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# Long-term behavior of an artificial market, composed of fundamentalists and noise traders

**Master Thesis**

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# Abstract

A modified version of an artificial market model designed by Kaizoji et al [1] is provided, stressing the significance of dividends and bringing a possibility of bias amongst noise traders. Noise traders and fundamentalists are the only types of traders involved in the market, which is composed of a risk-free asset and a risky asset. The risk-free asset can be thought as a government bond whereas the risky asset can be thought as a stock.

Fundamentalists follow dividends (which belong to the *fundamentals* of the risky asset) whereas noise traders are subject to social imitation and trend-following. The competition between those two kinds of traders is investigated through the long-term behavior of the market.

We find that dividends have a significant impact on the long-term behavior of the price of the risky asset since noise traders' strategy is not persistent in the long run, compared to the strategy of fundamentalists which mixes both short-term and long-term interesting approaches.

However, in absence of dividends and of fundamentalists, noise traders are able to have a consequent impact on the long-term behavior of the market. This ability depends a lot on the herding which they allow for trend-following and social imitation. We find a phase transition for the returns of the risky asset which can beat the *price of the money*, even if there is only noise within the market.

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# Contents

# Introduction

Hommes has argued [2] that in order to appreciate the features of financial markets, it is necessary to introduce some heterogeneity, in order to model the world economy as realistically as possible. This point corresponds to an important shift in Economics, leaving the traditional, rational agent approach to a behavioral agent-based approach. This traditional view comes from Microeconomics and the use of pure Mathematics. Most of the time, only one agent – or, equivalently, many identical agents – is used to model the whole economy. Such an agent is called *representative* agent. In this sense, the behavior of the economic system is a scaled up version of the behavior of one micro-agent. She is often taken to be perfectly *rational*, so that her decisions are ruled by constrained optimization. Nevertheless, optimization does not seem adequate to represent human behavior. Indeed, how could agents form fully rational expectations in a complex, non-linear world ?

Thus, much of evolutionary economics has been focused on the behavior of heterogeneous, boundedly rational, agents who are able to interact with each other. This conceptual change has been motivated by many authors [3, 4, 5, 6], convinced that Economics is a complex system, so that it can be viewed as network structures of elements and connections. A natural framework for exploring such systems is presented by the agent-based models (ABMs), based on the aggregation of simple interactions at the micro level, leading to sophisticated structures at the macro level. In statistical physics, it is said that those systems exhibit emergent phenomena.

Agent-based modeling has been an important tool for providing support to analytical propositions. Even simple agent-based models can explain significant observed stylized facts which rational agent models cannot, as excess volatility, firstly stressed by Shiller[7], high trading volumes, temporary bubbles or trend following. The following quotation illustrates well the needs for heterogeneity:

“ *One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn't have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.* ” Arrow, 2004 [8].

As an example of such heterogeneity, there are two main philosophies of trading in Economics. The first one corresponds mostly to the traditional, rational agent approach, driven by the *fundamentals* of a given asset, such as dividends, earnings of the firm, macroeconomic growth or unemployment rates. Such investors are called *fundamentalists*. The second one, the *technical analysis*, is more recent and is thought to be responsible of speculation. Technical analysts – also called *chartists* – do not take market fundamentals into account but, instead, they base their trading strategies upon observed price patterns in past prices. For that purpose, they use various technical indicators, trying to extrapolate observed price patterns, such as trends. What is interesting is that financial practitioners have learned to use both strategies. Indeed, Frankel and Froot [9] have provided survey data of the techniques used by some forecasting services between 1978 and 1988. The results are reported in Figure 1. One can observe that, during this period of time, uses of technical analysis became more and more important, compared to fundamentals models.

| Year | Total | Chartist | Fund. | Both |
|---|---|---|---|---|
| 1978 | 23 | 3 | 19 | 0 |
| 1981 | 13 | 1 | 11 | 0 |
| 1983 | 11 | 8 | 1 | 1 |
| 1984 | 13 | 9 | 0 | 2 |
| 1985 | 24 | 15 | 5 | 3 |
| 1988 | 31 | 18 | 7 | 6 |

Figure 1: Techniques used by forecasting services. Taken from Frankel and Froot [9]. Source: Euromoney, August issues. Total = number of services surveyed; Chart. = number who reported using technical analysis; Fund. = number who reported using fundamentals models; and Both = number reporting a combination of the two. When a forecasting firm offers more than one service, each is counted separately.

In fact, at short horizons, financial practitioners tend to use chartists' trading rules whereas, at longer horizons, they tend to look carefully at fundamentals [10]. This heterogeneity has stimulated much work on agent-based models with chartists against fundamentalists.

In the present report, we shall study the behavior of an interesting agent-based model, proposed by Kaizoji [1], which shows transient super-exponential bubble growth. This model has been studied and slightly modified by other authors [11, 12]. The corresponding artificial market is composed of one risky asset and one risk-free asset, with two different competing trading strategies. The first group of traders is composed of fundamentalists; they use dividends to perceive investment opportunities. Broadly speaking, they are rational risk averse investors, maximizing a given utility function at each time step. The second group of traders, called *noise traders*, are driven both by trend-following (chartists' trading rules) and by social imitation. The latter induces some feedback, sometimes leading to the creation of financial bubbles. Those are defined in [1] as transient super-exponential growth of prices. In contrast to the original model [1] but in the same spirit than in[11, 12], we introduce an exogeneous stochastic dividend process, accounting for non trivial economic determinants, and a possibility of bias among noise traders. In the original model, the impact of dividends was not so realistic, since fundamentalists used to invest a constant fraction of their wealth in the risky asset, with respect to time.

More specifically, we shall study the long-term behavior of this modified artificial market. It has been shown in [13] that dividends can be seen as an external field – in a phase transitions meaning – so that they drive prices. This result was obtained in a static perspective and in absence of speculation. Besides, in absence of external field, that is without dividends, a phase transitions approach has been developed for prices [13]. We shall try to investigate if those stylized facts hold with our dynamic model.

In Chapter 1, a detailed explanation of the artificial market model is provided. An effort has been made to put the chosen exogeneous dividend process into perspective with traditional processes, which one can find in the litterature. The strategy of both types of traders is then developed, leading to the price equation of the risky asset. Chapter 2 focuses on the impact of dividends on the market model, using numerical simulations. The purpose of Chapter 3 is more theoretical, but most of the results obtained are then used to find what happens to the market when there is no dividend.

# Chapter 1

# The market model

## 1.1 The assets

The market model has only two assets: a risk-free asset and a risky asset. The risk-free asset can be thought of as a government bond. It has perfect elastic supply: it is guaranteed to pay a fixed *risk-free interest rate* $r_f$ at each time step, no matter how much is invested. For instance, $r_f$ could be equal to 2% per year. If one invests a wealth $W_{t-1}^{risk-free}$ in the risk-free asset at time $t-1$, one gets the wealth $W_{t-1}^{risk-free}\ (1+r_f)$ at time $t$. The risk-free interest rate $r_f$ characterizes the 'price of money': it is a source of certain returns, without any risk. As a consequence, if a trader does not invest in the risk-free asset, she expects that she will get return rates greater than $r_f$ – perhaps by taking some risk – otherwise, it is of no interest.

The risky asset can be represented by a stock. Investors buy and sell shares of this stock at a given price $P_t$, which is set, at time $t$, by supply and demand. If one invests a wealth $W_{t-1}^{risky}$ in the risky asset at time $t-1$, one obtains $\frac{W_{t-1}^{risky}}{P_{t-1}}$ shares of the stock. The price of the risky asset is, thus, a benchmark unit for the value of a share.

There are two sources of returns, so that $W_t^{risky} = W_t^a + W_t^b$. At time $t$, the price changes but the number of owned shares does not, so that one still has $\frac{W_{t-1}^{risky}}{P_{t-1}} = \frac{W_t^a}{P_t}$ shares of the stock, with a new wealth $W_t^a = W_{t-1}^{risky}\ \frac{P_t}{P_{t-1}}$. The risky asset also pays dividends $d_t$ at each time step: those are payments per owned share of the asset, chosen irrespective of the investors. Having $\frac{W_{t-1}^{risky}}{P_{t-1}}$ shares of the stock at time $t-1$, one receives the dividend payment $W_t^b = \frac{W_{t-1}^{risky}}{P_{t-1}}\ d_t$ at time $t$. Thus, being invested in the risky asset from $t-1$ to $t$ yields two kinds of returns: the *price return rate* $r_t := \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$ and the *dividend yield* $\frac{d_t}{P_{t-1}}$.

$$\begin{cases} W_t^{risk-free} & = W_{t-1}^{risk-free}\ (1 + r_f) \\ W_t^{risky} & = W_{t-1}^{risky}\ [1 + r_t + \frac{d_t}{P_{t-1}}] \end{cases} \tag{1.1}$$

In the special case where the price $P_t$ of the risky asset is lower than its previous value $P_{\mathrm{t}-1}$, that is the price return rate $r_t$ is negative, there is a competition between the price return rate and the dividend yield. Hence, even if prices have fallen, the risky asset can still be profitable, compared to the risk-free asset, if dividends are enough to compensate the 'loss' due to the change of prices.

In the general case, a trader $i$ chooses, at any time $t$, how much will be invested respectively in the risk-free and the risky assets. This notion is called the *risky fraction* $x_t^i$, that is the fraction of wealth invested in the risky asset at time $t$. Thus, the fraction of wealth invested in the risk-free asset at time $t$ is nothing but $(1 - x_t^i)$. From Equations 1.1, a trader $i$, having a wealth $W_{\mathrm{t}-1}^i$ at time $t-1$, gets the wealth $W_t^i$ at the next time step, defined by:

$$W_t^i = (1 - x_{\mathrm{t}-1}^i)\, W_{\mathrm{t}-1}^i\, (1 + r_f) + x_{\mathrm{t}-1}^i\, W_{\mathrm{t}-1}^i\, [1 + r_t + \frac{d_t}{P_{\mathrm{t}-1}}] \tag{1.2}$$

$$W_t^i = W_{\mathrm{t}-1}^i\, [1 + r_f + x_{\mathrm{t}-1}^i\, R_{\mathrm{excess,t}}] \quad \text{with} \quad R_{\mathrm{excess,t}} := r_t - r_f + \frac{d_t}{P_{\mathrm{t}-1}} \tag{1.3}$$

$R_{\mathrm{excess,t}}$ is called the *excess return* of the risky asset over the risk-free asset. As said before, the risk-free interest rate $r_f$ can be viewed as a lower base line for capital returns. The excess return then provides a measure for the profitability of the risk of buying the risky asset, instead of the risk-free one. From Equation 1.3, one could notice that what defines an investor $i$ are her initial wealth $W_0^i$ and her risky fraction $x_t^i$ process. In the latter equation (called the wealth dynamics of trader $i$), all quantities are known, except the risky fraction and the dividend processes, as well as the price dynamics.

## 1.2 The dividend process

When a firm makes profits, the resulting earnings are allocated to retained earnings or dividend payments by a financial decision. Dividends are generally distributed one or several times per year to its shareholders – the owners of shares in the company – to satisfy their need for liquidity or other uses. Dividend payment behavior – also called dividend policy – has been a strong research field in economics [14].

Unfortunately, there is no widely accepted theory of optimal dividend policy. Thus, many papers found in the litterature use empirical facts to establish specific dividend policy theories which can be incorporated in more general models. Many of these specific theories are based on John Lintner's model [15] which includes several stylized facts coming from interviews of managers about their dividend policies. Generally speaking, managers choose the dividends which will be paid by their firms to have a target payout ratio – the proportion of firm earnings paid out as dividends to shareholders – as a long-term objective. However, due to unanticipated changes in earnings of companies, those policies can deviate from their initial objective. In this case, managers choose policies which smooth the time path of the changes in dividends needed to meet that objective. One should notice that the earnings mentioned by Lintner are 'permanent' earnings, that is to say that all 'temporary' earnings, known to vanish in the future, are not taken into account.

In[16], Marsh and Merton develop a model of the dividend process which captures well the behavior described in the Lintner interviews. Let $d_t$ be the dividend per share paid by a firm to shareholders at time $t$ and $e_t$ the permanent earnings per share of this firm at time $t$. They introduced the following dividend policy process:

$$d_t \;=\; (1+r_d)\; d_{\mathrm{t-1}} + \sum_{k=0}^{N} \gamma_k \; [e_{\mathrm{t-k}} - (1+r_d)\; e_{\mathrm{t-k-1}}] \; . \tag{1.4}$$

It is assumed that $\gamma_k \geq 0$ for all $k \;=\; 0, 1, ..., N$. Put another way, managers choose dividends to grow at a constant growth rate $r_d$ and it happens at least when $e_{\mathrm{t-k}} = (1+r_d)\; e_{\mathrm{t-k-1}}$ for all $k \;=\; 0, 1, ..., N$. It means that, in order to get a constant target payout ratio, the earnings of their firm should follow the same growth behavior than the dividends paid to shareholders. Thus, a deviation from this growth rate for permanent earnings will impact future dividends. One can notice that the parameter $N$ characterizes the number of past periods which would be taken into account by managers to set dividends. The positivity of the coefficients $\gamma_k$ leads to a negative impact on dividends when earnings are less than expected. For the sake of simplicity, we shall denote the dividends per share by 'dividends' in all this thesis since the total amount of dividends will be of no use.

Since the Lintner interviews, other variables such as free cash flows or firm size have appeared to be determinants of dividend policy [14]. In addition, Dhrymes and Kurz showed that variations in dividend policy are primarily because of a combination of many endogenous and exogenous elements [17]. Adding more complexity on the subject, Shiller has stressed the fact that financial behavior is influenced by societal norms and attitudes and that social pressures are able to lead to errors in judgement and trading activities that defy logical explanation [7, 18]. Considering those new aspects makes it possible to enrich dividend policy theories so that dividend payouts can be viewed as the socioeconomic repercussion of corporate evolution [19]. However, it seems really difficult to introduce this kind of psychology behavior into traditional financial pricing models. As an illustration to this non-trivial subject, one should keep in mind the following quotation:

“ *The harder we look at the dividend picture, the more it seems like a puzzle with pieces that just don't fit together.* ” Black, 1976

As a consequence of this huge complexity to model dividend policy, it has become quite standard to take the dividend process as completely exogenous, adding some stochasticity accounting for all non-trivial time-dependent determinants of dividend policy. Such models include Markov switching[20] and autoregressive integrated moving average [21] models but also trend-stationary autoregressive [22], simple random walk [23] and stationary [24] processes. However, the most widely used process – and considered as the tradition in applied work in finance – remains the geometric random walk.

Although it is possible to find in the litterature uses of geometric log-normal random walk models [24, 25], we shall adopt, in this thesis, a geometric normal random walk process for the dividends, following several authors [26, 11, 12, 27]:

$$\tilde{d}_t \;=\; (1+r_t^d)\;\tilde{d}_{t-1} \;=\; \tilde{d}_0 \prod_{k=1}^{t} (1+r_k^d)\;. \tag{1.5}$$

Here, the growth rate $r_t^d$ of the dividends is stochastic and, more precisely, follows a normal distribution characterized by a mean value $r_d$ and a standard deviation $\sigma_d$:

$$r_t^d \;=\; r_d + \sigma_d\; u_t \qquad \text{where} \;\; u_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)\;. \tag{1.6}$$

One could notice an analogy between this dividend process and Marsh and Merton's endogenous model, defined in equation (1.4). From their perspective, all the information contained in the aggregate permanent earnings of the firm is now replaced by a stochastic term $\sigma_d\; u_t\; \tilde{d}_{t-1}$, considering thus that the aggregate unexpected changes in permanent earnings can be viewed as stochastic. This should not be surprising, given the difficulty of distinguishing 'permanent' from 'temporary' earnings and considering the large number – and their complexity – of the reasons for changes in earnings. From a more state-of-the-art perspective, this stochastic term accounts for all previously described determinants of dividend policy such as psychology and sociological elements, which are hardly quantifiable.

One should notice that, depending on the values of $r_d$ and $\sigma_d$, dividends can become negative. This seems unrealistic, so that the real dividend process is thought to be the following:

$$d_t \;=\; \begin{cases} \tilde{d}_t & \text{if} \quad \tilde{d}_t \;\geq\; 0 \\ 0 & \text{if} \quad \tilde{d}_t \;<\; 0 \end{cases} \tag{1.7}$$

In the previous equation, the quantity $\tilde{d}_t$ simply corresponds to the exogeneous process, defined in Equation 1.5. The dividends paid by the risky asset are then always positive.

## 1.3 Fundamentalists

Fundamentalists are rational risk averse value investors; this kind of traders comes directly from Microeconomics. Their risky fraction $x_t^f$, at time $t$, is the result of a maximization of the expected utility of their expected wealth $W_{t+1}^f$ at time $t+1$:

$$x_t^f \;=\; \max_{x_t^f} \mathbb{E}_t[U(W_{t+1}^f)] \tag{1.8}$$

Here, $\mathbb{E}_t[\cdot]$ is the mathematical expectation, given all information available at time $t$. Let us explain their strategy in details. In order to find their risky fraction $x_t^f$ at time $t$, they maximize the previous quantity. Their future wealth $W_{\mathrm{t+1}}^f$ is given by Equation 1.3, that is the wealth dynamics. In this equation, the following quantities are unknown at time $t$, given that they will be determined in the future: $x_t^f$, $r_{\mathrm{t+1}}$ and $d_{\mathrm{t+1}}$. Then, they compute the utility function of their future wealth $U(W_{\mathrm{t+1}}^f)$, still without knowing $x_t^f$, $r_{\mathrm{t+1}}$ and $d_{\mathrm{t+1}}$. Utility functions are a well-known concept in Economics; they characterize consumer's preference ordering over a choice set[28]. Thus, fundamentalists use an utility function $U(W_{\mathrm{t+1}}^f)$ – which will be detailed later – to quantify explicitly their preference for their future wealth, that is how much they would prefer getting the wealth $W_{\mathrm{t+1}}^f \;=\; 3\;W_t^f$ to getting the wealth $W_{\mathrm{t+1}}^f \;=\; 2\;W_t^f$ for instance.

Here, this choice is made under uncertainty (mainly about future returns $r_{\mathrm{t+1}}$) so that, as *risk averse* investors, they exhibit a *relative risk aversion* $\gamma(W_{\mathrm{t+1}}^f)$, which is nothing but a quantitative measure of how averse to risk and uncertainty they are. The most commonly used measure of relative risk aversion $\gamma(W_{\mathrm{t+1}}^f)$ for an investor, having a given utility function $U(W_{\mathrm{t+1}}^f)$, was developed by J.W. Pratt in the 1960s [29]:

$$\gamma(W_{\mathrm{t+1}}^f) \;=\; -W_{\mathrm{t+1}}^f\;\frac{U''(W_{\mathrm{t+1}}^f)}{U'(W_{\mathrm{t+1}}^f)} \tag{1.9}$$

Here, $U'(W_{\mathrm{t+1}}^f)$ (respectively $U''(W_{\mathrm{t+1}}^f)$) denotes the first (respectively the second) derivative of the utility function $U(W_{\mathrm{t+1}}^f)$. The utility function, which fundamentalists use to measure their preference for their future wealth, is chosen so that it would be compliant with their relative risk aversion $\gamma$ being constant [1]:

$$U(W_{\mathrm{t+1}}^f) \;=\; \begin{cases} \log(W_{\mathrm{t+1}}^f) & \text{for} \quad \gamma \;=\; 1 \\ \frac{{W_{\mathrm{t+1}}^f}^{1-\gamma}}{1-\gamma} & \text{for} \quad \gamma \;\neq\; 1 \end{cases} \tag{1.10}$$

Obviously, the utility function is an increasing function of their expected wealth – their preference corresponds to possible gains – but there is a notion of risk, when investing in the risky asset, characterized by $\gamma$. Since they do not know the quantities $x_t^f$, $r_{\mathrm{t+1}}$ and $d_{\mathrm{t+1}}$ in the expression of $U(W_{\mathrm{t+1}}^f)$, they now have to estimate them. They behave as *rational traders*, meaning that they are able to guess the *true* values of parameters, when computing an expectation. In this sense, their expectations coincide with mathematical expectations, which explains the expectation appearing in the quantity to maximize. Finally, the optimization gives the risky fraction $x_t^f$ they have to choose to get a future wealth that maximizes their utility function – at least, in terms of expectation – that is to say, their preference taking a constant relative risk aversion $\gamma$ into account. The derivation of this maximization has been done in [27, 1]:

$$x_t^f \;=\; \frac{1}{\gamma}\;\frac{\mathbb{E}_t[R_{\mathrm{excess,t+1}}]}{\mathrm{Var}_t[R_{\mathrm{excess,t+1}}]} \tag{1.11}$$

Here, $\mathrm{Var}_t[\cdot]$ is the mathematical variance, given all information available at time $t$. One could notice that the more they expect the risky asset to return, compared to the risk-free asset, the greater the risky fraction $x_t^f$ of fundamentalists – that is, the amount of wealth they invest in the risky asset. Besides, their risky fraction decreases with the relative risk aversion $\gamma$ to the risky asset and with the uncertainty they have about the future excess return. Using the definition of the excess return in Equation 1.3, one finds:

$$\mathbb{E}_t[R_{\text{excess,t+1}}] \;=\; \mathbb{E}_t(r_{\text{t+1}}) \;-r_f \;+\; \frac{\mathbb{E}_t(d_{\text{t+1}})}{P_t} \tag{1.12}$$

As rational traders, they are able to compute the mathematical expectation of dividends $d_{\text{t+1}}$, using the *true* parameters $r_d$ and $\sigma_d$ of Equations 1.5 and 1.6, but they cannot get a convincing value for their expectation of the price return rate $r_{\text{t+1}}$ since there is no such parameter: they make a simple guess. Since there is no obvious way of having a time-dependent expectation of price return rates, it is assumed, for the sake of simplicity, that this quantity is constant with respect to time: $\mathbb{E}_t(r_{\text{t+1}}) \;:=\; E_{r_t}$. It only means that fundamentalists make a guess, at the beginning of the market, about the price return rate $r_t$. This value can then be viewed as what they expect in the long-run. It is assumed that $E_{r_t} \;>\; r_f$, otherwise it means that, from the very beginning of the market, fundamentalists would always think that the risk-free asset is the best option, and consequently stay away from the risky asset, which is not an interesting case. Using the equations of the dividend process 1.5 and 1.6, one finds:

$$\mathbb{E}_t[R_{\text{excess,t+1}}] \;=\; E_{r_t} \;-r_f \;+\; \frac{d_t}{P_t}\,(1+r_d) \tag{1.13}$$

For the variance $\mathrm{Var}_t[R_{\text{excess,t+1}}]$ appearing in Equation 1.11, one should notice that fundamentalists face the same problem than with the expectation $\mathbb{E}_t(r_{\text{t+1}})$. They still do not have any information about the future price return rate $r_{\text{t+1}}$ and, unlike the expectation $\mathbb{E}_t[R_{\text{excess,t+1}}]$, the variance is non-linear. Thus, it is not possible to separate terms, as it has been done in Equation 1.12, except using the covariance, which depends on the future price return rate $r_{\text{t+1}}$, so that it is of no use. As a consequence, in the same spirit than with the expectation $\mathbb{E}_t(r_{\text{t+1}})$, fundamentalists make an initial guess about the variance of the excess return, which can then be viewed as a long-run value: $\mathrm{Var}_t[R_{\text{excess,t+1}}] \;:=\; \sigma^2_{\text{excess}}$. Using Equations 1.11 and 1.13, one finds the risky fraction of fundamentalists:

$$\begin{cases} x_t^f \;=\; x_{\text{min}}^f \;+\; \frac{d_t}{P_t}\,\frac{1\,+\,r_d}{\gamma\;\sigma^2_{\text{excess}}} \qquad > 0 \\[2ex] x_{\text{min}}^f \;=\; \frac{E_{r_t}\;-\;r_f}{\gamma\;\sigma^2_{\text{excess}}} \qquad > 0 \end{cases} \tag{1.14}$$

One should notice that, at each time step, fundamentalists invest at least the fraction $x_{\text{min}}^f$ of their wealth in the risky asset. This minimum fraction corresponds to a long-term strategy, depending on their initial guess of the long-term behavior. The other quantity appearing in the expression of the risky fraction $x_t^f$ is time-dependent and forms a short-term strategy, only depending on the current price-dividend ratio. For a better understanding of that short-term strategy, let us find how the risky fraction changes with respect to time. Using Equations 1.14, one finds:

$$x_{t+1}^f - x_t^f = \left(\frac{d_{t+1}}{P_{t+1}} - \frac{d_t}{P_t}\right) \frac{1 + r_d}{\gamma\, \sigma_{\text{excess}}^2} \tag{1.15}$$

Since, the long-term strategy, that is $x_{\min}^f$, is constant with respect to time, the change in the risky fraction $x_t^f$, which one could observe in the previous equation, is only due to the short-term strategy. In particular, it is now possible to compare the risky fraction $x_t^f$ of fundamentalists at any time $t$ with their initial risky fraction $x_0^f$:

$$x_t^f - x_0^f = \left(\frac{d_t}{P_t} - \frac{d_0}{P_0}\right) \frac{1 + r_d}{\gamma\, \sigma_{\text{excess}}^2} \tag{1.16}$$

As a consequence, to observe a constant risky fraction with respect to time ($x_t^f = x_0^f$ for all $t$), the price of the risky asset must follow the following specific behavior:

$$P_t^\star = P_0 \frac{d_t}{d_0} = P_0 \prod_{k=1}^{t} (1 + r_d + \sigma_d u_k) \tag{1.17}$$

In this sense, fundamentalists expect the long-term price growth to be due to the growth of dividends, that is $r_t \underset{+\infty}{\sim} r_d$. Thus, any deviations from this behavior are perceived as investment opportunities. This explains why they are called fundamentalists. This kind of traders is widely used in Economics; they base their trading strategies upon market *fundamentals* and economic factors, such as dividends, and they tend to invest in assets which are undervalued, that is, whose prices are below a benchmark *fundamental value*, and sell assets which are overvalued, that is, whose prices are above the fundamental value[2]. In this case, the fundamental value is the price following the same growth than the dividends, that is $P_t^\star$.

For the sake of simplicity, fundamentalists are assumed to be identical, so that we can consider, in the following, the behavior of one representative fundamental trader, having an initial wealth $W_0^f$ and a risky fraction process as defined in Equations 1.14. The concept of *representative agent* is well-known and has been widely used in Economics [30]. In this case, it does not seem very restrictive since fundamentalists are maximizers so, assuming that they have the same relative risk aversion $\gamma$ and the same opinion about the long-term behavior of the risky asset, they all have the same strategy.

## 1.4 Noise traders

In contrast to the fundamentalists, noise traders have different opinions. Besides, they embody the lack-of-diversification puzzle [31, 32], so that, in this model, they are always either fully invested in the risky or in the risk-free asset. Among a total of $N_n$ noise traders, the number of noise traders invested in the risky asset (respectively invested in the risk-free asset) at time $t$ is $N_t^+$ (respectively $N_t^-$), so that the conservation of noise traders gives: $N_t^+ + N_t^- = N_n$ for all $t$. At each time step, all noise traders decide whether to keep their current position or to change it, in a probabilistic manner. Let $p_{\mathrm{t}-1}^+$ (respectively $p_{\mathrm{t}-1}^-$) be the probability that any of the $N_{\mathrm{t}-1}^+$ traders who are fully invested in the risky asset (respectively any of the $N_{\mathrm{t}-1}^-$ traders who are out of the risky asset) at time $t-1$ decides to sell it (respectively to buy it) at time $t$. Those probabilities are called *transition probabilities*; they characterize the time-dependent propensity for noise traders to move in a two-state system. At this point, one might wonder about what is able to affect those transition probabilities.

Noise traders do not take fundamentals into account but, instead, they base their trading strategies upon imitation and technical analysis [2, 33]. The latter – also called chartist strategy – is vast and includes lots of possible indicators, most of them based on observed historical patterns in past prices, giving the 'trend' of the asset. One of the most widely used indicators is the *price momentum* $H_t$, defined as follows:

$$H_t \;=\; \theta\, H_{\mathrm{t}-1} \;+\; (1-\theta)\, r_t \tag{1.18}$$

One should notice that it is nothing but an exponential moving average of past price return rates $r_t$. The parameter $\theta \in (0,1)$ is a measure of the noise trader memory length.

In order to use imitation as a strategy, it is necessary to know what to imitate. For that purpose, the model of Lux [34, 35] and Lux and Marchesi [36, 37] introduces a useful quantity for noise traders, the *opinion index* $s_t$, representing the average opinion among them:

$$s_t \;=\; \frac{N_t^+ \;-\; N_t^-}{N_n} \quad \in\; [-1, 1] \tag{1.19}$$

It is clear that the sign of the opinion index $s_t$ indicates whether the prevailing sentiments on the risky asset are optimistic ($s_t > 0$) or pessimistic ($s_t < 0$). In a nutshell, noise traders are trend-followers (using the price momentum $H_t$) and they tend to imitate each other (using the opinion index $s_t$). As a consequence, those quantities are the only ones affecting the transition probabilities $p_t^-(s_t, H_t)$ and $p_t^+(s_t, H_t)$. As in [1], the expressions of those probabilities are taken so that they depend linearly on $s_t$ and $H_t$, for simplicity, having an intrinsic non zero value. However, unlike[1], some bias is introduced among noise traders:

$$\begin{cases} \tilde{p}_t^- &= \frac{p_-}{2}\left[1+\kappa(s_t+H_t)\right] \quad \text{with} \quad p_- \in (0,1) \\ \tilde{p}_t^+ &= \frac{p_+}{2}\left[1-\kappa(s_t+H_t)\right] \quad \text{with} \quad p_+ \in (0,1) \end{cases} \tag{1.20}$$

$\kappa$ is called the *herding propensity*, giving the strength of trend and imitation among noise traders. For instance, for $\kappa > 0$, if the number of noise traders invested in the risky asset increases and/or its price has increased recently, the probability $\tilde{p}_t^-$ – that is the probability that any of the traders currently out of the risky asset decides to buy it – increases and the probability $\tilde{p}_t^+$ – that is the probability that any of the traders currently fully invested in the risky asset decides to sell it – decreases. One could notice that, in absence of herding ($\kappa = 0$), the transition probabilities are non zero and characterized by the constants $p_-$ and $p_+$. Any difference between those two constants induces some bias among noise traders: if $p_- > p_+$, noise traders are intrinsically more likened to buy the risky asset when they are not invested in it than to sell it when they are fully invested in it. One should notice that the quantities defined in Equations 1.20 are not real probabilities since they do not necessarily belong to the set $[0,1]$; we call them *pseudo transition probabilities* and they are denoted using the symbol $\sim$. The 'real' transitions probabilities $p_t^-(s_t, H_t)$ and $p_t^+(s_t, H_t)$ are then simply defined as saturations of the pseudo transition probabilities:

$$p_t^- = \begin{cases} \tilde{p}_t^- & \text{if} \quad \tilde{p}_t^- \in [0,1] \\ 0 & \text{if} \quad \tilde{p}_t^- \leq 0 \\ 1 & \text{if} \quad \tilde{p}_t^- \geq 1 \end{cases} \tag{1.21}$$

$$p_t^+ = \begin{cases} \tilde{p}_t^+ & \text{if} \quad \tilde{p}_t^+ \in [0,1] \\ 0 & \text{if} \quad \tilde{p}_t^+ \leq 0 \\ 1 & \text{if} \quad \tilde{p}_t^+ \geq 1 \end{cases} \tag{1.22}$$

Let us now derive the dynamics of the opinion index $s_t$. Noise traders' decision, about whether to keep their position or to change it, is represented by Bernoulli random variables, which depend themselves on the transition probabilities. In details, for a noise trader $k$ who owns the risky asset at time $t-1$, her specific decision at time $t$ is represented by the Bernoulli random variable $\xi_k(p_{t-1}^+)$, taking the value 1 with probability $p_{t-1}^+$ – that is she sells the asset which she owns – and the value 0 with probability $(1-p_{t-1}^+)$ – that is she keeps the asset. In the same spirit, for a noise trader $j$ who is out of the risky asset at time $t-1$, her specific decision at time $t$ is represented by the Bernoulli random variable $\xi_j(p_{t-1}^-)$, taking the value 1 with probability $p_{t-1}^-$ – that is she buys the asset – and the value 0 with probability $(1-p_{t-1}^-)$ – that is she stays away from the asset. The random variables $\{\xi_i\}$ are independent, so that noise traders make independent decisions. Now that the decision process of noise traders is defined, it is possible to find the dynamics of the number $N_t^+$ of noise traders fully invested in the risky asset at time $t$ and of the number $N_t^-$ who are out of it at time $t$:

$$\begin{cases} N_t^- &= \sum_{k=1}^{N_{t-1}^+} \xi_k(p_{t-1}^+) \ + \ \sum_{j=1}^{N_{t-1}^-} [1 - \xi_j(p_{t-1}^-)] \\ N_t^+ &= \sum_{k=1}^{N_{t-1}^+} [1 - \xi_k(p_{t-1}^+)] \ + \ \sum_{j=1}^{N_{t-1}^-} \xi_j(p_{t-1}^-) \end{cases} \tag{1.23}$$

Let us explain the equation giving $N_t^-$, that is the number of noise traders who are out of the risky asset at time $t$. The first sum corresponds to the number of noise traders who were fully invested in the risky asset at time $t-1$ and decided to sell it at time $t$. The second sum corresponds to the number of noise traders who were already out of the risky asset at time $t-1$ and decided to stay away from it at time $t$. An analogous explanation holds for $N_t^+$.

Summing both previous equations, one can get convinced that the conservation of noise traders holds for the previous specific dynamics of $N_t^-$ and $N_t^+$:

$$N_t^- \ + \ N_t^+ \ = \ N_{t-1}^- \ + \ N_{t-1}^+ \ = N_0^- \ + \ N_0^+ \ = \ N_n \tag{1.24}$$

Using Equations 1.23 in the definition of the opinion index $s_t$ in Equation 1.19, one finds the dynamics of the opinion index:

$$s_t \ = \ \frac{1}{N_n} \left( \sum_{k=1}^{N_{t-1}^+} [1 - 2\xi_k(p_{t-1}^+)] \ + \ \sum_{j=1}^{N_{t-1}^-} [2\xi_j(p_{t-1}^-) - 1] \right) \ \in \ [-1, 1] \tag{1.25}$$

As in [1], we do not aim at describing the heterogeneity between noise traders: only their aggregate impact will be considered, so that, as for the fundamentalists, they will be treated as one group with total wealth $W_t^n$. One can picture this situation as one group of noise traders, sharing a common total wealth and having equal weights in their decision of investing or not in the risky asset, so that their risky fraction $x_t^n$ is nothing but the following:

$$x_t^n \ = \ \frac{N_t^+}{N_n} \ = \ \frac{1 \ + \ s_t}{2} \quad \in \ [0, 1] \tag{1.26}$$

As a consequence, their risky fraction is only a rescaling of the opinion index $s_t$; it is stochastic and strongly dependent on the transition probabilities which are saturations of the pseudo transition probabilities, defined in Equations 1.20. Those probabilities lead to an imitative and trend-following strategy. This kind of traders results from an important paradigm shift in Economics, that is the transition between the representative, rational agent approach to the behavioral, agent-based approach [2]. One should keep in mind that this conceptual change appeared when Economics began to be thought as a complex evolving system.

## 1.5 Market clearing condition

The two considered kinds of traders are completely defined: they have initial wealths $W_0^f$ and $W_0^n$, risky fraction processes defined in Equations 1.14 and 1.26 and wealth dynamics defined in Equation 1.3. According to 1.1, at time $t$, one of those two representative traders, let say trader $i$, holds the following number of shares of the risky asset:

$$n_t^i \; = \; \frac{x_t^i \; W_t^i}{P_t} \tag{1.27}$$

Thus, from time $t-1$ to time $t$, the trader $i$ has an excess demand $\Delta D^i_{\mathrm{t-1\to t}} \; = \; n_t^i \; - \; n^i_{\mathrm{t-1}}$ , in terms of shares of the risky asset. Using Equations 1.27 and 1.3, one finds:

$$\begin{aligned}
\Delta D^i_{\mathrm{t-1\to t}} \; &= \; \frac{x_t^i \; W_t^i}{P_t} \; - \; \frac{x^i_{\mathrm{t-1}} \; W^i_{\mathrm{t-1}}}{P_{\mathrm{t-1}}} \\
&= \; W^i_{\mathrm{t-1}} \left[ \; \frac{x_t^i}{P_t} \; [1 + r_f + x^i_{\mathrm{t-1}} \; ( \; \frac{P_t}{P_{\mathrm{t-1}}} - 1 - r_f + \frac{d_t}{P_{\mathrm{t-1}}} \; )] \; - \; \frac{x^i_{\mathrm{t-1}}}{P_{\mathrm{t-1}}} \; \right] \\
&= \; W^i_{\mathrm{t-1}} \; \frac{x^i_{\mathrm{t-1}}}{P_{\mathrm{t-1}}} \; (x_t^i - 1) \; + \; W^i_{\mathrm{t-1}} \; \frac{x_t^i}{P_t} \left[ \; 1 + r_f + x^i_{\mathrm{t-1}} \; ( \; \frac{d_t}{P_{\mathrm{t-1}}} - 1 - r_f \; ) \right]
\end{aligned} \tag{1.28}$$

It has been explained in 1.1 that the price $P_t$ of the risky asset at time $t$ is set by supply and demand. This is a common way to obtain successive prices in Economics, called the *market clearing condition* or Walras' law. Sometimes considered as an 'equilibrium', it is nothing but the conservation of shares:

$$\Delta D^f_{\mathrm{t-1\to t}} \; + \; \Delta D^n_{\mathrm{t-1\to t}} \; = \; 0 \tag{1.29}$$

It only means that the price $P_t$ of the risky asset has to evolve in such a way that there is a trade of a given number of shares of the risky asset between fundamentalists and noise traders. In the following, the corresponding price equation is derived.

Using Equation 1.28 in the market clearing condition, one finds:

$$W^f_{t-1}\ \frac{x^f_{t-1}}{P_{t-1}}\ (x^f_t - 1)\ +\ W^f_{t-1}\ \frac{x^f_t}{P_t}\ \left[\ 1 + r_f + x^f_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ )\right]$$

$$+\ W^n_{t-1}\ \frac{x^n_{t-1}}{P_{t-1}}\ (x^n_t - 1)\ +\ W^n_{t-1}\ \frac{x^n_t}{P_t}\ \left[\ 1 + r_f + x^n_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ )\right]\ =\ 0 \tag{1.30}$$

The risky fraction $x^f_t$ of fundamentalists depends explicitly on the price of the risky asset. Equations 1.14 give:

$$W^f_{t-1}\ \frac{x^f_{t-1}}{P_{t-1}}\ (x^f_{\text{min}} + \frac{d_t}{P_t}\frac{1+r_d}{\gamma\ \sigma^2_{\text{excess}}} - 1) + \ \frac{W^f_{t-1}}{P_t}\ \ (x^f_{\text{min}} + \frac{d_t}{P_t}\frac{1+r_d}{\gamma\ \sigma^2_{\text{excess}}})\left[\ 1 + r_f + x^f_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ )\right]$$

$$+\ W^n_{t-1}\ \frac{x^n_{t-1}}{P_{t-1}}\ (x^n_t - 1)\ +\ W^n_{t-1}\ \frac{x^n_t}{P_t}\ \left[\ 1 + r_f + x^n_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ )\right]\ =\ 0 \tag{1.31}$$

This equation leads to the price equation, of second-order in the price:

$$a_t\ P^2_t\ +\ b_t\ P_t\ +\ c_t\ =\ 0 \tag{1.32}$$

The coefficients $a_t$, $b_t$ and $c_t$ are time dependent; their expression is given below:

$$a_t\ =\ \frac{1}{P_{t-1}}\ \left[\ \frac{W^n_{t-1}}{W^f_{t-1}}\ x^n_{t-1}\ (x^n_t - 1)\ +\ x^f_{t-1}\ \left(\ \frac{E_{r_t}\ -\ r_f}{\gamma\ \sigma^2_{\text{excess}}}\ - 1\right)\right] \tag{1.33}$$

$$b_t\ =\ \frac{x^f_{t-1}}{\gamma\ \sigma^2_{\text{excess}}}\ \frac{d_t\ (1+r_d)}{P_{t-1}}\ +\ \frac{E_{r_t}\ -\ r_f}{\gamma\ \sigma^2_{\text{excess}}}\ \left[\ x^f_{t-1}\ \left(\ \frac{d_t}{P_{t-1}} - 1 - r_f\ \right) + 1 + r_f\ \right] +\ \frac{W^n_{t-1}}{W^f_{t-1}}\ x^n_t\ \left[x^n_{t-1}\ \left(\ \frac{d_t}{P_{t-1}} - 1 - r_f\ \right) + 1 + r_f\ \right] \tag{1.34}$$

$$c_t\ =\ \frac{d_t\ (1+r_d)}{\gamma\ \sigma^2_{\text{excess}}}\ \left[\ x^f_{t-1}\ \left(\ \frac{d_t}{P_{t-1}} - 1 - r_f\ \right) + 1 + r_f\ \right] \tag{1.35}$$

In the expression of $a_t$ – see Equation 1.33 – one could notice that, if $\gamma\ \sigma^2_{\text{excess}}\ \geq\ E_{r_t}\ -\ r_f$, the sign of $a_t$ is known: $a_t\ \leq\ 0$. Indeed, by definition, the risky fraction $x^n_t$ of noise traders satisfies $x_t^n \leq 1$ and all other quantities appearing in the expression of $a_t$ are positive. Using Equations 1.14, it is possible to define the quantity $\gamma\ \sigma^2_{\text{excess}}$ as a function of the initial risky fraction $x_0^f$:

$$\gamma\ \sigma^2_{\text{excess}} = \frac{1}{x_0^f}\ \left[\ E_{r_t} - r_f + \frac{d_0}{P_0}(1 + r_d)\ \right] \tag{1.36}$$

This is how the product $\gamma\ \sigma^2_{\text{excess}}$ is implemented in the numerical simulations. Then, the condition to obtain $a_t \leq 0$, that is $\gamma\ \sigma^2_{\text{excess}} \geq E_{r_t} - r_f$, is equivalent to:

$$\frac{1}{x_0^f}\ \left[\ E_{r_t} - r_f + \frac{d_0}{P_0}(1 + r_d)\ \right] \geq E_{r_t} - r_f \tag{1.37}$$

Let us recall that it has been supposed in 1.3 that $E_{r_t} > r_f$. Hence, one finds the following condition on the initial risky fraction $x_0^f$ of fundamentalists:

$$x_0^f \leq 1 + \frac{d_0}{P_0}\ \frac{1 + r_d}{E_{r_t} - r_f} \tag{1.38}$$

As a consequence, if $x_0^f < 1$, then the product $\gamma\ \sigma^2_{\text{excess}}$ satisfies $\gamma\ \sigma^2_{\text{excess}} > E_{r_t} - r_f$, so that $a_t \leq 0$ for all $t$. In the following of this thesis, it will be assumed that $x_0^f < 1$ to fix the sign of the coefficient $a_t$. It is not really restrictive and seems even realistic since it means that, at the beginning of the market, fundamentalists do not invest all their wealth in the risky asset.

In fact, it is even possible to show that $a_t < 0$ for all $t$. Indeed, since the risky fraction $x_t^n$ of noise traders satisfies $0 \leq x_t^n \leq 1$, one finds the following inequality, using the expression of $a_t$ in Equation 1.33:

$$a_t \leq \frac{1}{P_{\text{t}-1}}\ x^f_{\text{t}-1} \left( \frac{E_{r_t} - r_f}{\gamma\ \sigma^2_{\text{excess}}} - 1 \right) \tag{1.39}$$

Let us recall that, by definition, fundamentalists invest at least the fraction $x^f_{\text{min}}$ of their wealth in the risky asset at any time $t$, so that: $x^f_{\text{t}-1} \geq x^f_{\text{min}}$ i.e. $x^f_{\text{t}-1} \geq \frac{E_{r_t} - r_f}{\gamma\ \sigma^2_{\text{excess}}} > 0$. Besides, since it is assumed that $x_0^f < 1$, we have $\frac{E_{r_t} - r_f}{\gamma\ \sigma^2_{\text{excess}}} - 1 < 0$. Using those inequalities in the previous inequality on $a_t$, one finds:

$$a_t \leq \frac{1}{P_{\text{t}-1}}\ \frac{E_{r_t} - r_f}{\gamma\ \sigma^2_{\text{excess}}} \left( \frac{E_{r_t} - r_f}{\gamma\ \sigma^2_{\text{excess}}} - 1 \right) < 0 \tag{1.40}$$

In a nutshell, once it is assumed that $E_{r_t} > r_f$ and that $x_0^f < 1$, the coefficient $a_t$ is negative for all $t$.

Furthermore, one could notice, in Equations 1.34 and 1.35, that the signs of the coefficients $b_t$ and $c_t$ are much more difficult to find. However, the only quantity, appearing in their expression, which is not clearly positive is the following: $x^i_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ ) + 1 + r_f$, where $i$ denotes the fundamentalists or the noise traders. Thus, if $b_t$ or $c_t$ is negative, it means that this quantity is negative too. One should notice that there is no equivalence between those two propositions and the latter proposition is only a necessary condition to obtain $b_t$ or $c_t$ negative. Let us find what happens when the previous quantity is negative:

$$x^i_{t-1}\ (\ \frac{d_t}{P_{t-1}} - 1 - r_f\ ) + 1 + r_f\ \leq\ 0$$

$$i.e.\quad (1 + r_f)\ (1 - x^i_{t-1})\ +\ x^i_{t-1}\ \frac{d_t}{P_{t-1}}\ \leq\ 0 \tag{1.41}$$

The second term of the latter inequality is positive. As a consequence, if $b_t$ or $c_t$ is negative, it means that the first term is negative, that is $x^i_{t-1}\ >\ 1$. As said before, this condition cannot be satisfied for the risky fraction $x^n_t$ of noise traders but it might be possible for the risky fraction $x^f_t$ of fundamentalists. Thus, the coefficients $b_t$ and $c_t$ are most of the time positive and a necessary condition for them to be negative is that the risky fraction $x^f_t$ of fundamentalists is greater than 1, which should rarely occurs. It means that fundamentalists borrow some money to invest in the risky asset, thinking that it is profitable.

In a nutshell, once we choose an initial risky fraction for the fundamentalists such that $x^f_0 < 1$, and if their risky fraction $x^f_t$ at time $t$ is less than 1, one is able to find the sign of the discriminant of Equation 1.32:

$$\Delta\ =\ b_t^2\ -\ 4a_tc_t\ =\ b_t^2\ +\ 4|a_t|c_t\ \geq\ 0 \tag{1.42}$$

As a consequence, in those conditions, the only positive solution for the price of the risky asset is the following:

$$P_t\ =\ \frac{b_t\ +\ \sqrt{b_t^2\ +\ 4|a_t|c_t}}{2|a_t|} \tag{1.43}$$

Among the two conditions required to obtain this result ($x^f_0\ <\ 1$ and $x^f_t\ \leq\ 1$), the only one over which there is no possible control is the risky fraction $x^f_t$ being less than 1. Thus, for theoretical study, we shall use Equation 1.29 or 1.32 whereas, for numerical simulations, we shall use Equation 1.43 with a verification of the positivity of the discriminant and of the price, at each time step.

# Chapter 2

# The impact of dividends

In this chapter, the herding propensity $\kappa$ is time-dependent and undergoes a discretized Ornstein-Uhlenbeck process, as in [1]:

$$\begin{cases} \kappa_t \;-\; \kappa_{\mathrm{t}-1} \;=\; \eta(\mu_\kappa \;-\; \kappa_{\mathrm{t}-1}) \;+\; \sigma_\kappa\, \nu_t \\ \nu_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)\ . \end{cases} \tag{2.1}$$

It seems to be a more realistic feature for noise traders; the strength of social imitation and trend influence varies in time. It adds a second input – in addition to the stochastic dividend process – from the external world, corresponding to a varying economy, geopolitical climate, psychology for instance.

## 2.1 Choice of parameters

For the following numerical simulations, it is necessary to fix the values of the constant parameters. For that purpose, a *basic parameter set* is provided by Table 2.1. The values within this set are mostly imported from [1], where it is shown that they correspond to realistic values on a daily basis. This means that they come from empirical economic studies and that they are rescaled, so that they remain realistic when the time step of the model, described in Chapter 1, is equal to 1 day. In cases where other values than those in Table 2.1 are used instead, differences will always be highlighted.

## 2.2 Transition probabilities

In this section, a quantitative study of how the transition probabilities depend on each other is provided. The following discussion first considers the simple case of no bias ($p_+ = p_- = p$) and then the general case ($p_+ \neq p_-$). Let us recall the expressions of the

Table 2.1: Basic parameter set used for numerical simulations. Those values are mostly imported from [1], where it is shown that they correspond to realistic values on a daily basis.

| Assets | Fundamentalists | Noise traders | Transition probabilities |
|---|---|---|---|
| $r_f = 8 \times 10^{-5}$ | $W_0^f = 1 \times 10^6$ | $W_0^n = 1 \times 10^6$ | $p_+ = 0.199375$ |
| $d_0 = 1.6 \times 10^{-4}$ | $x_0^f = 0.3$ | $x_0^n = 0.3$ | $p_- = 0.200625$ |
| $P_0 = 1$ | $E_{r_t} = 1.6 \times 10^{-4}$ | $N_n = 1000$ | $\mu_\kappa = 0.98$ |
| $r_d = 1.6 \times 10^{-4}$ | | $\theta = 0.95$ | $\eta = 0.11$ |
| $\sigma_d = 1.6 \times 10^{-5}$ | | $H_0 = 0$ | $\sigma_\kappa = 1 \times 10^{-3}$ |

transition probabilities $p_t^-$ and $p_t^+$ in absence of bias, which are saturations of the pseudo transition probabilities $\tilde{p}_{\mathrm{t}}^-$ and $\tilde{p}_{\mathrm{t}}^+$:

$$p_{\mathrm{t}}^- = \begin{cases} \tilde{p}_{\mathrm{t}}^- & \text{if } \tilde{p}_{\mathrm{t}}^- = \frac{p}{2}\left[1 + \kappa_t\,(s_t + H_t)\right] \in [0,1] \\ 0 & \text{if } \tilde{p}_{\mathrm{t}}^- \leq 0 \\ 1 & \text{if } \tilde{p}_{\mathrm{t}}^- \geq 1 \end{cases}$$

$$p_{\mathrm{t}}^+ = \begin{cases} \tilde{p}_{\mathrm{t}}^+ & \text{if } \tilde{p}_{\mathrm{t}}^+ = \frac{p}{2}\left[1 - \kappa_t\,(s_t + H_t)\right] \in [0,1] \\ 0 & \text{if } \tilde{p}_{\mathrm{t}}^+ \leq 0 \\ 1 & \text{if } \tilde{p}_{\mathrm{t}}^+ \geq 1 \end{cases} \tag{2.2}$$

One could notice that the sum of the pseudo transition probabilities $\tilde{p}_{\mathrm{t}}^-$ and $\tilde{p}_{\mathrm{t}}^+$ is constant with respect to time:

$$\tilde{p}_t^- + \tilde{p}_t^+ = p \in (0,1) \tag{2.3}$$

As a consequence, it proves the following result:

**Proposition 1.** The pseudo transition probabilities are symmetrical with respect to $\frac{p}{2}$.

This explains how the pseudo transition probabilities depend on each other. Let us now focus on the real transition probabilities and, for that purpose, let us consider all possible cases on the pseudo transition probabilities.

- $\tilde{p}_t^- \in [0,1] \iff p_t^- = \tilde{p}_t^-$

  In this case, the real transition probability $p_t^-$ is known. Using Equation 2.3, one finds the following condition on the pseudo transition probability $\tilde{p}_t^+$:

  $$p - 1 \leq \tilde{p}_t^+ \leq p < 1$$

To know the value of the real transition probability $p_t^+$, it is necessary to distinguish the cases where the corresponding pseudo transition probability $\tilde{p}_t^+$ is in the set $[0, 1]$ and where it is not.

- $0 \leq \tilde{p}_t^+ \leq p \quad \Rightarrow \quad p_t^+ = \tilde{p}_t^+$

  This restriction on $\tilde{p}_t^+$ implies a restriction on $\tilde{p}_t^-$, according to Equation 2.3: $0 \leq \tilde{p}_t^- \leq p$. In brief, the conditions $0 \leq \tilde{p}_t^- \leq p$ and $0 \leq \tilde{p}_t^+ \leq p$ give $p_t^- = \tilde{p}_t^-$ and $p_t^+ = \tilde{p}_t^+$. Thus, one finds the following solutions:

  $$\begin{cases} 0 \leq p_t^- \leq p \\ p_t^+ = p - p_t^- \end{cases}$$

- $p - 1 \leq \tilde{p}_t^+ \leq 0 \quad \Rightarrow \quad p_t^+ = 0$

  This restriction implies, according to Equation 2.3: $p \leq \tilde{p}_t^- \leq 1$. Thus, using $p_t^- = \tilde{p}_t^-$ and $p_t^+ = 0$, one finds:

  $$\begin{cases} p \leq p_t^- \leq 1 \\ p_t^+ = 0 \end{cases}$$

- $\tilde{p}_t^- \leq 0 \iff p_t^- = 0$

  Equation 2.3 gives the following condition on the pseudo transition probability $\tilde{p}_t^+$:

  $$\tilde{p}_t^+ \geq p > 0$$

  Using Equations 2.2, one finds the following solutions:

  $$\begin{cases} p_t^- = 0 \\ p \leq p_t^+ \leq 1 \end{cases}$$

  This solution is degenerate: for one single value of $p_t^-$, many values for $p_t^+$ are possible. Indeed, once $\tilde{p}_t^- \leq 0$ is considered, $p_t^-$ becomes fixed, equal to 0. However, the lower $\tilde{p}_t^-$, the greater $p_t^+$ until its saturation to 1. The problem is that the information on how much $\tilde{p}_t^-$ is lower than 0 vanishes when considering $p_t^-$ – which is equal to 0.

- $\tilde{p}_t^- \geq 1 \iff p_t^- = 1$

  Equation 2.3 gives the following condition on the pseudo transition probability $\tilde{p}_t^+$:

  $$\tilde{p}_t^+ \leq p - 1 < 0$$

Using Equations 2.2, one finds the following solution:

$$\begin{cases} p_t^- &= 1 \\ p_t^+ &= 0 \end{cases}$$

In order to get a clearer idea of how the transition probabilities $p_t^-$ and $p_t^+$ depend on each other, the previous results are represented by a plot $p_t^+ = f(p_t^-)$, provided by Figure 2.1. The corresponding plot is depicted by the green curve. The parameter $p$ takes the value 0.2 and it is represented both on the x-axis and the y-axis by a red dashed line. Inside the domain where both transition probabilities are lower than $p$, the transition probabilities $p_t^-$ and $p_t^+$ are equal to their corresponding pseudo transition probability $\tilde{p}_t^-$ or $\tilde{p}_t^+$. Then, according to Equation 2.3, the following relation holds inside this domain: $p_t^- + p_t^+ = p$. As for Proposition 1, it implies that the real transition probabilities are symmetrical with respect to $\frac{p}{2}$ inside this domain. The two remaining domains ($p_t^- = 0$ and $p_t^+ = 0$) are characterized by the saturation of at least one of the transition probabilities. For instance, when $p_t^- = 0$, the values taken by $p_t^+$ are degenerate; they can be equal to any value in the set $[p, 1]$, depending on the value of the pseudo transition probability $\tilde{p}_t^-$. Inside those saturated domains, there is not symmetry with respect to $\frac{p}{2}$ anymore for the real transition probabilities whereas this symmetry holds for the pseudo transition probabilities, according to Proposition 1.

Let us now consider the general case, that is with bias ($p_- \neq p_+$). The transition probabilities are the following:

$$p_{\text{t}}^- = \begin{cases} \tilde{p}_{\text{t}}^- & \text{if} \quad \tilde{p}_{\text{t}}^- = \frac{p_-}{2} \left[1 + \kappa_t \left(s_t + H_t\right)\right] \in [0,1] \\ 0 & \text{if} \quad \tilde{p}_{\text{t}}^- \leq 0 \\ 1 & \text{if} \quad \tilde{p}_{\text{t}}^- \geq 1 \end{cases}$$
$$p_{\text{t}}^+ = \begin{cases} \tilde{p}_{\text{t}}^+ & \text{if} \quad \tilde{p}_{\text{t}}^+ = \frac{p_+}{2} \left[1 - \kappa_t \left(s_t + H_t\right)\right] \in [0,1] \\ 0 & \text{if} \quad \tilde{p}_{\text{t}}^+ \leq 0 \\ 1 & \text{if} \quad \tilde{p}_{\text{t}}^+ \geq 1 \end{cases} \tag{2.4}$$

Then, the sum of the pseudo transition probabilities is not constant with respect to time anymore:

$$\tilde{p}_t^- + \tilde{p}_t^+ = \frac{p_- + p_+}{2} + \frac{p_- - p_+}{2} \kappa_t \left(s_t + H_t\right) \tag{2.5}$$

Let us now compare both terms appearing at the right side of the previous equation. According to the definition of the opinion index $s_t$, it is known that $|s_t| \leq 1$. Furthermore, it is shown in[1] that the Ornstein-Uhlenbeck process for the herding propensity $\kappa_t$, described in 2.1, admits the following stationary distribution:

$$\kappa_t \sim \mathcal{N}\left(\mu_\kappa, \frac{\sigma_\kappa}{\sqrt{2\eta}}\right) \tag{2.6}$$

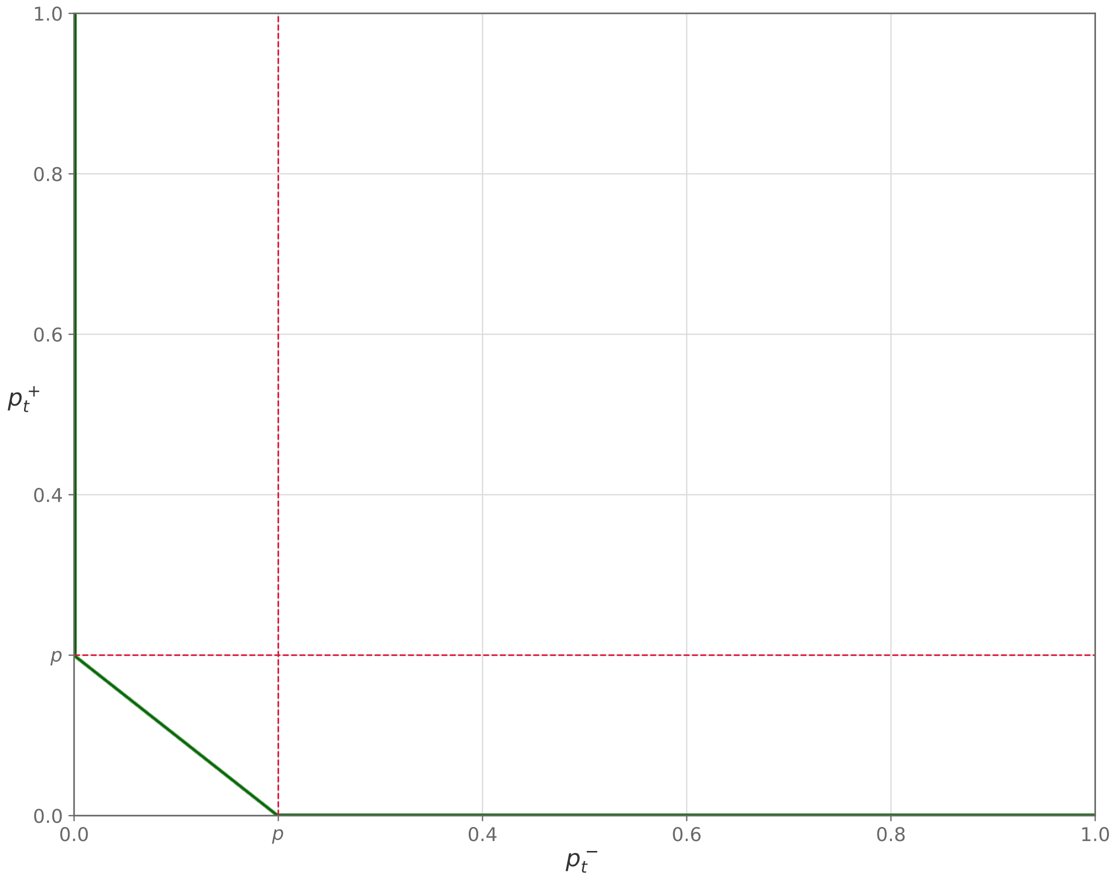


Figure 2.1: Possible values for the transition probability $p_t^+$ as functions of the other transition probability $p_t^-$, in absence of bias. The corresponding plot is depicted by the green curve. The parameter $p$ takes the value 0.2 and it is represented both on the x-axis and the y-axis by a red dashed line. Inside the domain where both transition probabilities are lower than $p$, the transition probabilities $p_t^-$ and $p_t^+$ are equal to their corresponding pseudo transition probability $\tilde{p}_t^-$ or $\tilde{p}_t^+$. The two remaining domains ($p_t^- = 0$ and $p_t^+ = 0$) are characterized by the saturation of at least one of the transition probabilities.

As a consequence, in average, the following inequality holds for the herding propensity: $\kappa_t \leq \mu_\kappa + 10\, \frac{\sigma_\kappa}{\sqrt{2\eta}}$. Let us now focus on the price momentum $H_t$, whose expression is the following:

$$\begin{cases} H_0 = 0 \\ H_t = \theta H_{t-1} + (1-\theta)\, r_t \quad \text{for} \quad t \geq 1 \end{cases} \tag{2.7}$$

Let us recall that it is an exponential moving average of the successive past return rates $r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$.

**Proposition 2.** If $|r_t| \leq \alpha$ with $\alpha \geq 0$ for any $t$, then $|H_t| \leq \alpha$ for any $t$.

*Proof.* Let us proceed by induction.

- $H_0 = 0 \le \alpha$
- Let us suppose that $H_{t-1} \le \alpha$. Using the definition of the price momentum $H_t$, one finds:

$$\begin{aligned} |H_t| &\le \theta\, |H_{t-1}| + (1-\theta)\, |r_t| \\ &\le \theta\, \alpha + (1-\theta)\, \alpha \\ &\le \alpha \end{aligned}$$

□

For a realistic purpose, it is expected – and verified in the following simulations – that $|r_t| << \frac{1}{2}$. Indeed, it means that the price $P_t$ of the risky asset cannot increase or decrease by more than 50% of its current value during one time step only. Thus, according to Proposition 2, one finds: $|H_t| \le \frac{1}{2}$.

Thus, in average, the last term at the right side of Equation 2.5 satisfies:

$$|\frac{p_- - p_+}{2}\, \kappa_t\, (s_t + H_t)| \le \frac{|p_- - p_+|}{2}\, (\mu_\kappa + 10\, \frac{\sigma_\kappa}{\sqrt{2\eta}})\, \frac{3}{2} \tag{2.8}$$

Considering the values used for the simulations, which are given in the basic parameter set provided by Table 2.1, one finds:

$$\begin{cases} |\frac{p_- - p_+}{2}\, \kappa_t\, (s_t + H_t)| \le 9.207 \times 10^{-4} \\ |\frac{p_- + p_+}{2}| = 0.2 \end{cases} \tag{2.9}$$

As a consequence, in those conditions, the following relation holds:

$$|\frac{p_- - p_+}{2}\, \kappa_t\, (s_t + H_t)| << |\frac{p_- + p_+}{2}| \tag{2.10}$$

Thus, according to Equation 2.5, the sum of the pseudo transition probabilities is approximately constant with respect to time, in those conditions:

$$\tilde{p}_t^- + \tilde{p}_t^+ \approx \frac{p_- + p_+}{2} \tag{2.11}$$

Hence, the previous discussion for the simpler case of no bias holds for the general case. The parameter $p$ is played by $\frac{p_- + p_+}{2}$: when both real transition probabilities $p_t^-$ and $p_t^+$ are in the set $[0, \frac{p_- + p_+}{2}]$, they are approximately equal to their corresponding pseudo transition probabilities $\tilde{p}_t^-$ and $\tilde{p}_t^+$, so that they are roughly symmetrical with respect to $\frac{p_- + p_+}{4}$. If one of the transition probabilities is greater than $\frac{p_- + p_+}{2}$, there is saturation of at least one of them, thus no symmetry anymore.

## 2.3 General time series

In order to get a clearer idea of the dynamics of the market, time series of some relevant variables are plotted on Figure 2.2. The simulation is computed until $t = 5000$ days and uses the basic parameter set provided in Table 2.1. The herding propensity $\kappa_t$ undergoes the Ornstein-Uhlenbeck process, defined in Equations 2.1, as one can observe on the corresponding panel of Figure 2.2. The mean reversion level $\mu_\kappa$ is represented by a dashed line.

The price $P_t$ of the risky asset is displayed on a log-linear scale, so that a straight line expresses an exponential behavior. Between $t = 2000$ and $t = 3000$ days, one could observe a *financial bubble*, defined in [1] as a super-exponential growth of the price. Because of their herding behavior, based on feedback and imitation, noise traders outperform fundamentalists during the bubble but fail to maintain their competitive advantage because of the subsequent crash, as one can observe on the plot corresponding to the wealth ratio $\frac{W_t^n}{W_t^f}$. The price momentum $H_t$ series is plotted on the same panel than the price return rate $r_t$ series to stress the fact that $H_t$ is nothing but an exponential moving average of $r_t$.

On Figure 2.2, one could notice that the risky fraction $x_t^f$ of fundamentalists, defined in Equations 1.14, depends well linearly on the dividend-price ratio. The transition probabilities are roughly symmetrical with respect to $\frac{p_- + p_+}{4}$, which is represented by a dashed line. This behavior can be predicted, using the definitions of the pseudo transition probabilities in Equations 2.4 and the fact that the bias introduced in the basic parameter set of Table 2.1 is small: $p_- \sim p_+$. When one of the transition probabilities becomes greater than $\frac{p_- + p_+}{2} = 0.2$ (see Table 2.1), there is saturation to 0 for the other transition probability. The dependence of the transition probabilities on the risky fraction $x_t^n$ of noise traders (imitation) and on the price momentum $H_t$ (trend) appears clearly on Figure 2.2.

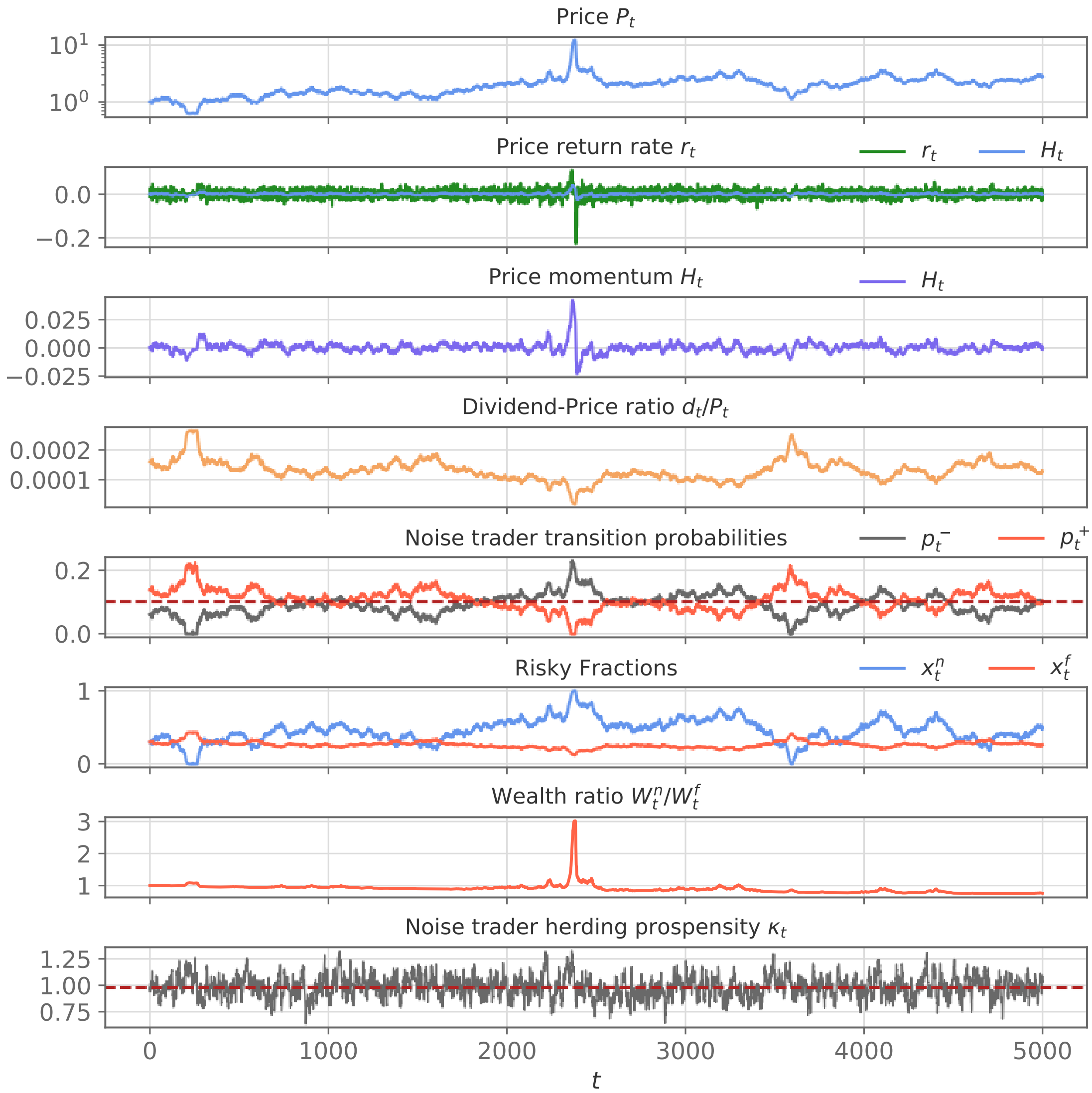


Figure 2.2: Time series of the market over 5000 days. All the parameters used for the simulation are taken from the basic parameter set provided in Table 2.1. The herding propensity undergoes a discretized Ornstein-Uhlenbeck process. Its mean reversion level $\mu_\kappa$ is represented by a dashed line. One could observe on the price series a *financial bubble* between $t = 2000$ and $t = 3000$ days, defined in [1] as a super-exponential growth of the price $P_t$. Because of their herding behavior, noise traders outperform fundamentalists during the bubble but fail to maintain their competitive advantage in the end. The transition probabilities are roughly symmetrical with respect to $\frac{p_- + p_+}{4}$, which is represented by a dashed line. It is possible to observe their almost linear dependence (the herding propensity $\kappa_t$ is not constant with respect to time) on the price momentum $H_t$ and on the risky fraction $x_t^n$ of noise traders.

## 2.4 Long-term cumulative price return rate

It has been shown in [13] that dividends can be seen as an external field, in a phase transitions meaning. Indeed, positive dividends make a stock desirable whereas negative dividends – corresponding to a premium that must be paid regularly to own the asset – make it not attracting if it does not provide other benefits. Thus, dividends drive the price of the asset. However, the discussion in [13] has been made in a static view for the fundamental price of the asset, using the celebrated formula of Gordon and Shapiro [38], and in absence of speculation. In a dynamic perspective, corresponding to our model, we expect the deterministic long-term price return rate $r_\infty$ to be the mean growth rate $r_d$ of dividends. Indeed, the speculation brought by noise traders seems to have a non-persistent impact on the market, according to the wealths' dynamics on Figure 2.2, so that fundamentalists drive the price $P_t$ of the risky asset to its fundamental value in the long run.

In the model described in Chapter 1, the notion of 'infinite time' is difficult to catch. The only simple characteristic time which is present corresponds to the dividend process. Using Equations1.5 and 1.6, the mathematical expectation gives an estimate of the deterministic evolution of dividends, in the special case where $r_d >> \sigma_d$: $\mathbb{E}(d_t) \sim (1 + r_d)\ \mathbb{E}(d_{\mathrm{t}-1})$. Thus, one finds the approximate characteristic time $\tau_d$ for the exponential behavior of the expectation of dividends:

$$\tau_d = \frac{1}{\log(1 + r_d)} \tag{2.12}$$

This characteristic time can act as a benchmark for the time of our simulations. However, one should notice that it is not sufficient to provide any information on the minimum time of simulation required to obtain a hypothetical steady state: it is only a lower bound. As a consequence, the simulations have been run until the maximum possible time of simulation $t_{\max}$. This maximum corresponds to a technical limit of storage of high numbers (typically greater than $10^{308}$).

Due to the stochastic aspect of our model, the cumulative price return rate $\langle r_t \rangle$ is considered, instead of the price return rate $r_t$. It is nothing but the cumulative moving average of the successive price return rates, making it possible to smooth the stochasticity. In all following simulations in this thesis, stochasticity is represented by initial seeds which generate random numbers. On Figure2.3, time series of the price return rate $r_t$ and of the cumulative price return rate $\langle r_t \rangle$ have been plotted until the maximum possible time of simulation $t_{\max} = 431733$, depicted by a green dashed line on each panel. All parameters used for the simulation are taken from the basic parameter set, provided by Table 2.1, except the mean growth rate $r_d$ of dividends ($r_d = 0.0016$). For greater clarity, only one seed has been used. The first two panels suggest that both $r_t$ and $\langle r_t \rangle$ seem to converge over time, after a given transient regime. The last two panels are nothing but zooms of the previous panels around their 'convergent' value. On the third panel, one could observe that stochasticity has a significant impact on the price return rate $r_t$, even when $t$ is near its final value $t_{\max}$, so that there is not really convergence of $r_t$ over time. However, from $t \sim 25000$ to $t_{\max}$, the price return rate $r_t$ of the risky asset is centered around a mean value approximately equal to 0.0016, exhibiting variations about 6% from this value ($\frac{0.00165-0.00155}{0.0016} \sim 6\%$). The large period of time for which those observations are verified suggests that, from $t \geq 25000$, the variations of the price return rate $r_t$ are only due to stochasticity, so that there would be no significant change for times greater than $t_{\max}$. As a consequence, it seems difficult to define a deterministic long-term price return rate $r_\infty$ – let us recall that the purpose of this section is to find such a deterministic long-term value, to compare with the mean growth rate $r_d$ of dividends – using only the stochastic price return rate $r_t$. On the last panel of Figure 2.3, one could notice that the cumulative price return rate $\langle r_t \rangle$ is much more stable than the price return rate $r_t$ for $t \geq 25000$. Furthermore, it converges to a value approximately equal to 0.0016, that is the previous mean value of the price return rate $r_t$. Thus, the variations, observed on the time series of the price return rate $r_t$, seem to be offset when using a simple average, meaning that they are caused by stochasticity only. Hence, a deterministic long-term price return rate $r_\infty$ can be defined using the long-term cumulative price return rate. In addition, one should notice that the convergent value of the cumulative price return rate $\langle r_t \rangle$ is exactly the mean growth rate $r_d$ of dividends, used for this simulation (0.0016). It suggests that the long-term cumulative price return rate is well equal to the mean growth rate $r_d$ of dividends.

Even if the cumulative price return rate $\langle r_t \rangle$ seems to converge over time, it has the disadvantage of keeping in memory transient past return rates. Thus, its long-term value $\langle r_\infty \rangle$ will only lead to an estimate of the deterministic long-term price return rate $r_\infty$. Having said that, let us focus on the numerical process to get the value $\langle r_\infty \rangle$. To find how this value depend on the mean growth rate $r_d$ of dividends, the computation of $\langle r_\infty \rangle$ is made in the following, for many values of $r_d$.

Because of the significant corresponding times of simulation, only the simulation of the first seed is run until $t_{\max}$. Taking the final value of the cumulative price return rate $\langle r_{\mathrm{t_{max}}} \rangle$ for this specific seed, we find the time $t_c$ from which the cumulative price return rate $\langle r_t \rangle$ stays into a convergent interval of 1% of the final value $\langle r_{\mathrm{t_{max}}} \rangle$. Then, the simulations of all seeds are computed until this 'convergent' time $t_c$. The process of computation of the time $t_c$ is illustrated on Figure 2.4. As for Figure 2.3, the time series have been plotted until the maximum possible time of simulation $t_{\max} = 431733$ for one specific seed, depicted by a green dashed line on each panel. All parameters used for the simulation are taken from the basic parameter set, provided by Table 2.1, except the mean growth rate $r_d$ of dividends ($r_d = 0.0016$). The horizontal blue dashed lines correspond to the convergent interval of 1% of the final value $\langle r_{\mathrm{t_{max}}} \rangle$. Then, it is possible to find the time $t_c$ from which the cumulative price return rate $\langle r_t \rangle$ stays into this convergent interval. This time $t_c$ is depicted by a vertical blue dashed line on each panel.

Finally, we take the average of the values $\langle r_{\mathrm{t_c}} \rangle$ over the seeds to get the long-term cumulative price return rate $\langle r_\infty \rangle$. The standard deviation of the set of values $\langle r_{\mathrm{t_c}} \rangle$, each corresponding to one seed (that is before taking the average $\langle r_\infty \rangle$), is denoted by $\sigma[\langle r_\infty \rangle]$. In the following, 100 seeds are considered. Assuming that all those seeds lead exactly to the same simulations, the uncertainty of 1% over the value $\langle r_{\mathrm{t_c}} \rangle$ of the first seed propagates to a global uncertainty $\sigma_{\mathrm{err}}$ over the long-term cumulative price return rate $\langle r_\infty \rangle$, whose value is the following:

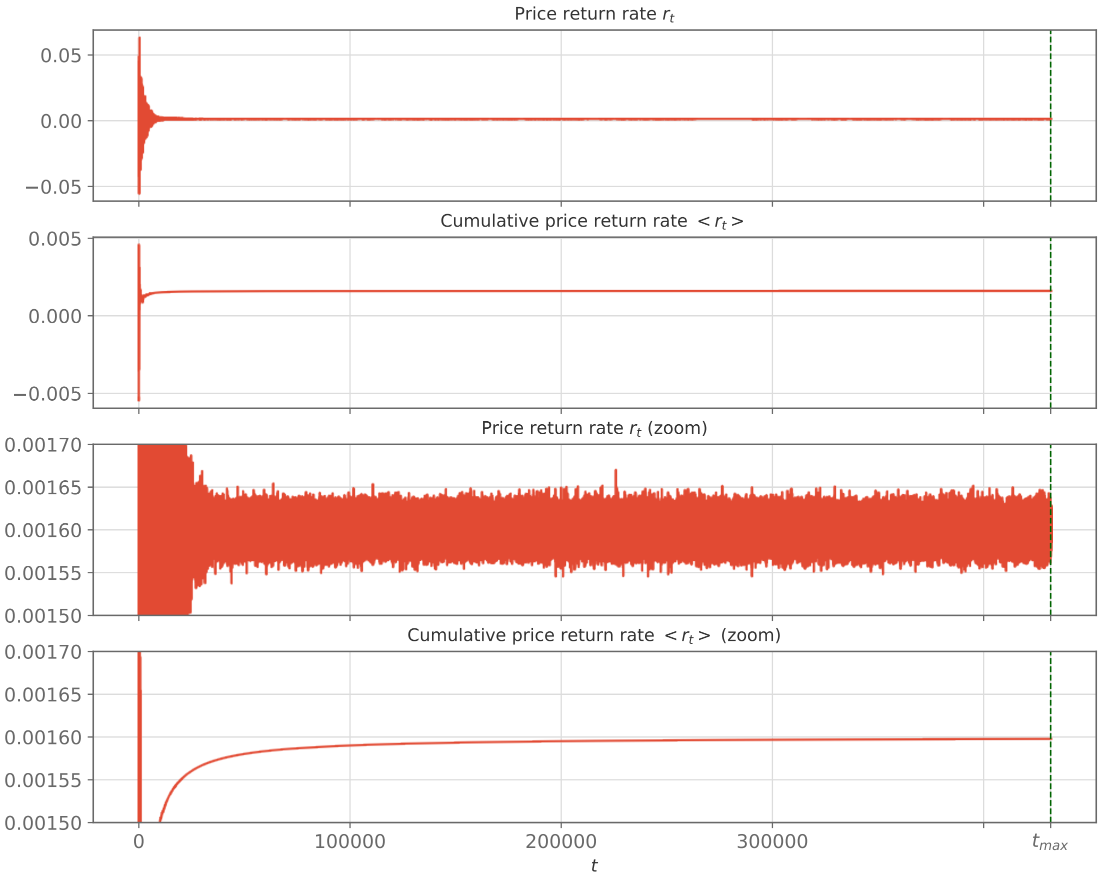


Figure 2.3: Comparison between time series of the price return rate $r_t$ and time series of the cumulative price return rate $\langle r_t \rangle$ of the risky asset. The time series have been plotted until the maximum possible time of simulation $t_{\max} = 431733$, depicted by a green dashed line on each panel. All parameters used for the simulation are taken from the basic parameter set, provided by Table 2.1, except the mean growth rate $r_d$ of dividends ($r_d = 0.0016$). For greater clarity, only one seed has been used. The first two panels suggest that both $r_t$ and $\langle r_t \rangle$ seem to converge over time, after a given transient regime. The last two panels are nothing but zooms of the previous panels around their 'convergent' value. On the third panel, one could observe that stochasticity has a significant impact on the price return rate $r_t$, even when $t$ is near its final value $t_{\max}$, so that there is not really convergence of $r_t$ over time. The last panel shows that the cumulative price return rate $\langle r_t \rangle$ is much more stable than the price return rate $r_t$ for $t \geq 25000$. Furthermore, it converges to a value approximately equal to 0.0016, that is the mean growth rate $r_d$ of dividends, used for this simulation.

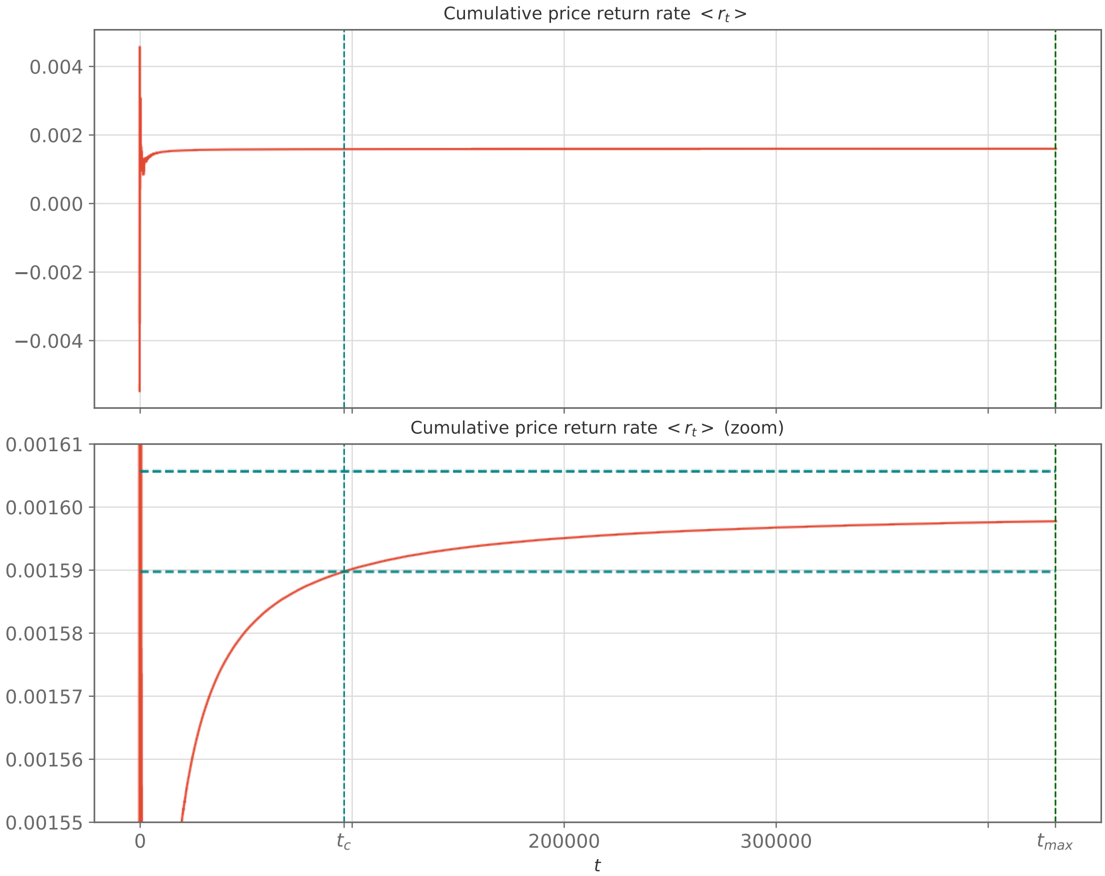


Figure 2.4: Process of computation of the convergent time $t_c$, which will be the maximum time of simulations for all seeds. The time series have been plotted until the maximum possible time of simulation $t_{\max} = 431733$, depicted by a green dashed line on each panel. All parameters used for the simulation are taken from the basic parameter set, provided by Table 2.1, except the mean growth rate $r_d$ of dividends ($r_d = 0.0016$). One specific seed is used to obtain a simulation until $t_{\max}$. Taking the final value of the cumulative price return rate $\langle r_{t_{\max}} \rangle$ for this specific seed, we find the time $t_c$ from which the cumulative price return rate $\langle r_t \rangle$ stays into a convergent interval of 1% of the final value $\langle r_{t_{\max}} \rangle$. The horizontal blue dashed lines correspond to the convergent interval of 1% of the final value $\langle r_{t_{\max}} \rangle$. Then, it is possible to find the time $t_c$ from which the cumulative price return rate $\langle r_t \rangle$ stays into this convergent interval. This time $t_c$ is depicted by a vertical blue dashed line on each panel.

$$\sigma_{\text{err}} \;=\; \frac{0.01}{\sqrt{100}} \;=\; 1\times 10^{-3} \tag{2.13}$$

One should notice that $\sigma_{\text{err}}$ is a lower bound for the real uncertainty over $\langle r_\infty\rangle$, since each seed leads to a unique behavior, due to the stochasticity. Doing the simulations for many values of the mean growth rate $r_d$ of dividends makes it possible to find the relation between $\langle r_\infty\rangle$ and $r_d$. The results are plotted on Figure 2.5.

100 seeds have been used and all parameters – except $r_d$ which varies along the x-axis – are taken from the basic parameter set, provided in Table 2.1. On the first panel, the values of the long-term cumulative price return rate $\langle r_\infty\rangle$ are plotted along with the corresponding values of $r_d$ (blue line) used to compute them. As expected, the long-term cumulative price return rate $\langle r_\infty\rangle$ seems to be equal to the mean growth rate $r_d$ of dividends. On the third panel, the ratio $\frac{t_c}{t_{\max}}$ gives some information about the convergence of the cumulative price return rate $\langle r_t\rangle$. For values of $r_d$ lower than $10^{-3}$, this ratio becomes significant, meaning that there is not really convergence. Thus, the corresponding values of $\langle r_\infty\rangle$ are not relevant, so that lower values than $10^{-4}$ for $r_d$ have not been represented. One should notice that low values for the ratio $\frac{t_c}{t_{\max}}$ do not mean that there has been a real convergence of the cumulative price return rate. On the fourth panel, the ratio $\frac{t_c}{\tau_d}$ shows that the time $t_c$ used to compute $\langle r_\infty\rangle$ is almost always greater than 100 times the characteristic time of dividends' dynamics, except for values of $r_d$ around $10^{-3}$. Its use is justified since the considered values of $r_d$ are greater than $10^{-3}$ and that the standard deviation $\sigma_d$ of dividends is roughly equal to $10^{-5}$ (see Table 2.1). On the second panel, the relative difference between $\langle r_\infty\rangle$ and $r_d$ is represented. There is a peak which we cannot explain but it appears for values of $r_d$ for which the time $t_c$ is less than 100 times the characteristic time $\tau_d$. On this panel, the value of $\sigma_{\text{err}}$ is displayed by a dashed line. For the relevant values of $r_d$ (greater than $10^{-3}$), one could observe that the relative difference between $\langle r_\infty\rangle$ and $r_d$ is greater than $\sigma_{\text{err}}$ (around $4\times 10^{-3}$ in average). The last panel provides some information about this difference: the standard deviation of the values used to compute the average $\langle r_\infty\rangle$ over all the seeds, rescaled by $\langle r_\infty\rangle$. One could notice that its values are too small to explain the previous difference. Then, we conclude by remarking that the memory of the transient past cumulative price return rates $\langle r_t\rangle$ may have an impact of our results. However, it may be possible that there has not been a real convergence. In all cases, even if $\langle r_\infty\rangle$ is not exactly equal to $r_d$, dividends have still a significant impact on the long-term price return rate of the risky asset, supporting [13] about their external field aspect.

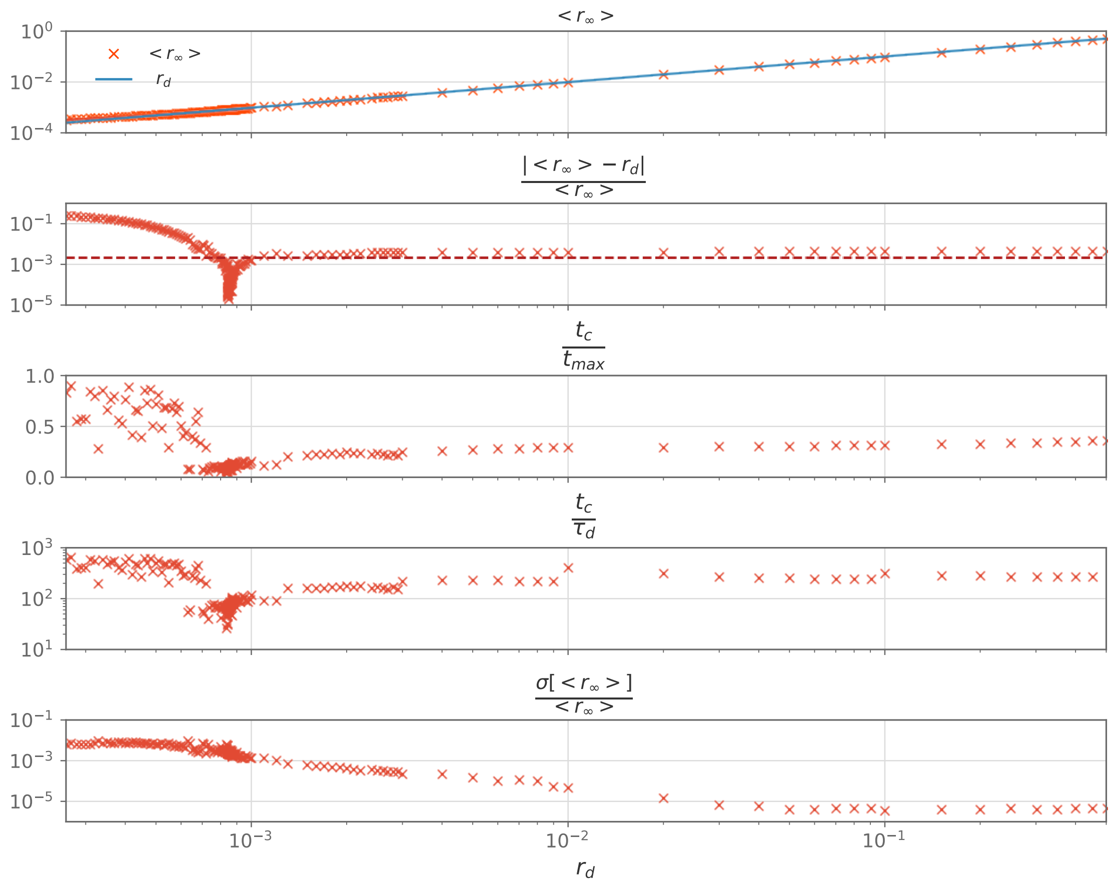


Figure 2.5: Long-term cumulative price return rate $\langle r_\infty \rangle$ as a function of the mean growth rate $r_d$ of dividends. The cumulative price return rate $\langle r_t \rangle$ is the cumulative moving average of the successive price return rates $r_t$. 100 different seeds have been used for the simulations. All parameters are taken from the basic parameter set, provided in Table 2.1, except the mean growth rate $r_d$ of dividends, which varies along the x-axis. To obtain the long-term value $\langle r_\infty \rangle$ for one given value of $r_d$, the simulation corresponding to one specific seed is run until the maximum possible time of simulation $t_{\max}$ (technical limit of storage of numbers greater than $10^{308}$) – see Figure 2.3 for more details. The final value $\langle r_{t_{\max}} \rangle$ of the cumulative price return rate for the first seed is used to compute a 'convergent' time $t_c$ from which the cumulative price return rate $\langle r_t \rangle$ stays into an interval of 1% of its final value $\langle r_{t_{\max}} \rangle$ (see Figure 2.4). The simulations of all 100 seeds are then run until $t_c$. The average of the values $\langle r_{t_c} \rangle$ over all the seeds is the long-term cumulative price return rate $\langle r_\infty \rangle$, plotted on the first panel along with the value of $r_d$ used for its computation (blue straight line). The standard deviation of the values $\langle r_{t_c} \rangle$ over all the seeds is denoted by $\sigma[\langle r_\infty \rangle]$ and a rescaled version is plotted on the last panel. On the third panel, the ratio $\frac{t_c}{t_{\max}}$ shows that the results obtained for values of $r_d$ lower than $10^{-3}$ are not relevant. The fourth panel compares the 'convergent' time $t_c$ to the approximate characteristic time $\tau_d$ of dividends, defined in Equation2.12. Its use is legitimate since the considered values of $r_d$ are greater than $10^{-3}$ and the standard deviation $\sigma_d$ of dividends is around $10^{-5}$ (see Table 2.1). The relative difference between $\langle r_\infty \rangle$ and $r_d$ is plotted on the second panel. The dashed line corresponds to $\sigma_{\text{err}}$, defined in Equation 2.13, which is a lower bound for the uncertainty on $\langle r_\infty \rangle$.

# Chapter 3

# Theoretical analysis of the long-term behavior

## 3.1 The price momentum

Let us recall the expression of the price momentum, as a function of the price return rate $r_t$:

$$\begin{cases} H_0 \\ H_t \;=\; \theta \; H_{\mathrm{t-1}} \;+\; (1-\theta) \; r_t \quad \text{if} \quad t \geq 1 \end{cases} \tag{3.1}$$

**Proposition 3.** For $t \geq 1$, $\;H_t \;=\; (1-\theta) \; \sum_{i=1}^{t} \theta^{t-i} \; r_i \;+\; \theta^t \; H_0.$

*Proof.* We have $H_1 \;=\; (1-\theta) \; r_1 \;+\; \theta \; H_0$. Then, if we suppose that the proposition stands for $t \geq 1$, we have :

$$\begin{aligned} H_{\mathrm{t+1}} &= \theta \; [\; (1-\theta) \; \sum_{i=1}^{t} \theta^{t-i} \; r_i \;+\; \theta^t \; H_0 \;] \;+ (1-\theta) \; r_{\mathrm{t+1}} \\ &= (1-\theta) \; \sum_{i=1}^{t} \theta^{t+1-i} \; r_i \;+\; \theta^{t+1} \; H_0 \;+ (1-\theta) \; r_{\mathrm{t+1}} \\ &= (1-\theta) \; \sum_{i=1}^{t+1} \theta^{t+1-i} \; r_i \;+\; \theta^{t+1} \; H_0 \end{aligned}$$

□

This expression clearly shows how the price momentum depends on the successive price return rates. Now, let us suppose that the price return rate $r_t$ converges to a given value $r_\infty$. We would like to know what is the long-term behavior of the price momentum $H_t$.

**Proposition 4.** If $r_t$ converges to a given value $r_\infty$, the price momentum $H_t$ converges to the same value.

*Proof.* We shall use the following identity: $1-\theta^t = (1-\theta)\sum_{i=1}^{t}\theta^{t-i}$. The long-term price return rate $r_\infty$ can then be written as follows:

$$\begin{aligned} r_\infty &= r_\infty\,(1-\theta^t+\theta^t) \\ &= r_\infty\,(1-\theta)\sum_{i=1}^{t}\theta^{t-i} \;+\; \theta^t\, r_\infty \end{aligned}$$

Then, one finds easily the difference between $H_t$ and $r_\infty$:

$$H_t \;-\; r_\infty \;=\; (1-\theta)\sum_{i=1}^{t}\theta^{t-i}\,[r_i \;-\; r_\infty] \;+\; \theta^t\,(H_0 \;-\; r_\infty)$$

The last term tends to 0 when $t \to +\infty$, since $\theta \in\, ]\,0,1\,[$. When $t \to +\infty$, $r_t \to r_\infty$, so the term in square brackets in the previous sum becomes as small as we want it to be. It does not prove why the sum converges to 0 but it gives a general idea. For a complete mathematical proof of this point, see Appendix A. □

In fact, the relation between $H_t$ and $r_t$ is even more important than that. Let us suppose that $H_t$ converges to a given value $H^*$. Then, $(H_t - \theta\, H_{t-1})$ converges to $(1-\theta)\,H^*$. From equation 3.1, one finds that $H_t - \theta\, H_{t-1} = (1-\theta)\,r_t$. As a consequence, $r_t$ converges to a given value $r_\infty$ which verifies $(1-\theta)\,r_\infty = (1-\theta)\,H^*$, that is $r_\infty = H^*$. This proves the following result.

**Proposition 5.** The price momentum $H_t$ converges if and only if the price return rate $r_t$ converges. If one of them converges to a given value, the other one converges to the same value.

## 3.2 The average opinion index

The opinion index $s_t$ is stochastic because of the randomness of the variables $\xi_k(p_t^+)$ and $\xi_j(p_t^-)$, which appear in the dynamics of the number $N_t^+$ of noise traders invested in the risky asset and of the number $N_t^-$ of those invested in the risk-free asset instead:

$$\begin{cases} N_t^+ \;=\; \sum_{k=1}^{N_{t-1}^+}\,[1-\xi_k(p_{t-1}^+)] \;+\; \sum_{j=1}^{N_{t-1}^-}\,\xi_j(p_{t-1}^-) \\ N_t^- \;=\; \sum_{k=1}^{N_{t-1}^+}\,\xi_k(p_{t-1}^+) \;+\; \sum_{j=1}^{N_{t-1}^-}\,[1-\xi_j(p_{t-1}^-)] \\ \\ s_t \;=\; \frac{N_t^+ \;-\; N_t^-}{N_c} \;\in\; [-1,1] \end{cases} \tag{3.2}$$

It is nevertheless possible to get the average deterministic behavior, using the mathematical expectation and considering $\overline{s}_t$, a linear expansion of $\mathbb{E}(s_t)$ instead.

**Proposition 6.** For $t \geq 1$, $\overline{s}_t \ = \ \overline{s}_{\mathrm{t}-1} \ + \ p^-_{\mathrm{t}-1} \ (1-\overline{s}_{\mathrm{t}-1}) \ - p^+_{\mathrm{t}-1} \ (1+\overline{s}_{\mathrm{t}-1})$

*Proof.* The random variable $\xi_k(p^+_t)$ (resp. $\xi_j(p^-_t)$) takes the value 1 with probability $p^+_t$ (resp. $p^-_t$) and the value 0 with probability $1-p^+_t$ (resp. $1-p^-_t$). Then, one easily finds their expectation:

$$\begin{cases} \mathbb{E}(\xi_k(p^+_t)) \ = \ 1 \times \mathbb{E}(p^+_t) \ + \ 0 \times (1-\mathbb{E}(p^+_t)) \ = \ \mathbb{E}(p^+_t) \\ \mathbb{E}(\xi_j(p^-_t)) \ = \ 1 \times \mathbb{E}(p^-_t) \ + \ 0 \times (1-\mathbb{E}(p^-_t)) \ = \ \mathbb{E}(p^-_t) \end{cases}$$

One should notice that the transition probabilities $p^-_t$ and $p^+_t$ are linear functions of the stochastic index opinion $s_t$ (and of the price momentum $H_t$). As a consequence, the transition probabilities $\mathbb{E}(p^-_t)$ and $\mathbb{E}(p^+_t)$ appearing as results of the previous calculus are deterministic versions of the *true* transition probabilities, so that $p^-_t\ (\overline{s}_t, H_t)$ and $p^+_t\ (\overline{s}_t, H_t)$ are now considered. For the sake of simplicity, their respective writing symbols are not changed.

Using a linear expansion in Equations 3.2, it is possible to find a deterministic version of the dynamics of $N^+_t$ and $N^-_t$:

$$\begin{cases} \mathbb{E}(N^+_t) \ = \ \mathbb{E}(N^+_{\mathrm{t}-1})\ (1-p^+_{\mathrm{t}-1}) \ + \ \mathbb{E}(N^-_{\mathrm{t}-1})\ p^-_{\mathrm{t}-1} \\ \mathbb{E}(N^-_t) \ = \ \mathbb{E}(N^+_{\mathrm{t}-1})\ p^+_{\mathrm{t}-1} \ + \ \mathbb{E}(N^-_{\mathrm{t}-1})\ (1-p^-_{\mathrm{t}-1}) \end{cases}$$

One finds the corresponding value of $\mathbb{E}(s_{\mathrm{t}}) \ \sim \ \overline{s}_{\mathrm{t}}$, using Equations 3.2:

$$\overline{s}_{\mathrm{t}} \ = \ \frac{\mathbb{E}(N^+_{\mathrm{t}-1})\ (1-2p^+_{\mathrm{t}-1}) + \ \mathbb{E}(N^-_{\mathrm{t}-1})\ (2p^-_{\mathrm{t}-1}-1)}{N_c}$$

The conservation of noise traders $N^+_t + N^-_t \ = \ N_c$ gives the relations between $N^+_t$ (or $N^-_t$) and $s_t$, which are: $\frac{N^+_t}{N_c} \ = \ \frac{1}{2}(1+s_t)$ and $\frac{N^-_t}{N_c} \ = \ \frac{1}{2}(1-s_t)$. It is then possible to use the corresponding relations between $\mathbb{E}(N^+_{\mathrm{t}-1})$ (or $\mathbb{E}(N^-_{\mathrm{t}-1})$) and $\mathbb{E}(s_{\mathrm{t}-1}) \ \sim \ \overline{s}_{\mathrm{t}-1}$:

$$\overline{s}_{\mathrm{t}} \ = \ \frac{\frac{N_c}{2}\ (1+\overline{s}_{\mathrm{t}-1})\ (1-2p^+_{\mathrm{t}-1}) + \ \frac{N_c}{2}\ (1-\overline{s}_{\mathrm{t}-1})\ (2p^-_{\mathrm{t}-1}-1)}{N_c}$$

$$= \ \frac{1+\overline{s}_{\mathrm{t}-1}}{2} \ - \ \frac{1-\overline{s}_{\mathrm{t}-1}}{2} \ + \ p^-_{\mathrm{t}-1}\ (1-\overline{s}_{\mathrm{t}-1}) \ - \ p^+_{\mathrm{t}-1}\ (1+\overline{s}_{\mathrm{t}-1})$$

$$= \overline{s}_{\mathrm{t}-1} \ + \ p^-_{\mathrm{t}-1}\ (1-\overline{s}_{\mathrm{t}-1}) \ - \ p^+_{\mathrm{t}-1}\ (1+\overline{s}_{\mathrm{t}-1})$$

□

The fact that $s_t \in [-1,1]$ stands for its expectation, that is: $\overline{s}_t \in [-1,1]$. One can notice that this is true because $p^+_{\mathrm{t}-1}$ and $p^+_{\mathrm{t}-1}$ are in the set $[0,1]$.

Proposition 6 can be rewritten, using the average risky fraction of noise traders $\overline{x}^n_t = \frac{1+\overline{s}_t}{2}$, instead of the average index opinion:

$$\overline{x}^n_t - \overline{x}^n_{\mathrm{t}-1} = p^-_{\mathrm{t}-1}\,(1-\overline{x}^n_{\mathrm{t}-1}) - p^+_{\mathrm{t}-1}\,\overline{x}^n_{\mathrm{t}-1} \tag{3.3}$$

Considering the average risky fraction $\overline{x}^n_t$ as the probability for noise traders to be invested in the risky asset at time $t$, this equation is exactly a discrete-time master equation. The first term represents the change, in successive time steps, of the probability of being invested in the risky asset. Two reasons are given. The first (positive) contribution appears when noise traders are not invested in the risky asset at time $t-1$ and decide to buy at time $t$. The second (negative) contribution appears when noise traders are well invested in the risky asset at time $t-1$, but decide to sell at time $t$. The change of position is then characterized by the transition probabilities.

In the expression of $\overline{s}_t$, the only link with the market remains in the transition probabilities $p^-_{\mathrm{t}-1}\,(\overline{s}_{\mathrm{t}-1}, H_{\mathrm{t}-1})$ and $p^+_{\mathrm{t}-1}\,(\overline{s}_{\mathrm{t}-1}, H_{\mathrm{t}-1})$. In order to obtain the long-term behavior of the average opinion index $\overline{s}_t$, it is thus necessary to look at their respective behavior. Let us recall that they are defined as saturations of the pseudo transition probabilities $\tilde{p}^-_{\mathrm{t}-1}$ and $\tilde{p}^+_{\mathrm{t}-1}$:

$$p^-_{\mathrm{t}} = \begin{cases} \tilde{p}^-_{\mathrm{t}} & \text{if } \tilde{p}^-_{\mathrm{t}} = \frac{p_-}{2}\,[1+\kappa\,(\overline{s}_t + H_t)] \in [0,1] \\ 0 & \text{if } \tilde{p}^-_{\mathrm{t}} \le 0 \\ 1 & \text{if } \tilde{p}^-_{\mathrm{t}} \ge 1 \end{cases}$$

$$p^+_{\mathrm{t}} = \begin{cases} \tilde{p}^+_{\mathrm{t}} & \text{if } \tilde{p}^+_{\mathrm{t}} = \frac{p_+}{2}\,[1-\kappa\,(\overline{s}_t + H_t)] \in [0,1] \\ 0 & \text{if } \tilde{p}^+_{\mathrm{t}} \le 0 \\ 1 & \text{if } \tilde{p}^+_{\mathrm{t}} \ge 1 \end{cases} \tag{3.4}$$

Here, the herding propensity $\kappa$ is chosen to be constant, for the sake of simplicity. From Equations3.4, one finds: $\tilde{p}^-_{\mathrm{t}} - \frac{p_-}{2}\,\kappa\,\overline{s}_t = \frac{p_-}{2}[1+\kappa\,H_t]$ and $\tilde{p}^+_{\mathrm{t}} + \frac{p_+}{2}\,\kappa\,\overline{s}_t = \frac{p_-}{2}[1-\kappa\,H_t]$. Then, if the price return rate $r_t$ converges to a given value $r_\infty$, the price momentum $H_t$ converges to the same value according to Proposition 4, so that $(\tilde{p}^-_{\mathrm{t}} - \frac{p_-}{2}\,\kappa\,\overline{s}_t)$ and $(\tilde{p}^+_{\mathrm{t}} + \frac{p_+}{2}\,\kappa\,\overline{s}_t)$ converge too. Nevertheless, there is no proof that the average opinion index $\overline{s}_t$ or the pseudo transition probabilities $\tilde{p}^-_{\mathrm{t}}$ and $\tilde{p}^+_{\mathrm{t}}$ converge.

When the herding propensity $\kappa$ (constant) is equal to 0, the problem becomes much simpler, since $\tilde{p}^-_{\mathrm{t}} = \frac{p_-}{2} \in (0,1)$ and $\tilde{p}^+_{\mathrm{t}} = \frac{p_+}{2} \in (0,1)$, so that $p^-_{\mathrm{t}} = \frac{p_-}{2}$ and $p^+_{\mathrm{t}} = \frac{p_+}{2}$. Proposition 6 gives:

$$\overline{s}_t = [1 - \frac{p_+ + p_-}{2}]\,\overline{s}_{\mathrm{t}-1} + \frac{p_- - p_+}{2} \tag{3.5}$$

If $p_- = p_+ := p$, that is there is no bias amongst noise traders, one finds a simple expression for the average opinion index:

$$\overline{s}_t = [1-p]^t \, s_0 \tag{3.6}$$

Since $p \in (0,1)$, the average opinion index converges to 0 when there is no herding and no bias.

If $p_- \neq p_+$, that is there is some bias amongst noise traders, one could notice that:

$$\frac{p_- - p_+}{p_- + p_+} = [1 - \frac{p_+ + p_-}{2}] \, [\frac{p_- - p_+}{p_- + p_+}] + \frac{p_- - p_+}{2} \tag{3.7}$$

Subtracting equation 3.7 from equation 3.5 gives:

$$[\overline{s}_t - \frac{p_- - p_+}{p_- + p_+}] = [1 - \frac{p_+ + p_-}{2}] \, [\overline{s}_{t-1} - \frac{p_- - p_+}{p_- + p_+}]$$

As a consequence, one finds the following expression for the average opinion index $\overline{s}_t$:

$$\overline{s}_t = [1 - \frac{p_+ + p_-}{2}]^t \, [s_0 - \frac{p_- - p_+}{p_- + p_+}] + \frac{p_- - p_+}{p_- + p_+} \tag{3.8}$$

Since $p_- \in (0,1)$ and $p_+ \in (0,1)$, the average opinion index converges to $\frac{p_- - p_+}{p_- + p_+}$ when there is no herding (but some bias). In those conditions, if noise traders are more likened to buy the risky asset than to sell it, that is $p_- > p_+$, the average opinion index converges to a positive value. One could notice that the case with no bias, that is $p_- = p_+ := p$, can be solved using the previous equation.

To conclude, the average opinion index $\overline{s}_t$ always converge when there is no herding ($\kappa = 0$). When it is no more the case, the problem of convergence becomes much more difficult. Thus, for the following theoretical study of the long-term behavior of the market, it will be assumed that there is convergence of the average opinion index, in addition to the convergence of the price return rate $r_t$.

## 3.3 Fixed points of the average opinion index without bias

Let us assume that, in addition to the convergence of the price return rate $r_t$ to the long-term value $r_\infty$, the average opinion index $\overline{s}_t$ converges to a given value $s^*$. Then, in absence of bias, the pseudo transition probabilities converge too and Equations 3.4 give:

$$\begin{cases} \tilde{p}_t^- \underset{+\infty}{\longrightarrow} \frac{p}{2}\left[1+\kappa\left(s^* + r_\infty\right)\right] := \tilde{p}_\infty^- \\ \tilde{p}_t^+ \underset{+\infty}{\longrightarrow} \frac{p}{2}\left[1-\kappa\left(s^* + r_\infty\right)\right] := \tilde{p}_\infty^+ \end{cases} \tag{3.9}$$

As a consequence, the true transition probabilities $p_t^-$ and $p_t^+$ converge too, since they are nothing but saturations of the pseudo transition probabilities. Let us denote their respective limits by $p_\infty^-$ and $p_\infty^+$. The expression of the average opinion index $\overline{s}_t$, defined in Proposition 6, gives when $t \to +\infty$:

$$p_\infty^- \left(1-s^*\right) = p_\infty^+ \left(1+s^*\right) \tag{3.10}$$

One could notice that, in the special case where $p_\infty^- \left(1-s^*\right) = p_\infty^+ \left(1+s^*\right) = 0$, Equation3.10 is satisfied. This case corresponds to the detailed balance of the master equation 3.3. As it is well known, it provides a sufficient condition to obtain the equilibrium. However, this is not a necessary condition, as we shall see later.

Now that the limits of the pseudo transition probabilities are obtained by Equations 3.9, it is possible to find the limits $p_\infty^-$ and $p_\infty^+$ of the true transition probabilities. The relations between them are given in Equations 3.4. To find those new limits, it is thus necessary to find whether $\tilde{p}_\infty^-$ and $\tilde{p}_\infty^+$ are in the set $[0,1]$ or not. In the following, we distinguish all possible cases.

- $\begin{cases} \tilde{p}_\infty^- < 0 \\ \tilde{p}_\infty^+ \in [0,1] \end{cases} \iff \begin{cases} \frac{p}{2}\left[1+\kappa\left(s^* + r_\infty\right)\right] < 0 \\ \frac{p}{2}\left[1-\kappa\left(s^* + r_\infty\right)\right] \in [0,1] \end{cases}$

  According to the definition of the transition probabilities in Equations 3.4, one finds: $p_\infty^- = 0$ and $p_\infty^+ = \frac{p}{2}\left[1-\kappa\left(s^* + r_\infty\right)\right]$. Equation 3.10, providing the equilibrium, gives in this particular case:

  $$\begin{cases} \left[1-\kappa\left(s^* + r_\infty\right)\right]\left(1+s^*\right) = 0 \\ \frac{p}{2}\left[1+\kappa\left(s^* + r_\infty\right)\right] < 0 \\ \frac{p}{2}\left[1-\kappa\left(s^* + r_\infty\right)\right] \in [0,1] \end{cases}$$

  The solutions for this set of equation and inequalities are the following:

  - $\kappa < 0$

    $\begin{cases} 1-\frac{1}{\kappa} < r_\infty \leq 1+\frac{1}{\kappa}-\frac{2}{\kappa p} \\ s^* = -1 \end{cases}$

  - $\kappa > 0$

    $\begin{cases} 1+\frac{1}{\kappa}-\frac{2}{\kappa p} \leq r_\infty < 1-\frac{1}{\kappa} \\ s^* = -1 \end{cases}$

- $\begin{cases} \tilde{p}^{-}_{\infty} < 0 \\ \tilde{p}^{+}_{\infty} > 1 \end{cases} \iff \begin{cases} \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right] < 0 \\ \frac{p}{2}\left[1-\kappa\left(s^{*} + r_{\infty}\right)\right] > 1 \end{cases}$

  In this case, one finds: $p^{-}_{\infty} = 0$ and $p^{+}_{\infty} = 1$. Equation 3.10 gives:

  $$\begin{cases} s^{*} = -1 \\ \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right] < 0 \\ \frac{p}{2}\left[1-\kappa\left(s^{*} + r_{\infty}\right)\right] > 1 \end{cases}$$

  The solutions for this set of equation and inequalities are the following:

  - $\kappa < 0$

  $$\begin{cases} r_{\infty} > 1 + \frac{1}{\kappa} - \frac{2}{\kappa p} \\ s^{*} = -1 \end{cases}$$

  - $\kappa > 0$

  $$\begin{cases} r_{\infty} < 1 + \frac{1}{\kappa} - \frac{2}{\kappa p} \\ s^{*} = -1 \end{cases}$$

- $\begin{cases} \tilde{p}^{-}_{\infty} \in [0,1] \\ \tilde{p}^{+}_{\infty} < 0 \end{cases} \iff \begin{cases} \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right] \in [0,1] \\ \frac{p}{2}\left[1-\kappa\left(s^{*} + r_{\infty}\right)\right] < 0 \end{cases}$

  The transition probabilities are thus: $p^{-}_{\infty} = \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right]$ and $p^{+}_{\infty} = 0$. Equation 3.10 gives:

  $$\begin{cases} \left[1+\kappa\left(s^{*} + r_{\infty}\right)\right]\left(1-s^{*}\right) = 0 \\ \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right] \in [0,1] \\ \frac{p}{2}\left[1-\kappa\left(s^{*} + r_{\infty}\right)\right] < 0 \end{cases}$$

  The solutions for this set of equation and inequalities are the following:

  - $\kappa < 0$

  $$\begin{cases} \frac{2}{\kappa p} - \frac{1}{\kappa} - 1 \leq r_{\infty} < \frac{1}{\kappa} - 1 \\ s^{*} = +1 \end{cases}$$

  - $\kappa > 0$

  $$\begin{cases} \frac{1}{\kappa} - 1 < r_{\infty} \leq \frac{2}{\kappa p} - \frac{1}{\kappa} - 1 \\ s^{*} = +1 \end{cases}$$

- $\begin{cases} \tilde{p}^{-}_{\infty} \in [0,1] \\ \tilde{p}^{+}_{\infty} \in [0,1] \end{cases} \iff \begin{cases} \frac{p}{2}\left[1+\kappa\left(s^{*} + r_{\infty}\right)\right] \in [0,1] \\ \frac{p}{2}\left[1-\kappa\left(s^{*} + r_{\infty}\right)\right] \in [0,1] \end{cases}$

One should notice that this case is different than the detailed balance of the master equation 3.3. Here, the transition probabilities are: $p^-_\infty = \frac{p}{2}\,[1+\kappa\,(s^* + r_\infty)]$ and $p^+_\infty = \frac{p}{2}\,[1-\kappa\,(s^* + r_\infty)]$. Equation 3.10 gives:

$$\begin{cases} (\kappa-1)\,s^* + \kappa\,r_\infty = 0 \\ \frac{p}{2}\,[1+\kappa\,(s^* + r_\infty)] \in [0,1] \\ \frac{p}{2}\,[1-\kappa\,(s^* + r_\infty)] \in [0,1] \end{cases}$$

The solutions for this set of equation and inequalities are the following:

- $\kappa < 0$

$$\begin{cases} \frac{1}{\kappa} - 1 \le r_\infty \le 1 - \frac{1}{\kappa} \\ s^* = \frac{\kappa\,r_\infty}{1-\kappa} \end{cases}$$

- $\kappa = 0$

$$\begin{cases} r_\infty \in \mathbb{R} \\ s^* = 0 \end{cases}$$

- $0 < \kappa < 1$

$$\begin{cases} 1 - \frac{1}{\kappa} \le r_\infty \le \frac{1}{\kappa} - 1 \\ s^* = \frac{\kappa\,r_\infty}{1-\kappa} \end{cases}$$

- $\kappa = 1$

$$\begin{cases} r_\infty = 0 \\ s^* \in [-1,1] \end{cases}$$

- $\kappa > 1$

$$\begin{cases} \frac{1}{\kappa} - 1 \le r_\infty \le 1 - \frac{1}{\kappa} \\ s^* = \frac{\kappa\,r_\infty}{1-\kappa} \end{cases}$$

- $\begin{cases} \tilde{p}^-_\infty > 1 \\ \tilde{p}^+_\infty < 0 \end{cases} \iff \begin{cases} \frac{p}{2}\,[1+\kappa\,(s^* + r_\infty)] > 1 \\ \frac{p}{2}\,[1-\kappa\,(s^* + r_\infty)] < 0 \end{cases}$

In this case, the transition probabilities are: $p^-_\infty = 1$ and $p^+_\infty = 0$. Equation 3.10 gives:

$$\begin{cases} s^* = 1 \\ \frac{p}{2}\,[1+\kappa\,(s^* + r_\infty)] > 1 \\ \frac{p}{2}\,[1-\kappa\,(s^* + r_\infty)] < 0 \end{cases}$$

The solutions for this set of equation and inequalities are the following:

- $\kappa < 0$

$$\begin{cases} r_\infty < \frac{2}{\kappa p} - \frac{1}{\kappa} - 1 \\ s^* = 1 \end{cases}$$

- $\kappa > 0$

$$\begin{cases} r_\infty > \frac{2}{\kappa p} - \frac{1}{\kappa} - 1 \\ s^* = 1 \end{cases}$$

The following cases are not possible, since Equations 3.9 give: $\tilde{p}_\infty^- + \tilde{p}_\infty^+ = p \in (0,1)$.

- $\begin{cases} \tilde{p}_\infty^- < 0 \\ \tilde{p}_\infty^+ < 0 \end{cases}$
- $\begin{cases} \tilde{p}_\infty^- \in [0,1] \\ \tilde{p}_\infty^+ > 1 \end{cases}$
- $\begin{cases} \tilde{p}_\infty^- > 1 \\ \tilde{p}_\infty^+ \in [0,1] \end{cases}$
- $\begin{cases} \tilde{p}_\infty^- > 1 \\ \tilde{p}_\infty^+ > 1 \end{cases}$

Now that all cases have been studied, it is necessary to regroup all of them, depending on the values of the herding propensity $\kappa$ and the long-term price return rate $r_\infty$.

- $\kappa < 0$
  - $r_\infty \leq \frac{1}{\kappa} - 1 \quad \rightarrow s^* = 1$
  - $\frac{1}{\kappa} - 1 \leq r_\infty \leq 1 - \frac{1}{\kappa} \quad \rightarrow s^* = \frac{\kappa r_\infty}{1-\kappa}$
  - $r_\infty \geq 1 - \frac{1}{\kappa} \quad \rightarrow s^* = -1$
- $\kappa = 0$
  - $r_\infty \in \mathbb{R} \quad \rightarrow s^* = 0$
- $0 < \kappa < 1$
  - $r_\infty \leq 1 - \frac{1}{\kappa} \quad \rightarrow s^* = -1$
  - $1 - \frac{1}{\kappa} \leq r_\infty \leq \frac{1}{\kappa} - 1 \quad \rightarrow s^* = \frac{\kappa r_\infty}{1-\kappa}$
  - $r_\infty \geq \frac{1}{\kappa} - 1 \quad \rightarrow s^* = 1$
- $\kappa = 1$
  - $r_\infty < 0 \quad \rightarrow s^* = -1$

- $r_\infty = 0 \quad \to s^* \in [-1, 1]$
- $r_\infty > 0 \quad \to s^* = 1$

- $\kappa > 1$
  - $r_\infty < \frac{1}{\kappa} - 1 \quad \to s^* = -1$
  - $\frac{1}{\kappa} - 1 \leq r_\infty \leq 1 - \frac{1}{\kappa} \quad \to s^* \in \{-1, \frac{\kappa r_\infty}{1-\kappa}, 1\}$
  - $r_\infty > 1 - \frac{1}{\kappa} \quad \to s^* = 1$

In order to get a clear idea of how the fixed points $s^*$ behave when the long-term price return rate $r_\infty$ change, this evolution is shown in Figure 3.1. All the cases on the value of the herding propensity $\kappa$, discussed above, are present, except the simple case $\kappa = 0$, for which no matter the value of $r_\infty$, the fixed point $s^*$ will always stay equal to 0. This is not surprising, since it has been shown in 3.2 that, if $\kappa = 0$, the average opinion index $\overline{s}_t$ is sure to converge and, if there is no bias, it converges to 0. From the derivation of the fixed points $s^*$ above and from Figure3.1, one could notice two significant characteristics of the average opinion index $\overline{s}_t$. On the one hand, for $\kappa = 1$ and $r_\infty = 0$, all the values in $[-1, 1]$ are fixed points of $\overline{s}_t$. On the other hand, for $\kappa > 1$ and $r_\infty$ sufficiently near 0, there is coexistence of 3 different fixed points. In this sense, the case $\kappa = 1$ can be viewed as an Ising phase transition for the average opinion index $\overline{s}_t$. The case $\kappa > 1$ corresponds to the hysteresis cycle in the presence of a magnetic field, played either by $r_\infty$ or the long-term value of the price momentum $H_t$, since they have the same behavior in the long run.

## 3.4 A market without dividends

As it has been shown in Chapter 2, the long-term price return rate $r_\infty$ of the risky asset seems to be controlled by the growth rate $r_d$ of the dividends. Thus, a significant question would be: What happens to the long-term price return rate when there is no dividend at all?

To answer that question, we shall take $r_d = 0$ and $d_t = 0$ for all $t$. Doing so, the risky fraction $x_t^f$ of the fundamentalists becomes a constant:

$$x_t^f = x_{\min}^f = \frac{E_{r_t} - r_f}{\gamma \sigma^2} \tag{3.11}$$

This constant risky fraction characterizes the optimism of the fundamentalists about the risky asset. Indeed, the term $E_{r_t} - r_f$ is about how much more than the risk-free asset they think the risky asset would return. This 'pure' optimism is created at the beginning of the market ($t = 0$) and remains constant in the future. It is rescaled by the global risk they take by investing in the risky asset, that is the constant relative risk-aversion $\gamma$

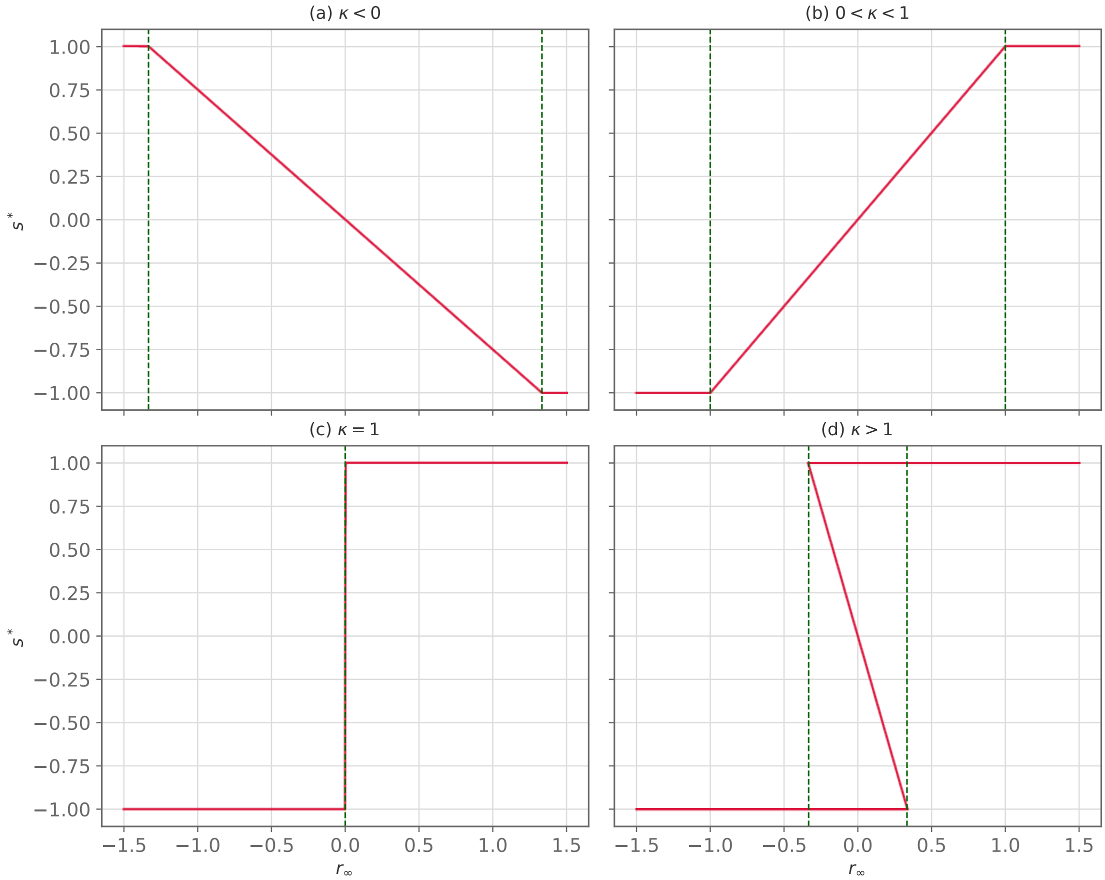


Figure 3.1: Fixed points of the average opinion index $\overline{s}_t$ as functions of the constant herding propensity $\kappa$ and the long-term price return rate $r_\infty$. The specific values taken for the herding propensity are the following: (a) $\kappa = -3$, (b) $\kappa = 0.5$, (c) $\kappa = 1$, (d) $\kappa = 1.5$. Those values have been chosen so that all possible fixed points appear when $r_\infty \in [-1.5, 1.5]$. The green dashed lines correspond to the threshold values of the long-term price return rate $r_\infty$, derived in 3.3. For $\kappa = 1$ and $r_\infty = 0$, all the values in $[-1, 1]$ are fixed points of the average opinion index $\overline{s}_t$. For $\kappa > 1$ and $r_\infty$ sufficiently near 0, there is coexistence of 3 different fixed points.

and the uncertainty they have about their prediction of the excess return rate, which is $\sigma^2 := Var_t[r_t - r_f]$. Once again, those quantities are constant with respect to time. They are simple guess made by fundamentalists who expect those to be accurate, at least in the long-run, so that they keep investing a minimum fraction of their wealth in the risky asset. In the special case where there is no dividend, they cannot adjust their risky fraction since they do not have any external information about the market – modeled by the dividends – so that they keep investing a constant fraction of their wealth for all future trading days. Then, their strategy is completely determined by their initial optimism about the risky asset, at the beginning of the market.

When there is no dividend, price returns are the only source of returns for the risky asset. In this sense, in terms of evolution of the traders' wealths, there is a clear competition between the price return rate $r_t := \frac{P_t}{P_{t-1}} - 1$ and the risk-free return rate $r_f$. The latter is a good benchmark for the price return rate $r_t$, since it is the rate which one trader can get without any risk and uncertainty. If the risky asset has a return rate $r_t$ less or equal than $r_f$ in average, it is of no interest since traders can earn more or equivalent without any risk. One should notice that, usually, dividends are there to compensate some losses. However, since here is considered a market without dividend, that is without external information for fundamentalists, one finds the following dynamics of the traders' wealths, showing a strict competition between $r_t$ and $r_f$:

$$\left\{ \begin{array}{lcl} W_t^f &=& W_{t-1}^f \; [1 + r_f + x_{\min}^f (r_t - r_f)] \\ W_t^n &=& W_{t-1}^n \; [1 + r_f + x_{t-1}^n (r_t - r_f)] \end{array} \right. \tag{3.12}$$

## 3.5 A market without fundamentalists

For the sake of simplicity, let us focus on the simpler case where fundamentalists do not show any optimism about the risky asset, at the beginning of the market. It means that their long-run minimum constant risky fraction $x_{\min}^f$ is equal to 0, and, in absence of dividends, one finds using Equation 3.11:

$$x_t^f \;=\; 0 \quad \text{for all } t \tag{3.13}$$

In absence of external information about the market and of 'a priori' optimism about the risky asset at the beginning of the market, fundamentalists do not have any reason to invest in it. Then, they invest, at each time step, all of their wealth in the risk-free asset. This fact can also be observed in Equations 3.12 using $x_{\min}^f = 0$:

$$W_t^f \;=\; W_{t-1}^f \; [1 + r_f] \quad \Rightarrow \quad W_t^f \;=\; W_0^f \; [1 + r_f]^t \tag{3.14}$$

Thus, fundamentalists stay away from the risky asset, which means that there is no possible trade: noise traders cannot exchange their shares of the stock to anyone. As a consequence, they possess a constant number of shares with respect to time. Let us recall that the number of shares $n_t^i$ that a trader $i$ possess at time $t$ is nothing but the amount of wealth they have invested in the risky asset, divided by the current price of the risky asset:

$$n_t^i \;=\; \frac{x_t^i \; W_t^i}{P_t} \tag{3.15}$$

Obviously, since their risky fraction is constant equal to 0, fundamentalists do not possess any share. The no-trade situation can also be seen using the excess demands $\Delta D_{t-1\to t}^f$ and $\Delta D_{t-1\to t}^n$ of fundamentalists and noise traders, in terms of shares. The conservation of shares gives:

$$\Delta D^f_{t-1\to t} \ + \ \Delta D^n_{t-1\to t} \ = \ (n^f_t - n^f_{t-1}) + (n^n_t - n^n_{t-1}) \ = \ n^n_t - n^n_{t-1} \ = \ 0$$

Then, noise traders keep a constant number of shares over time:

$$n^n_t \ = \ n^n_{t-1} \ = \ n^n_0 \ = \ \frac{x^n_0 W^n_0}{P_0} \tag{3.16}$$

One should keep in mind that those shares are nothing but a matter of initialization. If the initial risky fraction $x^n_0$ of noise traders is equal to 0, then no one has invested in the risky asset. As a consequence, the conservation of shares cannot provide future prices for the risky asset, which seems reasonable. Let us recall that the risky fraction of noise traders is just a rescaling of their opinion index $s_t \in [-1, 1]$, so that $x^n_t \in [0, 1]$ for all $t$. If $0 < x^n_0 \leq 1$, it means that noise traders have invested a part of their wealth, at the beginning of the market, to get some shares of the risky asset, and once it is done, they are the only share-holders of the stock in the market, so that they cannot buy more shares or sell their current shares: they are completely stuck and cannot do anything if it happens that they lose too much of their wealth because of the risky asset.

As long as prices are well defined – and it should always be the case, otherwise the market stops –, Equation 3.16 gives, when using the dynamics of the noise traders' wealth in Equations 3.12:

$$\frac{x^n_t \ [1 + r_f + x^n_{t-1}(\frac{P_t}{P_{t-1}} - 1 - r_f)]}{P_t} \ W^n_{t-1} \ = \ \frac{x^n_{t-1}}{P_{t-1}} \ W^n_{t-1} \tag{3.17}$$

At this point, it seems relevant to know whether the noise traders' wealth can become 0 or not.

**Proposition 7.** The noise traders' wealth cannot reach 0, except if prices have fallen to 0, if current dividends are equal to 0 and if noise traders were completely invested in the risky asset:

$$\begin{cases} W^n_{t_0-1} \ \neq \ 0 \\ W^n_{t_0} \ = \ 0 \\ P_{t_0-1} \ \neq \ 0 \end{cases} \quad \Rightarrow \quad \begin{cases} P_{t_0} \ = \ 0 \\ d_{t_0} \ = \ 0 \\ x^n_{t_0-1} \ = \ 1 \end{cases}$$

*Proof.* Using the general dynamics of the noise traders' wealth, one finds:

$$W^n_{t_0} \ = \ W^n_{t_0-1} \ [1 + r_f + x^n_{t_0-1} \ (\frac{P_{t_0}}{P_{t_0-1}} - 1 - r_f + \frac{d_{t_0}}{P_{t_0-1}})] \ = \ 0$$

Then, as $W^n_{t_0-1} \neq 0$, the term in brackets must be equal to 0. Denoting $r_{t_0} = \frac{P_{t_0}}{P_{t_0-1}} - 1$ as the current price return rate leads to:

$$x^n_{t_0-1} \ (r_{t_0} - r_f + \frac{d_{t_0}}{P_{t_0-1}}) \ = \ -1 - r_f$$

The term in parenthesis cannot be equal to 0 since it would lead to $0 = -1 - r_f$, which is not possible when $r_f > 0$ (the realistic case which is considered in this thesis). As a consequence, it provides an expression for the risky fraction at $t_0 - 1$:

$$x^n_{t_0-1} = \frac{-1 - r_f}{r_{t_0} - r_f + \frac{d_{t_0}}{P_{t_0-1}}}$$

Let us recall that, for any $t$, $x^n_t \in [0,1]$. This implies that $r_{t_0} - r_f + \frac{d_{t_0}}{P_{t_0-1}} < 0$. Let us now focus on the fact that $x^n_{t_0-1} \leq 1$, that is:

$$\begin{aligned} r_{t_0} - r_f + \frac{d_{t_0}}{P_{t_0-1}} &\leq -1 - r_f \\ r_{t_0} &\leq -1 - \frac{d_{t_0}}{P_{t_0-1}} \end{aligned}$$

The lower bound for the price return rate $r_t$ is $-1$. If $r_t = \frac{P_t}{P_{t-1}} - 1 = -1$, it means that $P_t = 0$. In words, it says that prices have fallen by 100% of their past value. Having said that and recalling that dividends are always non negative, the previous inequality leads to:

$$\begin{cases} r_{t_0} = -1 \\ d_{t_0} = 0 \\ x^n_{t_0-1} = \frac{-1-r_f}{r_{t_0}-r_f+\frac{d_{t_0}}{P_{t_0-1}}} \end{cases} \Rightarrow \begin{cases} P_{t_0} = 0 \\ d_{t_0} = 0 \\ x^n_{t_0-1} = 1 \end{cases}$$

□

This result could seem to be obvious since as long as noise traders have a fraction of their wealth invested in the risk-free asset, they get a security against losses. Besides, as long as they receive a dividend or that they have some shares, still valuable, of the risky asset, they should keep a total wealth non equal to 0. Nevertheless, the real reason which proves this result is the fact that the risky fraction $x^n_t$ of noise traders cannot exceed 1. For instance, it is possible that the risky fraction $x^n_t$ of fundamentalists be more than 1. One could find this point surprising, but if it happens, it only means that they have borrowed some money to invest into the risky asset, thinking that this strategy would be profitable. Thus, it is no longer necessary that prices fall by 100% to see the fundamentalists' wealth vanish. Following the previous proof, the conditions leading the fundamentalists' wealth to vanish at time $t_0$ are:

$$\begin{cases} r_{t_0} < r_f - \frac{d_{t_0}}{P_{t_0-1}} \\ x^n_{t_0-1} = \frac{-1-r_f}{r_{t_0}-r_f+\frac{d_{t_0}}{P_{t_0-1}}} \end{cases} \tag{3.18}$$

Let us come back to where the conservation of shares, only held by noise traders, has led us, that is Equation 3.17. In absence of dividends, as long as prices are well defined (strictly positive prices for instance), Proposition 7 states that the noise traders' wealth cannot suddenly become equal to 0. Thus, since their initial wealth $W_0^n$ is strictly positive – otherwise, they would not exist, at the market level at least –, their successive wealth values stay non zero, as long as prices are well defined. As a consequence, Equation 3.17 leads to:

$$\frac{x_t^n\ (1+r_f)\ (1-x_{t-1}^n)}{P_t} \;=\; \frac{x_{t-1}^n\ (1-x_t^n)}{P_{t-1}} \quad \text{for all } t \geq 1 \tag{3.19}$$

From this equation, one could observe that, if either $x_t^n$ or $x_{t-1}^n$ is equal to 0 or 1, the price $P_t$, at time $t$, of the risky asset is not clearly defined. In particular, it is true for $t = 0$. The case $x_0^n = 0$ is not relevant and has been discussed earlier: if no trader invest in the risky asset at time $t = 0$, it is not surprising that the price at the next time step is not defined. However, what is more surprising is that if fundamentalists stay away from the risky asset (no dividends and no optimism) and if noise traders decide to invest all of their wealth in the risky asset at the beginning of the market (that is $x_0^n = 1$), future prices are not defined. Even taking $x_0^n \in (0,1)$ is not enough to get all future prices well defined; it depends completely on the stochastic behavior of noise traders, who are able to choose suddenly, at any time, a risky fraction equal to 0 or 1. Nevertheless, it is assumed in the following that $x_t^n \neq 0$ and $x_t^n \neq 1$ for any time $t$, in order to study the behavior of the market in those conditions. It may seem to be a strong assumption, but one should notice that, when it is not the case, the market just stops, as prices are no longer defined. In this sense, it provides an excellent control parameter of the latter assumption whereas it may sometimes lead to a finite-time market. Using the latter hypothesis, one finds:

$$\frac{P_t}{P_{t-1}} \;=\; (1+r_f)\ \frac{x_t^n\ (1-x_{t-1}^n)}{x_{t-1}^n\ (1-x_t^n)} \tag{3.20}$$

Now that the price equation has been established in absence of fundamentalists, let us focus on the long-term price return rate $r_\infty$. As said before, it is assumed that there is convergence of the price return rate $r_t = \frac{P_t}{P_{t-1}} - 1 \underset{+\infty}{\longrightarrow} r_\infty$. Consequently, there is convergence of the fraction at the right side of Equation 3.20. We denote this fraction by $F_t$, which satisfies $F_t > 0$. If it converges to 1, the long-term price return rate $r_\infty$ is equal to the risk-free interest rate $r_f$. The latter represents the 'price of the money' and should be a minimum for the long-term price return rate $r_\infty$; otherwise, it is better to invest in the risk-free asset in order to be sure to earn a constant return $r_f$, without any risk or uncertainty. But what happens if $F_t$ converges to a value different than 1 ?

**Proposition 8.** $F_t \underset{+\infty}{\longrightarrow} \alpha > 1$ if and only if $x_t^n \underset{+\infty}{\longrightarrow} 1$ and $x_t^n \underset{+\infty}{=} \frac{1}{\alpha}\ x_{t-1}^n + 1 - \frac{1}{\alpha} + o(1-x_t^n)$.
Besides, $F_t \underset{+\infty}{\longrightarrow} \alpha < 1$ if and only if $x_t^n \underset{+\infty}{\longrightarrow} 0$ and $x_t^n \underset{+\infty}{\sim} \alpha\ x_{t-1}^n$.

*Proof.* Let $F_t = \frac{x_t^n\ (1-x_{t-1}^n)}{x_{t-1}^n\ (1-x_t^n)} \underset{+\infty}{\longrightarrow} \alpha$ with $\alpha \neq 1$. Denoting $y_t := \frac{x_t^n}{1-x_t^n}$, one finds:

$$y_t \underset{+\infty}{\sim} \alpha\ y_{t-1}$$

In addition, as long as $x_0^n \in (0,1)$, it has been previously shown that $x_t^n \in (0,1)$ for all $t$, so that $y_t > 0$ for all $t$.

- $\alpha > 1$

  Since $y_t > 0$, one finds, using the asymptotic relation between $y_t$ and $y_{t-1}$, that $y_t = \frac{x_t^n}{1-x_t^n} = \frac{1}{1-x_t^n} - 1 \underset{+\infty}{\longrightarrow} +\infty$. Thus, the risky fraction $x_t^n$ of noise traders converges and $x_t^n \underset{+\infty}{\longrightarrow} 1$. Thus, it leads to a simpler equivalent of $y_t$: $y_t = \frac{x_t^n}{1-x_t^n} \underset{+\infty}{\sim} \frac{1}{1-x_t^n}$. Comparing $y_t$ to $y_{t-1}$ then gives:

$$1 - x_t^n \underset{+\infty}{\sim} \frac{1}{\alpha}\,(1 - x_{t-1}^n)$$
$$x_t^n \underset{+\infty}{=} \frac{1}{\alpha}\,x_{t-1}^n + 1 - \frac{1}{\alpha} + o(1 - x_t^n)$$

  The converse follows the same proof.

- $\alpha < 1$

  As said before, the fraction $F_t$ is strictly positive, so that $0 \leq \alpha < 1$ in this special case. Once again, since $y_t > 0$, the asymptotic relation between $y_t$ and $y_{t-1}$ gives: $y_t = \frac{x_t^n}{1-x_t^n} = \frac{1}{1-x_t^n} - 1 \underset{+\infty}{\longrightarrow} 0$. Thus, the risky fraction $x_t^n$ of noise traders converges and $x_t^n \underset{+\infty}{\longrightarrow} 0$. In the same spirit than the previous case, one finds a simpler equivalent: $y_t = \frac{x_t^n}{1-x_t^n} \underset{+\infty}{\sim} x_t^n$. Comparing $y_t$ to $y_{t-1}$ then gives:

$$x_t^n \underset{+\infty}{\sim} \alpha\, x_{t-1}^n$$

  The converse follows the same proof. □

As a direct consequence of Proposition 8, the long-term price return rate $r_\infty$ is almost always equal to the risk-free interest rate $r_f$, and if it is not the case, it means that the risky fraction $x_t^n$ of noise traders has converged to 0 or to 1. This condition of convergence to 0 or 1 is not a sufficient condition, as states Proposition 8: a special asymptotic behavior of the risky fraction $x_t^n$ is required. Nevertheless, it is a necessary condition. From Equation 3.20, one could notice that, if the risky fraction $x_t^n$ of noise traders converges to a value different than 0 or 1, the long-term price return rate $r_\infty$ is equal to the risk-free interest rate $r_f$. If the fraction $F_t$ converges to $\alpha < 1$, it means that the risky fraction $x_t^n$ converges to 0 and that $r_\infty$ is lower than $r_f$. Similarly, if $F_t$ converges to $\alpha > 1$, the risky fraction converges to 1 and $r_\infty$ is greater than $r_f$. Let us recall that it has been supposed that $x_t^n \neq 0$ and that $x_t^n \neq 1$ for all $t$. One should notice that there is a significant difference between converging to 0 or 1 and taking the value 0 or 1 at a given time $t_0$. Indeed, there are many ways in which the risky fraction $x_t^n$ can converge to 0 or 1, without ever taking one of those values.

In the following, we shall try to find what happens when the risky fraction $x_t^n$ of noise traders converges to 0 or 1 and hope that it is enough to get $F_t \underset{+\infty}{\longrightarrow} \alpha < 1$ (that is $r_\infty < r_f$) or $F_t \underset{+\infty}{\longrightarrow} \alpha > 1$ (that is $r_\infty > r_f$).

The problem of convergence of the average opinion index $\overline{s}_t$ (and thus the average risky fraction $\overline{x}_t^n$ of noise traders) has been discussed in 3.2. It has been seen that, if the herding propensity $\kappa = 0$, then the average risky fraction $\overline{x}_t^n$ always converge. If there is no bias amongst noise traders, that is $p_+ = p_- = p \in (0,1)$, it converges to $\frac{1}{2}$; otherwise, it depends on the bias. Even if it is not the 'true' risky fraction $x_t^n$, it can provide a good first approximation. Let us first focus on the special case of a null herding propensity $\kappa$. As explained before, in order to obtain the convergence of the average risky fraction $\overline{x}_t^n$ to 0 or 1, it is now necessary to introduce some bias. Let us recall the two expressions of the average opinion index $\overline{s}_t$ in presence of bias, derived in 3.2:

$$\begin{cases} \overline{s}_t = [1 - \frac{p_+ + p_-}{2}]\ \overline{s}_{t-1} + \frac{p_- - p_+}{2} \\ \overline{s}_t = [1 - \frac{p_+ + p_-}{2}]^t \quad [s_0 - \frac{p_- - p_+}{p_- + p_+}] + \frac{p_- - p_+}{p_- + p_+} \end{cases} \tag{3.21}$$

In this special case, the average opinion index $\overline{s}_t$ converges to the value $\frac{p_- - p_+}{p_- + p_+}$. Since the average risky fraction $\overline{x}_t^n = \frac{1 + \overline{s}_t}{2}$ is only a rescaling of $\overline{s}_t$, one finds the following conditions on $p_-$ and $p_+$ to obtain the convergence of $\overline{x}_t^n$ to 0 or 1:

$$\begin{cases} \overline{x}_t^n \underset{+\infty}{\longrightarrow} 0 \quad \text{if and only if} \quad p_- = 0 \quad \text{and} \quad p_+ \in (0,1) \\ \overline{x}_t^n \underset{+\infty}{\longrightarrow} 1 \quad \text{if and only if} \quad p_+ = 0 \quad \text{and} \quad p_- \in (0,1) \end{cases} \tag{3.22}$$

One should notice that it has been supposed since 3.2 that both $p_-$ and $p_+$ belong to the set $(0,1)$. But one of those values can be equal to 0, as long as the other one is non zero. Besides, the fact that they cannot be greater than 1 does not change much, as long as they are less than 2 in this special case. If $p_- = 0$, one finds, using Equations 3.4, that $p_t^-(s_t, H_t) = 0$ for all $t$. Let us recall that the latter transition probability characterizes the ability of one noise trader to buy the risky asset, when fully invested in the risk-free asset. Consequently, if $p_- = 0$, no noise trader is able to buy the risky asset, so that, from an initial value $x_0^n \in (0,1)$, the average risky fraction $\overline{x}_t^n$, that is the amount of wealth invested in the risky asset, converges to 0. Indeed, from Equations 3.4, one finds, in the case of a null herding propensity $\kappa$, the expression of the transition probability $p_t^+(s_t, H_t)$ of selling the risky asset: $p_t^+ = \frac{p_+}{2} \neq 0$. Thus, there is a null probability of buying the asset and a non zero probability of selling it, leading to a risky fraction converging to 0. Similarly, if $p_+ = 0$, one finds a null probability of selling the asset and a non zero probability of buying it, when invested in the risk-free asset, leading to a risky fraction converging to 1. One might think that the meaning of those transition probabilities is ambiguous, given that noise traders possess a constant number of shares of the risky asset and cannot trade them with fundamentalists, who stay away from the risky asset. Nevertheless, one should notice that those transition probabilities do not affect the number of shares, held by noise traders, but only their risky fraction, that is the fraction of wealth that they invest in the risky asset. In presence of fundamentalists, the change in their risky fraction can lead to the acquisition of new shares, as one can observe, regarding the conservation of shares. In this particular case, this is not possible. In this sense, noise traders cannot trade any share, but they decide at each time step how much wealth they will invest in the constant number of shares they possess. Since they share their total wealth, each noise trader has the same weight in that decision. One can picture this situation as a committee, holding the only shares of the stock and stuck with them, which vote equally at each time step whether they want to stay invested in the risky asset or not. Obviously, if no one of them wants to stay invested in the risky asset, that is $x_t^n = 0$, there is a problem, as it has been underlined before, given that they cannot sell any share. In a nutshell, the transition probabilities affect the risky fraction of noise traders, which, in turn, affect the price of the risky asset in such a way that noise traders always keep a constant total number of shares.

Let us first focus on the case $\overline{x}_t^n \underset{+\infty}{\longrightarrow} 1$, that is $p_+ = 0$ and $p_- \in (0,1)$. Using Equations 3.21 and the definition of the average risky fraction $\overline{x}_t^n = \frac{1+\overline{s}_t}{2}$, one finds:

$$\begin{cases} \overline{x}_t^n = [1 - \frac{p_-}{2}]\, \overline{x}_{t-1}^n + \frac{p_-}{2} \\ \overline{x}_t^n = [1 - \frac{p_-}{2}]^t \;\; [x_0^n - 1] + 1 \end{cases} \tag{3.23}$$

Thus, the average risky fraction $\overline{x}_t^n$ of noise traders converges exponentially to 1, with a characteristic time:

$$\tau_1 = \frac{1}{\log(\frac{1}{1-\frac{p_-}{2}})} \tag{3.24}$$

Such a process converges to 1, is always different 0 or 1 as long as $x_0^n \in (0,1)$ and satisfies the asymptotic behavior required by Proposition 8: $\overline{x}_t^n \underset{+\infty}{=} \frac{1}{\alpha}\, \overline{x}_{t-1}^n + 1 - \frac{1}{\alpha}$ with $\alpha = \frac{1}{1-\frac{p_-}{2}} > 1$. Hence, in those conditions, Proposition 8 states that the fraction $F_t\,(\overline{x}_t^n, \overline{x}_{t-1}^n)$ converges to $\alpha = \frac{1}{1-\frac{p_-}{2}}$. As a consequence, Equation 3.20 gives the long-term price return rate $r_\infty$:

$$r_\infty^1 = \frac{r_f + \frac{p_-}{2}}{1 - \frac{p_-}{2}} > r_f \tag{3.25}$$

In brief, in absence of fundamentalists and of herding ($\kappa = 0$), if noise traders have a null probability of not being interested in the risky asset and a non zero probability of being interested in it, they manage to outperform the price of the money, characterized by the risk-free interest rate $r_f$.

One should notice that this result is obtained using the average risky fraction $\overline{x}_t^n$, instead of the stochastic risky fraction $x_t^n$, so that it is an approximate solution. Indeed, while the average risky fraction has previously been useful to better understand the market behavior, one could observe that the fraction $F_t$ is of second-order in the risky fraction. Then, the stochasticity cannot be ignored and will have a certain impact on the market. To see this impact, we shall use a numerical simulation, given by the algorithm described in 2.4. Let us recall that it takes an initial configuration, then does the simulation of the first seed until the maximum possible time of simulation $t_{\max}$. In this case, the initial configuration is given, as usual, by Table 2.1 of Chapter2, except the following quantities: $d_t = 0$ for all $t$, $r_d = 0$, $\sigma_d = 0$, $\kappa = 0$ (constant process), $E_{r_t} = r_f$ and $p_+ = 0$. It corresponds exactly to the case described above, that is a market without dividend, without fundamentalists, without herding and with an average risky fraction $\overline{x}_t^n$ of noise traders converging to 1. Here, the time limit of simulation is due to the risky fraction $x_t^n$ taking the value 1 since it has been seen before that, if it happens, the market stops. Indeed, the average $\overline{x}_t^n$ converges to 1 so it seems really likely that the real risky fraction takes the value 1 at some time. Furthermore, using the complete simulation of the first seed, the algorithm finds the time $t_c$ from which the cumulative moving average of the price return rate, that is $\langle r_t \rangle := \frac{1}{t} \sum_{i=1}^{t} r_i$, stays inside a convergent interval of 1% of the final value $\langle r_{t_{\max}} \rangle$, then does the simulations, until $t_c$, for 100 different seeds, takes the new final values of the cumulative price return rate $\langle r_{t_c} \rangle$ of each seed and computes the average over the seeds, denoted by $\langle r_\infty \rangle$. On Figure 3.2, the first plot represents the 'infinite' cumulative price return rate $\langle r_\infty \rangle$ as a function of $p_-$, in the conditions described above. The error bars correspond to the range of values taken by the final values of $\langle r_{t_c} \rangle$ of each seed, that is before taking the average $\langle r_\infty \rangle$. On the same plot, one can also observe the behavior of $r_\infty^1$, to compare with $\langle r_\infty \rangle$. To get a better idea, the second plot shows the relative difference between those two quantities. One can notice that, even if their respective behavior look more or less alike, the stochasticity induces significant differences between the values taken by them. The third plot compares the characteristic time $\tau_1$, defined in Equation 3.24, to the time $t_c$ of convergence of the cumulative price return rate. From this plot, one can conclude that there is no problem of convergence for the average risky fraction $\overline{x}_t^n$ of noise traders in the simulations. The last plot shows the rescaled standard deviation of the final values of the cumulative price return rate $\langle r_{t_c} \rangle$ for the 100 considered seeds, that is before taking the average $\langle r_\infty \rangle$ over those seeds.

In the case of a non zero herding propensity ($\kappa \neq 0$) and in absence of bias ($p_- = p_+ := p$), it has been seen in 2.2 that, in order to get an average risky fraction $\overline{x}_t^n$ of noise traders converging to 1, it is necessary to choose first the range of $\kappa$. The simplest case is $0 < \kappa < 1$, for which $\overline{x}_t^n$ converges to 1 – for simplicity, it has been assumed since 2.2 that the average risky fraction $x_t^n$ of noise traders converges to its fixed point – if the long-term price return rate $r_\infty$ satisfies: $r_\infty \geq \frac{1}{\kappa} - 1$. However, we do not have any control on the long-term price return rate. Assuming that it would be positive, one should notice that, if $r_\infty < \frac{1}{\kappa} - 1$ (that is an average risky fraction converging to a value lower than 1), then the long-term price return rate $r_\infty$ would be equal to the risk-free interest rate $r_f$, according to the previous discussion. Then, it is possible to force the average risky fraction $\overline{x}_t^n$ to converge to 1, by setting $r_f > \frac{1}{\kappa} - 1$ for the considered range of $\kappa$ values. The previous inequality gives in our special case: $\frac{1}{1+r_f} < \kappa < 1$. Usual values for $r_f$, according to Table 2.1, are around $10^{-4}$, giving a really small interval for $\kappa$ values. To address this issue, we choose an arbitrary value of 1 for $r_f$ (and thus, $E_{r_t} = 1$ since $x^f_{\min} = 0$). One should notice that it does not change much the general behavior of the market but makes it possible to get a larger panel of $\kappa$ values. One must keep in mind that $\frac{1}{1+r_f} < \kappa < 1$ is not a necessary condition to obtain the convergence of the average risky fraction $\overline{x}_t^n$ to 1: it is only a sufficient condition. The 'real' condition which determines completely the convergence to 1 is $r_\infty \geq \frac{1}{\kappa} - 1$, as it has been highlighted earlier.

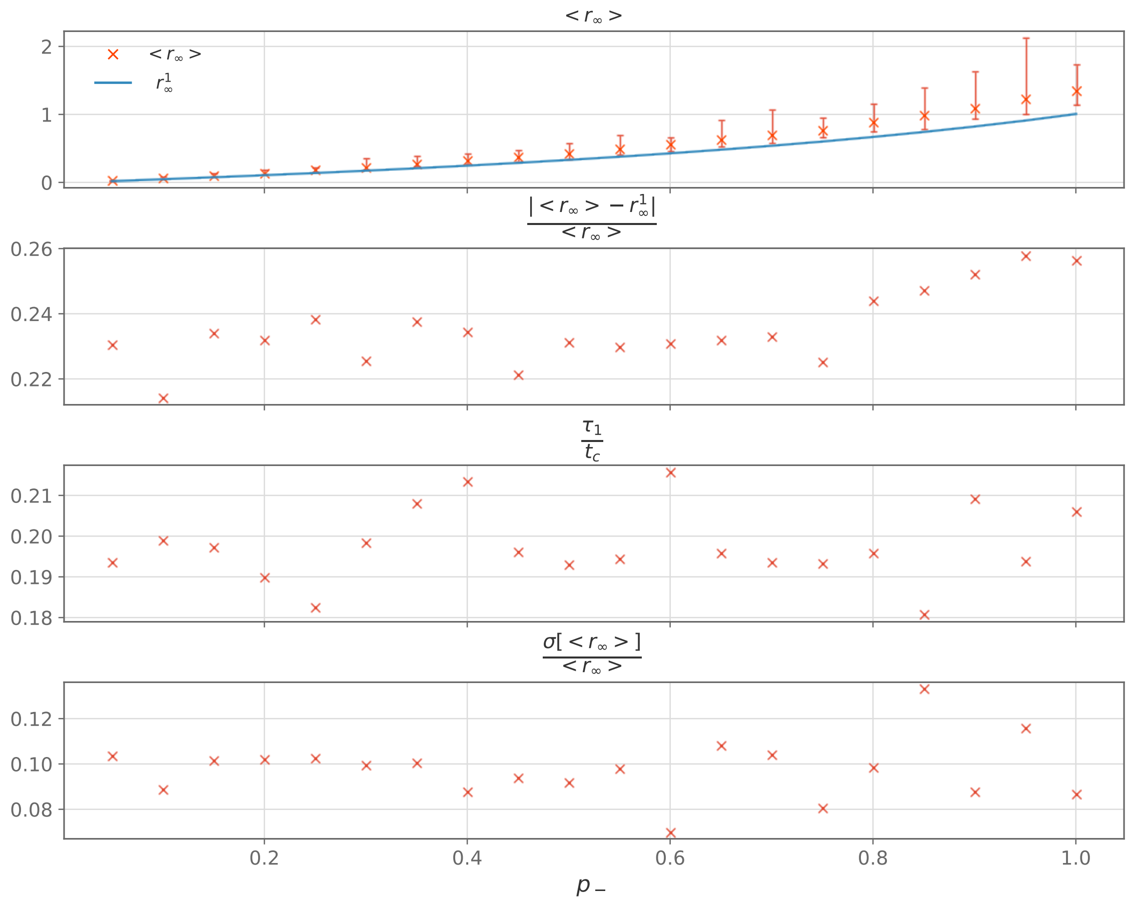


Figure 3.2: Long-term cumulative price return rate $\langle r_\infty \rangle$ in a market without dividends, without fundamentalists, without herding ($\kappa = 0$) and with an average risky fraction $\overline{x}_t^n$ of noise traders converging to 1. The quantity $\langle r_\infty \rangle$ has been computed by the algorithm described in 2.4, using 100 different seeds. All parameters are taken from Table 2.1 of Chapter 2, except the following quantities: $d_t = 0$ for all $t$, $r_d = 0$, $\sigma_d = 0$, $\kappa = 0$ (constant process), $E_{r_t} = r_f$ and $p_+ = 0$. Those values are necessary to obtain the particular considered market. The parameter $p_-$ corresponds to the x-axis: it characterizes the probability for noise traders of being interested in the risky asset. The first plot shows its impact on the long-term cumulative price return rate $\langle r_\infty \rangle$ and on the theoretical long-term price return rate $r^1_\infty$ of Equation 3.25, derived using the average risky fraction $\overline{x}_t^n$ of noise traders, instead of the stochastic risky fraction $x_t^n$. The error bars correspond to the range of values taken by the final values of $\langle r_{t_c} \rangle$ of each seed, that is before taking the average $\langle r_\infty \rangle$. The second plot shows the relative difference between $\langle r_\infty \rangle$ and $r^1_\infty$, suggesting that the stochasticity has a significant impact on the market. The third plot compares the characteristic time $\tau_1$, defined in Equation 3.24, to the time $t_c$ of convergence of the cumulative price return rate, used by the algorithm of Chapter 2. It shows that the time of simulation is sufficient, compared to the time of convergence of the average risky fraction $\overline{x}_t^n$ of noise traders. The last plot shows the rescaled standard deviation of the final values of the cumulative price return rate $\langle r_{t_c} \rangle$ for the 100 considered seeds, that is before taking the average $\langle r_\infty \rangle$ over those seeds.

On Figure 3.3, we have represented the long-term cumulative price return rate $\langle r_\infty \rangle$ as a function of the herding propensity $\kappa$, in the conditions described above. Its computation follows the same process than the one described in 2.4. For each value of $\kappa$, 100 different seeds have been used for the simulation. The parameters are all taken from the basic parameter set, provided in Table 2.1, except the following which corresponds to our particular considered conditions: $d_t = 0$ for all $t$, $r_d = 0$, $\sigma_d = 0$, $E_{r_t} = r_f = 1$ and $p_- = p_+ = 0.2$. On both panels of Figure 3.3, the error bars correspond to the uncertainty over the seeds. The red straight line represents the value of $r_f$ whereas the blue dashed line depicts the threshold value $\frac{1}{\kappa} - 1$, derived in 2.3. One could observe the transition between $\langle r_\infty \rangle = r_f$ and $\langle r_\infty \rangle > r_f$ depending on the values of $\kappa$. The second panel is nothing but a zoom of the first panel around the transition. Thus, one could notice that there exists some values of $\kappa$ for which $r_f < \langle r_\infty \rangle < \frac{1}{\kappa} - 1$. It suggests that the stochasticity have a certain impact on the fraction $F_t$. Nevertheless, the threshold value gives a quite good estimate of the true one. For $r_f < \langle r_\infty \rangle$, the maximum time $t_{\max}$ of computation is due to the risky fraction $x_t^n$ taking the value 1. The average risky fraction $\overline{x}_t^n$ converging to 1, it seems very likely that the stochastic risky fraction takes the value 1 at some time. According to Figure 3.3, it is thus possible to view the herding propensity as a control parameter leading to a phase transition in the special case of a market without dividends and without fundamentalists.

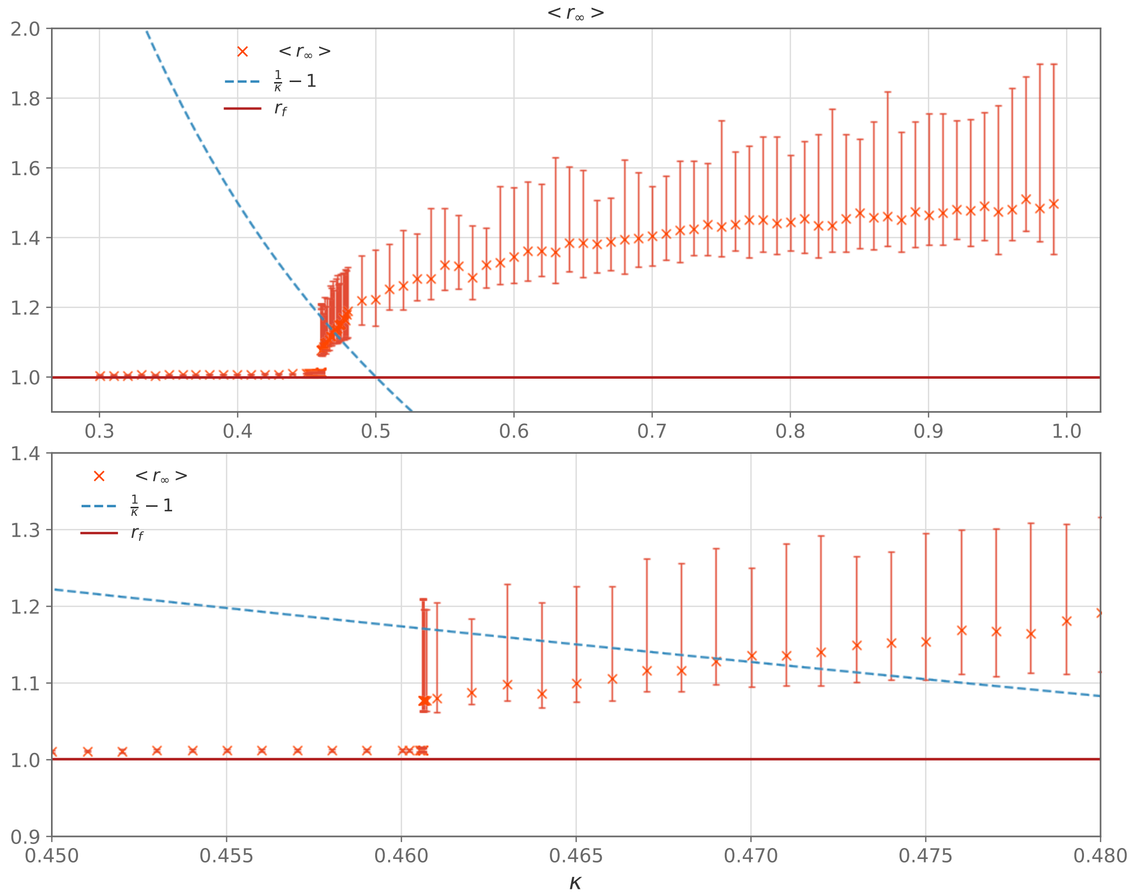


Figure 3.3: Long-term cumulative price return rate $\langle r_\infty \rangle$ in a market without dividends, without fundamentalists, with herding but no bias. The quantity $\langle r_\infty \rangle$ has been computed by the process described in 2.4, using 100 different seeds. All parameters are taken from the basic parameter set provided in Table 2.1, except the following quantities: $d_t = 0$ for all $t$, $r_d = 0$, $\sigma_d = 0$, $E_{r_t} = r_f = 1$ and $p_+ = p_- = 0.2$. Those values are necessary to obtain the particular considered market. On both panels, the error bars correspond to the uncertainty over the seeds. The red straight line represents the value of $r_f$ whereas the blue dashed line depicts the threshold value $\frac{1}{\kappa} - 1$, derived in 2.3. One could observe the transition between $\langle r_\infty \rangle = r_f$ and $\langle r_\infty \rangle > r_f$ depending on the values of $\kappa$. The second panel is a zoom of the first panel around the transition. For $r_f < \langle r_\infty \rangle$, the maximum time $t_{\max}$ of computation is due to the risky fraction $x_t^n$ taking the value 1.

# Conclusion

The goal of the present thesis was to study the long-term behavior of an artificial market, composed of fundamentalists and noise traders. The model has two assets, a constant interest rate risk-free asset and a dividend paying risky asset, whose price is determined by the market clearing condition. Fundamentalists have a long-term strategy, corresponding to their optimism about the risky asset, and a short-term one, based on the dynamics of the dividend-price ratio. The price following the same growth than the dividends is thus a benchmark, called the *fundamental value*. They buy the risky asset mostly when it is undervalued, that is whose price is below that fundamental value, and they sell it mostly when it is overvalued, that is whose price is above the fundamental value. Noise traders are completely different investors. They are trend-followers and subject to social imitation.

In presence of dividends, noise traders do not have a persistent impact on the market. During bubbles, they outperform fundamentalists but fail to maintain their advantage, because of the subsequent crash. They are only 'noise' compared to the strategy of fundamentalists. Thus, the latter manage to drive the price of the risky asset to its fundamental value. The long-term price return rate then follows the mean growth rate of dividends. The corresponding simulations in 2.4 may not be fully accurate, mainly since we did not find any trustful indicator of convergence and since simulations until a maximum time of computation might not be pertinent. Nevertheless, there is still a strong dependence on the mean growth rate of dividends for the risky asset, which supports their external field aspect, as explained in [13]. One solution to get clearer ideas about the exact relation between the mean growth rate $r_d$ of dividends and the long-term price return rate $r_\infty$ would be to do a prior theoretical study. It is in that spirit that the following chapter (3.2 especially) was meant to be more theoretical, in addition to the curiosity of finding what drives the risky asset when there is no external field.

In absence of dividends and of fundamentalists, the prior study of the fixed points of the average risky fraction of noise traders has been of significant help. Without herding, only trivial solutions lead to a long-term price return rate $r_\infty$ different than the 'price of the money' $r_f$. However, when there is some herding, that is imitative feedback and trend-following, noise traders manage to outperform this benchmark $r_f$, depending on the strength they allow for this herding. In this sense, the lonely noise can become constructive.

A lot of remaining questions about this model are still to be answered. For instance, still without dividends, what happens if fundamentalists invest their constant risky fraction $x^f_{\min}$ of wealth ? They might perhaps force the long-term price return rate $r_\infty$ to its lower base line value $r_f$. But it is also possible that they enrich the long-term behavior of the risky asset.

In this sense, the issue covered by the present thesis has still much to reveal, so that, since the questions involved are rich in terms of complexity, there should be a second more intensive study, given the broad ramifications. This future work should be quite theoretical (in the same spirit than Chapter 3), given the difficult numerical simulations involved, but it would benefit from all results one can find in this thesis.

# Appendix A

# Convergence of the price momentum

Let recall the ending result of the proof of Proposition 4:

$$H_t \; - \; r_\infty \; = \; (1-\theta) \sum_{i=1}^{t} \theta^{t-i} \; [r_i \; - \; r_\infty] \; + \; \theta^t \; (H_0 \; - \; r_\infty) \quad \text{for } t \geq 1$$

Let $\epsilon > 0$. We have supposed that $r_t \to r_\infty$ when $t \to +\infty$. So, there exists $t_0$ such that, for all $t \geq t_0$, $|r_t \; - \; r_\infty| \; \leq \epsilon$. We how have a quantification of the term in brackets appearing in the sum. One finds:

$$\begin{aligned} |H_t \; - \; r_\infty| \; &\leq \; (1-\theta) \sum_{i=1}^{t} \theta^{t-i} \; |r_i \; - \; r_\infty| \; + \; \theta^t \; |H_0 \; - \; r_\infty| \\ &\leq \; \theta^t \; [ \; (1-\theta) \sum_{i=1}^{t_0-1} \frac{1}{\theta^i} \; |r_i \; - \; r_\infty| + |H_0 - r_\infty| \; ] \; + \; (1-\theta) \sum_{i=t_0}^{t} \theta^{t-i} \; |r_i \; - \; r_\infty| \\ &\leq \; \theta^t \; A \; + \; \epsilon \; (1-\theta) \sum_{i=t_0}^{t} \theta^{t-i} \end{aligned}$$

$A$ is a constant, defined as follows:

$$A = \; (1-\theta) \sum_{i=1}^{t_0-1} \frac{1}{\theta^i} \; |r_i \; - \; r_\infty| + |H_0 - r_\infty|$$

Then, one finds easily:

$$\begin{aligned} |H_t \; - \; r_\infty| \; &\leq \; \theta^t \; A \; + \; \epsilon \; (1 - \theta^{t+1-t_0}) \\ &\leq \; \theta^t \; A \; + \; \epsilon \; - \epsilon \; \theta^{t+1-t_0} \end{aligned}$$

The first and the last terms at the right side of the previous inequality tend to 0 when $t \to +\infty$. As a consequence, we have proved that $H_t \to r_\infty$ if $r_t \to r_\infty$.

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