Model error from the Time@Risk perspective

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Abstract

Two option pricing models are discretized and simulated data from each is used to compare the resulting hedges of an option's payoff when the assumed underlying is correct vs. is misspecified. These models are the JLS and the BS models. We distinguish two factors that hinder perfect replication and are very much relevant in daily trading: discretization error and model error. In view of studying model error, we introduce a quantity measured in the simulation study that represents model error as observed in a hedging situation. This quantity is then analysed and quantified when the underlying is of BS-type and when it is of JLS-type.

Keywords: Johansen-Ledoit-Sornette (JLS) model, Black Scholes (BS) model, tracking error, model error and model risk, finite-time singularity, option hedging.

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1 Introduction

1.1 Sources of imperfect hedging

Our aim in this paper is to provide a comparison between two continuous-time share price processes and their respective option price models in a hedging context. These are the Black-Scholes (BS) and the Johansen-Ledoit-Sornette (JLS) models. To be more precise we treat a slightly generalized BS model, one with a time-varying drift as opposed to the classical constant return percentage on stock. In both models we assume to work in units of the bank account numeraire with no discounting needed. This amounts to assuming interest rates are risk free and set to zero. For the sake of simplicity, only European call options are considered. The option payoffs have the form $(S_{T_C} - K)_+$ where $(S_t)_{t\in[0,T]}$ is the underlying stock price process, $T_C \in]0,T]$ is the option's expiration date and K is the strike price. Several factors can prevent the trader from eliminating all risk from a sold option's underlying fluctuations up to maturity. Assuming self-financing strategies are the only ones allowed, one such factor is market incompleteness (when present in the model considered). It is relevant in a market whose stock prices follow the JLS process since the latter has jumps, one property of market incompleteness. In fact complete markets are rare in reality since they require restrictive assumptions such as unlimited liquidity of its traded assets, absence of transaction costs, deterministic volatility and no discontinuous behavior of stock prices. The BS model works under such assumptions. One can see for instance how, in a market with transaction costs, the latter can hinder the effectiveness of a strategy that intends to approximate the BS hedge as good as possible. We will call this error the *incompleteness error* (see section 2.2).

Yet two other important factors involved in inaccurate hedging are *model error* and *discretization error*.[1] We discuss them next (for a proper definition see section 2.2). If the option is priced according to its true underlying, discretization error is then the difference between the value process of a hedging strategy with discretized (piecewise constant) hedging ratios and the value process of the continuously hedged strategy. It arises as a result of the departure from continuous time hedging, through discretization of the time interval $[0, T_C]$. By doing this we slice the entire span of time to trade into subintervals over which we hold the portfolio proportion in the underlying stock fixed despite option price fluctuations, incurring a hedging error. It is clear that continuous-time models can only serve as approximations of hedging in practice since traders can only hedge at discrete time intervals, thus being forced to hold their portfolio hedge ratios constant even though the underlying keeps moving in the meantime. In comparison, model error is a hedging error of a very different kind. It stems from making a wrong assumption on the underlying price process, or in other words the price of the option we intend to hedge has a payoff determined by an underlying price process unlike the one we suppose it to have. Due to model and parameter uncertainty being present in realistic situations, the model choice made to forecast stock movements may not fully capture actual stock price behavior either structurally or parametrically. Structural misspecifications, better known as model uncertainty, hail from a wrong model choice whereby fundamental observed behavior is absent in the modelling, eg. a purely continuous process model notwithstanding discontinuities being present in the in-sample or historical data of the stock price. Parametrical misspecifications, better known as parameter uncertainty, are instead a matter of ill-informed parameter values whereby no fundamental change in the behavior predicted by the model is observed when changing the parameters. In what concerns us here model error is restricted to option (mis)pricing using an inaccurate underlying model with respect to structural properties only. All handled data is synthetic – that is, generated by simulations – and thus studying paramet-

[1] Sometimes called tracking error.

rical model error with respect to real stock price movements is infeasible in our setting. In the aggregate, these three sources of error cause the portfolio value to deviate from the option price. In this project we will suppose trading strategies to be self-financing and such deviations will be understood with respect to the expected squared residual hedging error (see section 2.2), which is a benchmark used in theory (called the mean variance approach) to minimize hedging errors over self-financing strategies. Generally speaking, we assume throughout this work that the reader is familiar with basic stochastic calculus and the BS model, but unfamiliar with the JLS model and its embedding in the time@risk framework.

As mentioned above, the source of discretization error is the discretization over the time inverval of the hedging problem at hand. In comparison thereto, model error is a general concept in statistical modeling, but will take a very concrete form under the two-model framework we study. Indeed we will see that model error in our specific situation also has a discrete property as its source, that of the (possible) discontinuity of the sample paths of the price process.[2] In essence, model error will amount to accounting or not for a single jump of a certain size in the model used, given the true stock price having or not having a non-null probability of making that jump. In fact, adding jumps to a continuous Brownian process renders the implied market incomplete, which can be seen to be tantamount to the inability to perfectly hedge and thus to perfectly replicate an option in such scenarios. Market risk for a trader in such a market is never fully eliminated by hedging strategies as a consequence. This is to say that the risk of jumps is fundamentally different to volatility risk as expressed through Brownian fluctuations and should be carefully analysed if sound risk management is to be carried out. In the results presented below we clearly see how this two errors add up to yield a worse portfolio hedge of the call option than considered separately, instead of canceling out – which would be unexpected as they are errors of different nature. Indeed, assuming we are using a BS hedge for the option, the JLS model assumption for the underlying bears both of the above errors and reveals greater portfolio value deviations from the BS option price than when the underlying evolves according to the BS model, in which case model specification is correct.

1.2 Time@Risk

Closely tied to sound risk management is the time@risk concept that we now outline. Companies invested in financial markets strive to remain financially sound while at the same time being exposed to market risk, ie. the risk of asset price fluctuations. To take action in order to mitigate the impact of financial crises or general instability, companies must first predict their arrival in as much precise a future time interval as possible. That is, they must first evaluate the risk of an event before taking precautionary measures to curb its potential detrimental effects or, put differently, before increasing *resilience* to such event. Resilience in our context can be understood as the effectiveness of a company's (buffer) strategies to withstand financial shocks unscathed or at least in as good of a shape as possible, and in the latter case to swiftly restore the company's full pre-crisis financial health (see [3] for a broader out-of-our-scope definition of resilience). With this in mind, the time@risk framework encompasses all the concrete methods to be used to flag future time intervals where significant drawdowns or instability seem likely to occur – in terms of probability weighting – based on available data and related indicators. It can be thought of as computational forecasting framework and infrastructure that builds up resilience through preventive action against, most notably, systemic risk and stressors in the financial system. Once the JLS is presented and expounded, we will see how it clearly represents

[2] Out of our setting model error can be conceived in more general terms along the lines of the description above. For instance it may quantify the error made by assuming some process is deterministic when the true process is stochastic, eg. it fluctuates locally like a multiple of Brownian motion or has a stochastic volatility process instead of a constant volatility parameter. We called these differences in behavior *fundamental differences*.

one such method of forecasting, using the characteristic movements that log-returns exhibit in the pre-crash period. We concentrate on model error and related measurable quantities which we believe to be the directly relevant to the time@risk framework. Indeed the incompleteness error is reduced to the minimum in theory and can therefore not be further acted upon, whilst the discretization error is simply reduced by rehedging more frequently – with issues such as transaction costs to be taken into account (not in our work). Model error on the other hand can be difficult to optimize and interesting to try to quantify, with plenty of room for freedom of choice as compared to the other two sources of errors. Therefore it will be model error concretely (and the risk of it taking place) that we will focus on in this project.

1.3 The JLS process

As mentioned, the specific models considered in our analysis of model error are the BS and JLS models. The BS model is arguably the most known and documented option price model in the literature and we will thus forego further discussion on its workings. Nevertheless we point out one of its shortcomings in modeling logarithmic returns, namely its normal distribution assumption on the latter, ie. the log-normally distributed stock prices in the model. Such light-tailed distribution assumption leads to an underestimation of extreme (eg. crash) event likelihood. In contrast, it is well documented that log-return SDEs with jump processes lead to heavier tails. The earliest idea along these lines came from Merton [1]. He proposed a compound Poisson process to model jumps of different sizes, thereby extending the BS model to discontinuous stock prices. The goal in [1] consists in a generic modelling of upward or downward discontinuous fluctuations of any size in log-returns of assets as a reaction to relevant important news. We intend here to be more specific by concentrating on a single (severe) discontinuous downward move in log-returns. To this end we consider the JLS model for the share price process as an alternative to the BS model and proceed now with its definition.

Consider a stochastic process $(\mathrm{J}_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with state space $E = \{0, 1\}$. Define $\mathbb{F} = \{\mathcal{F}_t \,|\, t \in [0, T]\}$ and let $\tau_\mathrm{J} := \inf\{t \in [0, T] \,|\, \mathrm{J}_t = 1\}$ be the random time of the first jump from 0 to 1 taking place, associated to a crash materializing. Let $h : [0, T] \longrightarrow \mathbb{R}_+$ given by

$$h(t) = B_1(T - t)^{-\alpha} + B_2(T - t)^{-\alpha}\cos(\omega\ln(T - t) - \psi). \tag{1}$$

where B_1, B_2 are non-negative constants, $\alpha \in]0, 1[$ controls the power law behavior of the function h, $\omega \in \mathbb{R}_+$ is the angular frequency of the cosine wave and $\psi \in [0, 2\pi[$ is a constant phase. Define $\Gamma_t = \int_0^t h(s)\mathrm{d}s$. The function h is chosen to be the hazard rate (also called intensity function) of the random time τ_J and $(\Gamma_t)_{t\in[0,T]}$ is the hazard process of τ_J. In particular $\mathbb{E}[\mathrm{dJ}_t] = h(t)$. We will measure probabilities with the physical measure $\mathbb{P}$ only. Let $(W_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted Brownian motion started from zero. The JLS model consists in positing the stock price process follows the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = (\mu_0 + \kappa h(t))\mathrm{d}t + \sigma \mathrm{d}W_t - \kappa \mathrm{dJ}_t \tag{2}$$

where $\mu_0 \in \mathbb{R}$ is called the excess drift for reasons explained below, $\sigma \in \mathbb{R}_+$ denotes the volatility of the stock and $\kappa \in [0, 1]$ is a constant weight factor of the jump process measuring the severity or relative size of the crash. With no further assumptions on the process resulting from (2) we have that, if $\tau_\mathrm{J} = t^\star$, then $S_{t^*} = (1 - \kappa)S_{t^*-}$ and for $t > t^*$ the stock price continues to behave like the geometric Brownian motion part of (2) (starting from the crashed value S_{t^*}) until T is reached. In section 2.1 we will simplify the above setting in order to modify (2) to treat only processes which are constant after the crash.

Formally, the JLS model for an underlying stock is the Doleans-Dade exponential $\mathcal{E}(\mathrm{X}_t)$ of the semimartingale $\mathrm{X}_t = (\mu_0 + \kappa h(t))\mathrm{d}t + \sigma \mathrm{d}W_t - \kappa J_t$. Its form is the outcome of a specific set of postulates on the market environment we trade in. First, the existence of traders in the market following rational expectation theory, called rational traders. Rational traders determine stock prices according to the market information available up until the present time. The risk appetite of these traders is matched in the model to μ_0, with $\mu_0 > 0$ denoting risk aversion, $\mu_0 = 0$ risk neutrality and $\mu_0 < 0$ risk-seeking behavior. These conditions imply the Efficient Market Hypothesis, ie. that prices reflect all available information at a given time. It is not difficult to see that $S_t = M_t + V_t$ where

$$M_t = \int_0^t S_s(\kappa h(s)\mathrm{d}s + \sigma \mathrm{d}W_s - \kappa \mathrm{d}J_s) \tag{3}$$

is a martingale and $V_t = \mu_0 \int_0^t S_s \mathrm{d}s$ is of bounded variation. This gives meaning to the terminology *excess drift*; a consistent upward or downward trend in the data throughout the time interval $[0, T]$ is present in the case $\mu_0 \neq 0$, and equally likely "fair game" outcomes are present otherwise. Second, the existence of so-called "noise traders" in the market operating under a certain hierarchical structure.[3] Unlike rational traders, whose actions draw exclusively from exogenous news, the trading behavior of noise traders is driven by an endogenous mechanism of state-interdependence. This mechanism is bound to create instability and crashes when the initial local influence of noise traders on one another happens to lead to a final coordinated sell-off on a global scale. Vaguely speaking, inspired by a particular hierarchical structure (see appendix), the possibility of a finanical crash is defined in the model exogenously through the distribution of τ. The exogenous property of the crash implies traders cannot earn excess profit by foreseeing the crash. Additionally, under the rational expectation assumption taken, feedback loops on prices within the network of traders do not affect the probability or occurence of a crash – which can be unrealistic in practice. The intuition behind this postulate on the market is deep and not relevant to the task this project focuses on. Therefore we relegate a discussion of this point to the appendix and refer the interested reader to the papers pointed at in it.

Contrary to many common market models, a market whose prices follow the JLS model loses the Lévy property of the stochastic process describing logarithmic returns. Because of the nature of the process J, future log-price increments depend on whether the (single) jump has ocurred already or not which forces log-returns to have dependent, non-stationary increments based on whether the crash happened or not. More generally, any process with jumps with a time-dependent hazard rate will violate the stationarity assumption of a Lévy process. This is in sharp contrast to the more common Merton model, which is an (exponential) Lévy jump-diffusion as sum of two (exponential) Lévy processes – a compound poisson and a Brownian motion – plus a bounded variation term. Our setup is thus deprived of the convenient machinery of Lévy processes. For instance the Lévy-Khintchine formula, which assists in finding criteria to determine when a complex underlying asset model is a martingale. On closer inspection, one may argue the stationarity requirement of the log-price process is too restrictive to model crashes because it implies the stochastic law of returns remains the same from start to end of the trading period. By doing so, it excludes a regime switch to periods close to a financial shock with their particular circumstances – for instance given by positive feedback loop interactions

[3] See the appendix for details, understanding of the underlying structure of the JLS model will not be necessary in what follows.

between traders. Stationarity excludes any new information from affecting the distribution of future market price movements, which is a strong assumption. This single-regime, uni-period approach is a limitation for us and hence non-stationarity may be regarded as a strength, not a weakness, of the JLS model.

Intuitively, the JLS model was conceived with financial crash modeling and forecasting in mind. From (2) we see that the JLS model simply adds a single jump to the diffusive behavior of the Geometric Brownian motion in the classical Black-Scholes model for the underlying. It is therefore a jump-diffusion model. Jump diffusion models treat prices as (usually large) jumps interspersed with small continuous movements. They account for the crashes with jumps without sacrificing continuous behaviour. Unlike the standard setting for the BS model, the JLS has a time-dependent drift exploding at the finite time horizon T. The jump dynamics of the JLS are given by an inhomogeneous Poisson process whose rate has the same explosive behavior as the drift. The rational expectation hypothesis (through martinagle considerations) is behind this identical behavior; near a region of space with high likelihood of the jump obtaining, the sample paths of the price process ought to have a strong drift upwards as a reward to investors for incurring the higher risk. In other words an investor must be compensated with higher return on investment in order not to be enticed to immediately rid himself of an asset in his portfolio that shows signs of possibly crashing. Moreover, again from an intuitive, non-rigorous perspective, we note that the idea of thinking of crashes as violent instantaneous drawdowns instead of the more gentle, progressive drawdowns of diffusion processes (such as in the BS model) has clear advantages over the latter models. For one thing, it is in fact how many crashes unravel.[4] Moreover, it recognises that, in the midst of a sharp decrease in stock prices, liquidity falters and traders are forced to keep a larger proportion of their portfolios in these stocks than desired. Hence a jump maps the situation the trader (or company willing to mitigate the risk of option contracts through hedging) faces as the situation unfolds more realistically than assuming he can sell stocks progressively (and continuously in the case of the BS model) all the way through the crash.

2 Setup and trading errors

2.1 The BS and JLS models

The setting we work with is a simplified version of the dynamics for the JLS and BS models. In the notations of section 2.1 and 2.2, we let $T = T_C$, $\mu_0 = 0$, $B_2 = 0$. Respectively, this is equivalent to the expiration date being at the finite time horizon, the price processes behaving as martingales (and hence for the EMH to hold we require that the rational investors in the market be risk neutral), and log prices exhibiting no log-periodic oscillatory movements speeding up near a possible crash (see appendix). Additionally, in the case of the JLS model, we assume stock prices remain constant after a crash, which implies $S_T = S_{\tau_J}$ ($\mathbb{P}$-almost surely given that a crash occurs).

The time evolution of a Black-Scholes underlying stock price is given by a geometric Brownian motion, which under the above conditions has the form

$$\frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_t} = \mu(t)\mathrm{d}t + \sigma \mathrm{d}W_t \tag{4}$$

[4] A quick look at the S&P 500 price index during the 2008-2009 period exemplifies such claim. Pure diffusion models pretending to capture (approximately) such rough behavior must account for it with unrealistically high volatility.

with $\mu(t) = \kappa h(t)$. The parameters and function h involved are as above. In the sequel, by "BS model" we implicitly convey that the plain vanilla assumptions of deep liquidity and frictionless market hold, as well as the simplifications discussed above. The solution to (4) can be computed with Itô's formula for continuous semimartingales and reads

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\kappa\Gamma_t - \frac{\sigma^2 t}{2} + \sigma W_t\right) \tag{5}$$

with $\Gamma_t = \int_0^t h(s)\mathrm{d}s = B_1 \left(T^{1-\alpha} - (T-t)^{1-\alpha}\right) / (1-\alpha)$.

Similarly, the time evolution of a JLS underlying stock price under the above conditions (and on the same probability space as the BS model, spelled out in the introduction) is given by

$$\mathrm{d}S_t = S_t \left(\kappa h(t)\mathrm{d}t + \sigma \mathrm{d}W_t - \kappa \mathrm{dJ}_t\right) \mathbb{1}_{\{t \leq \tau_\mathrm{J}\}}. \tag{6}$$

Recall that the differential here stands as notation for the corresponding integrals – stochastic or Riemann-Stieltjes – on the right hand side. Given that a jump takes place, the interval $[\tau_\mathrm{J}, T]$ has breadth and we have $S_T = S_{\tau_\mathrm{J}}$ on $[\tau_\mathrm{J}, T]$. To solve (6) on $[0, \tau_\mathrm{J}]$ we can use Itô's lemma (for general) semimartingales (in particular then for jump-diffusions), which in our case boils down to

$$\begin{aligned} f(t, S_t) = {} & f(0, S_0) + \int_0^t \frac{\partial f(s, S_s)}{\partial S} \mathrm{d}S + \int_0^t \frac{\partial f(s, S_s)}{\partial s} \mathrm{d}s \\ & + \frac{1}{2} \int_0^t \frac{\partial^2 f(s, S_s)}{\partial S^2} \mathrm{d}\langle S \rangle_s + \sum_{\substack{s \in [0,t] \\ \Delta S_s \neq 0}} \left(f(s, S_s) - f(s, S_{s-}) - \Delta S_s \frac{\partial f(s, S_s)}{\partial S} \right) \end{aligned} \tag{7}$$

where $f : [0, t] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\langle S \rangle$ is the quadratic variation of the continuous part of the semimartingale S. Roughly speaking, formula (7) simply accounts for jumps by adding first order taylor expansions over the jumps to the usual second order expansion over the continuous semimartingale component. We can solve discontinuous Doleans-Dade exponentials in a conceptually similar way – by setting $f(t, S_t) = \log S_t$ – to the solution of continuous Doleans-Dade exponentials through the continuous version of Itô's lemma which corresponds to (7) with the sum vanishing. The only difference is in the treatment of jumps through the appropriate theory. Direct computation will yield the solution

$$S_t = S_0 \exp\left(\kappa\Gamma_t - \frac{\sigma^2 t}{2} + \sigma W_t\right) (1 - \kappa \mathrm{J}_t) \qquad \text{on } [0, \tau_\mathrm{J}]. \tag{8}$$

With some thought one might have spared the calculations. After all the JLS model just introduces a (possible) jump to the geometric Brownian motion of the BS model, and hence the process will evolve exactly like the geometric Brownian motion with the exception of a possible sudden drawdown expressed by the factor $(1 - \kappa\mathrm{J})$ until the crash materializes if it does. Hence treating the simpler geometric Brownian motion in the stochastic interval $[0, \tau_\mathrm{J})$ and then accounting for the jump separately yields the same solution.

We have now a closed analytical form for the stock prices of the BS and JLS versions of each model that we consider here. It is also possible to compute explicitly the expected (terminal) payoff of the European call option. Let $(S_T - K)_+$ and $(\tilde{S}_T - K)_+$ be the call option payoffs with respect to a JLS and a BS underlying respectively, in line with the notation of section 2.1. Suppose $T = T_C$ along with the additional assumptions needed for an equivalent martingale measure $\widetilde{\mathbb{Q}} \approx \mathbb{P}$ of $\tilde{S}$ to exist. Then an option price at $t \in [0, T]$ is given by the expected value of the (discounted) payoff $V_t^H = \mathbb{E}_\mathbb{P}[(S_T - K)_+ | \mathcal{F}_t]$ in the case of a JLS underlying and $\widetilde{V}_t^H = \mathbb{E}_{\widetilde{\mathbb{Q}}}[(\tilde{S}_T - K)_+ | \mathcal{F}_t]$ in the case of a BS underlying. From here onwards we drop the subscripts of expectations taken with respect to the physical measure $\mathbb{P}$. Note that $\mathbb{P}$ is a martingale measure of S by definition (2) and working assumption $\mu_0 = 0$. If it exists $\widetilde{\mathbb{Q}}$ is unique and so is the price of the option for underlying $\widetilde{S}$ (complete market). The option price in a JLS underlying is not unique however, and we choose one option price formula among the many approaches (incomplete market). In this paper we will provide neither of the two explicit formulas of the option prices we deal with. For the BS underlying the formula is standard and for the JLS underlying we refer to [12] for a formula derived under a few weak additional assumptions to the ones we work with.

2.2 Deviations from perfect payoff replication

Let the above assumptions on the JLS and BS market hold. Unless otherwise stated, a mention to either model is implicitly accompanied by the acceptance of the assumptions in section 2. The notation S will initially designate a generic stock price process and later, when explicitly stated, the JLS price process. Suppose we discretize the time interval $[0, T]$ into N equally spaced time points, among which we can only hedge in the $N - 1$ first. Thus we trade at

$$t \in \left\{0, \frac{T}{N-1}, \frac{2T}{N-1}, \cdots, \frac{(N-2)T}{N-1}\right\} =: I. \tag{9}$$

In the two models the market is composed of a stock S and a riskless security B, which can be thought of as a bank account. The risk free interest rate being equal to zero translates into $B_t = 1$ for all t. The proportion of holdings in stock and riskless asset at time t are denoted by θ_t and η_t respectively. The strategy we look for can be defined as a $\mathbb{F}$-predictable process $\Theta = (\theta, \eta)$ of pairs on $[0, T]$. Its value process is given by

$$V_t(\Theta) = \theta_t S_t + \eta_t. \tag{10}$$

Both in the JLS and BS market we assume from now onwards that our trading strategy is self-financing so that

$$V_t(\Theta) = V_0 + \int_0^t \theta_s \mathrm{d}S_s \tag{11}$$

with $V_0 \geq 0$ known at $t = 0$. In this case we denote $\Theta = (V_0, \theta)$.

In complete markets like in the BS framework we can find self-financing strategies satisfying $V_T(\Theta) = X$ where X is a contingent claim. In other words there exist self financing strategies that perfectly replicate any contingent claim. Such ideal scenario is no longer true in incomplete markets such as the JLS framework. There are several ways to treat option replication problems of this type, including superhedging and minimizing the hedging error. Our optimal replication criterion will fall under the latter choice, in that our goal is to minimize (globally) the expected quadratic hedging error with a self financing strategy. More precisely we mean by this to find

$$(\widehat{V}_0, \hat{\theta}) = \underset{(V_0,\theta) \in \mathbb{R}_+ \times \mathcal{X}}{\operatorname{argmin}} \left\{ \mathbb{E}\left[\left((S_T - K)_+ - (V_0 + \int_0^T \theta_s \mathrm{d}S_s)\right)^2\right]\right\} \tag{12}$$

where $\mathcal{X}$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-predictable processes on $[0, T]$. We call

$$\varepsilon^{\mathcal{I}} = \mathbb{E}\left[\left((S_T - K)_+ - (\widehat{V}_0 + \int_0^T \hat{\theta}_s \mathrm{d}S_s)\right)^2\right] \tag{13}$$

the incompleteness error at $t = 0$ (we assume $\mathcal{F}_0$ to be trivial), where the expectation is taken with respect to the physical measure $\mathbb{P}$. In the BS complete market framework it holds that $V_T(\widehat{\Theta}) = (S_T - K)_+$ for the well-known optimal strategy and hence the expected quadratic deviation from a perfect hedge $\varepsilon^{\mathcal{I}} = 0$. But in a JLS market option payoffs are not attainable in general which implies $\varepsilon^{\mathcal{I}} > 0$. For such situations optimal replication strategies have been found; in [9], a solution to the optimization problem (12) under a more general framework than ours is proved and reads

$$\hat{\theta} = \frac{\mathrm{d}\langle V^H, S\rangle_t}{\mathrm{d}\langle S\rangle_t} \qquad \widehat{V}_t = \mathbb{E}[(S_T - K)_+ | \mathcal{F}_t]. \tag{14}$$

In particular we can compute $\widehat{V}_0$, as well as the risk-free security holdings derived via (10). The process $\hat{\theta}$ can be derived in the JLS framework and used to hedge optimally in the mean-variance sense. We use this strategy for the JLS hedge in our results presented in section 3. For a detailed derivation of it see [12].

As stated in section 1.1, because of the finite number of times $N-1$ we update our portfolio hedge meanwhile the option price changes in between two updates, the continuous trading assumption is necessarily violated, introducing the following source of error in option replication. Let S be some stock price process, $\Theta = (V_0, \theta)$ a trading strategy on $[0,T]$ and $(\theta_t^{disc})_{t\in I}$ the discrete predictable process resulting from the discretization on time grid I of $(\theta_t)_{t\in[0,T]}$. The initial investment V_0 remains unchanged in the discretized portfolio $\Theta^{disc} = (V_0, \theta^{disc})$. Since we look at European call options and not American ones, it makes sense to only look at the hedging error at the finite time horizon T, when the option expires. The discretization error is defined by

$$\varepsilon_T^D = \left| V_T(\Theta^{disc}) - V_T(\Theta) \right| = \left| (V_0 + \int_0^T \theta_s^{disc} \mathrm{d}S_s) - (V_0 + \int_0^T \theta_s \mathrm{d}S_s) \right|. \tag{15}$$

In particular for the BS model $\varepsilon_T^D = \left| V_T(\widehat{\Theta}^{disc}) - (S_T - K)_+ \right|$ when the optimal strategy is used. Notice that ε_T^D depends on I, which in turn is determined by the finite time horizon T and the number of discretized time points N.

The remaining source of error in payoff replication is the one we seek to quantify or at least find comparable quantitites for: model error. Let us use from here onwards the notation $S, \tilde{S}$ as in section 2.1. Denote by $\widehat{\Theta}^{(i)} = (\widehat{V_0}^{(i)}, \hat{\theta}^{(i)})$ the optimal self-financing strategy – as in (12) – that tracks some stock price process $S^{(i)}$, where $i \in \{1,2\}$ and $S^{(1)} \neq S^{(2)}$ in law (ie. the finite dimensional marginal distributions differ). Next, suppose we lack information on the dynamics of some stock in the market whose true price process is given by $S^{(1)}$. With the limited information at our disposal we assume the prices follow the process $S^{(2)}$. Since the stochastic laws of the two processes do not match, our strategy hedging ratios will track the wrong underlying and hence our portfolio value process will deviate from the one tracking the true underlying, thus incurring a non-zero model error as defined by

$$\varepsilon_T^M = \varepsilon_T^M(S^{(1)}, S^{(2)}) = \left| (\widehat{V_0}^{(1)} + \int_0^T \hat{\theta}_s^{(1)} \mathrm{d}S_s^{(1)}) - (\widehat{V_0}^{(2)} + \int_0^T \hat{\theta}_s^{(2)} \mathrm{d}S_s^{(1)}) \right|. \tag{16}$$

In our context, $S^{(1)}, S^{(2)} \in \{S, \tilde{S}\}$. Note that we will deal with model uncertainty, because the difference between S and $\tilde{S}$ is of structural nature as mentioned in section 1.1. Once again in the particular case of complete market we can write the above as $\varepsilon_T^M = \left| (S_T^{(1)} - K)_+ - V_T^{(1)}(\widehat{\Theta}^{(2)}) \right|$.

The incompleteness error $\varepsilon^{\mathcal{I}}$ can be thought of as the bedrock replicating deviation with which we invariably have to deal and to which ε_T^M and ε_T^D possibly add on depending on our choices.

3 Results

The remainder of this paper will first explain the simulation study carried out and then present the key results of it, while commenting on the principal aspects. Fix the time horizon T and let $S^{(1)}$ designate the true underlying price process in what follows, so that picking $S^{(2)}$ to replicate the option leads to a non-zero model error. In order to address model risk by attempting to study ε_T^M, we simulate sample paths of a BS and JLS underlying and compute the theoretical optimal hedges (which are not explicitly given in this paper, as already mentioned in section 2.1) in the BS and the JLS market environment. These two ingredients are then suitably fed into the simulated errors that we measure, which are

$$\varepsilon_N^{\mathsf{true}} = \left|(\widehat{V_0}^{(1)} + \int_0^T \hat{\theta}_s^{(1),disc} \mathrm{d}S_s^{(1)}) - (S_T^{(1)} - K)_+\right| \tag{17}$$

$$\varepsilon_N^{\mathsf{false}} = \left|(\widehat{V_0}^{(2)} + \int_0^T \hat{\theta}_s^{(2),disc} \mathrm{d}S_s^{(1)}) - (S_T^{(1)} - K)_+\right|. \tag{18}$$

Both errors are considered at the time horizon T – which is the relevant choice for European options. Here N stands for the granularity of the time grid on which the portfiolio $\widehat{\Theta}^{disc,(1)}$ or $\widehat{\Theta}^{disc,(2)}$ (depending on if we simulate $\varepsilon_N^{\mathsf{true}}$ or $\varepsilon_N^{\mathsf{true}}$ respectively) is discretized. Note that $\varepsilon_N^{\mathsf{true}}$ bears the incompleteness and discretization error whereas $\varepsilon_N^{\mathsf{false}}$ bears model error in addition to the former two. Simulating $\varepsilon_N^{\mathsf{true}}$ implies $\varepsilon_T^M = 0$ and thus presupposes we know the true underlying model to begin with. We can distinguish two case scenarios; either we trade in a complete market, in which case, provided N be large, $\varepsilon_N^{\mathsf{true}} \approx 0$ and $\varepsilon_N^{\mathsf{false}} \approx \varepsilon_T^M$ (since $\varepsilon^{\mathcal{I}} = 0$ and $\varepsilon_T^D \approx 0$), or the market is incomplete, in which case the situation is more delicate. Indeed we have $\varepsilon^{\mathcal{I}} > 0$, which implies we are not able to extract the theoretical value (16) of model error from (18) as before because the incompleteness and model errors, when deprived of the square and absolute value in their respective expressions (13) and (16) and for N large, may have opposite signs and hence partly or fully cancel out in the overall hedging error (18). Therefore model error as defined in (16) is not an observable quantitiy for the trader. We will instead look at the following quantities, based on the L^2 norm of $\varepsilon_N^{\mathsf{true}}$ and $\varepsilon_N^{\mathsf{false}}$, also called the root mean square error (RMSE). We define

$$\mathsf{RMSE}_N^{\mathsf{true}} = \sqrt{\mathbb{E}\left[(\varepsilon_N^{\mathsf{true}})^2\right]} \tag{19}$$

$$\mathsf{RMSE}_N^{\mathsf{false}} = \sqrt{\mathbb{E}\left[(\varepsilon_N^{\mathsf{false}})^2\right]} \tag{20}$$

$$\Delta_N = \left|\mathsf{RMSE}_N^{\mathsf{true}} - \mathsf{RMSE}_N^{\mathsf{false}}\right|. \tag{21}$$

Note that (21) can only be computed if one knows the true underlying's price process, by choosing the portfolio $\widehat{\Theta}^{disc,(1)}$ that matches the true underlying $S^{(1)}$ to compute $\mathsf{RMSE}_N^{\mathsf{true}}$. In theory, like in our setting, the true underlying model may well be exactly known, but in practice it rarely is (estimates can be drawn from data however, though we still deal with $\varepsilon_N^{\mathsf{false}}$ in such case since estimates are not exact). This paper attempts in part to give an idea of the relevance of lacking this knowledge, in the sense that if the true model is known, then the trader can simply handle $\varepsilon_N^{\mathsf{true}}$ and this paper serves no purpose to such trader. Unlike (16), the difference Δ_N is a measurable quantity, ie. it can be observed when hedging. It is a measure of the quadratic loss of accuracy the trader incurs in practice when choosing the wrong underlying model to hedge. It is the key quantity we wish to focus on in each simulation instance performed under a given set of parameter values, with N large relative to the time horizon T to make ε_T^D vanish. The quantity Δ_N is computed as follows; as a first step we fit a shape constrained additive model (SCAM) to the RMSE data for simulated trajectories as a function of the number of discretized points N – with smaller interval length between two hedges for higher N. Then we use the end value of the fitting function at the finest time grid we considered (largest N we chose) as the best estimate of the asymptotic limit of the RMSE as N tends to infinity. We call this value γ^{BS} if the BS hedge is used and γ^{JLS} if the JLS hedge is used. For convenience we also define the crash probability on the whole trading interval $[0, T]$ by δ. Our objective is to quantify using Δ_N how much additional error in option replication should one expect when the underlying model assumed is incorrect, given that the time intervals over which we held our portfolio constant are small enough to result in a negligible discretization error. From this perspective, the quantity Δ_N will serve to pinpoint model risk as a function of the parameter values chosen.

3.1 Complete market: the BS scenario

Recall that under the assumptions of this paper the geometric Brownian motion $\tilde{S}$ of the BS underlying has a time-varying drift $\mu(t) = \kappa h(t) = \kappa B_1(T-t)^{-\alpha}$. Notice h explodes at the finite time horizon (see the appendix for the motivation behind this choice), and does so faster when α tends to one (when the power law is more accentuated). However, for an equivalent martingale measure to exist in the particular BS framework we work in, the drift in the SDE of the underlying must satisfy $\mathbb{P}\{\int_0^T \mu(s)^2 \mathrm{d}s < \infty\} = 1$. This condition will not be satisfied for α close to one, which is why we will impose the restriciton $\alpha < 0.5$ throughout section 3.1. If real data is used instead of simulated data, the parameter α is unknown and estimated from the data, so that an a priori restriction of its value is no longer necessary to quantify model error. We may also fix $B_1 = 1$ and vary only κ, since in the absence of jumps both play the exact same role as constant factors in the drift term – varying one suffices to obtain all possible behaviours of the RMSE. We fix $\sigma = 0.4, T = 1, K = 1$ for all results in this section. All price processes start at $\tilde{S}_0 = 1$. It is redundant to add the BS hedge here since this corresponds to simulating (17), which by the well known result of Black and Scholes converges to zero in the limit of continuous trading independently of parameter value choices (as we discussed in the beginning of section 3). Nevertheless we add the RMSE for the sake of comparison with the JLS hedge as well as to look at the speed of convergence of the RMSE to zero. Here are the principal results we found for a BS market.

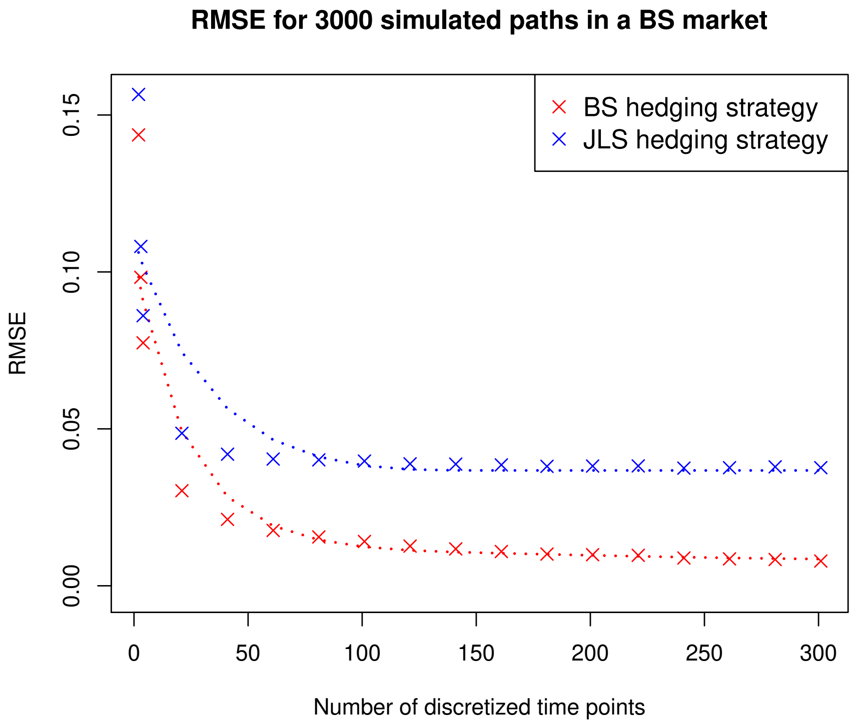
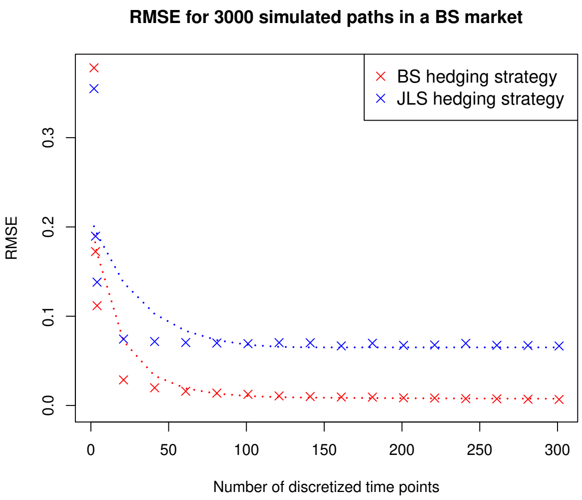


Fig. 1: Parameter values: $\alpha = 0.2, B_1 = 1, \kappa = 0.1$. Discretization-error-free estimates: $\gamma^{BS} = 0.00851 \approx 0, \gamma^{JLS} = 0.0368$. Total number of data points for the fit: 18.

In **Fig. 1** we consider to start with an expected (but almost surely not ocurring) crash of 10% of the stock price value, along with a relatively small crash probability $\delta = 1 - \exp\{-\Gamma_T\} = 0.713$. We emphasize that the crash probability and size for the BS model simply give an indication of the stock returns (captured by the drift of the SDE) that rational traders require, with no actual crash ensuing. As in the vast majority of the plots presented in sections 3.1-3.2, we consider hedging a maximum of 300 times regularly within $[0, 1]$, which corresponds to a maximum of $N = 301$ discretized points and the time $1/300$ between two hedges. We note that, throughout section 3.1, $\mathsf{RMSE}_N^{\mathsf{true}}$ will correspond to the BS hedge whereas $\mathsf{RMSE}_N^{\mathsf{false}}$ will correspond to the JLS hedge in our plots. For **Fig. 1** we obtain that $\varepsilon_T^M \approx \Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.02829$. Alternatively we may assume that with $N = 301$ discretized points $\varepsilon_T^D \approx 0$ which implies $0 \approx \mathsf{RMSE}_N^{\mathsf{true}} \approx \gamma^{BS}$ and thus to estimate model error one can simply take the limiting value γ^{JLS}.



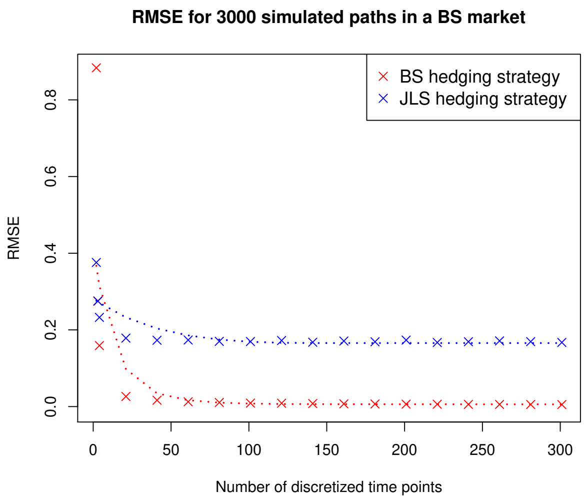


Fig. 2 (left): Parameter values: $\alpha = 0.2, B_1 = 1, \kappa = 0.5$. Discretization-error-free estimates: $\gamma^{BS} = 0.00760 \approx 0, \gamma^{JLS} = 0.0651$. Total number of data points for the fit: 18.
Fig. 3 (right): Parameter values: $\alpha = 0.2, B_1 = 1, \kappa = 0.9$. Discretization-error-free estimates: $\gamma^{BS} = 0.00589 \approx 0, \gamma^{JLS} = 0.166$. Total number of data points for the fit: 18.

In **Fig. 2** and **Fig. 3** we increase the (non-ocurring) crash to 50% and 90% of the stock price value respectively, keeping the same crash probability $\delta = 0.713$. For **Fig. 2** we obtain that $\varepsilon_T^M \approx \Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.0575$ whereas in the case of **Fig. 3** we have $\Delta_{301} \approx 0.16011$.

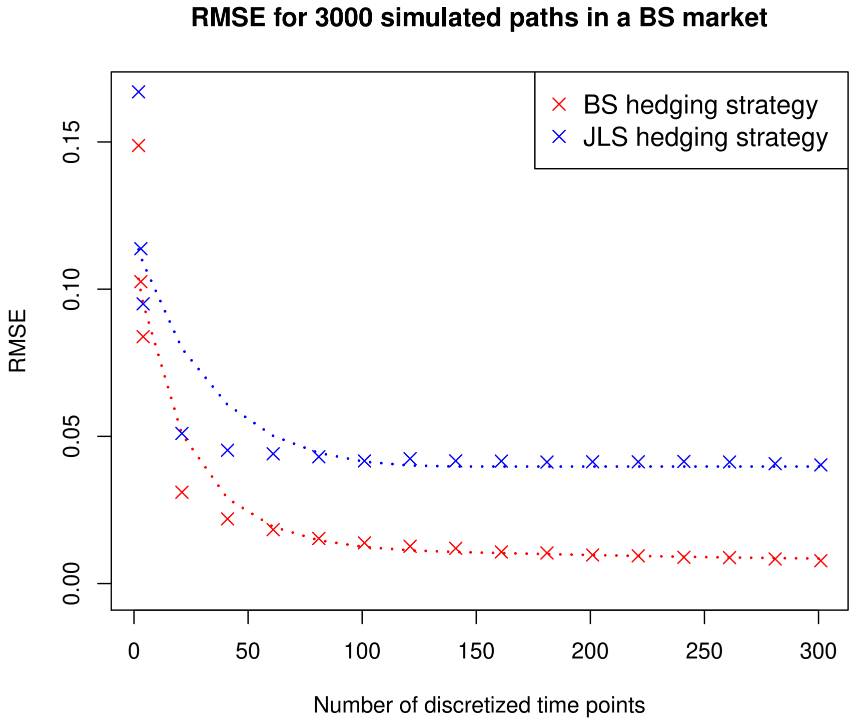
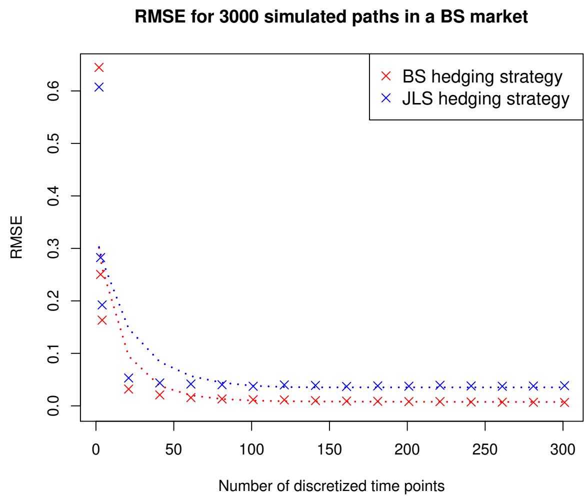


Fig. 4: Parameter values: $\alpha = 0.45, B_1 = 1, \kappa = 0.1$. Discretization-error-free estimates: $\gamma^{BS} = 0.00850 \approx 0, \gamma^{JLS} = 0.0400$. Total number of data points for the fit: 18.

In **Fig. 4** we change the crash probability to $\delta = 0.838$ and reduce the (non-ocurring) crash size to 10% of the stock price value to compare with **Fig. 1**. This amounts to a more significant upward trend in stock prices than before, recall from the introduction to the JLS process in section 1.3 that this behaviour is desired to comply with the rational expectation hypothesis whereby traders demand higher return for a higher crash probability (the latter being mainly controlled by α). For **Fig. 4** we obtain that $\varepsilon_T^M \approx \Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.0315$.



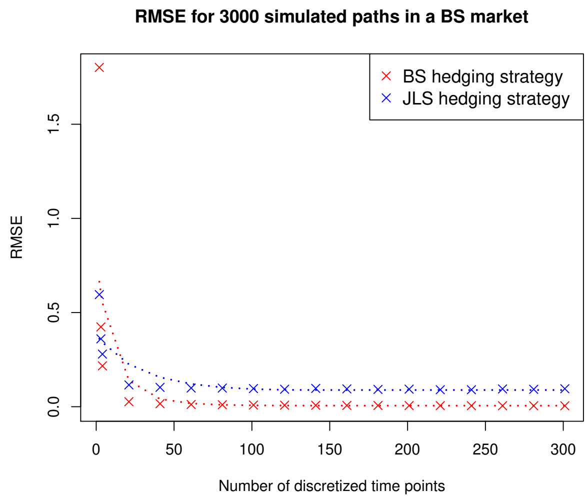


Fig. 5 (left): Parameter values: $\alpha = 0.45, B_1 = 1, \kappa = 0.5$. Discretization-error-free estimates: $\gamma^{BS} = 0.00736 \approx 0, \gamma^{JLS} = 0.0352$. Total number of data points for the fit: 18.
Fig. 6 (right): Parameter values: $\alpha = 0.45, B_1 = 1, \kappa = 0.9$. Discretization-error-free estimates: $\gamma^{BS} = 0.00562 \approx 0, \gamma^{JLS} = 0.0886$. Total number of data points for the fit: 18.

Analogous to the case $\alpha = 0.2$ we increase the (again non-ocurring) crash size to 50% and 90% of the stock price value in **Fig. 5** and **Fig. 6** respectively, whilst holding the crash probability at the same level $\delta = 0.838$. For **Fig. 5** we obtain $\varepsilon_T^M \approx \Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.02784$ whereas for **Fig. 6** we obtain $\Delta_{301} \approx 0.08298$.

3.2 Incomplete market: the JLS scenario

Next, we introduce the possibility of a crash taking place and causing a discontinuous drop of the stock price. We will now vary B_1 as well as the size of the crash (relative to the stock) κ since their effect is clearly distinct in the JLS SDE. Again we keep $\sigma = 0.4, T = 1.K = 1$, fixed and let all simulated price processses start at $S_0 = 1$. In contrast to a BS market, for the JLS market α can be raised as close to one as we wish in theory. We should point out, however, that the numerical integration method we used – namely the integrate(f,lower, upper,...) function in R – cannot handle functions which do not behave nicely (eg. are nearly zero in almost all of their range, see the R specification for details). We solved this issue by significantly increasing the relative tolerance at the cost of accuracy. There may well be other numerical integrators that will perform better in integrating the specific function we deal with (which is found in the computation of the price of an option for the JLS setup, see [12]). This being said and taken into consideration, we present now the main results in a JLS market.

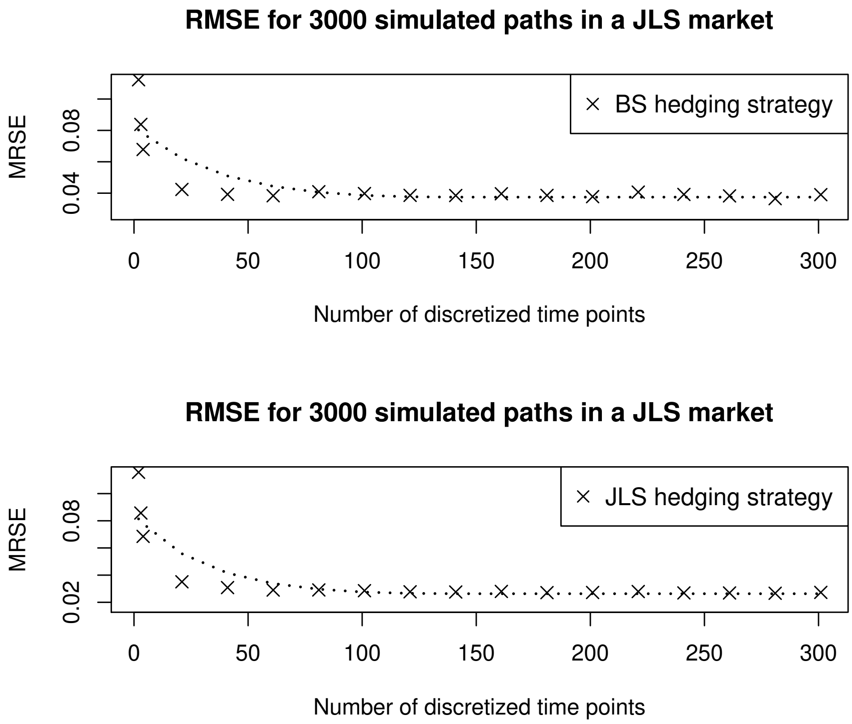


Fig. 7: Parameter values: $\alpha = 0.4, B_1 = 0.4, \kappa = 0.3$. Discretization-error-free estimates: $\gamma^{BS} = 0.0375, \gamma^{JLS} = 0.0263$. Total number of data points for the fit: 18.

We begin in **Fig. 7** with a small crash size – relative to the ones we treat – of 30% of the stock price value and a relatively small crash probability $\delta = 1 - \exp\{-\Gamma_T\} = 0.487$. The constant B_1 is small as well so that overall the upward drift of the price process is not pronounced. We note that, throughout section 3.2, $\mathsf{RMSE}_N^{\mathsf{true}}$ will correspond to the JLS hedge whereas $\mathsf{RMSE}_N^{\mathsf{false}}$ will correspond to the BS hedge in our plots. In **Fig. 7** we obtain that $\Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.0112$. Note that Δ_N is no longer a good approximation of the theoretical value ε_T^M because of incompleteness of the market. Two plots are used here to highlight that both hedges converge in a almost identical manner to their respective bedrock errors (recall $\varepsilon^{\mathcal{I}} > 0$ and model error is non-zero for the BS hedge in the JLS scenario).

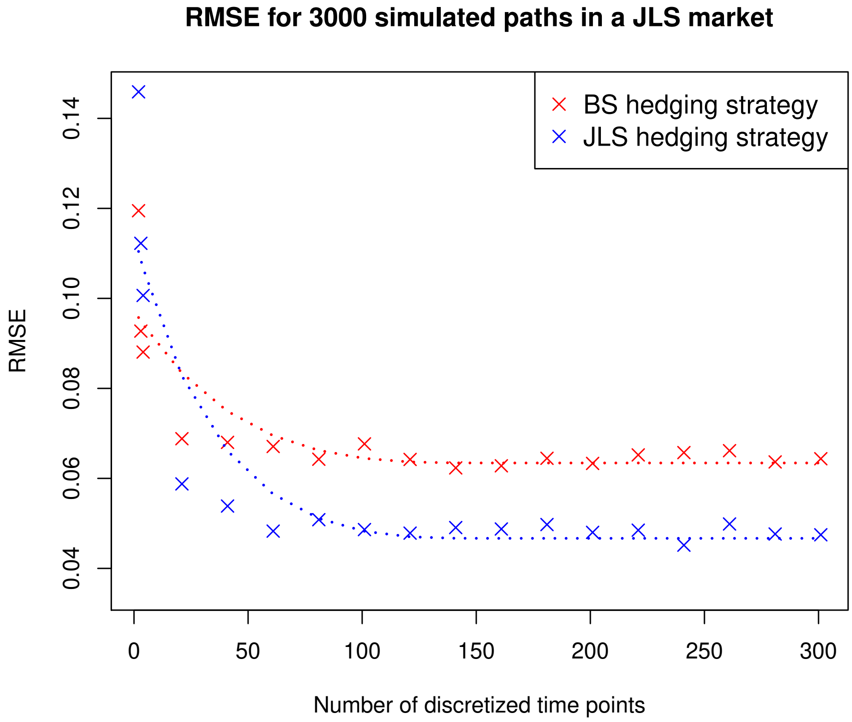


Fig. 8: Parameter values: $\alpha = 0.8, B_1 = 0.4, \kappa = 0.3$. Discretization-error-free estimates: $\gamma^{BS} = 0.0634, \gamma^{JLS} = 0.0467$. Total number of data points for the fit: 18.

In **Fig. 8** we significantly increase the crash probability to $\delta = 0.865$, which necessarily comes along with stronger positively biased drifts of the simulated sample paths. For **Fig. 8** we obtain that $\Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.0167$. Notice our estimate is higher now that the probability of a crash is higher.

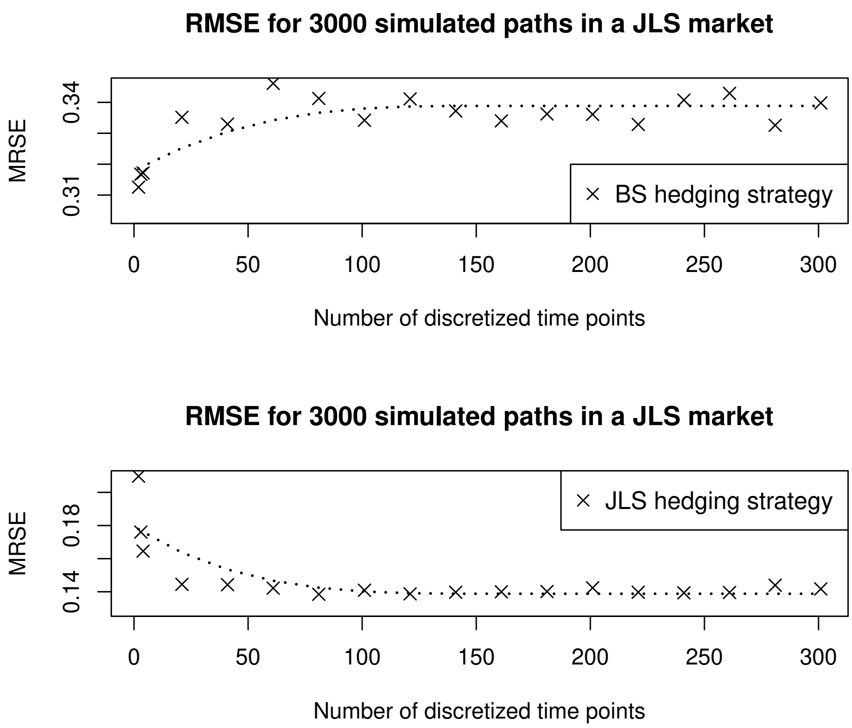


Fig. 9: Parameter values: $\alpha = 0.4, B_1 = 0.4, \kappa = 0.8$. Discretization-error-free estimates: $\gamma^{BS} = 0.339, \gamma^{JLS} = 0.139$. Total number of data points for the fit: 18.

The behavior we obtained in **Fig. 9** for the BS hedge was unexpected and we were not able to predict it from theory at first. In section 4 we will elaborate on our thoughts with respect to the initial increase of the RMSE in the JLS market for the BS hedge given the parameter values specified for the figure. The crash probability is kept (relatively) low at $\delta = 0.487$ like in **Fig. 7**, but the crash size is large compared to the latter figure *ceteris paribus*, which allows a direct comparison of both. For **Fig. 9** we obtain that $\Delta_{301} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.2$ which is significant. Surprised by this behavior, we increase the time steps and look for parameter values that will illustrate it more clearly.

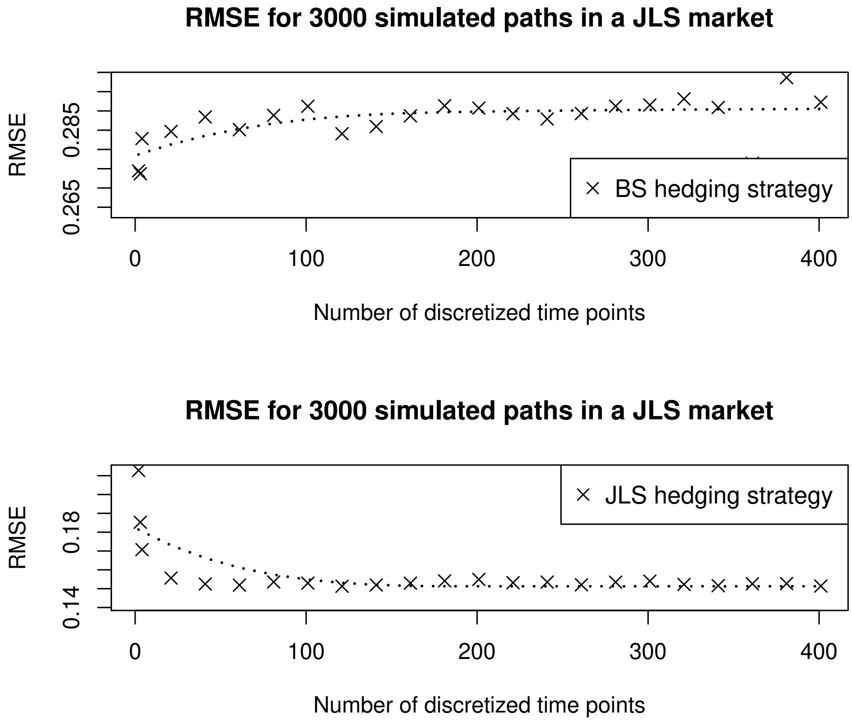


Fig. 10: Parameter values: $\alpha = 0.2, B_1 = 0.4, \kappa = 0.8$. Discretization-error-free estimates: $\gamma^{BS} = 0.291, \gamma^{JLS} = 0.151$. Total number of data points for the fit: 23.

Hedging now 400 times with $N = 401$ we retrieve this behaviour slightly more perceptibly in **Fig. 10**. In particular it now becomes visible that the BS hedge keeps increasing slightly well above $N = 50$ which was not evident in **Fig. 9**. The only difference with respect to the latter is the smaller crash probability $\delta = 0.393$ and drift induced by a smaller α. It is now apparent that hedging more often will not rid us of this behaviour, but we will discuss in section 4 whether more simulations may. The estimate for **Fig. 10** is $\Delta_{401} \approx 0.14$.

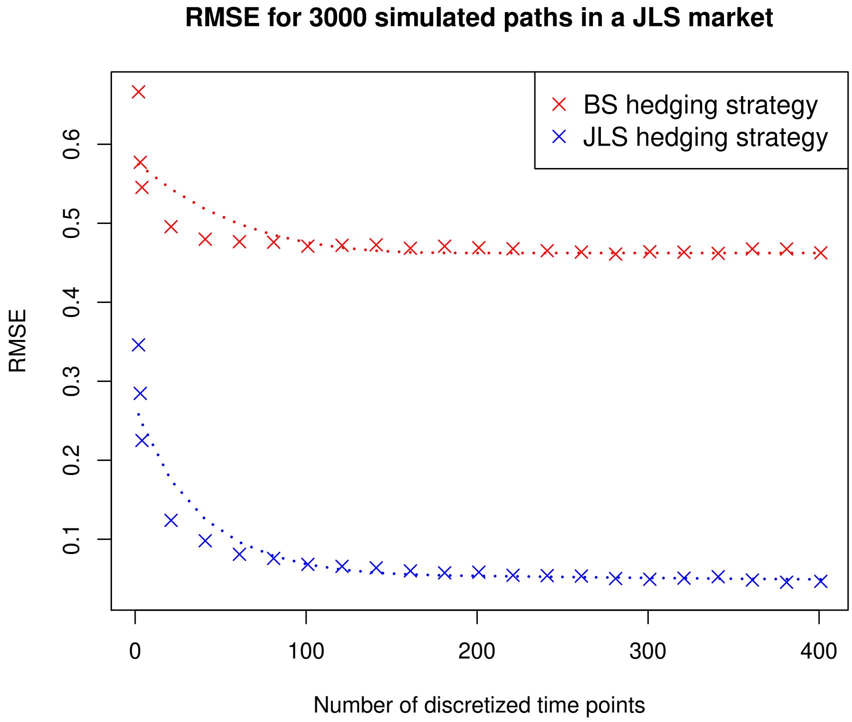
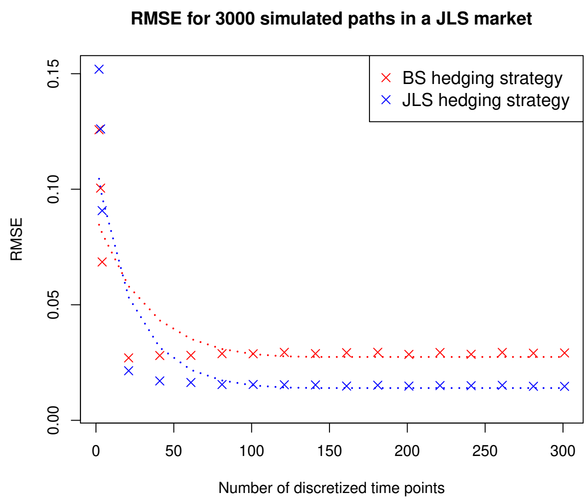


Fig. 11: Parameter values: $\alpha = 0.8, B_1 = 0.4, \kappa = 0.8$. Discretization-error-free estimates: $\gamma^{BS} = 0.462, \gamma^{JLS} = 0.0492$. Total number of data points for the fit: 23.

The result of **Fig. 11** is one of the most important in this project and will also be discussed in Section 4. What is of interest is the difference between the two hedges, resulting in a sizeable error estimate of $\Delta_{401} \approx \left|\gamma^{BS} - \gamma^{JLS}\right| = 0.4128$. Given that the trading interval is $[0, 1]$ and the initial share price value $S_0 = 1$, this quantity is significant. The crash probability is $\delta = 0.865$.



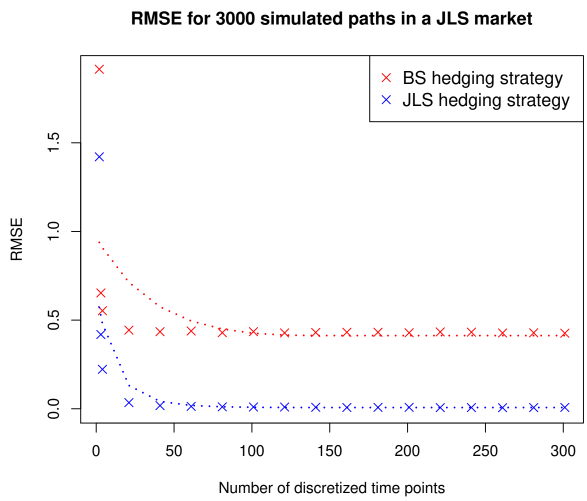


Fig. 12 (left): Parameter values: $\alpha = 0.4, B_1 = 3, \kappa = 0.4$. Discretization-error-free estimates: $\gamma^{BS} = 0.0274, \gamma^{JLS} = 0.0140$. Total number of data points for the fit: 18.
Fig. 13 (right): Parameter values: $\alpha = 0.4, B_1 = 3, \kappa = 0.8$. Discretization-error-free estimates: $\gamma^{BS} = 0.413, \gamma^{JLS} = 0.007$. Total number of data points for the fit: 18.

As a last step we considerably increase the positive constant B_1 in figures **Fig. 12** and **Fig. 13** relative to the previous plots for the JLS underlying. The crash probabilities for both figures is the same since only the crash size is modified, $\delta = 0.993$. We see how B_1 can have an important influence in deciding how likely are stock price sample paths to crash, in this case the vast majority of them do so. Notice the increase of the RMSE for a BS hedge given a larger crash size, as well as a decrease in the RMSE for a JLS hedge given a larger crash size. The error estimates are $\Delta_{301} \approx 0.0134$ for **Fig. 12** and $\Delta_{301} \approx 0.406$ for **Fig. 13**, the latter being an important deviation. The parameter choices and resulting Δ_N of **Fig. 12** and **Fig. 13** are to be compared to the pair of figures **Fig. 8** and **Fig. 11**. A priori one assigns little importance to constant B_1 since the power law of the hazard rate is determined by α, with B_1 only entering the formula as a prefactor. Likewise for the drift of the JLS SDE. However this underestimates the role of B_1 in leading to the behaviour of the Black-Scholes hedging error for a JLS underlying found in **Fig. 9** and **Fig. 10**. Indeed for larger B_1 – as in **Fig. 13** – we see a decreasing and not increasing behavior of the RMSE in the region of small N. We discuss possible explanations to this in section 4.

4 Discussion

4.1 Model error in the BS scenario

For the BS underlying case there is no surprise: the BS hedge perfectly hedges (up to a discretization error that is negligible if enough time granularity is given) the call option and the JLS hedge doesn't because it tracks a misspecified underlying. Hence the latter must underperform the former and indeed the RMSE is greater for the JLS hedge. A few minor statements are in order for this setting. The behaviour of the RMSE may resemble a square root function to the reader; and indeed it has been shown [11] that, under certain (non-trivial) assumptions, the RMSE for a BS hedge in a BS market satisfies $\mathsf{RMSE}_N^{\mathsf{true}} = \left(g/\sqrt{N}\right) + o(1/\sqrt{N})$ where g is a parameter that we were able to estimate with simulations. In addition, we see that for the JLS underlying case the speed of convergence to some non-zero limit is similar and are therefore led to think that the asymptotic convergence for the JLS case may bear resemblance to the one of the BS hedge. Studying the distribution of the RMSE for a JLS underlying in the same manner as [11] studied the case for the BS underlying RMSE should provide insight into this question. In the BS scenario, we notice by looking at **Fig. 2** and **Fig. 3** that increasing the crash size (from $\kappa = 0.5$ to $\kappa = 0.9$) results in estimates of model error that are considerably higher (from $\Delta_{301} \approx 0.0575$ to $\Delta_{301} \approx 0.16011$).

4.2 Model error in the JLS scenario

The analysis for the JLS underlying on the other hand is more interesting. Let us focus first on **Fig. 11**. It conveys precisely one of the key ideas that this project intended to investigate. The error estimate Δ_N for the parameter choices of **Fig. 8** is $\Delta_{301} \approx 0.0167$. If we increase the size of the crash from $\kappa = 0.3$ to $\kappa = 0.8$ with all else remaining equal except that we add cases of more rehedging than 300 times for more precision, then the error estimate we obtain rises up sharply to $\Delta_{401} \approx 0.4128$ as shown in **Fig. 11**. This significant difference (relative to our parameter choices) in estimates can serve to exemplify the importance of model risk as part of the risk management considerations of businesses, in particular when invested in markets whose stock prices jump (which is commonplace in real markets as mentioned in section 1.3). Thus it largely pays off to make use of a JLS hedge when we suspect the market prices to behave like a JLS process of sizeable drawdown instead of sticking to a BS hedge for the sake of simplicity. By looking at **Fig. 1**–**Fig. 6** we realize the deviation we face by erring on the flip side of the coin, ie. choosing a JLS hedge when the underlying is in fact a geometric Brownian motion, is comparatively lower – and in most cases we considered significantly so, eg. in **Fig. 4**. Statistically, if we designate by H_0 the null hypothesis "JLS underlying therefore use JLS hedge" and for the alternative hypothesis H_1 we choose "BS underlying therefore use BS hedge" (which is not the negation of the former statement), we would then say in vague terms that type I error can be severe relative to type II error. We can say that the crash size is the most influential parameter when it comes to estimate Δ_N and in turn to model error, but changing the probability of the crash via α also has an effect on Δ_N; in view of **Fig. 9** – **Fig. 11**, which have different values of α *ceteris paribus* (again excluding the value N), lowering α will decrease the BS hedge error but increase the JLS hedge error. Intuitively this is clear: less crashes taking place implies more simulated trajectories behave as a geometric Brownian motion up to arrival to the time horizon T in which case the BS hedge is the suitable hedge. In the discussion of section 4.3 concerning the unexpected increase in the RMSE for increasing N we come back to this fact. Lastly, in hindsight, one notices that the hedging error in portfolio replication is larger in general for the JLS underlying than for the BS underlying. This is expected since $\varepsilon^{\mathcal{I}} > 0$ for a JLS underlying.

4.3 A note on an unexpected simulation result

Let us now address the behavior observed in **Fig. 9** and **Fig. 10**. We expected that for all four scenarios in the statistical testing of hypotheses – these are JLS hedge chosen correctly or falsely, BS chosen correctly or falsely – we would encounter a decreasing RMSE as the portfolio hedging times $N-1$ increase. This is because when raising N we expected the discretization error to decrease while the model and incompleteness errors remain constant. For a BS hedge given a JLS underlying we found this not to be the case. **Fig. 9** and **Fig. 10** show a rapid increase of the RMSE for the smallest N values possible of a hedging strategy, before stabilizing around δ^{BS} as N increases in the case of **Fig. 9** or entering a new phase of slow increasing progression before stabilizing in the case of **Fig. 10**. Before diving into potential reasons behind such growth, we might believe it is just a matter of not enough simulations for such few time steps being performed in order for the decreasing behavior of the RMSE (that we falsely forecasted) to appear. Let us realize 3×10^4 simulations for the parameter values of **Fig. 9** instead of the 3000 we considered throughout the study.

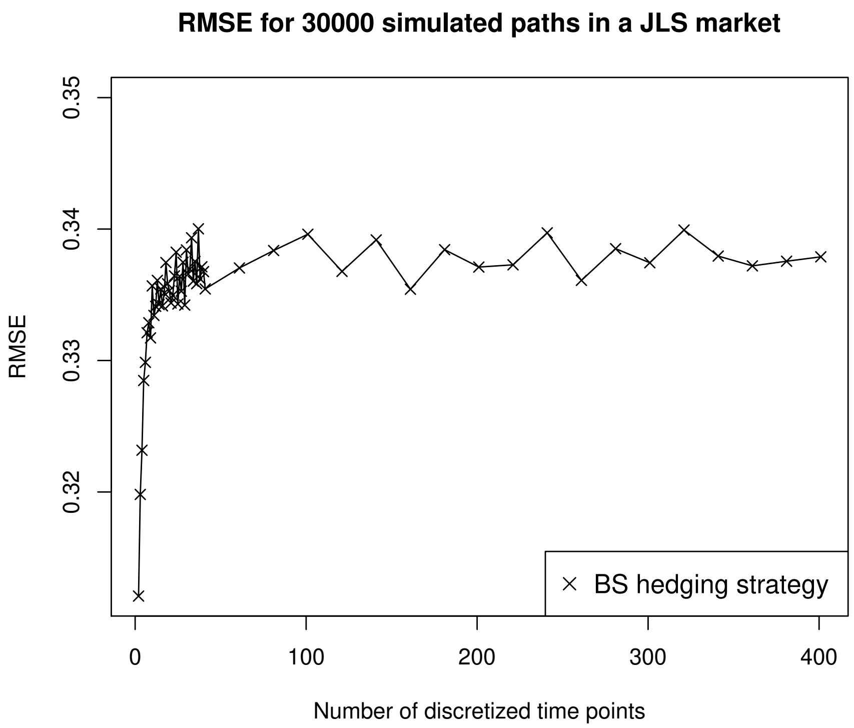
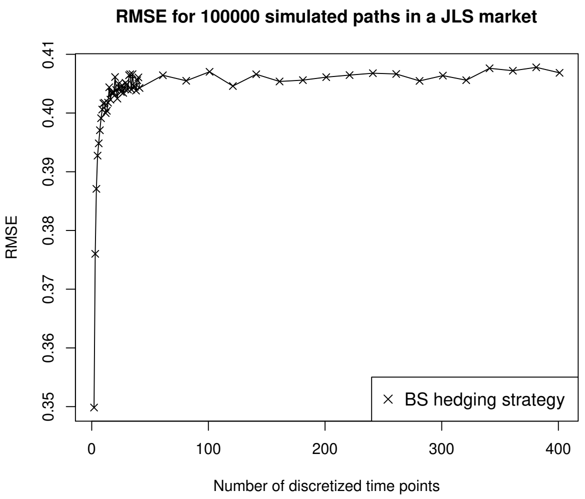


Fig. 14: Parameter values: $\alpha = 0.4, B_1 = 0.4, \kappa = 0.8$. Total number of data points for the fit: 57

The solid lines do not correspond to a fit of the data but simply join the points to show the progression of the RMSE as N increases. We can see how the unexpected behaviour persists despite higher accuracy through simulated trajectories. Below we show two addtional graphs for specific parameter values that illustrate especially well this behaviour we are attempting to understand, each with 10^5 simulations and a maximum of $N-1=400$ hedge points for the sake of reliability.



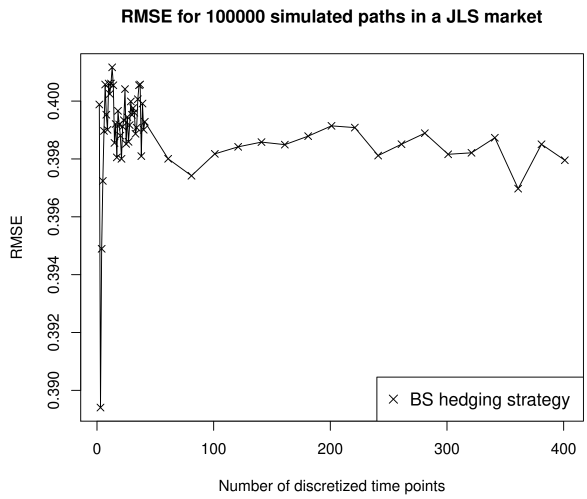


Fig. 15 (left): Parameter values: $\alpha = 0.2, B_1 = 0.5, \kappa = 0.9$. Total number of data points for the fit: 57.
Fig. 16 (right): Parameter values: $\alpha = 0.5, B_1 = 0.5, \kappa = 0.8$. Total number of data points for the fit: 57.

The unexpected growth in the RMSE seems to prevail roughly (at least) on the condition of choosing parameter values $\kappa \in [0.6, 0.9], \alpha \in [0.1, 0.4], B_1 \in [0.1, 0.5]$. That is, it takes place when the crash size is large but the probability of a crash taking place is not. We note that, as already mentioned in the comparison of **Fig. 9** and **Fig. 13**, for high B_1 (we checked $B_1 = 3$ and $B_1 = 10$) the initial sharp increase in the RMSE is replaced by an initial sharp decrease as seen in the figures for the BS underlying of section 3.1. Likewise for $\alpha > 0.7$. Only one of the two needs to be high for the increasing RMSE behaviour to vanish (one simply needs a high crash probability).

Let us now attempt to explain this behaviour. We warn the reader that what follows is intended to be a discussion of the observed simulated results – in which vague, hand-waving explanations are used – and by no means a rigorous claim whose proof we decide to omit. We believe this behaviour may be due to the overall effect of individual factors affecting the error. The easiest one to identify is encapsulated in the definition of ε_T^D, namely the greater hedging error incurred when the portfolio is held constant for longer periods of time without hedging the option payoff in between. This source of error makes for larger values of the RMSE for N small. But a second factor operates against this in the case of the BS hedge given a JLS underlying. By definition of the delta hedge, the hedge ratios $\hat{\theta}^{disc}$ that we will hold in our discretized portfolio depend on the infinitesimal variation of the price of the option with respect to the value of the underlying. Higher underlying value at a given time t will lead the trader that delta hedges the option to increase the proportion of stock holdings in the portfolio. This implies that if the number of hedging times $N-1$ is large and the underlying stock price soars, then $\hat{\theta}^{disc}$ will become larger after less time because of rehedging being done more often, which makes the trader vulnerable should a crash obtain. In particular if the size of the crash is large, we will suffer a greater financial loss than if we had considered coarser time grids and hence not rehedged that often to increase the stock holdings before the crash takes place (if it does). This may explain why for larger κ the increasing RMSE behavior seems to prevail over the effect of larger discretization error in the region of lower N. However, this should imply that if the crash – governed by α – is more probable in our model, then the increasing RMSE behavior for increasing N such reveal itself stronger. An attempt to offer an explanation why this does not happen is a third factor coming into play: the fact that lower crash probability means in particular the crash is less likely to occur far away from T (if it happens). A plot of the hazard rate for low and high α can serve to better understand this factor. In particular this implies the delta hedging strategy is allowed to be dynamically implemented for longer before the possible crash, increasing the hedging ratios to levels that would be higher than if α where large (despite the fact that the drift is larger for α large, yielding a higher hedge ratio, which could yet be another factor against the third one).

With the above factors in mind, the following provides only a partial explanation to the observed results. We believe the third factor coupled with the second make for an overall effect on the RMSE that is greater than the opposed effect of the first factor in the case of lower α, so that the effect of the former two dominates as seen in the plots with an increasing RMSE. But for larger α only the second factor interfers with the first and this may not be enough to counteract the effect of discretization error becoming smaller for large N, so that it is the latter effect that dominates. **Fig. 16** provides a good example of the tug of war between the factors above that we believe cause the observed behaviour. We see how for the first data point $N = 2$ the discretization error is big enough to make the RMSE be high despite the influence of the second and third factor, but as N grows the latter two dominate and we see an increase of the RMSE, which then remains at a value higher than the limiting value γ^{BS} for very large N. Eventually discretization error tapers off as N increases and the RMSE leaves this transition phase at the high value to come down and oscillate around the limiting value γ^{BS}. The reason why this is only a partial explanation is that there are other factors to consider apart from the former three, and the strength of each in determining the overall RMSE behavior is not clear to us.

5 Concluding remarks

The goal of this project was to investigate and quantify how much hedging error results in practice from a wrong underlying stock price process assumption, under considerable simplification of the two underlying models possible: the BS and JLS models. For the BS world the BS clearly hedges better but the difference in option replication accuracy is relatively small compared to the JLS world case. The latter adds jump risk to the Brownian fluctation risk as expressed through volatility, and hence represents a higher chance of severe losses for the trader. Indeed our figures show that in the JLS world the error estimates representing observed model error are signifcantly larger than in the BS case regardless of the hedge used. Therefore in case of doubt over the possibility of discountinuous drawdowns in a given market one should arguably err on the safe side by choosing a JLS hedge despite the added complexity (including the time complexity of the simulations to study it). This is especially true if the relative size of the crash is large, as has been elucidated in section 4 and especially well illustrated by **Fig. 11** and **Fig. 13**. The only exception to envisage is if the expected size of the jump is small enough to not generate a significant added hedge error to the BS hedge. In addition, our simulation results suggest that in a JLS market model, for the specific parameter choices that are roughly given in section 4.3, the BS hedge underperforms the trivial hedge (ie. the strategy that assumes the full market risk of the underlying stock by not hedging).

There are plenty of possible ways to carry a more advanced analysis of model risk for the JLS and BS models. One can concentrate on simulation studies that estimate risk measures such as VaR. Or introduce real market data in order to extend model error to parameter uncertainty considerations on top of model uncertainty, using statistical techniques from time series analysis to minimize the total (observed) model error on the parametric side. In view of mimicking real stock price behavior more closely, another direction of futher study is to reconsider one or many of the simplifying assumptions that were made in section 2.1 and held true throughout the paper. For example the $B_2 = 0$ assumption deprives the JLS model of a key feature with which it was endowed: the sinusoidal oscillations with increasing frequency that explode at the time horizon, which add to the power behaviour that was treated in this paper. This feature was primarily credited with making crashes predictable in advance in real markets (see the appendix and the references therein). Better yet is to realize that the behavior of the JLS price dynamics through our choice of hazard rate only accurately matches the theory (presented in [2]) that motivates its analytical form in the neighbourhood of the critical time T. Using the more complex form for log-prices in equation (22) of [2] should give deviations in option replicaton that more precisely describe markets with hierarchical interactions between traders of the type that motivates the JLS model especially when considering American options, since in this case hedging errors need to be considered far away from the time horizon, in regions where the Taylor-approximation-derived hazard rate we defined in (1) is not giving values in agreement with the noise trader network theory that motivated the JLS model.

We close off by reminding ourselves of the overarching purpose of the JLS model and more generally of the time@risk concept. It is the ambitious objective of modeling and forecasting the largest-scale financial crashes through precursors indicating if and when it arrives, then in case of affirmative feedback anticipating the crash by building resilience to it in vulnerable systems. The JLS process contributes to such a framework in at least two ways. Concerning the forecast component of time@risk, the JLS models financial market instability assuming super exponential growth of stock prices and so called log-periodic oscillations (see appendix) characteristic of financial crashes. Concerning the build-up of resilience, the JLS model can be used to quantify model risk as was done in this project in a rather simple way (more refined analysis of the kind mentioned above, eg. via risk measures, can be performed). The growth and oscillation signatures preceding the crash in the JLS process are handy in practice as a concrete indicator of the progressive build-up of "stress" in a system, which in turn increases the likelihood of a collapse, by which in our case it is meant a financial crash. Stress here is to be understood as greater sensitivity of the network of noise traders to sudden shifts in the state (ie. buying or selling mood) of a single given member of the network. Through the posited interconnectedness of the network, if that member decided for instance to start selling, it would propagate the decision to switch from buying to selling globally through the network and lead to a possible collapse of the system. The JLS model in this context is a tool that helps us cope with such systemic risk. Now diving out of theory and into concrete practice, it is worthwhile to ponder over two statements relevant to actual crashes within the global financial system, in particular the systemic series of crashes that took place in 2008-2009. The first statement can be read off an article of the Financial Times the 12th of June 2011, where former Secretary of the US Treasury L. Summers states that
"The central irony of the financial crisis is that while it is caused by too much confidence, borrowing and lending, and spending, it is only resolved by increases in confidence, borrowing and lending, and spending".
The second is a possible reply to such statement, a remark credited to Einstein:
"We cannot solve our problems with the same thinking we used when we created them".

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A The hazard rate and its implications

At first sight the convoluted structure of the intensity h of the model may not seem intuitive. The set of assumptions on the market environment behind this model feature are perhaps what distinguishes it most from other jump diffusion models such as Merton's model. The analytical form of the intensity function h in (1) reveals power law growth with added oscillatory perturbations to the trend in the form of sinusoidal waves. One shows that a similar growth pattern then follows for logarithmic returns in the model, which can be then used in practice as a precursor for a (possible) looming crash. Key to this choice is the assumption of proportionality between the *susceptibility* of (suitable) complex microscopic systems – in a handwaving manner, we mean by this the sensitivity of the overall state of the system's constituents to infinitesimal fluctuations of a global influence on them (see [2]) – and the intensity function h. Positing the proportionality of this two quantities allows for the transfer of the question "how does one best model a market vulnerable to a bubble and subsequent potential crisis?" to the question "which complex system of microscopic particles (in our case: traders) out there in the sciences best captures how a bubble originates, grows, propagates and bursts in a global financial shock?" to which the initial answer of [2] is to make use of a hierarchical diamond lattice system as first used in statistical physics [4] [5]. This answer has been refined a posteriori [7]. Vaguely speaking, in the mentioned papers, the features of such system are mapped to our financial market setting by visualizing a network of traders interacting locally in a hierarchy in a way they mutually influence their peers' decision either to buy or to sell stocks. Approaching critical time T the imitation of their direct neighbours' decision reaches its peak and coordinated action (initially on the buy side) cascades through larger and larger scales into everywhere-similar global action. At this point the risk of crash is at its highest since even the slightest perturbation, for instance in the form of bad news, will cause the system to react similarly (most likely on the sell-side) on a global scale. At a more fundamental level, the choice of h is not only motivated by the behavior of the above mentioned physical systems, but is also the result of far-reaching insight from scale invariance properties [6] and renormalization group theory. For details and deeper understanding we refer the interested reader to the cited papers.

Under the JLS model assumptions on the market environment, the following events are allowed to unfold. First a price bubble forms near a given critical time t_c as the interdependent system of noise traders gradually alineate themselves under a bullish (buying) mood. Bubble growth is described by a power law punctuated by sinusoidal wave oscillations whose frequency diverges to infinity when approaching t_c. This is easily seen in (1), where one also understands why the oscillations are termed "log-periodic". Reaching the time horizon T, either a discontinuous drawdown signals the occurrence of a crash (if it had not happened already) or the build-up tapers off more gently to return to pre-bubble values. The latter scenario may seem unrealistic but the model derives a non-zero probability of it happening and in practice there have been cases reported falling under such description. One such example supplied by [2] is the instability period end October 1997 in US equity markets. The characteristic signatures in log-price fluctuations present in the JLS framework as an omen for an imminent financial shock were measured and public warnings were issued but no significant crash event ensued. We thus note the crash need not take place under the model assumptions of the JLS, nor should it occur at the time horizon. Rather, with non-zero probability the crash may or may not happen, and if it does it will occur closer to the time horizon with a greater probability than farther away. Regardless of the case, in the JLS model signs of log-periodic and power law behavior appear in bubbles ending with a sudden crash as well as in bubbles landing smoothly.

It is important to notice that, even though the crash is treated as an exogenous event of the known unknown risk type – known probabilities of unsure occurrence – with prices incorporating the probability of its occurrence, the development and potential bursting of bubbles corresponds in the JLS model to endogenous event formation by the noise-trader network. Indeed the rise in stock prices is a consequence of the imitation of the noise traders at a micro-level which when propagated through the network end up leading to macro-level repercussions. This hints at a certain market reflixivity whereby noise traders are on a bearish or bullish mood based on the mood of the neighbours they're directly into contact with in the hierarchy. The rational traders of the market model in contrast are receptive only to external news – in particular to the probability of the crash, which is dynamically factored into stock prices by the above stated assumptions on a JLS market.

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