:

Strategy	CAPM Alpha	CAPM Beta	Fama Beta	Specific Risk	Systematic Risk	Total Risk
GHHI "buy-and-hold"	-0.0008	0.9634	0.9882	4.7462	20.8004	21.3350
GHHI "rolling window"	0.0033	0.9725	0.9973	4.7789	20.9950	21.5320

Table 15: CAPM measures and risk values for two distinct GHHI portfolios; the first row is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of each ten years (1990 - 1999 and 2000 - 2009, respectively), whereas the porftolio from the second row rebalances the weight distribution every year using a rolling window approach; as hoped for, the rolling window approach dominates the "buy-and-hold" strategy in terms of the CAPM Alpha intersect

The CAPM measures confirm the conjecture that both GHHI approaches actually behave very similar to the $\frac{1}{N}$-strategy, because the alpha intersect for both optimizing portfolios is close to zero and their beta slope is almost one. Plotting the GHHI portfolio's return versus the return of the $\frac{1}{N}$-strategy results in an individual scatterplot for each optimization approach. Laying a LOWESS (locally weighted scatterplot smoothing) line fit over the return scatterplot points out the relation between the optimizing portfolio and the $\frac{1}{N}$-strategy:

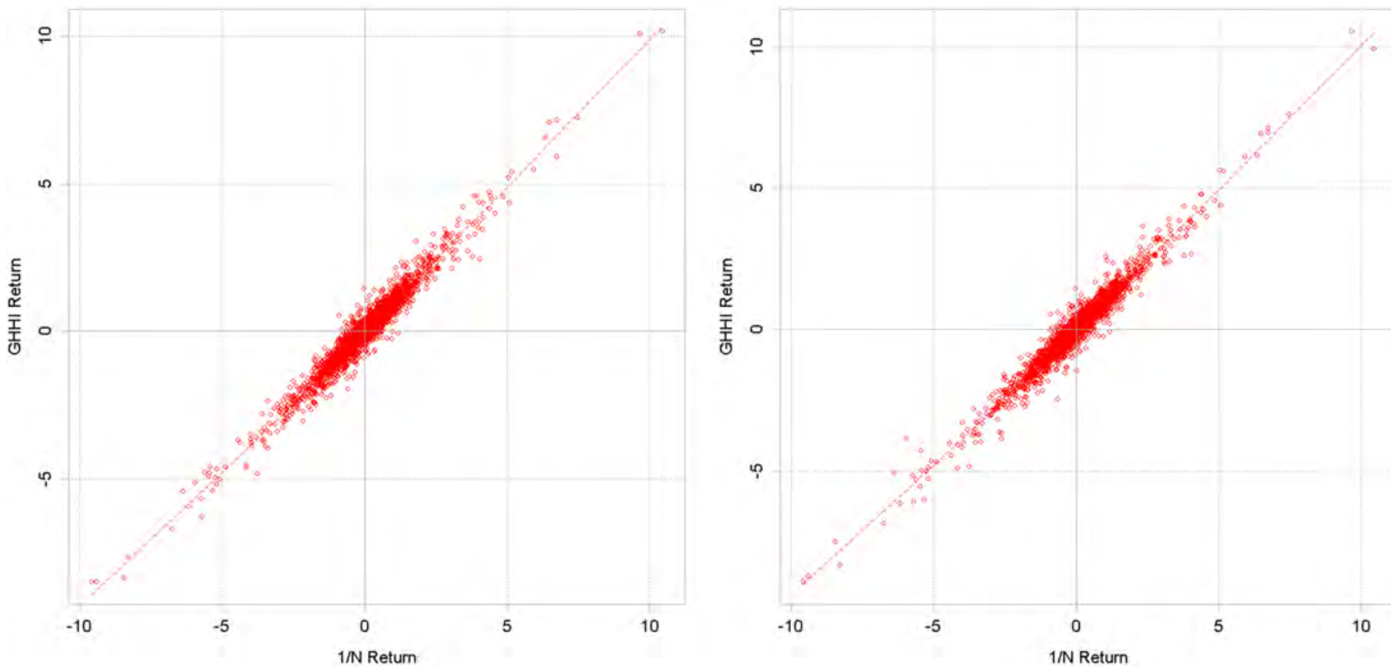


Figure 16: diagram showing a return scatter for two GHHI distinct portfolios (y-axis) compared to the naive $\frac{1}{N}$-rule (x-axis); the left panel is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of each ten years (1990 - 1999 and 2000 - 2009, respectively), whereas the porftolio in the right panel rebalances the weight distribution every year using a rolling window approach; overlaying a LOWESS (locally weighted scatterplot smoothing) line illustrates the relation between the separate optimizing portfolios and the $\frac{1}{N}$-benchmark; evidently, both optimizing portfolios behave very much like the benchmark, as the scatter points almost lie on a straight line.

Obviously, both plots demonstrate a strong correlation between both optimizing portfolios and the $\frac{1}{N}$-strategy, since the correlation coefficients for both linear fits are 0.9505 and 0.9507, respectively. Therefore, we can negate the null hypothesis, that both strategies are the same, yet, they are very much alike. Finally, we computed the Modigliani-Modigliani measure (also called Modigliani risk-adjusted performance), which shifts the portfolio's standard deviation so that it fits the benchmark's standard deviation, which we already partly achieved by imposing the target risk and return values from the $\frac{1}{N}$-strategy. The M^2-measure is defined as follows:

$$MM_p = \frac{E\left[R_p - R_f\right]}{\sigma_p} = SR \cdot \sigma_b + E\left[R_f\right] \tag{58}$$

where:

σ_b: standard deviation of the benchmark

The M^2-measure is 0.84 and 1.26 percent for both portfolios, respectively. In contrast to the Sharpe Ratio, which is an abstract dimensionless measure, the M^2-measure takes units of percentage return, which is much more comprehensible. The M^2-measure enables us to realize that the rolling window approach is actually a better investment opportunity, because it yields a risk-adjusted return that is larger by more than 0.4 percent. The cumulative returns of all three strategies, namely $\frac{1}{N}$-benchmark, "buy-and-hold" strategy and rolling window approach are revealed below. Note that the return of the $\frac{1}{N}$-benchmark is associated with black color, while the "buy-and-hold" strategy and the rolling window approach are colored green and red, respectively. Also be aware that the y-axis shows the development over time of one money unit invested in the beginning:

Figure 17: cumulative return for two distinct GHHI portfolios compared to the $\frac{1}{N}$-benchmark over a period of ten years (2000-2009); black: $\frac{1}{N}$-benchmark, green: "buy-and-hold" strategy, red: rolling window approach; the rolling window approach often provides a higher cumulative return than the benchmark and only sometimes underperforms it, while the "buy-and-hold" strategy almost always yields smaller returns than the naive $\frac{1}{N}$-rule

As expected, the rolling window approach mostly outperforms the $\frac{1}{N}$-benchmark, while the "buy-and-hold" strategy leads to consistently lower returns, which brings us to the conclusion that the GHHI optimization is able to perform better than the naive $\frac{1}{N}$-rule, but only when implemented in combination with frequently rebalancing the portfolio weights.

Third Data Set: 1974 - 2011

In the following we deployed the third data set to estimate the distribution parameters over 26 years (01/01/1974 - 31/12/1999) before comparing the out-of sample performance of the resulting "buy-and-hold" strategy to the $\frac{1}{N}$-strategy for the next twelve years (01/01/2000 - 31/12/2011). Some key statistics for both portfolios are summarized below. Note again that the weaker of both values is colored red, while the stronger performance value is colored green:

Portfolio	$\frac{1}{N}$-strategy	GHHI	Portfolio	$\frac{1}{N}$-strategy	GHHI
Median	**0.0718**	**0.0559**	Ulcer Index	**19.5877**	**17.3071**
Mean	**0.0161**	**0.0148**	Sharpe Ratio	**-0.0468**	**-0.0492**
St. Dev.	**1.3093**	**1.2739**	Adjusted SR	**0.0880**	**0.0791**
Skewness	**-0.2539**	**-0.1721**	Omega SR	**-0.1306**	**-0.1377**
Kurtosis	**8.1322**	**8.3730**	Sortino Ratio	**-0.0622**	**-0.0659**
Pain Index	**13.9745**	**12.3449**	Upside Potential Ratio	**0.4128**	**0.4138**

Table 16: key statistics for a GHHI portfolio compared to the naive $\frac{1}{N}$-rule; the second column is associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of 26 and twelve years (1974 - 1999 and 2000 - 2011), respectively; expectedly, the "buy-and-hold" strategy does not outperform the $\frac{1}{N}$-strategy

The performances of both portfolios match each other in most categories except for median, Pain Index and Ulcer Index. Apparently, the $\frac{1}{N}$-strategy exhibits a larger mean, while the two drawdown metrics speak in favor of the GHHI portfolio. Regarding the remaining performance measures only minor differences are observed among both strategies. A closer look at the Up/Down Ratios provides more information on whether there is a significant performance gap between both portfolios:

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
GHHI	0.9231	0.9235	0.8953	0.9039	0.4009	0.6105

Table 17: Up/Down Ratios for a GHHI portfolio associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of 26 and twelve years (1974 - 1999 and 2000 - 2011), respectively; no significant performance difference can be observed between the GHHI portfolio and the $\frac{1}{N}$-strategy

The Percentage Ratios show that the GHHI portfolio tends to underperform the $\frac{1}{N}$-portfolio, when the market experiences an up-swing, but outperforms it when the market is falling. Since the Up Capture Ratio as well as Up Number Ratio are almost the same as their Down counterparts, the GHHI portfolio does not outperform the $\frac{1}{N}$-rule. In fact, the "buy and hold" strategy even performs slightly worse, which is a observation that was already suggested by the results obtained from the second data set above. A similar picture is drawn by the following the CAPM measures:

CAPM Measure	CAPM Alpha	CAPM Beta	Fama Beta
GHHI	-0.0005	0.9427	0.9730

Table 18: CAPM measures and risk values for a GHHI portfolio associated with a "buy-and-hold" portfolio with an estimation period and an out-of-sample period of 26 and twelve years (1974 - 1999 and 2000 - 2011), respectively; all three measures suggest that the GHHI portfolio performs very similar to the $\frac{1}{N}$-rule; the CAPM Alpha intersect illustrates that investing into a GHHI portfolio and holding it for the entire period is not a good idea.

As anticipated the CAPM measures provide no evidence for a significant performance deviation. Furthermore the CAPM Alpha intersect demonstrates that an investor deploying a "buy-and-hold" GHHI approach exhibits weaker investment skills than investors that put their money into the benchmark portfolio. Finally, we present the cumulative return of the "buy-and-hold" strategy compared to the naive $\frac{1}{N}$-rule: Note that the line associated with the $\frac{1}{N}$-benchmark is colored black, whereas the cumulative returns of the GHHI "buy-and-hold" strategy are shown in red. Again, the y-axis indicates the development of one money unit:

Figure 18: cumulative return for a GHHI portfolio compared to the $\frac{1}{N}$-benchmark over a period of twelve years (2000-2011); black: $\frac{1}{N}$-benchmark, red: "buy-and-hold" strategy; the GHHI "buy-and-hold" strategy permanently fails to generate superior returns

It does not come as a surprise that the GHHI "buy-and-hold" strategy is not a suitable approach, because its cumulative return is consistently lower than the return of the benchmark.

6.2 State-Dependent Allocation versus 1/N-Strategy

First Data Set: 2000 - 2009

In the following we will examine different characteristics of the state-dependent allocation strategy: First of all, the effect of the rebalancing frequency is evaluated by contrasting two rebalancing portfolios (every six and twelve months) and the "buy-and-hold" strategy. In order to assure statistical pertinence we deploy the particular form of the state-dependent allocation strategy that makes use of random numbers during the decision making process such that the strategy is switched once the indicator value is larger than a daily generated random number. Taking the average of the key metrics for ten different sets of random numbers leads to the table below. Note again that the weakest value for each metric is colored red, whereas the strongest one is branded green:

Rebalancing Frequency	6 months	12 months	“buy-and-hold”
Median	**0.0098**	0.1028	**0.1032**
Mean	**0.0111**	**0.0111**	**0.0108**
St. Dev.	1.4259	**1.4246**	**1.4319**
Skewness	**-0.3113**	-0.3116	**-0.3137**
Kurtosis	**10.9796**	10.9624	**10.6952**
Pain Index	**12.7206**	12.7059	**12.6574**
Ulcer Index	**21.0136**	20.9911	**20.9272**

Table 19: key statistics for state-dependent allocation portfolios using different rebalancing frequencies (6 months, 12 months, never) over a period of ten years (2000-2009); note that the strongest (weakest) value per each meatric is colored green (red); surprisingly, none of the portfolio strategy dominates the others.

Considering the fact that frequently rebalancing the weights does not yield significantly superior returns compared to the “buy-and-hold” strategy, it appears reasonable to stay with the less computationally intensive method. In fact, the “buy-and-hold” strategy even provides stronger performance regarding some important measures. This might be due to some potential regime changing. Accordingly, throughout the rest of the paper we will rely on the “buy-and-hold” strategy only.

Now, the impact of the threshold size on the portfolio performance is evaluated by utilizing different indicator thresholds between 0.0 and 0.8. Again the key performance metrics are presented in a table. Note again that the weakest value for each performance measure is colored red, whereas the strongest performance per each metric is associated with green color.

Threshold	0.0	0.2	0.4	0.6	0.8
Median	0.0984	0.1022	**0.1064**	**0.0944**	0.0956
Mean	**0.0153**	0.0127	0.0129	0.0096	0.0074
St. Dev.	**1.4277**	1.4280	1.4295	1.4318	1.4340
Skewness	**-0.3187**	-0.3151	-0.3164	-0.3108	**-0.3093**
Kurtosis	**10.8458**	10.8343	10.7816	10.6960	**10.6178**
Pain Index	**11.6561**	12.0956	12.4830	12.7395	**13.4131**
Ulcer Index	**19.8943**	20.3284	20.8030	21.0110	**21.7725**
Sharpe Ratio	**-0.0140**	-0.0159	-0.0157	-0.0180	**-0.0195**
Adjusted SR	**0.0555**	0.0254	0.0281	-0.0086	**-0.0341**
Omega SR	**-0.0444**	-0.0502	-0.0496	-0.0564	**-0.0611**
Sortino Ratio	**-0.0189**	-0.0214	-0.0211	-0.0241	**-0.0262**
Upside Potential Ratio	**0.4067**	0.4052	0.4053	0.4041	**0.4028**

Table 20: key statistics for state-dependent allocation portfolios using five different bubble indicator thresholds (0.0, 0.2, 0.4, 0.6, 0.8) over a period of five years (2005-2009); note that the weakest value for each performance statistic is colored

red, while the strongest is colored green; it comes into notice that a smaller threshold leads to superior performance in almost every instance except for skewness and kurtosis (where a bigger threshold is desirable).

It becomes obvious that the portfolio obtained by selecting a zero threshold dominates the other investments regarding almost every performance measure. Surprisingly, for most metrics the performance seems to decline with increasing threshold. Apparently, the opposite applies to skewness and kurtosis, while no such assumption can be made with respect to the median. Besides, the zero threshold portfolio also outperforms the corresponding random number portfolio in terms of almost every performance parameter except for mean, skewness and kurtosis. For the purpose of verifying afore-mentioned suppositions we check the $\frac{1}{N}$-strategy as a benchmark against the two portfolios created by applying thresholds of 0.0 and 0.8. First of all, the Up/Down Ratios are presented below:

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
0.0 threshold	0.9651	0.9481	0.9803	0.9763	0.0818	0.1639
0.8 threshold	0.9659	0.9664	0.9887	0.9872	0.0465	0.1038

Table 21: Up/Down Ratios for two state-dependent allocation portfolios using two different bubble indicator thresholds (0.0, 0.8) over a period of five years (2005-2009); still, a portfolio resulting from a lower threshold dominates high threshold portfolios

Similar to the GHHI portfolios, the Capture and Number Ratios are almost one meaning that these portfolios work very much like the $\frac{1}{N}$-strategy. Plotting the Up/Down Capture Ratios for both optimizing portfolios yields the subsequent figure. Note again that the Capture Ratios corresponding to the $\frac{1}{N}$-strategy are located in the center of the plot, while a red dot indicates the Capture Ratios of the state-dependent allocation strategy. Also, note that the performance of the zero threshold can be found in the left half of the plot.

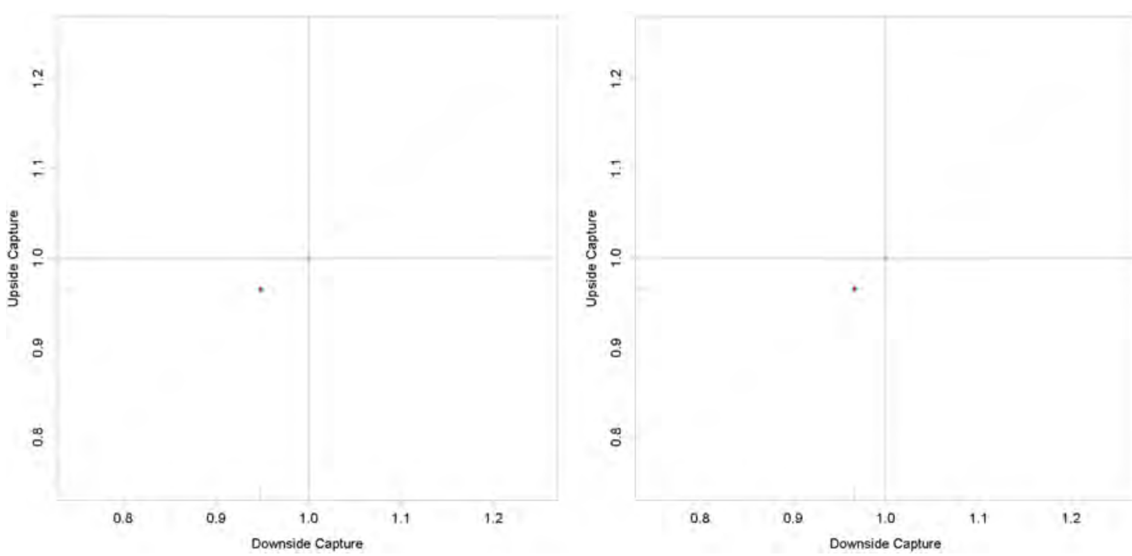


Figure 19: diagram showing the ratio of Up Capture Ratio (y-axis) and Down Capture Ratio (x-axis) for two distinct portfolios compared to the naive $\frac{1}{N}$-rule; the Up (Down) Capture Ratio is computed by dividing the returns of the

optimizing portfolios by the returns of the $\frac{1}{N}$-rule when the benchmark experiences an up (down) market; the left panel is associated with a portfolio that switches from the $\frac{1}{N}$-rule to a minimum variance strategy whenever the crises indicator gives values above zero, whereas the porftolio in the right panel only replaces the $\frac{1}{N}$-rule when the crisis indicator rises above 0.8; both portfolios deploy a minimum variance strategy that assigns weights according to an estimation period of five years (2000 - 2004) and then holds the respective weight distribution for another five years (2005-2009); apparently, choosing a smaller threshold yields Capture Ratios above the ordinary.

Obviously, the zero threshold portfolio manages to outperform the $\frac{1}{N}$-strategy, since the red dot lies above the central linear slope. On the contrary, neglecting indicator values less than 0.8 seemingly does not improve the performance compared to the $\frac{1}{N}$-strategy, as the red dot belonging to the 0.8 threshold portfolio lies almost exactly on the gray line. This makes sense since a large threshold means that the investor only rarely enters the market and that there is no room for significant improvement over the $\frac{1}{N}$-strategy provided that the bubble indicator sufficiently predicts crises and is thereby able to generate superior returns. The plot below further depicts the relative performance of the different strategies highlighting under- and outperformance. The performance of the $\frac{1}{N}$-strategy is represented by the zero line, such that under-/outperformance is displayed by deviations from this line. Realizing that the slope of the deviating line offers more valuable clues to the level of under-/outperformance than the actual value of the deviation is acutely important, since a positive slope is associated with outperformance and vice versa. Note that the red line corresponds to the portfolio deploying a zero threshold, while the green line is associated with the 0.8 threshold.

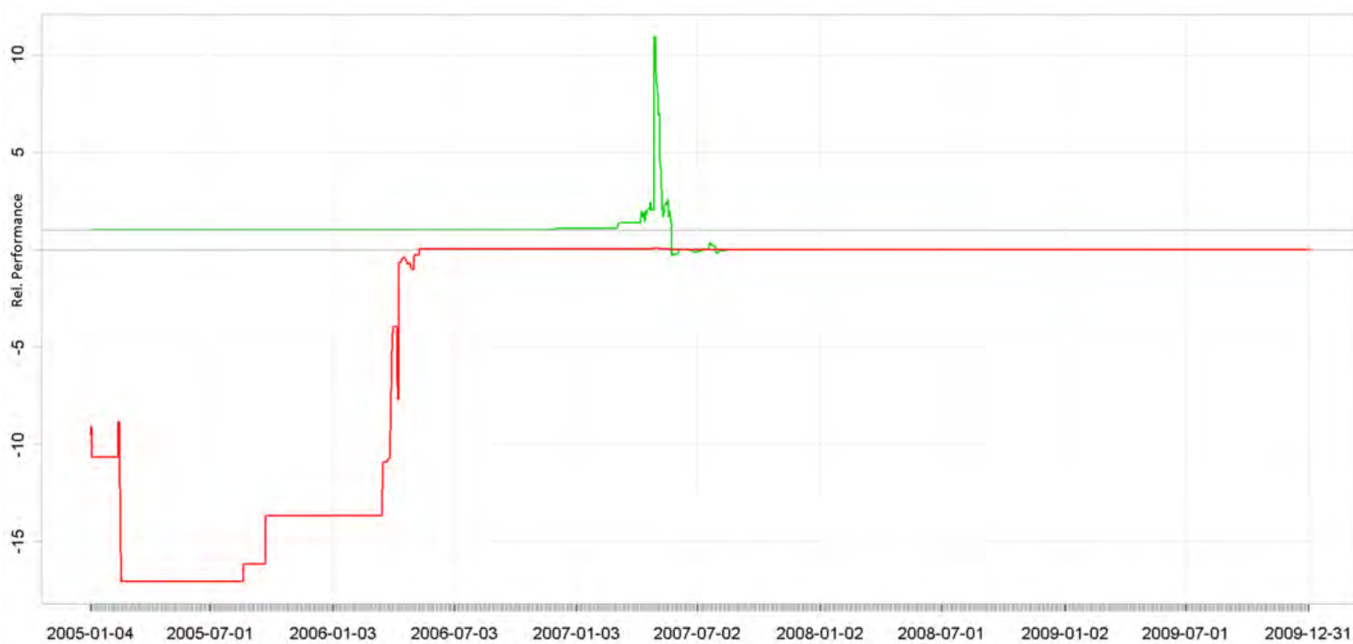


Figure 20: diagram showing the ratio of the cumulative performance (y-axis) of two distinct portfolios compared to the naive $\frac{1}{N}$-rule over time (x-axis); the red line is associated with a portfolio that switches from the $\frac{1}{N}$-rule to a minimum variance strategy whenever the crises indicator gives values above zero, whereas the green line is linked to a portfolio that only replaces the $\frac{1}{N}$-rule when the crisis indicator rises above 0.8; both portfolios deploy a minimum variance strategy that assigns weights according to an estimation period of five years (2000 - 2004) and then holds the respective weight

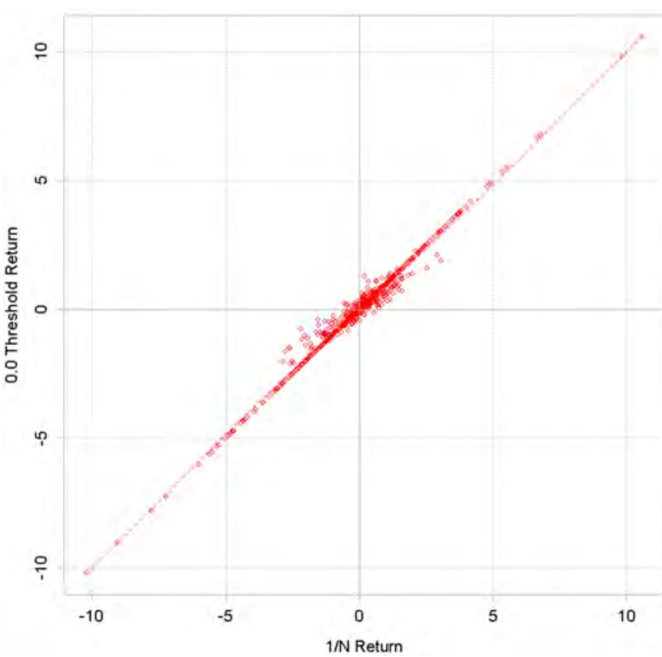
distribution for another five years (2005-2009); the benchmark performance is represented by the zero line; instead of examining the absolute size of the deviation from the benchmark performance, one has to look at the slope of the deviating graph, since a positive (negative) slope is linked with out-/(under-)performance; as the the relative performance line of the zero threshold portfolio starts in the negative and rises onto the benchmark / zero line after some time, it mainly undergoes periods of outperformance demonstrated by a positive slope.

This plot confirms the above-established conjecture that the zero threshold portfolio dominates the $\frac{1}{N}$-strategy, because it exhibits a relatively long period of nearly constant outperformance (positive slope) from the beginning of 2005 until late 2006. In opposition to this, the green line only suggests a brief period of out-/underperformance (i.e. a sharp peak) in early 2007. Again, a summary of the important CAPM measures and risk values is given below:

Strategy	CAPM Alpha	CAPM Beta	Fama Beta	Specific Risk	Systematic Risk	Total Risk
0.0 Threshold	0.0077	0.9716	0.9794	2.8486	22.4847	22.6644
0.8 Threshold	-0.0003	0.9787	0.9837	2.2988	22.6480	22.7644

Table 22: CAPM measures and risk values for two state-dependent allocation portfolios using two different bubble indicator thresholds (0.0, 0.8) over a period of five years (2005-2009); once again, a lower threshold gives far better returns as proven by the CAPM Alpha intersect value.

Judging from the alpha intersect value, it appears that somebody who invests in a portfolio that changes the allocation strategy, whenever the previously introduced crisis indicator is larger than zero, demonstrates higher investment skills, since the portion of return of their portfolio that is not attrituable to the benchmark is comparably large. Finally, the return scatter with respect to the $\frac{1}{N}$-strategy is displayed for both thresholds:



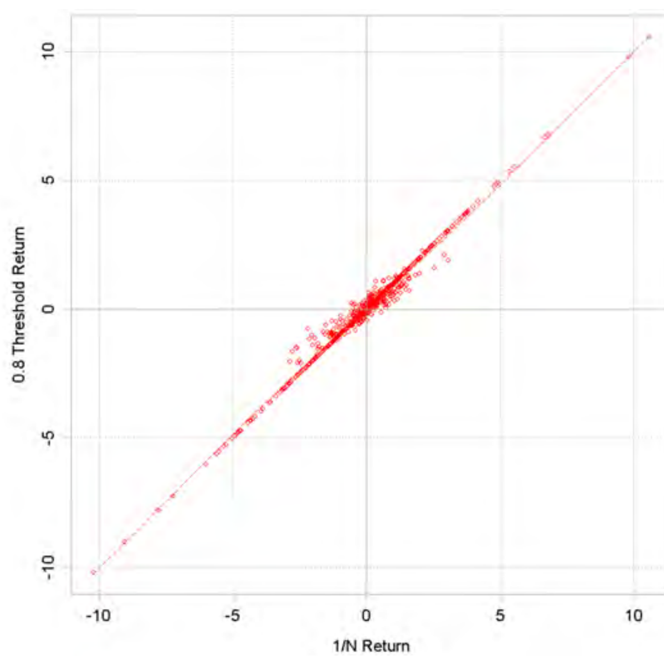


Figure 21: diagram showing a return scatter for two distinct portfolios (y-axis) compared to the naive $\frac{1}{N}$-rule (x-axis); the left panel is associated with a portfolio that switches from the $\frac{1}{N}$-rule to a minimum variance strategy whenever the crises indicator gives values above zero, whereas the porftolio in the right panel only replaces the $\frac{1}{N}$-rule when the crisis indicator rises above 0.8; both portfolios deploy a minimum variance strategy that assigns weights according to an estimation period of five years (2000 - 2004) and then holds the respective weight distribution for another five years (2005-2009); overlaying a LOWESS (locally weighted scatterplot smoothing) line illustrates the relation between the separate optimizing portfolios and the $\frac{1}{N}$-benchmark; again, both optimizing portfolios behave very much like the benchmark, as the scatter points almost lie on a straight line.

Clearly, both state-dependent portfolios behave very much like the $\frac{1}{N}$-strategy, as the correlation coefficients are 0.9842 and 0.9898, respectively. The accumulation of red dots in the center is mostly due to the switching procedure from the $\frac{1}{N}$-strategy to the minimum-variance portfolio. As before, we can deny the null hypothesis of both strategy being equal. Expectedly, the M^2-measure further documents that the zero threshold portfolio yields superior returns compared to portfolios using higher threshold, since it results in a risk-adjusted return that is 0.8 percent larger than that associated with a threshold of 0.8. As last performance comparison we compute the cumulative returns:

Figure 22: cumulative return for two state-dependent portfolios compared to the $\frac{1}{N}$-benchmark over a period of five years (2005-2009); black: $\frac{1}{N}$-benchmark, red: 0.0 indicator threshold portfolio strategy; green: 0.8 indicator threshold portfolio; the portfolio associated with a zero threshold consistently outperforms the benchmark, while the portfolio applying a threshold of 0.8 mostly falls onto the benchmark return and sometimes underperforms it

As anticipated, the portfolio, that switches from an $\frac{1}{N}$-strategy to a minimum variance portfolio whenenver the bubble indicator rises above zero, consistently provides larger cumulative returns than the naive $\frac{1}{N}$-rule, while the other portfolio deploying an indicator threshold of 0.8 tracks the $\frac{1}{N}$-benchmark very closely, but sometimes underperforms it.

Second Data Set: 1990 - 2009

In the case of the second data set we calculate the distribution parameters based on the asset's performance from 01/01/1990 to 31/12/1999 to compute the minimum variance portfolio weights before evaluating the out-of sample performance of the state-dependent allocation strategy in the subsequent ten years. In this case, instead of comparing rolling window approaches and "buy and hold" strategy, we only examine the effect of various indicator thresholds. In order to verify the conjecture that lower bubble indicator thresholds provide returns superior to portfolios applying higher thresholds we generate five different portfolios and compare them to the naive $\frac{1}{N}$-rule. The respective key statistics are presented in the following table. Note that the average drawdown and recovery length is given in days:

Threshold	$\frac{1}{N}$	0.0	0.2	0.4	0.6	0.8
Median	**0.0512**	0.0502	**0.0500**	**0.0500**	0.0502	**0.0500**
Mean	0.0095	0.0138	**0.0140**	0.0128	0.0103	**0.0083**
St. Dev.	**1.3600**	**1.3712**	1.3700	1.3696	1.3685	1.3685
Skewness	**-0.2073**	-0.2036	-0.2017	-0.2013	-0.1992	**-0.1975**
Kurtosis	**8.4689**	**8.1243**	8.1569	8.1705	8.2022	8.2028
Avg. Drawdown Depth	**9.6200**	**17.5239**	**17.5239**	**17.5239**	**17.5239**	**17.5239**
Avg. Drawdown Length	628	628	628	628	628	628
Avg. Recovery Length	**625**	**624**	**624**	**624**	**624**	**624**
Pain Index	14.2912	13.7188	**13.4458**	13.6839	14.0014	**14.4936**
Ulcer Index	20.4404	19.6638	**19.5368**	19.8214	20.1378	**20.7086**
Sharpe Ratio	-0.0414	-0.0379	**-0.0378**	-0.0387	-0.0406	**-0.0420**
Adjusted SR	-0.0002	0.0475	**0.0507**	0.0360	0.0071	**-0.0159**
Omega SR	-0.1175	-0.1073	**-0.1069**	-0.1096	-0.1147	**-0.1186**
Sortino Ratio	-0.0556	-0.0511	**-0.0509**	-0.0522	-0.0546	**-0.0565**
Upside Potential Ratio	**0.4177**	0.4255	**0.4256**	0.4239	0.4215	0.4200

Table 23: key statistics for state-dependent allocation portfolios using different bubble indicator thresholds (0.0, 0.2, 0.4, 0.6, 0.8) compared to the $\frac{1}{N}$-strategy over a period of ten years (2000-2009); note that the weakest value for each performance statistic is colored red, while the strongest is colored green; obviously, smaller thresholds (0.0, 0.2) outperform the $\frac{1}{N}$-strategy larger thresholds regarding almost every performance metric;

As opposed to the first data set, now selecting a threshold of 0.2 yields the best performance instead of 0.0. Nonetheless, both thresholds (0.0 and 0.2) lead to very similar performance with only slight disadvantages related with choosing a threshold of 0.0. Still, relying on higher thresholds leads to considerably weaker performances, which brings us to the conclusion that the minimum variance strategy works remarkably well for minor crises, but not so well for major crises associated with a high bubble indicator values. Aiming at further shedding light on the effect of thresholds sizes we once again calculate a plethora of Up / Down Ratios:

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
0.0 threshold	0.9996	0.9901	0.9641	0.9686	0.0973	0.0934
0.2 threshold	0.9994	0.9893	0.9648	0.9703	0.0891	0.0857
0.4 threshold	0.9987	0.9915	0.9701	0.9754	0.0793	0.0722
0.6 threshold	0.9965	0.9947	0.9746	0.9796	0.0689	0.0552
0.8 threshold	0.9944	0.9970	0.9760	0.9805	0.0614	0.0458

Table 24: Up/Down Ratios for two state-dependent allocation portfolios using five different bubble indicator thresholds (0.0, 0.2, 0.4, 0.6, 0.8) over a period of ten years (2000-2009); still, a portfolio resulting from a lower threshold dominates high threshold portfolios

Comparing Up and Down Capture Ratios we immediately realize that utilizing lower thresholds enables us to outperform the $\frac{1}{N}$-strategy. In fact, only the highest threshold yields a portfolio that performs worse than the benchmark strategy, which is also illustrated in the figure below, where we match the performance of two distinct portfolios (0.2 and 0.8 threshold) against the $\frac{1}{N}$-strategy. Note that in order to outperform the benchmark the red dot needs to be located above the grey benchmark line:

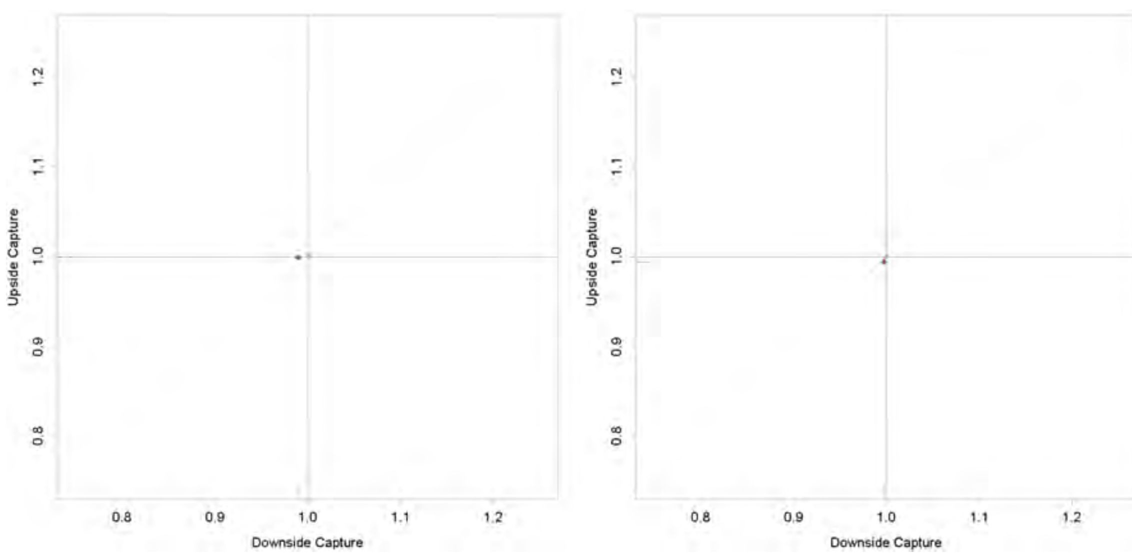


Figure 23: diagram showing the ratio of Up Capture Ratio (y-axis) and Down Capture Ratio (x-axis) for two distinct portfolios compared to the naive $\frac{1}{N}$-rule; the Up (Down) Capture Ratio is computed by dividing the returns of the optimizing portfolios by the returns of the $\frac{1}{N}$-rule when the benchmark experiences an up (down) market; the left panel is associated with a portfolio that switches from the $\frac{1}{N}$-rule to a minimum variance strategy whenever the crises indicator gives values above 0.2, whereas the porftolio in the right panel only replaces the $\frac{1}{N}$-rule when the crisis indicator rises above 0.8; both portfolios deploy a minimum variance strategy that assigns weights according to an estimation period of ten years (1990 - 1999) and then holds the respective weight distribution for another ten years (2000-2009); apparently, a low threshold ensures larger returns than the $\frac{1}{N}$-strategy, whereas portfolios utilizing a very large threshold underperform the $\frac{1}{N}$-strategy

Obviously, the making use of a low threshold results in a significantly better performance, while a portfolio applying a very high threshold even slightly underperforms the $\frac{1}{N}$-strategy. The same claim is brought up by the calculation of the following CAPM and Modigliani measures. Note once again that the Modigliani measure (also: M^2-measure) aims at adapting the covariance risk of the optimizing portfolio to the one of the benchmark and can thus be used to relatively compare portfolios:

Strategy	CAPM Alpha	CAPM Beta	Fama Beta	Modigliani
0.0 Threshold	0.0043	0.9946	1.0082	0.0137
0.2 Threshold	0.0046	0.9949	1.0073	0.0139
0.4 Threshold	0.0033	0.9954	1.0070	0.0127
0.6 Threshold	0.0008	0.9959	1.0062	0.0102
0.8 Threshold	-0.0012	0.9968	1.0063	0.0082

Table 25: CAPM measures and Modigliani measure for five state-dependent allocation portfolios using different bubble indicator thresholds over a period of ten years (2000-2009); the CAPM Alpha intersect demonstrates that investors selecting a lower threshold implies stronger investment skills; the same applies to the Modigliani measure

As expected, portfolios using the two lowest thresholds yield very similar performance, whereas the highest threshold generates deeply inferior returns. Finally, we will have a look at the cumulative returns of the portfolios associated with an indicator threshold of 0.2 and 0.8 and compare them to the $\frac{1}{N}$-strategy. Note that the lower (larger) threshold is associated with red (green) color, whereas the cumulative $\frac{1}{N}$-benchark is colored black:

Figure 24: cumulative return for two state-dependent portfolios compared to the $\frac{1}{N}$-benchmark over a period of ten years (2000-2009); black: $\frac{1}{N}$-benchmark, red: 0.2 indicator threshold portfolio strategy; green: 0.8 indicator threshold portfolio; the portfolio associated with a 0.2 threshold permanently generates larger cumulative returns than the benchmark, while the portfolio generated by using a threshold of 0.8 behaves a lot like the $\frac{1}{N}$-strategy although sometimes underperforming it

Once again, it is demonstrated that switching between $\frac{1}{N}$-rule and minimum variance portfolio at a lower indicator threshold yields superior returns in most cases. The total turnover of the strategy is 96.5585. Nevertheless, so far no clear proof has been found showing a correlation between the length of the estimation window and the performance of the optimizing portfolios, since both data sets (2000 - 2009 and 1990 - 2009, respectively) lead to performances that a very much alike, which is the reason why we consulted a third data set, which leans the maximum period for which the bubble indicator is available.

Third Data Set: 1974 - 2011

In the following we will compare five different state-dependent portfolios using indicator thresholds varying from 0.0 to 0.8 whose parameters are estimated over a period of 26 years (1974 - 1999). It follows a summary of the key performance metrics compared to the $\frac{1}{N}$-strategy based on an evaluation period of twelve years (2000 - 2011):

Threshold	$\frac{1}{N}$	0.0	0.2	0.4	0.6	0.8
Median	**0.0718**	**0.0728**	**0.0728**	**0.0728**	**0.0728**	**0.0728**
Mean	**0.0161**	**0.0173**	**0.0173**	**0.0173**	**0.0173**	**0.0173**
St. Dev.	**1.3092**	**1.3282**	**1.3281**	**1.3282**	**1.3281**	**1.3281**
Skewness	**-0.2539**	**-0.2267**	**-0.2268**	**-0.2266**	**-0.2268**	**-0.2269**
Kurtosis	**8.1322**	7.6407	7.6416	**7.6403**	7.6417	7.6426
Avg. Drawdown Depth	21.4762	21.4762	21.4762	21.4762	21.4762	21.4762
Avg. Drawdown Length	3019	3019	3019	3019	3019	3019
Avg. Recovery Length	3015	3015	3015	3015	3015	3015
Pain Index	**13.9745**	**14.4584**	**14.4584**	**14.4584**	**14.4584**	**14.4583**
Ulcer Index	**19.5877**	**20.1384**	**20.1384**	**20.1385**	**20.1384**	**20.1383**
Sharpe Ratio	-0.0467	-0.0453	**-0.0451**	-0.0453	-0.0460	**-0.0468**
Adjusted SR	0.0880	0.0975	**0.1014**	0.0982	0.0885	**0.0781**
Omega SR	-0.1306	-0.1257	**-0.1252**	-0.1260	-0.1278	**-0.1298**
Sortino Ratio	-0.0622	-0.0604	**-0.0602**	-0.0605	-0.0612	**-0.0623**
Upside Potential Ratio	**0.4138**	0.4202	**0.4203**	0.4197	0.4181	0.4176

Table 26: key statistics for state-dependent allocation portfolios using different bubble indicator thresholds (0.0, 0.2, 0.4, 0.6, 0.8) compared to the $\frac{1}{N}$-strategy over a period of twelve years (2000-2011); note that the weakest value for each performance statistic is colored red, while the strongest is colored green; increasing the indicator threshold mitigates the performance of the portfolio; also, applying a state-dependet strategy yields better performance

As before, a low threshold yields superior returns compared to high thresholds. Also, the $\frac{1}{N}$-strategy is consistently outperformed by the low threshold state-dependent portfolios as proven by the table above.

In order to assure whether the above-established suggestion is true we return to investigating the Up / Down Ratios for all indicator thresholds:

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
0.0 threshold	0.9903	0.9873	0.9602	0.9653	0.0967	0.1055
0.2 threshold	0.9929	0.9893	0.9627	0.9689	0.0936	0.0961
0.4 threshold	0.9946	0.9918	0.9718	0.9747	0.0796	0.0759
0.6 threshold	0.9966	0.9956	0.9786	0.9776	0.0618	0.0585
0.8 threshold	0.9952	0.9964	0.9792	0.9798	0.0551	0.0491

Table 27: Up/Down Ratios for five state-dependent allocation portfolios using two different bubble indicator thresholds (0.0, 0.2, 0.4, 0.6, 0.8) over a period of twelve years (2000-2011); still, lower portfolio thresholds provide superior returns compared to higher portfolio thresholds

As anticipated, the wide choice of Up / Down Ratios affirms the notion that decreasing the threshold size has a positive impact on the performance of the state-dependent portfolios. In addition we presented the following CAPM and Modigliani measures:

Strategy	CAPM Alpha	CAPM Beta	Fama Beta	Modigliani
0.0 Threshold	0.0012	0.9941	1.0144	0.0181
0.2 Threshold	0.0015	0.9946	1.0136	0.0184
0.4 Threshold	0.0012	0.9951	1.0126	0.0180
0.6 Threshold	0.0004	0.9971	1.0125	0.0172
0.8 Threshold	-0.0005	0.9969	1.0102	0.0162

Table 28: CAPM and Modigliani measures for five state-dependent allocation portfolios using different bubble indicator thresholds over a period of twelve years (2000-2011); again, there is evidence that the indicator threshold has a positive impact on the portfolio performance

In order to test the before-established conejcture that switching the allocating strategy improves the performance we will consult the cumulative returns of a portfolio, that alters the composition whenever the indicator assumes any positive value, and compare them to the $\frac{1}{N}$-benchmark:

Figure 25: cumulative return for a state-dependent portfolio compared to the $\frac{1}{N}$-benchmark over a period of twelve years (2000-2011); black: $\frac{1}{N}$-benchmark, red: 0.0 indicator threshold portfolio strategy; the state-dependent allocation strategy outperforms the benchmark

As expected, the state-dependent allocation strategy performs better over the whole evaluation period, which was already suggested by the perviously computed performance measures. The excess return comes with an increased turnover. Since switching the strategy once will generate a turnover of 1.268, we will face a total turnover of 185.125, because over the entire investment period the weight distribution has to be changed 146 times. Incorporating transaction costs will mitigate the return of our state-dependent strategy, which means we are dealing with a tradeoff in which the competing variables are presented by switching frequency and transaction costs.

6.3 Kelly Portfolio versus 1/N-Strategy

In the following we will compare the performances of three different Kelly portfolios generated from the three different data sets with the naive $\frac{1}{N}$-rule. Therefore, we calculated some key statistics for each of the six portfolios. As the stock portions of alle three portfolios are comparatively small, it is important to keep in mind that we include a risk-free asset that pays an interest rate based on the performance of US treasury bills. Note once again that the more favorable value for each metric is colored green, while the less advantageous value for each metric is colored red.

Data Set	2005 - 2009		2000 - 2009		2000 - 2011	
Strategy	$\frac{1}{N}$-strategy	Kelly	$\frac{1}{N}$-strategy	Kelly	$\frac{1}{N}$-strategy	Kelly
Median	**0.0944**	**0.0055**	**0.0512**	**0.0087**	**0.0718**	**0.0131**
Mean	**0.0079**	**-0.0001**	**0.0095**	**0.0001**	**0.0161**	**0.0037**
St. Dev.	**1.4578**	**0.1131**	**1.3600**	**0.2140**	**1.3093**	**0.2479**
Skewness	**-0.3034**	**-0.3699**	**-0.2073**	**-0.0720**	**-0.2539**	**-0.0670**
Kurtosis	**9.8454**	**8.4466**	**8.4689**	**7.4714**	**8.1322**	**8.5406**
Average Drawdown	**1.0486**	**0.1245**	**9.6197**	**0.5324**	**21.4762**	**0.7015**
Average Length	**1259**	**30**	**628**	**838**	**3019**	**1509**
Average Recovery	**1258**	**14**	**625**	**111**	**3015**	**1238**
Pain Index	**13.5662**	**1.2241**	**14.2912**	**2.5541**	**13.9745**	**2.2529**
Ulcer Index	**21.8689**	**2.0262**	**20.4404**	**3.5896**	**19.5877**	**3.2361**
Sharpe Ratio	**-0.0196**	**-0.0245**	**-0.0414**	**-0.0493**	**-0.0468**	**-0.0482**
Adjusted SR	**-0.0321**	**-0.0273**	**-0.0002**	**-0.0125**	**0.0880**	**0.2091**
Omega SR	**-0.0604**	**-0.0758**	**-0.1175**	**-0.1367**	**-0.1306**	**-0.1312**
Sortino Ratio	**-0.0263**	**-0.0329**	**-0.0556**	**-0.0666**	**-0.0622**	**-0.0647**
Upside Potential Ratio	**0.4084**	**0.4009**	**0.4177**	**0.4207**	**0.4138**	**0.4284**

Table 29: key statistics for Kelly portfolios generated from differen data sets compared to the $\frac{1}{N}$-strategy; note that the weakest value for each performance statistic is colored red, while the strongest is colored green; it becomes obvious, that the Kelly portfolios do not yield consistently superior returns

Since in general, the Kelly portfolio comes with a much smaller return, but also much smaller risk than the $\frac{1}{N}$-strategy, especially the last five rows containing performance measures are worth to have an eye on, because they incorporate both return and risk. Obviously, the Kelly portfolios generated from all three data sets fail to permanently outperform the $\frac{1}{N}$-strategy. At this point it is reasonable to check another data set for which the Kelly portfolio works better due to some sort of regime change. That is to say, it is possible that our three data sets cover a certain period over which a certain market behavior is dominating which makes Kelly portfolios impractical. For the time being we will focus only on the results from the third data set. First of all, we computed the Up / Down Ratios for the stock portion of the Kelly portfolio and presented them in the table below:

Ratio	Up Capture	Down Capture	Up Number	Down Number	Up Percent	Down Percent
Kelly Portfolio	0.1876	0.1861	0.9106	0.9220	0.0269	0.9653

Table 30: Up/Down Ratios for a Kelly portfolio compared to the naive $\frac{1}{N}$-rule over a period of twelve years (2000-2011); these performance measures give no clear hint regarding the investment decision between Kelly portfolio and $\frac{1}{N}$-strategy

Expectedly, the stock portion of the Kelly portfolio only seldomly provides larger returns than the $\frac{1}{N}$-strategy, when the market experiences an up-swing, but it often works better when the return of the benchmark decreases as shown by the Percentage Ratios. The Capture Ratios demonstrate slight advantages related with the Kelly Portfolio, since its return rises on average by 18.76 percent when the benchmark return goes up by 100 percent, but only decreases by 18.61 percent when the benchmark goes down by 100 percent. Nevertheless, the Number Ratios draw a different picture: The Kelly Portfolio exhibits fewer up periods compared to down periods than the $\frac{1}{N}$-benchmark portfolio. To sum up, these performance measures only demonstrate evidence for minor gaps differences, although both portfolios work fundamentally different. As before, we will further evaluate the portfolio performance by looking at the following CAPM measures:

Strategy	CAPM Alpha	CAPM Beta	Fama Beta
Kelly Portfolio	0.0009	0.1819	0.1887

Table 31: CAPM measures for a Kelly Portfolio compared to the naive $\frac{1}{N}$-rule over a period of twelve years (2000-2011); the CAPM Alpha intersect suggest a negligible advantage associated with the Kelly portfolio, while the Beta values illustrate that the Kelly portfolio follows the benchmark to a much weaker extent than the previous portfolios, since most of its return can not be attributed to the benchmark performance

Obviously, the return of the stock portion of the Kelly portfolio is attributable to the benchmark to a only very little extent as proven by the Beta measures. Furthermore, the Kelly Portfolio appears to have a negligible advantage over the $\frac{1}{N}$-strategy, because the CAPM Alpha intersect is presented by a small positive value. In what way the return of the Kelly portfolio is related to the benchmark return will be illustrated by the return scatter below:

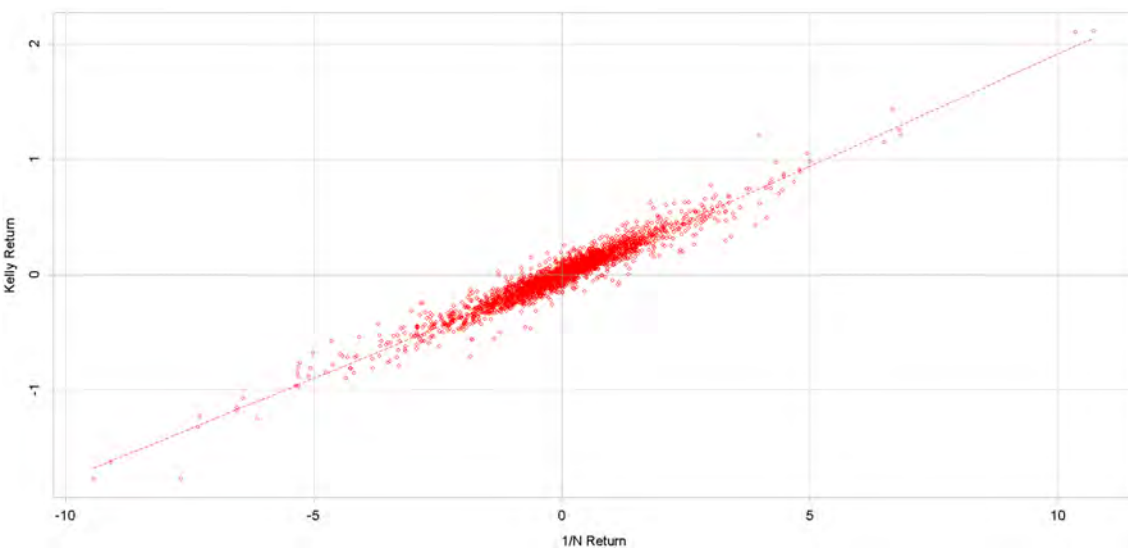


Figure 26: diagram showing a return scatter for the stock portion of a Kelly portfolio (y-axis) compared to the naive $\frac{1}{N}$-rule (x-axis); the Kelly portfolio comprises weights based on an estimation period of 26 years (1974 - 2000) and holds this particular weight distribution for twelve years (2000-2011); overlaying a LOWESS (locally weighted scatterplot smoothing) line illustrates the relation between Kelly portfolio and the $\frac{1}{N}$strategy; as anticipated, the Kelly portfolio follows the benchmark but to a much weaker extent which becomes apparent when taking a closer look at the scales

As we can see both portfolios behave in a very similar manner, but the return of the stock portion of the Kelly portfolio tracks the benchmark return to a weaker extent than the previous portfolios which is revealed when looking at how x- and y-axes are scaled differently meaning that we can negate the null hypothesis of equalness once again. As last step the cumulative return of the stock portion of the Kelly portfolio is contrasted with the cumulative return of the $\frac{1}{N}$-benchmark:

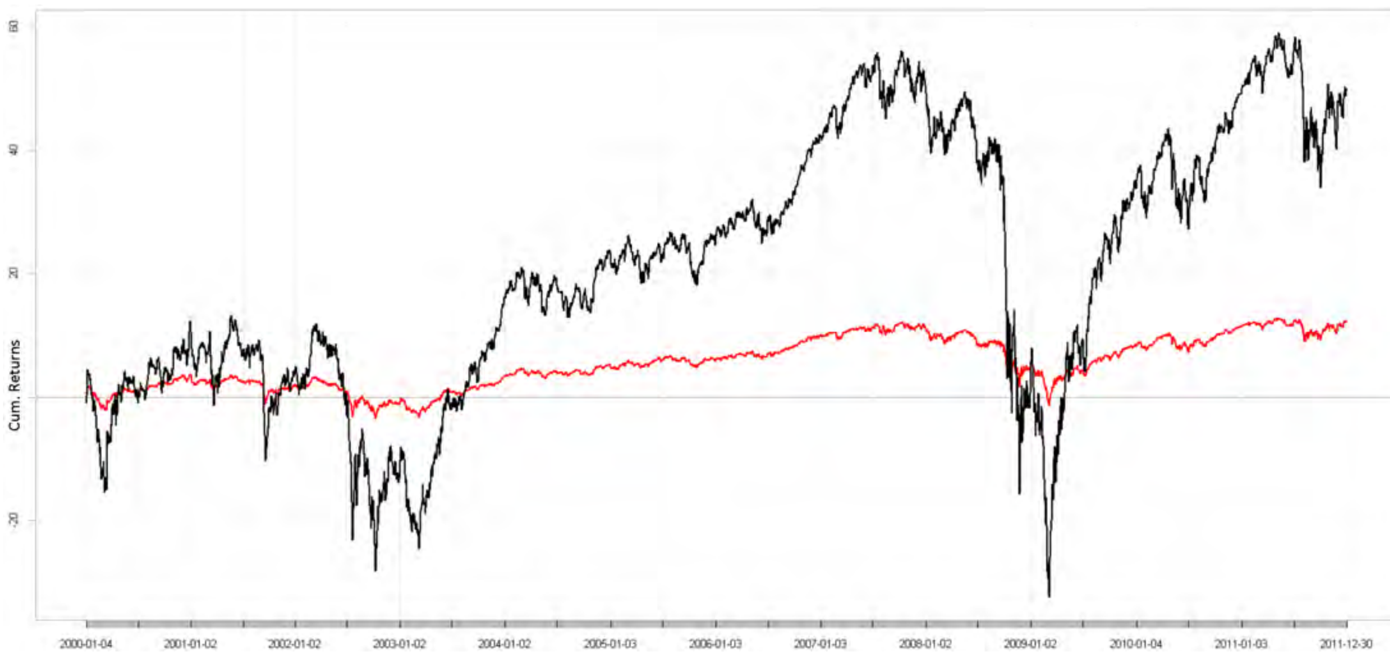


Figure 27: cumulative return for the stock portion of a Kelly portfolio compared to the $\frac{1}{N}$-benchmark over a period of twelve years (2000-2011); black: $\frac{1}{N}$-benchmark, red: Kelly portfolio; expectedly, the Kelly portfolio exhibits much smaller fluctuations

The plot above demonstrates that both portfolio approaches work very differently, since the Kelly portfolio exhibits much smaller fluctuations due to the large share attributed to the risk-free asset. Thus, the Kelly portfolio appears to be the better choice whenever the market experiences an extensive downswing, but not when market prices rise over a long period. Therefore, the Kelly portfolio might also serve as a suitable approach for the previously introduced state-dependent allocation strategy.

7. Summary

To start with, the theory part (Section 2) introduced the general Markowitz problem and reveals several weaknesses that come with the empirical estimation of portfolio weights. Therefore, three distinct concepts - namely the Generalized Herfindahl Index, a state-dependent allocation strategy and the Kelly portfolio - are presented in an effort to overcome these disadvantages by adding diversification constraints. The data part (Section 3) effectively disproves Markowitz' assumption that stock returns are normally distributed while illustrating how some assets follow a Gaussian distribution more closely than others. The implementation / results part (Section 4 and 5) exemplifies the technical peculiarities of the before-mentioned concepts and shows the respective numerical results.

Finally, the analysis part (Section 6) contrasts the performance of the three strategies of interest with the naive $\frac{1}{N}$-rule, which serves as a benchmark. In this regard, it is demonstrated that a portfolio utilizing the Generalized Herfindahl Index is able to slightly outperform the $\frac{1}{N}$-strategy, yet only when combined with a rolling window approach that rebalances at least every year. Furthermore, increasing the length of data window for parameter estimation did not improve the performance of the GHHI portfolio significantly. Secondly, the state-dependent allocation strategy, which switches from $\frac{1}{N}$-strategy to minimum variance portfolio depending on the value of the DS LPPLS^{TM} Confidence Indicator, consistently outperforms the $\frac{1}{N}$-benchmark provided that a threshold of less than 0.8 is applied regardless of the length of the data window and even without permanently rebalancing the portfolio weights. As opposed to the first two concepts, the Kelly portfolio works quite differently than the naive $\frac{1}{N}$-rule due to the not negligible contribution of the risk-free asset. Hence, applying a sufficiently long estimation window enables us to adopt the Kelly portfolio as an alternative to common portfolio strategies in the case of an extreme market downswing.

8. Conclusion

The paper demonstrated that neither the Kelly portfolio nor the GHHI portfolio are able to consistently outperform the $\frac{1}{N}$-portfolio, whereas the state-dependent allocation strategy based on the DS LPPLS^{TM} confidence indicator generates superior returns. Nevertheless, our analysis does not include transaction costs. Incorporating transaction costs is crucial when comparing investment strategies to the $\frac{1}{N}$-benchmark, because the $\frac{1}{N}$-portfolio represents a "buy-and-hold" strategy that does not cause transaction costs, while constantly switching between $\frac{1}{N}$-strategy and minimum variance portfolio might be very costly. Apart from the previously mentioned imperfect $\frac{1}{N}$-portfolio our approach also suffers from a variety of other limitations. First of all, the Markowitz portfolio requires parameter estimation, which comes with an estimation error. This phenomenon can be tackled by subsitituting sample mean and sample covariance by more robust return and risk estimators. Furthermore, the correlation matrix used for the Generalized Herfindahl Index is only vaguely defined so far, since Anand and Ramasubramanian leave its entries open for discussion. Understandably, different implementations of this matrix yield deviating results, which is why we advocate a consistent and standardized usage of this particular index.

As promised before the conclusion section will come up with a potential remedy for the issue of an imperfect $\frac{1}{N}$-portfolio without increasing the computational effort significantly. The imperfect $\frac{1}{N}$-portfolio poses as a thread to our work, since a stock selection bias might as well lead to a performance bias which gives some portfolio an edge over others. Instead of solely investing into 50 assets from the S&P500 we propose to invest into the subsectors defined in the Industry Classification Benchmark (ICB), which depends on stock from ten differen industries, whilst dividing them into 19 supersectors, which are further subdivided into 41 sectors and 119 subsectors. A summary of all subsectors can be found in the appendix (see Tables 58 - 67). By utilizing this kind of segmentation we ensure that our investments strategy covers the entire market. Accordingly, we obtained positive Sharpe Ratios for the entire observation period. Moreover, it also allows us to implement the Generalized Herfindahl Index by imposing for example a 119×119 GHHI correlation matrix with correlations of 0.75 to 1.0 for subsectors from the same sector, correlations of 0.5 to 0.75 for subsectors from the same supersector but sectors as well as correlations of 0.25 to 0.5 for subsectors from the same industry but supersectors and finally no correlations between subsectors from different industries. Obviously, this results in a much more elaborated matrix than the one we deployed in our case. Thereby we hope to generate a $\frac{1}{N}$-portfolio that is not outperformed by simple treasury bills which makes our results more meaningful than before.

In addition, we will suggest a way to incorporate the three above-exemplified concepts into a joint investment strategy. One way to achieve this is to add a couple more optimization approaches to our state-dependent allocation strategy, since we only tested two at the same time so far, but it might be possible that neither of these appropriately corresponds to a certain market condition. In the case of a small bubble indicator value one could for instance focus more on return maximization instead of only depending on a minimum variance portfolio, but when the bubble indicator rises we would emphasize risk minimization and finally if there is a severe crisis we would rely on the Kelly portfolio as it exhibits the lowest volatility. The $\frac{1}{N}$-portfolio could be sort of implemented by adapting the λ-value associated with diversification to the length of the estimation window. Here, de Miguel's analytically calculated estmiation window length could serve as a benchmark (i.e. if an estimation window of 600 months is needed to outperform the $\frac{1}{N}$-portfolio and our estimation window is only 60 months, we would impose a Herfindahl Index of 0.1 or similar). Of course, we can also substitute the Herfindahl Index by its generalized version at this point. Thus, the resulting overall investment decision is not discrete any more, because the various λ-values can be adjusted smoothly. Accordingly, the focus of future research regarding this topic has to be put on the question which strategy is most suitable for a certain market regime, while keeping in mind that the addition of too many strategies will induce disadvantageous transaction costs.

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10. Appendix

In the appendix several complementary tables and figures are presented. In addition, one can find the code regarding the optimization models used in this paper at the end.

10.1 Data

The data subsection first shows the stock symbols associated with the different data sets and then demonstrates additional statistics regarding the individual data sets that did not fit into the main part of the paper.

Stock Symbols

Sector	Companies
Consumer Discretionary (XLY)	Walt-Disney (DIS), McDonalds (MCD), Ford (F)
	Goodyear (GT), Nike (NKE)
Financials (XLF)	American-Express (AXP), JPMorgan (JPM), AON (AON)
	Wells-Fargo (WFC), American-Intl-Corp (AIG)
Health Care (XLV)	Johnson-Johnson (JNJ), Pfizer (PFE)
	Merck (MRK), Bristol-Myers-Squibb (BMY)
Industrials (XLI)	General-Electrics (GE), Boeing (BA)
	FedEx (FDX), Southwest-Airlines (LUV)
Materials (XLB)	E-I-du-Pont-de-Nemours (DD), Dow-Chemical (DOW)
	Newmont-Mining (NEM), International-Paper (IP)
Consumer Staples (XLP)	Procter-and-Gamble (PG), Coca-Cola (KO)
	Wal-Mart (WMT), Colgate (CL)
Energy (XLE)	Exxon-Mobile (XOM), Halliburton (HAL), Chevron (CVX)
	Schlumberger (SLB), Occidential-Petroleum (OXY)
Utilities (XLU)	American-Electric-Power (AEP), Duke-Energy (DUK)
	Southern-Company (SO), Exelon (EXC)
Technology (XLK)	IBM (IBM), Texas-Instruments (TXN)
	Apple (AAPL), Intel (INTC)

Table 32: nine out of eleven S&P500 sectors (first data set, 2000 - 2009) and 39 S&P500 companies (second data set, 1974 - 2011) from the respective sectors plus respective stock symbols

Sector	Companies
Consumer Discretionary (XLY)	Walt-Disney (DIS), McDonalds (MCD), Ford (F), Goodyear (GT)
	Footlocker (FL), Avon-Products (AVP), Altria-Group (MO)
Financials (XLF)	American-Express (AXP), Wells-Fargo (WFC), Bank-of-New-York-Mellon (BK)
Health Care (XLV)	Johnson-Johnson (JNJ), Pfizer (PFE), Merck (MRK)
	Bristol-Myers-Squibb (BMY), Eli-Lilly (LLY)
Industrials (XLI)	General-Electrics (GE), Boeing (BA), Sparton (SPA), Alcoa (AA), Caterpillar (CAT)
	John-Deere (DE), Eaton (ETN), Navistar (NAV), Ducommun (DCO)
Materials (XLB)	E-I-du-Pont-de-Nemours (DD), Dow-Chemical (DOW), International-Paper (IP)
Consumer Staples (XLP)	Procter-and-Gamble (PG), Coca-Cola (KO)
	Wal-Mart (WMT), Pepsico (PEP), Sysco (SYY)
Energy (XLE)	Exxon-Mobile (XOM), Halliburton (HAL), Chevron (CVX), Marathon-Oil (MRO)
Utilities (XLU)	American-Electric-Power (AEP), Consilidated-Edison (ED), DTE-Energy (DTE)
	Entergy (ETR), PC&E (PCG), Center-Point-Energy (CNP),
	Allete (ALE), Empire-District-Electric (EDE), SJW-Corp. (SJW)
Technology (XLK)	IBM (IBM), 3M (MMM), Unisys (UIS), United-Technologies (UTX), HP-Inc. (HPQ)

Table 33: nine out of eleven S&P500 sectors (first data set, 2000 - 2009) and 50 S&P500 companies (third data set. 1974 - 2011) from the respective sectors plus respective stock symbols

Additional Statistics

First Data Set (2000 - 2009)

Sector	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU
XLF	85.03286							
XLV	65.37189	101.18137						
XLI	52.98491	86.25566	58.44248					
XLB	71.76739	98.60876	77.64861	59.00767				
XLP	70.62401	104.36677	61.55139	65.16961	79.41578			
XLE	96.28730	114.48271	94.64808	85.10342	81.77816	93.42602		
XLU	80.37915	108.30239	69.51226	71.64061	82.27154	65.55001	86.42774	
XLK	79.95649	106.38981	80.31624	73.42108	95.69652	96.88011	111.71114	96.62310

Table 34: Euclidean distance measures for nine S&P500 sectors over a period of ten years (01/01/2000 - 31/12/2009)

Sector	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU	XLK
Minimum	-12.358	-19.069	-10.295	-9.877	-13.253	-6.213	-15.600	-8.914	-9.051
Quartile 1	-0.787	-0.845	-0.607	-0.715	-0.846	-0.522	-0.943	-0.603	-0.9612
Median	0.030	0.000	0.031	0.071	0.074	0.042	0.112	0.094	0.051
Mean	0.004	-0.009	0.006	0.005	0.019	0.014	0.037	0.019	-0.032
Quartile 3	0.810	0.829	0.655	0.758	0.940	0.560	1.163	0.731	0.868
Maximum	9.327	27.298	11.382	10.170	13.153	6.659	15.250	11.398	14.930
St. Dev.	1.646	2.392	1.269	1.504	1.739	1.054	1.975	1.375	1.964
Skewness	-0.165	0.385	-0.051	-0.171	-0.065	-0.055	-0.454	0.168	0.350
Kurtosis	4.867	17.650	9.256	5.323	5.380	4.534	8.969	8.189	4.664

Table 35: key statistics for nine S&P500 sectors over a period of ten years (2000-2009)

Sector	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU	XLK
Average Drawdown Depth	26.087	81.262	3.888	12.328	19.074	27.599	181.618	12.237	20.318
Average Drawdown Length	73.029	38.453	278.889	249.900	56.023	625.500	19.427	313.625	19.040
Average Recovery Length	59.912	3.109	272.889	229.500	49.614	618.750	6.452	294.750	14.397
Conditional Drawdown at Risk	25.783	81.046	6.671	19.407	54.455	74.373	34.108	27.681	37.638
Sterling Ratio	0.010	NaN	NaN	-0.008	157.890	-0.010	NaN	NaN	$5.1 \cdot 10^9$
Calmar Ratio	0.010	NaN	NaN	-0.008	157.954	-0.010	NaN	NaN	$5.1 \cdot 10^9$
Burke ratio	0.068	NaN	NaN	-0.009	431.673	-0.019	NaN	NaN	$1.9 \cdot 10^{10}$
Pain Index	18.120	21.247	16.248	19.313	19.076	14.752	18.624	17.733	7.005
Ulcer Index	23.117	31.789	18.342	24.913	24.232	17.471	25.663	23.363	7.226
Pain Ratio	0.316	NaN	NaN	-0.0345	2046.042	-0.068	NaN	NaN	$2.8 \cdot 10^{10}$
Martin Ratio	0.248	NaN	NaN	-0.027	1610.633	-0.057	NaN	NaN	$2.7 \cdot 10^{10}$

Table 36: drawdown statistics for nine S&P500 sectors over a period of ten years (2000-2009)

Metric	Formula	Explanation
Conditional Drawdown at Risk		mean of the worst p % drawdowns for some confindence level p
		reference: Chekhlov & Uryasev & Zabarankin, 2003
Calmar Ratio		annualized return over absolute value of max. drawdown of investment
Sterling Ratio		Calmar Ratio including excess risk to max. drawdown
Burke Ratio	$\frac{r_p - r_r}{\sqrt{\sum_{i=1}^{d} D_i^2}}$	r_p: investment return r_f: risk-free rate
Pain Index	$\sum_{i=1}^{d} D_i^{'}/n$	$D_i^{'}$: drawdown since the previous peak in period i
Ulcer Index	$\sqrt{\sum_{i=1}^{d} D_i^{'2}/n}$	$D_i^{'}$: drawdown since the previous peak in period i
Pain Ratio	$\frac{r_p - r_r}{PI}$	r_p: investment return, r_f: risk-free rate, PI: Pain Index
Martin Ratio	$\frac{r_p - r_r}{UI}$	r_p: investment return, r_f: risk-free rate, UI: Ulcer Index

Table 37: descriptions of drawdown metrics used in the previous table

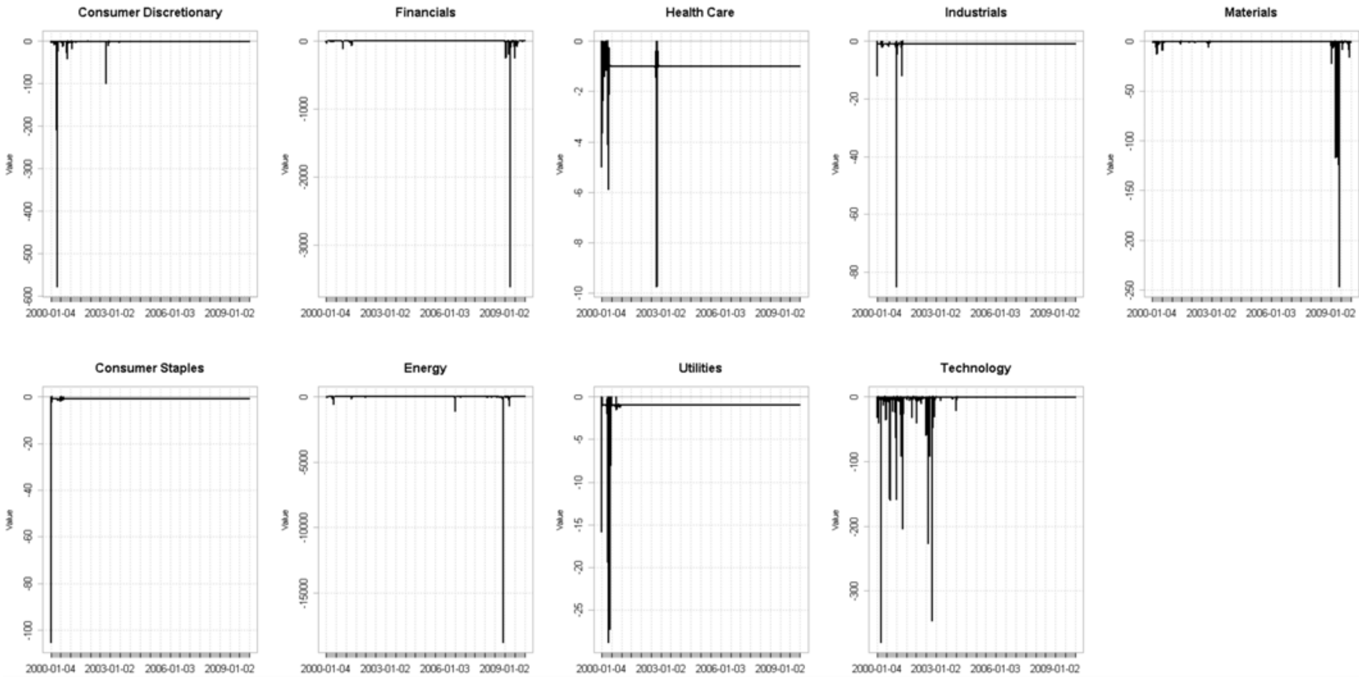

Figure 28: drawdown charts for nine out of eleven S&P500 sectors over a period of ten years (01/01/2000 - 31/12/2009); while some sectors experience long and shallow drawdowns, others exhibit short and deep drawdowns.

Sector	XLY	XLF	XLV	XLI	XLB	XLP	XLE	XLU	XLK
Sharpe Ratio	0.0022	-0.0039	0.0045	0.0034	0.0109	0.0130	0.0186	0.0138	-0.0163
Adjusted SR	-0.0944	-0.2265	-0.0297	-0.0657	0.0354	0.1231	0.1375	0.1106	-0.3680
Omega SR	0.0063	-0.0132	0.0131	0.0098	0.0313	0.0373	0.0541	0.0408	-0.0458
Sortino Ratio	0.0031	-0.0056	0.0062	0.0047	0.0154	0.0183	0.0255	0.0193	-0.0230
Upside Potential Ratio	0.4910	0.4202	0.4835	0.4832	0.5064	0.5095	0.4963	0.4928	0.4784
Bernardo Ledoit Ratio	1.0063	0.9868	1.0131	1.0097	1.0313	1.0373	1.0541	1.0408	0.9542
D Ratio	0.9551	0.9931	0.9404	0.8825	0.8989	0.8878	0.8409	0.8351	0.9704
Downside Deviation	1.1725	1.6730	0.9087	1.0858	1.2367	0.7465	1.4395	0.9783	1.3915
Downside Frequency	0.4821	0.4885	0.4805	0.4654	0.4749	0.4678	0.4654	0.4574	0.4730
Prospect Ratio	-0.6068	-0.5378	-0.5903	-0.5935	-0.5984	-0.5956	-0.5630	-0.5725	-0.6496

Table 38: key performance measures for nine S&P500 sectors over a period of ten years (2000-2009)

Sector	Formula	Explanation
Sharpe Ratio	$\frac{r_p - r_f}{\sigma_p}$	r_p: portfolio return r_f: riskfree rate
		σ_p: portfolio standard deviation
Adjusted SR	$SR\left[1+\left(\frac{S}{6}\right)SR-\left(\frac{K-3}{24}\right)SR^2\right]$	S: Skewness K: Kurtosis
Omega SR	$\frac{r_p - MAR}{\sum_{i=1}^{n}\frac{\max(MAR-r_i,0)}{n}}$	MAR: Minimum Acceptable Return n : number of observations
Sortino Ratio	$\frac{r_p - MAR}{\sqrt{\sum_{i=1}^{n}\frac{\max(MAR-r_i,0)^2}{n}}}$	
Upside Potential Ratio	$\frac{\sum_{i=1}^{n}\max(r_i-MAR,0)}{\sum_{i=1}^{n}\max(MAR-r_i,0)}$	
Bernardo Ledoit Ratio	$\frac{\sum_{i=1}^{n}\max(r_i,0)}{\sum_{i=1}^{n}\max(-r_i,0)}$	equivalent to Upside Potential Ratio if $MAR=0$
D Ratio	$\frac{n_d\sum_{i=1}^{n}\max(-r_i,0)}{n_u\sum_{i=1}^{n}\max(r_i,0)}$	n_d: number of observations < 0 n_u: number of observations > 0
Downside Deviation	$\sqrt{\sum_{i=1}^{n}\frac{\max(r_i-MAR,0)^2}{n}}$	
Downside Frequency	$\sum_{i=1}^{n}\frac{\max(r_i-MAR,0)}{r_i \cdot n}$	
Prospect Ratio	$\frac{\sum_{i=1}^{n}[\max(r_i,0)+2.25\min(r_i,0)-MAR]}{n \cdot \sigma_D}$	

Table 39: description of key performance measures used in the previous table

Second Data Set (1990 - 2009)

Company	Walt-Disney	McDonalds	Ford	Goodyear	Nike
Median	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0279	0.0445	0.0176	-0.0021	0.0608
St. Dev.	2.0354	1.6909	2.6398	2.8502	2.2437
Skewness	0.0417	-0.0347	0.0398	-0.3638	-0.1624
Kurtosis	7.0405	3.9485	14.1919	6.7125	6.6293
Pain Index	35.5819	23.7145	43.1759	52.8361	33.3385
Ulcer Index	43.6121	31.2872	54.4321	64.0482	38.7881
Sharpe Ratio	0.0137	0.0263	0.0067	-0.0007	0.0271
Adjusted SR	0.0567	0.2894	-0.1017	-0.2274	0.2563
Omega SR	0.0396	0.0743	0.0199	-0.0021	0.0809
Sortino Ratio	0.0199	0.0381	0.0096	-0.0010	0.0393
Upside Potential Ratio	0.5215	0.5513	0.4930	0.4686	0.5251

Table 40: key statistics for “Consumer Discretionary” S&P500 assets over a period of 20 years (1990-2009)

Company	American-Express	JPMorgan	AON	Wells-Fargo	American-Intl-Corp
Median	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0374	0.0426	0.0328	0.0562	-0.0332
St. Dev.	2.4289	2.6446	1.9927	2.4536	3.6480
Skewness	0.0590	0.2720	-2.2585	0.7848	-3.6933
Kurtosis	6.9344	10.1608	50.3617	23.3942	133.9169
Pain Index	28.1411	32.5713	33.2662	12.8470	28.6576
Ulcer Index	34.6634	39.8833	42.0232	20.2124	39.8895
Sharpe Ratio	0.0154	0.0161	0.0164	0.0229	-0.0091
Adjusted SR	0.0517	0.0467	0.0919	0.1744	0.0046
Omega SR	0.0461	0.0503	0.0541	0.0781	-0.0386
Sortino Ratio	0.0221	0.0236	0.0222	0.0341	-0.0119
Upside Potential Ratio	0.5023	0.4935	0.4331	0.4708	0.2960

Table 41: key statistics for “Financials” S&P500 assets over a period of 20 years (1990-2009)

Company	Johnson-Johnson	Pfizer	Merck	Bristol-Myers-Squibb
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0503	0.0460	0.0328	0.0271
St. Dev.	1.4893	1.8652	1.8874	1.8344
Skewness	-0.1566	-0.1684	-1.1078	-0.6846
Kurtosis	6.4850	3.0043	19.4714	12.4711
Pain Index	11.9861	31.5907	32.9829	38.4812
Ulcer Index	15.2067	39.5302	40.7012	48.3462
Sharpe Ratio	0.0338	0.0246	0.0174	0.0148
Adjusted SR	0.4113	0.2480	0.1209	0.0872
Omega SR	0.0983	0.0694	0.0509	0.0434
Sortino Ratio	0.0492	0.0353	0.0239	0.0206
Upside Potential Ratio	0.5494	0.5442	0.4939	0.4947

Table 42: key statistics for “Health Care” S&P500 assets over a period of 20 years (1990-2009)

Company	General-Electrics	Boeing	FedEx	Southwest-Airlines
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0304	0.0266	0.0391	0.0478
St. Dev.	1.8811	2.0059	2.1314	2.4875
Skewness	0.0183	-0.3434	0.0532	-0.3209
Kurtosis	8.1902	6.7855	4.1076	6.1245
Pain Index	27.2776	31.0609	24.5289	36.0345
Ulcer Index	37.6887	36.6433	29.5137	42.6001
Sharpe Ratio	0.0162	0.0133	0.0183	0.0192
Adjusted SR	0.1085	0.0504	0.1241	0.1075
Omega SR	0.0489	0.0381	0.0529	0.0542
Sortino Ratio	0.0231	0.0186	0.0269	0.0275
Upside Potential Ratio	0.4948	0.5078	0.5347	0.5340

Table 43: key statistics for “Industrials” S&P500 assets over a period of 20 years (1990-2009)

Company	E-I-du-Pont-de-Nemours	Dow-Chemical	Newmont-Mining	International-Paper
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0227	0.0183	0.0078	0.0095
St. Dev.	1.8578	2.0697	2.6929	2.2107
Skewness	-0.0727	-0.1758	0.4210	0.0901
Kurtosis	3.9510	7.7718	5.3099	9.2135
Pain Index	32.7080	25.1157	55.3973	35.1356
Ulcer Index	40.8192	30.6049	61.5255	42.6111
Sharpe Ratio	0.0122	0.0088	0.0029	0.0043
Adjusted SR	0.0462	-0.0249	-0.1580	-0.1050
Omega SR	0.0346	0.0261	0.0082	0.0127
Sortino Ratio	0.0174	0.0125	0.0043	0.0061
Upside Potential Ratio	0.5214	0.4925	0.5263	0.4895

Table 44: key statistics for “Materials” S&P500 assets over a period of 20 years (1990-2009)

Company	Procter-and-Gamble	Coca-Cola	Wal-Mart	Colgate
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0465	0.0424	0.0472	0.0543
St. Dev.	1.5820	1.5628	1.8362	1.6092
Skewness	-2.4623	0.0544	0.0954	-0.0111
Kurtosis	56.4295	5.0771	2.7550	9.3249
Pain Index	16.1451	30.5079	28.4776	14.0752
Ulcer Index	22.3191	38.383	32.5406	17.522
Sharpe Ratio	0.0294	0.0271	0.0257	0.0337
Adjusted SR	0.1987	0.3124	0.2715	0.3983
Omega SR	0.0902	0.0791	0.0732	0.1008
Sortino Ratio	0.0403	0.0394	0.0376	0.0491
Upside Potential Ratio	0.4873	0.5380	0.5511	0.5359

Table 45: key statistics for “Consumer Staples” S&P500 assets over a period of 20 years (1990-2009)

Company	Exxon-Mobile	Halliburton	Chevron	Schlumberger	Occidential-Petroleum
Median	0.0000	0.0000	0.0137	0.0000	0.0000
Mean	0.0457	0.0288	0.0437	0.0413	0.0491
St. Dev.	1.5571	2.8288	1.6070	2.3017	2.0474
Skewness	0.0579	-1.3667	0.1316	-0.2644	-0.1909
Kurtosis	8.8603	33.1449	9.6885	5.1776	8.6511
Pain Index	9.4422	53.033	12.1079	30.3531	18.7333
Ulcer Index	13.3208	58.7759	16.4493	37.2736	23.7164
Sharpe Ratio	0.0293	0.0102	0.0272	0.0180	0.0240
Adjusted SR	0.3416	-0.0734	0.3057	0.1027	0.2196
Omega SR	0.0856	0.0298	0.0785	0.0507	0.0707
Sortino Ratio	0.0423	0.0141	0.0392	0.0256	0.0341
Upside Potential Ratio	0.5363	0.4875	0.5382	0.5310	0.5169

Table 46: key statistics for “Energy” S&P500 assets over a period of 20 years (1990-2009)

Company	American-Electric-Power	Duke-Energy	Southern-Company	Exelon
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0243	0.0332	0.0478	0.0458
St. Dev.	1.5192	1.5412	1.2842	1.6486
Skewness	-0.4901	-0.2033	0.2375	-0.0689
Kurtosis	28.4567	10.9265	4.5154	8.1267
Pain Index	18.718	23.1093	7.6569	15.3547
Ulcer Index	24.7233	31.0532	10.0358	20.7832
Sharpe Ratio	0.0160	0.0215	0.0372	0.0278
Adjusted SR	0.1306	0.2181	0.4993	0.3106
Omega SR	0.0510	0.0655	0.1090	0.0830
Sortino Ratio	0.0226	0.0302	0.0550	0.0395
Upside Potential Ratio	0.4656	0.4918	0.5591	0.5147

Table 47: key statistics for “Utilities” S&P500 assets over a period of 20 years (1990-2009)

Company	IBM	Texas-Instruments	Apple	Intel
Median	0.0000	0.0000	0.0000	0.0285
Mean	0.0401	0.0514	0.0632	0.0604
St. Dev.	1.9112	2.9382	3.2630	2.6848
Skewness	0.0169	0.1156	-2.1362	-0.3816
Kurtosis	6.6011	2.8030	54.5246	5.2788
Pain Index	36.8785	50.1206	64.6656	45.8119
Ulcer Index	41.7821	61.2534	69.2508	57.7302
Sharpe Ratio	0.0210	0.0175	0.0194	0.0225
Adjusted SR	0.1843	0.0454	0.0179	0.1447
Omega SR	0.0619	0.0489	0.0581	0.0642
Sortino Ratio	0.0303	0.0255	0.0268	0.0317
Upside Potential Ratio	0.5202	0.5470	0.4872	0.5256

Table 48: key statistics for “Technology” S&P500 assets over a period of 20 years (1990-2009)

Third Data Set (1974 - 2011)

Company	Walt-Disney	McDonalds	Ford	Goodyear	Footlocker	Avon-Products	Altria-Group
Median	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0490
Mean	0.0480	0.0476	0.0385	0.0258	0.0387	0.0126	0.0680
St. Dev.	1.9935	1.6648	2.2491	2.4503	2.3641	2.1814	1.6679
Skewness	-0.6692	-0.3712	-0.0254	-0.5006	0.0201	-0.9794	-0.5406
Kurtosis	15.6364	8.8854	14.3747	10.6090	10.1625	23.6196	13.5846
Pain Index	33.0799	21.6464	40.8076	47.9325	48.8479	36.6802	13.7074
Ulcer Index	39.5928	28.7562	52.8881	60.0850	59.1216	43.2586	19.5479
Sharpe Ratio	0.0241	0.0286	0.0171	0.0105	0.0164	0.0058	0.0408
Adjusted SR	0.2159	0.3139	0.0932	-0.0296	0.0726	-0.0847	0.4289
Omega SR	0.0719	0.0842	0.0512	0.0318	0.0494	0.0178	0.1259
Sortino Ratio	0.0340	0.0409	0.0247	0.0148	0.0240	0.0080	0.0581
Upside Potential Ratio	0.5077	0.5267	0.5082	0.4787	0.5092	0.4560	0.5196

Table 49: key statistics for "Consumer Discretionary" S&P500 assets over a period of 38 years (1974-2011)

Company	American-Express	Wells-Fargo	Bank-of-New-York-Mellon
Median	0.0000	0.0000	0.0000
Mean	0.0398	0.0500	0.0401
St. Dev.	2.2186	2.0967	2.2980
Skewness	-0.2213	0.5116	0.0672
Kurtosis	9.6135	22.5254	182.8531
Pain Index	34.4683	46.7854	35.1361
Ulcer Index	39.2199	21.8210	43.8171
Sharpe Ratio	0.0180	0.0238	0.0175
Adjusted SR	0.1090	0.2140	0.0786
Omega SR	0.0536	0.0777	0.0604
Sortino Ratio	0.0256	0.0348	0.0253
Upside Potential Ratio	0.5035	0.4833	0.4434

Table 50: key statistics for "Financials" S&P500 assets over a period of 38 years (1974-2011)

Company	Johnson-Johnson	Pfizer	Merck	Bristol-Myers-Squibb	Eli-Lilly
Median	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0445	0.0455	0.0419	0.0509	0.0400
St. Dev.	1.4569	1.7842	1.6634	1.6873	1.7158
Skewness	-0.2920	-0.1751	-0.7822	-0.3726	-1.1125
Kurtosis	8.4285	4.0801	16.8903	10.0362	25.5483
Pain Index	15.4203	31.8440	30.1345	29.2697	34.2485
Ulcer Index	20.0091	38.5866	37.3283	39.4780	40.1908
Sharpe Ratio	0.0305	0.0255	0.0252	0.0302	0.0233
Adjusted SR	0.3576	0.2677	0.2513	0.3329	0.2135
Omega SR	0.0894	0.0739	0.0742	0.0891	0.0688
Sortino Ratio	0.0441	0.0367	0.0355	0.0433	0.0327
Upside Potential Ratio	0.5377	0.5337	0.5142	0.5293	0.5082

Table 51: key statistics for “Health Care” S&P500 assets over a period of 38 years (1974-2011)

Company	General-Electrics	Boeing	Sparton	Alcoa	Caterpillar	John-Deere	Eaton	Navistar	Ducommun
Median	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0428	0.0651	0.0300	0.0219	0.0311	0.0387	0.0502	-0.0304	0.0176
St. Dev.	1.7052	1.9529	2.8518	2.2195	1.9120	1.9997	1.7553	3.1415	3.0261
Skewness	-0.0812	-0.1085	0.1939	-0.2660	-0.3120	-0.1650	-0.2012	-0.0880	0.3377
Kurtosis	8.4587	4.8653	9.2555	9.3090	7.2925	5.1023	15.1043	10.8413	15.7726
Pain Index	29.8695	25.0160	70.4034	34.2689	26.3551	28.5410	16.2192	24.4716	72.6720
Ulcer Index	40.0949	31.4158	77.4103	45.0168	31.1532	33.2139	21.1506	25.6598	78.6463
Sharpe Ratio	0.0251	0.0333	0.0105	0.0098	0.0163	0.0193	0.0286	-0.0097	0.0058
Adjusted SR	0.2647	0.3809	-0.0584	-0.0210	0.1063	0.1500	0.3021	-0.3466	-0.1399
Omega SR	0.0753	0.0970	0.0344	0.0287	0.0473	0.0559	0.0862	-0.0289	0.0195
Sortino Ratio	0.0361	0.0484	0.0154	0.0139	0.0230	0.0276	0.0414	-0.0137	0.0085
Upside Potential Ratio	0.5159	0.5472	0.4624	0.4976	0.5108	0.5225	0.5209	0.4586	0.4433

Table 52: key statistics for “Industrials” S&P500 assets over a period of 38 years (1974-2011)

Company	E-I-du-Pont-de-Nemours	Dow-Chemical	International-Paper
Median	0.0000	0.0000	0.0000
Mean	0.0352	0.0355	0.0235
St. Dev.	1.7019	1.9652	2.0197
Skewness	-0.1873	-0.2659	-0.2849
Kurtosis	5.6442	7.7326	12.9022
Pain Index	30.0267	31.4883	37.8328
Ulcer Index	36.6714	36.9108	44.4617
Sharpe Ratio	0.0207	0.0181	0.0116
Adjusted SR	0.1945	0.1312	0.0229
Omega SR	0.0594	0.0530	0.0341
Sortino Ratio	0.0296	0.0257	0.0165
Upside Potential Ratio	0.5289	0.5101	0.5009

Table 53: key statistics for “Materials” S&P500 assets over a period of 38 years (1974-2011)

Company	Procter-and-Gamble	Coca-Cola	Wal-Mart	Pepsico	Sysco
Median	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0427	0.0445	0.0772	0.0517	0.0588
St. Dev.	1.4412	1.5553	1.8816	1.5974	1.9221
Skewness	-2.2734	-0.4221	0.1176	-0.0023	-0.2535
Kurtosis	65.4774	16.7900	4.5019	6.0962	13.3975
Pain Index	14.7619	28.1859	22.4245	12.7108	16.8999
Ulcer Index	19.3211	34.6209	27.3215	16.3156	21.7804
Sharpe Ratio	0.0296	0.0286	0.0410	0.0323	0.0306
Adjusted SR	0.1812	0.3073	0.5177	0.3889	0.3205
Omega SR	0.0922	0.0860	0.1239	0.0971	0.1059
Sortino Ratio	0.0411	0.0412	0.0608	0.0471	0.0442
Upside Potential Ratio	0.4873	0.5202	0.5512	0.5319	0.4616

Table 54: key statistics for “Consumer Staples” S&P500 assets over a period of 38 years (1974-2011)

Company	Exxon-Mobile	Halliburton	Chevron	Marathon-Oil
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0487	0.0229	0.0423	0.0187
St. Dev.	1.4563	2.4814	1.6438	2.1507
Skewness	0.4285	-1.0965	-0.0344	-1.5098
Kurtosis	18.6008	28.8362	7.2127	37.5474
Pain Index	11.4322	58.6558	17.4717	60.1186
Ulcer Index	15.4194	63.3055	23.1092	63.4209
Sharpe Ratio	0.0334	0.0092	0.0257	0.0087
Adjusted SR	0.3576	-0.0566	0.2803	-0.0389
Omega SR	0.0992	0.0268	0.0743	0.0256
Sortino Ratio	0.0479	0.0129	0.0370	0.0120
Upside Potential Ratio	0.5305	0.4925	0.5358	0.4784

Table 55: key statistics for "Energy" S&P500 assets over a period of 38 years (1974-2011)

Company	AEP	ED	DTE	ETR	PCG	CNP	ALE	EDE	SJW
Median	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0359	0.0502	0.0449	0.0362	0.0386	0.0338	0.0280	0.0286	0.0315
St. Dev.	1.3288	1.3408	1.2903	1.5423	1.7227	1.8090	1.4944	1.2914	1.7273
Skewness	-0.3423	-2.7434	0.1982	-0.5136	-3.7311	-2.0235	-10.1421	-0.8487	-6.1971
Kurtosis	25.0967	81.6862	8.6710	15.5054	153.2051	158.7629	481.1666	25.6691	246.2382
Pain Index	14.2513	12.2819	10.6405	17.8820	26.3350	33.5565	28.7244	17.6375	37.2476
Ulcer Index	19.9479	18.4010	14.7360	23.7171	35.6507	48.1486	38.6804	21.6983	44.8393
Sharpe Ratio	0.0271	0.0374	0.0348	0.0235	0.0224	0.0187	0.0188	0.0221	0.0183
Adjusted SR	0.2891	-0.0604	0.4413	0.2404	0.1229	0.1187	0.0402	0.2275	0.0907
Omega SR	0.0850	0.1231	0.1059	0.0731	0.0802	0.0648	0.0639	0.0687	0.066
Sortino Ratio	0.0387	0.0517	0.0507	0.0331	0.0305	0.0254	0.0242	0.0310	0.0243
Upside Potential Ratio	0.4935	0.4721	0.5292	0.4858	0.4108	0.4175	0.4034	0.4816	0.393

Table 56: key statistics for "Utilities" S&P500 assets over a period of 38 years (1974-2011)

Company	IBM	3M	Unisys	United-Technologies	HP-Inc.
Median	0.0000	0.0000	0.0000	0.0000	0.0000
Mean	0.0312	0.0383	-0.0310	0.0578	0.0348
St. Dev.	1.6596	1.4663	3.3260	1.6949	2.3010
Skewness	-0.3192	-0.3611	-0.3647	-0.5840	-0.2686
Kurtosis	13.0601	9.1373	31.3659	13.4897	7.0451
Pain Index	28.4267	14.7063	32.0496	14.5218	42.3955
Ulcer Index	34.8427	19.0058	39.4861	18.8600	50.6945
Sharpe Ratio	0.0188	0.0261	-0.0093	0.0341	0.0151
Adjusted SR	0.1651	0.2911	-0.3154	0.3665	0.0569
Omega SR	0.0555	0.0768	-0.0305	0.1003	0.0439
Sortino Ratio	0.0269	0.0374	-0.0130	0.0489	0.0214
Upside Potential Ratio	0.5117	0.5241	0.4123	0.5363	0.5093

Table 57: key statistics for "Technology" S&P500 assets over a period of 38 years (1974-2011)

10.2 Results

The results subsection deals with those risk-return diagrams that did not fit into the main part because we wanted to avoid redundancies. Nevertheless, they are worth too have a look at.

QP3 Markowitz Problem

Second Data Set: 1990 - 2009

The second data set containing 39 S&P500 assets from different sectors gives the graph below:

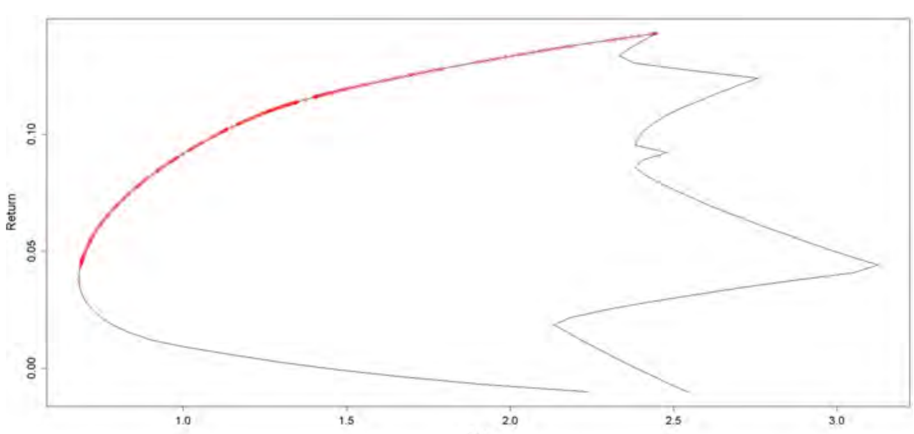

Figure 29: risk-return diagram showing the feasible set including red dots on the efficient frontier for 39 S&P500 assets over a period of ten years (01/01/1990 - 31/12/1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return maximization and risk minimization.

Again, the red dots gather around the efficient frontier. Because of the increased number of securities we receive more "spikes" on the right side of the risk-return diagram.

Third Data Set: 1974 - 2011

Finally, the risk-return diagram for the third data set is presented. The portfolio weights were estimated over a period of 26 years (01/01/1974 - 31/12/1999):

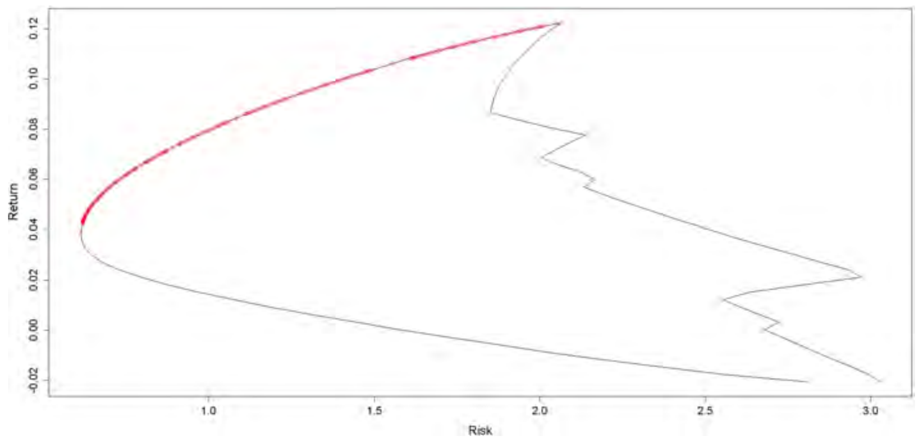

Figure 30: risk-return diagram showing the feasible set including red dots on the efficient frontier for 50 S&P500 assets over a period of 26 years (01/01/1974 - 31/12/1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return maximization and risk minimization.

Expectedly, the efficient frontier is highlighted by an accumulation of red dots and the once more increased number of assets leads to more spikes on the right part of the diagram.

QP3 Markowitz Problem including EWP Diversification

Second Data Set: 1990 - 2009

For the second data set the risk-return diagram including EWP diversification looks like this:

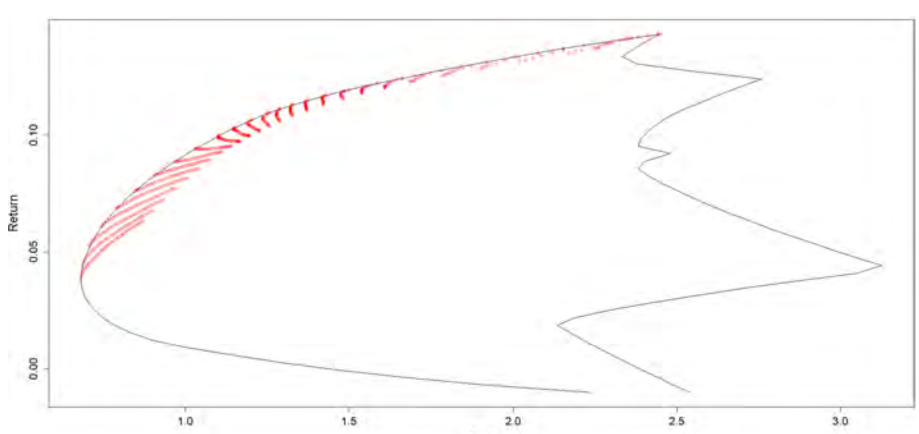

Figure 31: risk-return showing the feasible set including red dots indicating the optimization competition with respect to return, risk and Herfindahl index for 39 S&P500 assets over a period of ten years (1990-1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return / diversification maximization and risk minimization; the variation of red dots ranges from the minimum variance portfolio to the equal weights portfolio and the maximum return portfolio.

As opposed to the first data set, the red dots adapt more closely to the efficient frontier. Again zooming in and associating the degree of diversification with the dot color adds more detail to the graph. Note that the risk and return proxies are given by the average return and the standard deviation of the equal weights portfolio:

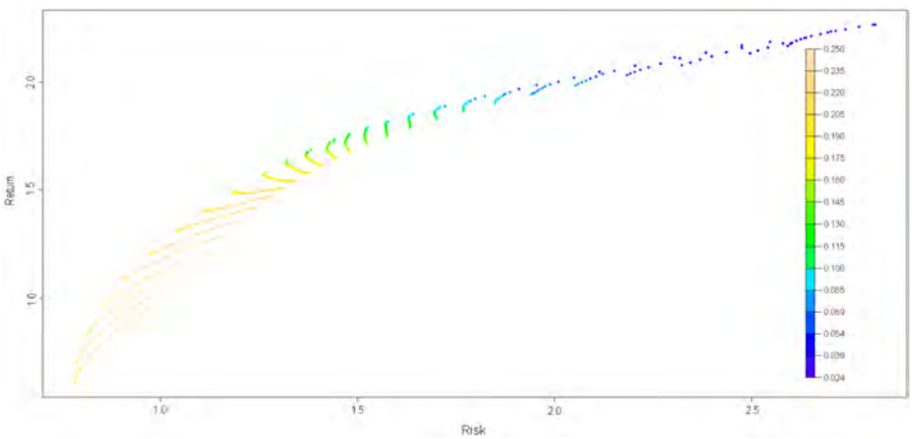

Figure 32: risk-return diagram showing the afore-exemplified variation of dots indicating the optimization competition with respect to return, risk and Herfindahl index for 39 S&P500 assets over a period of ten years (1990-1999); as opposed to before, the picture is zoomed in and the level of diversification for each single portfolio is branded by a color (skin-color: well diversified, blue color: poorly diversified); as expected, well diversified portfolios can be found in the proximity of the equal weights portfolio, while poorly diversified portfolios are located near the maximum return portfolio.

Again, skin color corresponds to strong diversification with respect to the equal weights portfolio, while blue color is associated with weak diversification.

Third Data Set: 1974 - 2011

Finally, the risk-return diagram showing the feasible set including (sub-)efficient frontier for the third data set is plotted:

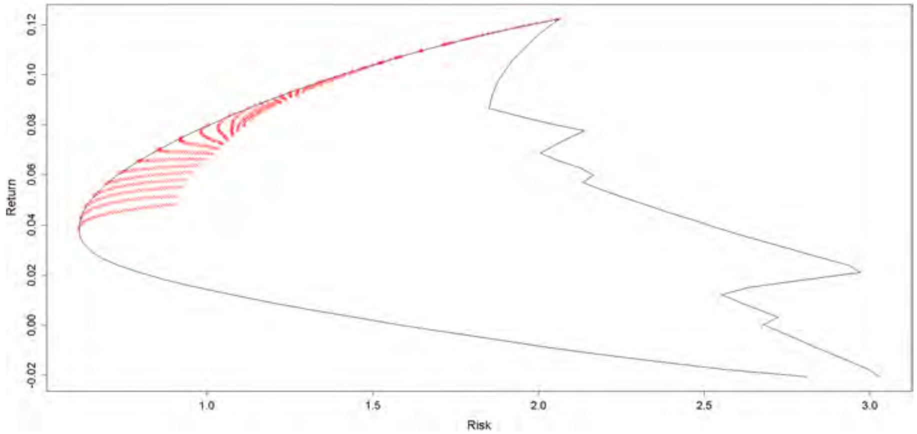

Figure 33: risk-return showing the feasible set including red dots indicating the optimization competition with respect to return, risk and Herfindahl index for 50 S&P500 assets over a period of 26 years (1974-1999); each red dot represents a single portfolio that is defined by the choice of lambdas which typifies the investor's fondness for return / diversification maximization and risk minimization; the variation of red dots ranges from the minimum variance portfolio to the equal weights portfolio and the maximum return portfolio.

Exptectedly, the sub-efficient frontier lies close to the efficient frontier while the rest of the feasible set remains untouched. Once more, we zoom in and brand the dots according to the level of diversification:

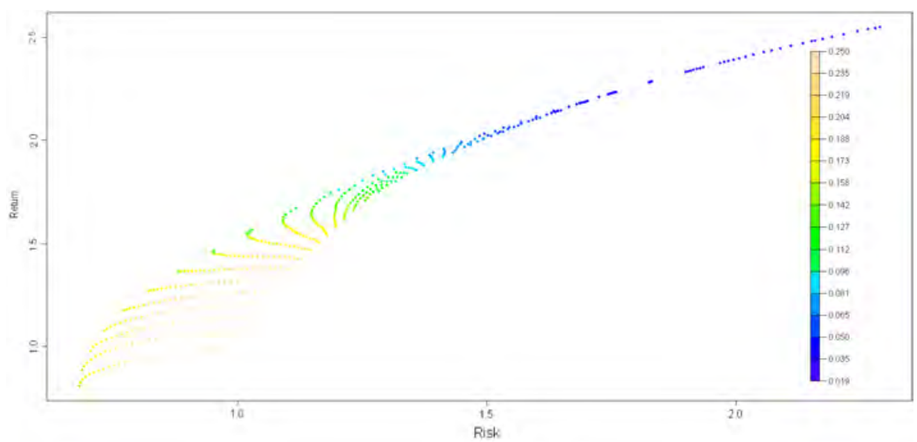

Figure 34: risk-return diagram showing the afore-exemplified variation of dots indicating the optimization competition with respect to return, risk and Herfindahl index for 50 S&P500 assets over a period of 26 years (1974-1999); as opposed to before, the picture is zoomed in and the level of diversification for each single portfolio is branded by a color (skin-color: well diversified, blue color: poorly diversified); as expected, well diversified portfolios can be found in the proximity of the equal weights portfolio, while poorly diversified portfolios are located near the maximum return portfolio.

As before, highly diversified portfolios can be found close to the minimum variance and the equal weights portfolio, whereas close to the maximum return portfolio we only find poorly diversified portfolios. With the help of the risk-return diagrams above we can conclude that the addition of the third goal, namely diversification with respect to equal weights, adds a cluster of portfolios to our feasible set, which is confined by the efficient and the sub-efficient frontier, while the corners of the cluster indicate the min-var, the EWP and the maximum return portfolio, respectively. Once again, considering a risk-free rate allows us to live outside the feasible set by drawing a line from any of our colored portfolios to the respective risk-free rate on the y-axis of the risk-return diagram.

10.3 Conclusion

In this subsection we are presenting the sectors and their subsectors mentioned in the conclusion section:

Industry	Supersector	Sector	Subsector
Oil & Gas	Oil & Gas	Oil & Gas Producers	Exploration & Production
			Integrated Oil & Gas
		Oil Equipment, Services & Distribution	Oil Equipment & Services
			Pipelines
		Alternative Energy	Renewable Energy Equipment
			Alternative Fuels

Table 58: Oil & Gas subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Basic Materials	Chemicals	Chemicals	Commodity Chemicals
			Specialty Chemicals
	Basic Resources	Forestry & Papers	Forestry
			Paper
		Industrial Metals & Mining	Aluminum
			Nonferrous Metals
			Iron & Steel
		Mining	Coal
			Diamonds & Gemstones
			General Mining
			Gold Mining
			Platinum & Precious Metals

Table 59: Basic Materials subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Industrials	Construction & Materials	Construction & Materials	Building Materials & Fixtures
			Heavy Construction
	Industrial Goods & Services	Aerospace & Defense	Aerospace
			Defense
		General Industrials	Containers & Packaging
			Diversified Industrials
		Electronic & Electrical Equipment	Electrical Components & Equ
			Electronic Equipment
		Industrial Engineering	Commercial Vehicles & Truck
			Industrial Machinery
		Industrial Transportation	Delivery Services
			Marine Transportation
			Railroads
			Transportation Services
			Trucking
		Support Services	Business Support Services
			Business Training & Employn
			Financial Administration
			Industrial Suppliers
			Waste & Disposal Services

Table 60: Industrials subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Consumer Goods	Automobiles & Parts	Automobiles & Parts	Automobiles
			Auto Parts
			Tires
	Food & Beverage	Beverages	Brewers
			Distillers & Vin
			Soft Drinks
		Food Producers	Farming & Fish
			Food Products
	Personal & Household Goods	Household Goods & Home Construction	Durable Househ
			Nondurable Hou
			Furnishings
			Home Construct
		Leisure Goods	Consumer Elect
			Recreational Pr
			Toys
		Personal Goods	Clothing & Acc
			Footwear
			Personal Produ
		Tobacco	Tobacco

Table 61: Basic Materials subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Health Care	Health Care	Health Care Equipment & Services	Health Care Providers
			Medical Equipment
			Medical Supplies
		Pharmaceuticals & Biotechnology	Biotechnology
			Pharmaceuticals

Table 62: Health Care subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Consumer Services	Retail	Food & Drug Retailers	Drug Retailers
			Food Retailers & Wholesalers
		General Retailers	Apparel Retailers
			Broadline Retailers
			Home Improvement Retailers
			Specialized Consumer Services
			Specialty Retailers
	Media	Media	Broadcasting & Entertainment
			Media Agencies
			Publishing
	Travel & Leisure	Travel & Leisure	Airlines
			Gambling
			Hotels
			Recreational Services
			Restaurants & Bars
			Travel & Tourism

Table 63: Consumer Services subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Telecommunications	Telecommunications	Fixed Line Telecommunications	Fixed Line Telecommunication
		Mobile Telecommunications	Mobile Telecommunications

Table 64: Telecommunications subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Utilities	Utilities	Electricity	Conventional Electricity
			Alternative Electricity
		Gas, Water & Multiutilities	Gas Distribution
			Multiutilities
			Water

Table 65: Telecommunications subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Financials	Banks	Banks	Banks
	Insurance	Non Life Insurance	Full Line Insurance
			Insurance Brokers
			Property & Casualty Insurance
			Reinsurance
		Life Insurance	Life Insurance
	Real Estate	Real Estate Investment & Services	Real Estate Holding & Development
			Real Estate Services
		Real Estate Investment Trusts	Industrial & Office REITs
			Retail REITs
			Residential REITs
			Diversified REITs
			Specialty REITs
			Mortgage REITs
			Hotel & Lodging REITs
	Financial Services	Financial Services	Asset Managers
			Consumer Finance
			Specialty Finance
			Investment Services
			Mortgage Finance
		Equity Investment Instruments	Equity Investment Instruments
		Nonequity Investment Instruments	Nonequity Investment Instruments

Table 66: Financials subsectors according to the Industrial Classification Benchmark

Industry	Supersector	Sector	Subsector
Technology	Technology	Software & Computer Services	Computer Services
			Internet
			Software
		Technology Hardware & Equipmen	Computer Hardware
			Electronic Office Equipment
			Semiconductors
			Telecommunications Equipment

Table 67: Technology subsectors according to the Industrial Classification Benchmark

10.4 Code

The code subsection presents the code used to generate the risk-return diagrams from the results section in the main part of the paper.

QP3 Markowitz Problem

```
modelQP3 <- c( N = "param N ;",
               lambda = "param lambda ;",
               mu = "param mu{1..N} ;",
               Sigma = "param Sigma{1..N,1..N} ;",
               mumax = "param mumax ;",
               Sigmamax = "param Sigmamax ;",
               Var = "var w{1..N} >= 0;",
               Objective = "maximize M: lambda * (sum{i in 1..N} mu[i] * w[i])/mumax-
                    (1-lambda)*(sum{i in 1..N}sum{j in 1..N}w[i]*Sigma[i,j]*w[j])/Sigmamax;",
               Budget = "subject to Budget: sum{i in 1..N} w[i] = 1 ;")
amplModelFile(modelQP3, project)
```

Box 1: model file for QP3 mean-variance portfolio optimization

```
modelQP3 <- c( N = "param N ;",
               lambda1 = "param lambda1 ;",
               lambda2 = "param lambda2 ;",
               mu = "param mu{1..N} ;",
               Sigma = "param Sigma{1..N,1..N} ;",
               mumax = "param mumax ;",
               Sigmamax = "param Sigmamax ;",
               Var = "var w{1..N} >= 0;",
               Objective = "maximize M: lambda1 * ((sum{i in 1..N} mu[i] * w[i])/
                    mumax- (lambda2/lambda1) * (sum{i in 1..N}
                    sum{j in 1..N}w[i]*Sigma[i,j] * w[j])/Sigmamax) ;",
               Budget = "subject to Budget: sum{i in 1..N} w[i] = 1 ;")
amplModelFile(modelQP3, project)
```

Box 2: model file for QP3 mean-variance portfolio optimization using a different λ-implementation

QP3 Markowitz Problem plus EWP Diversification

```
modelEWPherfindahl <- c( N = "param N ;",
                  lambda1 = "param lambda1 ;",
                  lambda2 = "param lambda2 ;",
                  mu = "param mu{1..N} ;",
                  mumax = "param mumax ;",
                  Sigmamax = "param Sigmamax ;",
                  Sigma = "param Sigma{1..N,1..N} ;",
                  Identity = "param Identity{1..N,1..N} ;",
                  Var = "var w{1..N} >= 0;",
                  Objective = "minimize M: - lambda1*(sum{i in 1..N} mu[i] * w[i])/mumax
                  + lambda2*(sum{i in 1..N} sum{j in 1..N} w[i]*Sigma[i,j]*w[j])/Sigmamax
                  + (1-lambda1-lambda2)*(sum{i in 1..N} sum{j in 1..N} w[i] * Identity[i,j] * w[j]) ;",
                  Budget = "subject to Budget: sum{i in 1..N} w[i] = 1 ;")
amplModelFile(modelEWPHerfindahl, project)
```

Box 3: model file for QP3 mean-variance optimization plus EWP diversification using Herfindahl Index

```
modelEWPentropy <- c( N = "param N ;",
                  lambda1 = "param lambda1 ;",
                  lambda2 = "param lambda2 ;",
                  mu = "param mu{1..N} ;",
                  mumax = "param mumax ;",
                  Sigmamax = "param Sigmamax ;",
                  Sigma = "param Sigma{1..N,1..N} ;",
                  Var = "var w{1..N} >= 0;",
                  Objective = "minimize M: - lambda1*(sum{i in 1..N} mu[i] * w[i])/mumax
                  + lambda2*(sum{i in 1..N} sum{j in 1..N} w[i]*Sigma[i,j]*w[j])/Sigmamax
                  + (1-lambda1-lambda2)*(sum{i in 1..N} sum{j in 1..N} -w[i] * ln(w[j])) ;",
                  Budget = "subject to Budget: sum{i in 1..N} w[i] = 1 ;")
amplModelFile(modelEWPentropy, project)
```

Box 4: model file for QP3 mean-variance optimization plus EWP diversification using entropy concentration

Minimum Variance Portfolio

```
modelMinVar <- c( N = "param N ;",
                  mu = "param mu{1..N} ;",
                  Sigma = "param Sigma{1..N,1..N} ;",
                  Var = "var w{1..N} >= 0;",
                  Objective = "minimize risk: (sum{i in 1..N} sum{j in 1..N} w[i]*Sigma[i,j]*w[j]);",
                  Budget = "subject to Budget: sum{i in 1..N} w[i] = 1 ;")
amplModelFile(modelMinVar, project)
```

Box 5: model file for minimum-variance portfolio