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A Nonuniformly Integrable Martingale Bubble with a Crash*

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Abstract. We investigate a deterministic criterion to determine whether a diffusive local martingale with a single jump ("crash") is a uniformly integrable martingale. We allow the jump hazard rate and the relative jump size to depend on the state and prove that the process is a uniformly integrable martingale if and only if the relative jump size is bounded away from one. The result helps to classify seemingly explosive behavior in diffusive local martingales compensated by the existence of a jump and provides natural examples of nonuniformly integrable martingales. Local martingales that fail to be uniformly integrable martingales have been used to model financial bubbles in stock prices as deviation from the fundamental value. Our result extends this classification to a comprehensive and relevant model class that explicitly models the financially relevant situation of a crash.

Key words. uniformly integrable martingales, local martingales, single jump diffusions, explosive diffusion processes, financial bubbles

AMS subject classifications. 60G44, 60G48, 60G55, 60H10, 60J60

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1. Introduction.

Local martingales that are not uniformly integrable martingales have recently gained increased attention in the stochastic processes and mathematical finance literature, being linked to special cases in arbitrage pricing theory (Elworthy, Li, and Yor (1999), Delbaen and Schachermayer (1998b), Guasoni and Rásonyi (2015)) and to the occurence of bubbles (Loewenstein and Willard (2000); Cox and Hobson (2005); Heston, Loewenstein, and Willard (2007), Jarrow, Protter, and Shimbo (2007), (2010); Herdegen and Schweizer (2016); Biagini and Nedelcu (2015)). Based on the seminal paper of Loewenstein and Willard (2000), Theorem 4.1 in (Jarrow, Protter, and Shimbo, 2010) characterizes three types of local martingales that can be used to model bubbles,[1] depending on the model horizon:

(a) general local martingales on an infinite time horizon,
(b) local martingales that are not uniformly integrable martingales on a stochastically unbounded but finite time horizon,
(c) strict local martingales on a bounded time horizon.

The result is based on the fact that local martingales, while being *instantaneous fair games*, may show a drop in expectation in the long term. Table 1 summarizes such a classification of local martingales. To date most of the literature is limited to a finite time horizon, thereby immediately excluding processes in (a) and (b). While processes in (a) rely on an infinite time horizon and seem somewhat ill-suited for financial modeling, processes in (b) are readily

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[1]Below, we refer to such processes as *mathematical (finance) bubbles*.

Table 1

Characterization of RCLL nonnegative local martingales by martingale property and uniform integrability (UI). The uniform integrability property has explanatory power on the loss of mass only for true martingales. Strict local martingales are true supermartingales even if they are uniformly integrable.

	Non-UI	UI
Strict local martingale	$\exists t \in [0,\infty)\colon \mathbb{E}[M_t] < M_0$	
Martingale	$\mathbb{E}[M_\infty] < M_0$	$\mathbb{E}[M_\infty] = M_0$

conceivable.[2] In the present paper we introduce a natural class of candidates for bubble processes on a finite but stochastically unbounded time horizon. We combine a homogeneous diffusion with a single jump (characterized by state dependent hazard rate and jump size) and provide a necessary and sufficient deterministic criterion to decide whether they are uniformly integrable martingales. While many models of mathematical bubbles lack a well-defined empirical basis, a single jump has a straightforward interpretation as a financial drawdown.

In particular, we look at processes $(S_t)_{t\in[0,\infty)}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ that satisfy a homogeneous version of the stochastic differential equation

$$dS_t = b(t,S_t)\mathbb{1}\{t<\tau_J\}dt + \sigma(t,S_t)1_{\{t<\tau_J\}}dW_t - \frac{b(t,S_{t-})}{h(t,S_{t-})}dJ_t, \tag{1}$$

with coefficient functions $b, \sigma, h : [0,\infty)\times\mathbb{R} \to [0,\infty)$, where $(W_t)_{t\in[0,\infty)}$ is an $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, $(J_t)_{t\in[0,\infty)}$ is a $\{0,1\}$-valued single jump process with

$$\mathbb{P}\left[dJ_t = 1|\mathcal{F}_{t-}, J_{t-} = 0\right] = h(t,S_{t-}), \tag{2}$$

such that h is the *hazard rate* (also known as *intensity process*) of J, and τ_J denotes the time of the jump. For most reasonable choices of b, σ, and h such that $b(t,x) \leqslant h(t,x)x$ for all $(t,x) \in [0,\infty)\times[0,\infty)$ (see specific examples in section 5 and the detailed setting in section 3 below) the resulting process S is nonnegative and a local martingale. Intuitively, the local martingale property can be seen from the fact that S is the sum of a Brownian integral and a single jump process that grows instantaneously by $b(t,S_{t-})$ and has an expected instantaneous decline of $-(b(t,S_{t-})/h(t,S_{t-}))\mathbb{P}\left[dJ_t = 1|\mathcal{F}_{t-}, J_{t-} = 0\right] = -b(t,S_{t-})$. For $\tau_J < t$, it remains constant at S_{τ_J}. The above questions on uniform integrability and strict local martingality have been answered in various special cases of (1). As a simple example, assume that for constants $\lambda \in [0,\infty)$ and $\kappa \in (0,1]$ we have

$$b(t,S_t) = \kappa\lambda S_t,\ \sigma(t,S_t) \equiv 0,\ \text{and}\ h(t,S_t) \equiv \lambda. \tag{3}$$

One can directly calculate the expected value of $S_\infty = S_{\tau_J}$ as

$$\mathbb{E}[S_\infty] = \mathbb{E}[S_{\tau_J}] = (1-\kappa)S_0\mathbb{E}\left[e^{\kappa\lambda\tau_J}\right] = (1-\kappa)S_0\int_0^\infty \lambda e^{-\lambda t}e^{\kappa\lambda t}dt = S_0 \tag{4}$$

[2] One may argue, for example, that the finite model horizon for a stock price of a large corporation is more realistically described by an unbounded random rather than a bounded deterministic lifetime.

for $\kappa < 1$ and $\mathbb{E}[S_\infty] = \mathbb{E}[0] = 0$ for $\kappa = 1$. In this simple case S is a uniformly integrable martingale if and only if $\kappa < 1$. Similarly, it is easy to check that S is indeed a martingale. Let us present three variants of (1) in somewhat increasing generality, the last illustrating the setting in this article, where the question of interest is whether a process can be used as a model of mathematical bubbles in (stochastic) finite time.

Linear characteristics and time-dependent hazard rate. Based on their examination of single jump processes with a deterministic hazard rate in Herdegen and Herrmann (2016), Herdegen and Herrmann (2019) consider (within a more general setting) a solution to the SDE (1) assuming a finite time horizon $T \in [0, \infty)$ and coefficients

$$b(t, S_t) = \phi'(t) S_t,\ \sigma(t, S_t) = \sigma_0 S_t,\ \text{ and } h(t, S_t) = h(t) \tag{5}$$

for $\sigma_0 \in (0, \infty)$ and continuously differentiable functions $\phi, h : [0, T) \to (0, \infty)$. They show, in particular, that the process $(S_t)_{t \in [0,T]}$ is a strict local martingale if and only if

$$\int_0^T h(t) dt = \infty \quad \text{and} \quad \int_0^T \big(h(t) - \phi'(t)\big)\, dt < \infty. \tag{6}$$

Due to the finite time window ($S_\infty = S_T$), any true martingale in this setting is immediately uniformly integrable.

Driftless homogeneous diffusion. There has been a lot of interest in the strict local martingale property of stochastic exponentials based on diffusions; see, e.g., Delbaen and Shirakawa (2002), Kotani (2006), Hulley and Platen (2008), Mijatović and Urusov (2012), and references therein. One can apply those results to a special case of (1) with a homogeneous diffusion function and zero drift,

$$b(t, S_t) \equiv 0,\ \sigma(t, S_t) = \sigma(S_t),\ \text{ and } h(t, S_t) \equiv 0. \tag{7}$$

In particular, for diffusion coefficients σ with $\sigma(\cdot) \neq 0$ and $\sigma^{-2}(\cdot)$ locally integrable on $(0, \infty)$, one can show that the process $(S_t)_{t \in [0,\infty)}$ is

1. a strict local martingale on any interval $[0, T]$ or $[0, \infty)$ if $\int_c^\infty x/\sigma^2(x) dx < \infty$ for some $c \in (0, \infty)$ and
2. a martingale that is not uniformly integrable if $\int_c^\infty x/\sigma^2(x) dx = \infty$ for all $c \in (0, \infty)$;

see, e.g., Corollary 4.3 in Mijatović and Urusov (2012). Note that, as in the last example, for such pure diffusion processes the question whether $(S_t)_{t \in [0,\infty)}$ is a uniformly integrable martingale is trivial, as almost surely we have $S_\infty = 0$.[3]

Objectives of the present paper—state-dependent drift and diffusion coefficients. Below we consider homogeneous, state-dependent coefficient functions

$$b(t, S_t) = b(S_t),\ \sigma(t, S_t) = \sigma(S_t),\ \text{ and } h(t, S_t) = h(S_t) \tag{8}$$

for locally Lipschitz continuous b, σ and locally Hölder continuous h. As such we (partly) extend the homogeneous, state-dependent setting of a pure diffusion as in Mijatović and Urusov (2012) and others to a single jump framework as in Herdegen and Herrmann (2019).

[3]One can see this by applying the classification of Chapters 2 and 4 in Cherny and Engelbert (2005) to the case of a driftless diffusion.

Our main result in section 4 below is concerned with a deterministic necessary and sufficient criterion on b, σ, and h to decide whether $(S_t)_{t\in[0,\infty)}$ is a uniformly integrable martingale.

If one accepts that local martingales that are not uniformly integrable martingales are suitable processes to model bubbles, our result contributes to the financial literature on bubbles in several dimensions:

1. Single jump processes as in (1) (where the single jump J represents a financial crash of relative size b/h) are a simple and tractable alternative to include crash risk in financial models and serve as a simple tool to integrate empirical features of bubbly markets into mathematical models. The main result below allows us to bridge one of the gaps between
   (a) the literature on bubbles based on explosive processes and a crash as in Sornette and Andersen (2002) with
   (b) the mathematical finance notion of bubbles as nonuniformly integrable martingales or strict local martingales discussed in section 5.2.

   See section 5.1.2 below for a specific example.
2. The classification of mathematical bubbles can be extended from single jump processes with deterministic intensity as in Herdegen and Herrmann (2019) to jumps whose hazard rate is random (state-dependent), allowing for a more realistic description of crash risk. Moreover, we (partly) extend the setting of a pure diffusion as in Mijatović and Urusov (2012), covering various models in the literature,[4] to include the financially relevant case of a crash; see sections 5.1.1 and 5.1.3 below for examples.
3. Equation (6) implies that single jump models with a deterministic hazard rate as in Herdegen and Herrmann (2019) can be mathematical bubble models only if there is an almost sure jump on a finite time interval $[0, T]$. Models based on a homogeneous diffusion as considered below feature a crash distributed on $[0, \infty)$. For an investor with deterministic finite investment horizon (as is standard in the literature) there is a nonzero probability that the crash does not happen within his investment horizon, a reasonable assumption in financial problem settings.

We close with a discussion of assumptions and open questions in section 5.3.

2. Notation. The following notation is used throughout the paper. Unless stated otherwise, we consider stochastic processes *unique up to indistinguishability* and require stochastic integral equations on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to hold $\mathbb{P}$-a.s. We assume familiarity with the notions of a *martingale*, *supermartingale*, and *local martingale*.[5] A stochastic process $(X_t)_{t\in[0,\infty)}$ is *uniformly integrable* if

$$\lim_{n\to\infty} \sup_{t\in[0,\infty)} \mathbb{E}\left[|X_t|\, \mathbb{1}\{n < |X_t|\}\right] = 0. \tag{9}$$

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random times $\sigma, \tau \colon \Omega \to [0, \infty]$ we use the *stochastic interval* notation

[4] For example, the CEV model or geometric Brownian motion.

[5] For an introduction, see, e.g., Chapter 1 in Protter (2010).

$$[[\sigma,\tau]] = \{(\omega,t)\in\Omega\times[0,\infty]\colon \sigma(\omega)\leqslant t\leqslant\tau(\omega)\},$$
$$[[\sigma,\tau)) = \{(\omega,t)\in\Omega\times[0,\infty]\colon \sigma(\omega)\leqslant t<\tau(\omega)\}. \tag{10}$$

For a stochastic process $(X_t)_{t\in[0,\infty)}$ we denote its left-continuous version by $(X_{t-})_{t\in[0,\infty)}$, that is, the process with the property that $X_{t-}=\lim_{s\nearrow t}X_s$ for all $t\in[0,\infty]$. For a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty)},\mathbb{P})$ with a right-continuous filtration and an $(\mathcal{F}_t)_{t\in[0,\infty)}$-stopping time $\tau:\Omega\to[0,\infty)$ we define the (itself right-continuous) filtration $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$ consisting of the σ-algebras $\mathcal{F}_{t\wedge\tau-}$ given by

$$\mathcal{F}_{t\wedge\tau-}=\sigma\left(\{A\cap\{s<\tau\}\colon 0\leqslant s\leqslant t, A\in\mathcal{F}_s\}\cup\mathcal{F}_0\right). \tag{11}$$

3. Setting.

3.1. Definitions. Let $b:[0,\infty)\to[0,\infty)$ and $\sigma:[0,\infty)\to[0,\infty)$ be locally Lipschitz continuous functions with $\sigma^{-1}(0)=\{0\}$, let $B_0\in(0,\infty)$, $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty)},\mathbb{P})$ be a filtered probability space with a right-continuous and $\mathbb{P}$-complete filtration, let $(W_t)_{t\in[0,\infty)}$ be a real valued $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $B\colon[0,\infty)\times\Omega\to(0,\infty]$ be the unique strictly positive process with the property that

1. for $(\mathcal{F}_t)_{t\in[0,\infty)}$-stopping times $(\tau_n)_{n\in\mathbb{N}}\colon\Omega\to[0,\infty)$ given by $\tau_n=\inf\{t\geqslant 0\colon B_t\geqslant n\}$, for all $n\in\mathbb{N}$ on $[[0,\tau_n]]$ we have

$$\int_0^t \left(b(B_s)ds+\sigma^2(B_s)\right)ds<\infty \text{ and}$$
$$B_t=B_0+\int_0^t b(B_s)ds+\int_0^t\sigma(B_s)dW_s, \tag{12}$$

   and
2. for the predictable $(\mathcal{F}_t)_{t\in[0,\infty)}$-stopping time $\tau\colon\Omega\to[0,\infty]$ given by $\tau=\sup_{n\in\mathbb{N}}\tau_n$ it holds that B is $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$ -adapted,

let $h:[0,\infty)\to[0,\infty)$ be a locally Hölder continuous function with the property that[6]

$$\frac{b(x)}{h(x)x}\in[0,1] \text{ for all } x\in(0,\infty) \text{ and}$$
$$\lim_{x\to\infty}\frac{b(x)}{h(x)x} \text{ exists,} \tag{13}$$

let $J:[0,\infty]\times\Omega\to\{0,1\}$ be an RCLL stochastic ("single jump") process with the property that for all $t\in[0,\infty)$ we have

$$\mathbb{P}[J_t=1|\mathcal{F}_{t\wedge\tau-}]=1-e^{-\int_0^{t\wedge\tau}h(B_s)ds}, \tag{14}$$

let $\tau_J:\Omega\to\mathbb{R}$ be the $\mathbb{P}$-a.s. unique random time with the property that for all $t\in[0,\infty)$ it holds that $\mathbb{P}\left[J_{t\wedge\tau}=1\right]=\mathbb{P}\left[\tau_J\leqslant t\wedge\tau\right]$, let $(\mathcal{G}_t)_{t\in[0,\infty)}$ be the filtration generated

[6]In the following, we employ the convention that $0/0=0$ to allow for $b(x)=h(x)=0$, for some $x\in[0,\infty)$, while retaining notational convenience. The quantity $b(x)/h(x)x$ is the relative jump size and thus not relevant in cases where $h(x)=0$. Let us also note here that assumption (A) below excludes the case $b\equiv h\equiv 0$.

by $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$ and J, and let $S\colon [0,\infty)\times\Omega\to[0,\infty)$ be the $(\mathcal{G}_t)_{t\in[0,\infty)}$-adapted RCLL process with the property that for all $t\in[0,\infty)$ we have

$$S_t = B_0 + \int_0^t b(S_s)\mathbb{1}\{s<\tau_J\}ds + \int_0^t \sigma(S_s)\mathbb{1}\{s<\tau_J\}dW_s - \int_0^t \frac{b(S_{s-})}{h(S_{s-})}dJ_s. \tag{15}$$

3.2. Assumptions. Moreover, we assume that

(A) for all $n\in\mathbb{N}\cap[B_0,\infty)$: $\mathbb{P}[\tau_n<\infty]=1$, and

(B) $\lim_{x\to\infty} h(x)x^2/\sigma^2(x)$ exists and is finite.

3.3. Comments to the setting. Uniqueness of solutions of SDEs is understood as pathwise uniqueness. The local Lipschitz assumption on b and σ ensures that the integral equation (12) has a unique strong solution up to the random time τ; see, e.g., Theorem 4.3 in Protter (1977). The time τ is called the explosion time of B. Local Lipschitz conditions on b and σ, $b\geqslant 0$ and $\sigma(0)=0$ ensure strict positivity of B; see, e.g., Theorem 4.1 in Chapter 9 of Friedman (1975). The σ-algebra $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$ includes precisely the information of the trajectories of B up to its explosion time τ. Uniqueness holds in law and pathwise; see, e.g., Chapter 1 in Cherny and Engelbert (2005).

The process J is a single jump process that jumps from 0 to 1 at time τ_J; for construction (and thus existence) see section 6.5 in Bielecki and Rutkowski (2002). For a measurable function $h:[0,\infty)\to[0,\infty)$ the process $(h(B_t)\mathbb{1}\{t<\tau\})_{t\in[0,\infty)}$ can be understood as the $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-martingale intensity process of J (cf., e.g., Chapter 6 of Bielecki and Rutkowski (2002)). The additional requirement of local Hölder continuity is used in the application of the Feynman–Kac formula. We show below ((34) in the proof of Theorem 1) that strict positivity of B and assumption (A) imply that $\mathbb{P}[\tau_J<\tau]=1$ and thus in particular that $J_\infty=1$. Similarly, assumption (B) is needed in the proof and clearly restricts the choice of σ and h, whereas (13) merely implies the natural condition that the relative jump size is in $[0,1]$ and excludes the special case of periodic behavior at infinity (and is therefore not listed as a distinct assumption in section 3.2).

Existence and uniqueness of the solution to (15) is guaranteed by the semimartingale property of the integrator (cf. section 3 of Cheridito, Filipovic, and Yor (2005) for a discussion of the semimartingale property of a stopped, time-inhomogeneous jump diffusion). Proposition 3.2 in Cheridito, Filipovic, and Yor (2005) (for $X=S$ and $T_\Delta=\tau_J$) implies that $(S_t)_{t\in[0,\infty)}$ is a nonnegative local martingale and thus, by Fatou's lemma, a nonnegative supermartingale with the property that for all $(\mathcal{G}_t)_{t\in[0,\infty)}$-stopping times $\rho\colon\Omega\to[0,\infty)$ it holds that $\mathbb{E}[S_\rho]\leqslant S_0$. Alternatively, one can

1. check that W is still a Brownian motion with respect to $(\mathcal{G}_t)_{t\in[0,\infty)}$ and
2. use the local martingale property of the compensated jump process $(J_t-\int_0^{t\wedge\tau}h(B_s)ds)_{t\in[0,\infty)}$ (cf. section 6.5 of Bielecki and Rutkowski (2002)).

Then S can be expressed as integrals with respect to the local martingales $J-\int_0^{\cdot\wedge\tau}h(B_s)ds$ and W and is itself a local martingale. The process S can be called a *single jump local martingale* as it follows the diffusion B and has a single jump at τ_J, thus obeying the equation

$$S = B\mathbb{1}\{\cdot<\tau_J\} + \left(1-\frac{b(B_{\tau_J})}{h(B_{\tau_J})B_{\tau_J}}\right)B_{\tau_J}\mathbb{1}\{\tau_J\leqslant\cdot\}\,. \tag{16}$$

4. Main result.

Proposition 1. *Assume the setting in section 3, $\mathbb{P}[\tau_J < \tau] = 1$, and let $f : [0, \infty) \to [0, 1]$ be a measurable function. Then it holds that*

$$\mathbb{E}\left[(1 - f(B_{\tau_J}))\, B_{\tau_J}\right] = \mathbb{E}\left[\int_0^\tau h(B_t)\,(1 - f(B_t))\, B_t e^{-\int_0^t h(B_s)ds} dt\right]. \tag{17}$$

Proof. We observe that $(\tau_n)_{n\in\mathbb{N}}$ and τ are $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-stopping times and thus τ an $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-predictable stopping time. Moreover, B has continuous trajectories on $[[0, \tau))$ and is $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-adapted. We can conclude that

$$(1 - f(B))\, B\ \mathbb{1}\{\cdot < \tau\} \colon [0, \infty) \times \Omega \to [0, \infty) \tag{18}$$

is an $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-predictable process. Then the claim follows from

$$\mathbb{E}\left[(1 - f(B_{\tau_J}))\, B_{\tau_J}\ \mathbb{1}\{\tau_J < \tau\}\right] = \mathbb{E}\left[(1 - f(B_{\tau_J}))\, B_{\tau_J}\right] \tag{19}$$

and part (ii) of Corollary 6.3. in Jeanblanc and Rutkowski (2000). The proof of Proposition 1 is thus completed. ∎

Lemma 1. *Assume the setting and the assumptions in section 3, and let $v\colon (0, \infty) \to [0, \infty)$ be a twice differentiable function with the property that it satisfies the ordinary differential equation*

$$\frac{1}{2}\sigma(x)^2 \frac{\partial^2 v}{\partial x^2}(x) + b(x)\frac{\partial v}{\partial x}(x) = h(x)v(x), \quad x \in (0, \infty), \tag{20}$$

with boundary condition $\lim_{n\to\infty} v(n) = \infty, n \in \mathbb{N}$. Then it holds that

$$\lim_{n\to\infty} \frac{n}{v(n)} = 0 \iff \left(\lim_{x\to\infty} \frac{b(x)}{h(x)x} < 1\right) \text{ and } \left(\lim_{x\to\infty} \frac{h(x)x^2}{\sigma^2(x)} > 0\right). \tag{21}$$

Proof. First we note that

$$\frac{b(x)x}{\sigma^2(x)} = \frac{b(x)}{h(x)x}\frac{h(x)x^2}{\sigma^2(x)}. \tag{22}$$

Using assumption (B) and the existence of $\lim_{x\to\infty} b(x)/(h(x)x) \in [0, 1]$, there are constants $p_0, q_0 \in [0, \infty)$ with the property that

$$\lim_{x\to\infty}\left|\frac{b(x)x}{\sigma^2(x)} - p_0\right| = 0, \quad \lim_{x\to\infty}\left|\frac{h(x)x^2}{\sigma^2(x)} - q_0\right| = 0. \tag{23}$$

Equation (22) implies that $p_0 \leqslant q_0$ and

$$p_0 < q_0 \iff \left(\lim_{x\to\infty} \frac{b(x)}{h(x)x} < 1\right) \text{ and } \left(\lim_{x\to\infty} \frac{h(x)x^2}{\sigma^2(x)} > 0\right). \tag{24}$$

As elaborated in Chapter 9.12 of Birkhoff and Rota (1989), a substitution $y = 1/x$ transforms the solution of the ODE (20) to a solution $w\colon (0,\infty) \to [0,\infty)$ of the ODE

$$\frac{\partial^2 w}{\partial y^2}(y) + \left(\frac{2}{y} - \frac{2}{y^2}\frac{b\left(\frac{1}{y}\right)}{\sigma^2\left(\frac{1}{y}\right)}\right)\frac{\partial w}{\partial y}(y) - \frac{2}{y^4}\frac{h\left(\frac{1}{y}\right)}{\sigma^2\left(\frac{1}{y}\right)}w(y) = 0, \quad y \in (0,\infty), \tag{25}$$

with boundary condition

$$\lim_{n\to\infty} w\left(\frac{1}{n}\right) = \lim_{n\to\infty} v(n) = \infty. \tag{26}$$

The continuity theorem for solutions of ODEs (see Theorem 3 on p. 177 in Birkhoff and Rota (1989)) and (23) ensures that the behavior of the solution w of (25) at 0 does not change if we substitute the coefficient functions using p_0 and q_0 to arrive at the ODE

$$\frac{\partial^2 w}{\partial y^2}(y) + \left(\frac{2}{y} - \frac{2}{y}p_0\right)\frac{\partial w}{\partial y}(y) - \frac{2}{y^2}q_0 w(y) = 0, \quad y \in (0,\infty), \tag{27}$$

which has a *regular singular point* at 0. To analyze the behavior of w around 0, we have to look at the solutions $r_2 < r_1 \in \mathbb{R}$ of the *indicial equation* of the ODE (27), $r(r-1) + 2(1-p_0)r - 2q_0 = 0$, with solutions

$$r_{2,1} = p_0 - \frac{1}{2} \pm \frac{1}{2}\sqrt{4p_0^2 - 4p_0 + 8q_0 + 1}. \tag{28}$$

Applying the corollary after Theorem 7 and Theorem 8 of Chapter 9 in Birkhoff and Rota (1989), we get that in a small enough neighborhood of 0, the function w can be written as

$$\begin{aligned} & w(y) = \alpha w_1(y) + \beta w_2(y) \text{ with} \\ & w_1(y) = y^{r_1}\left(1 + \sum_{k=1}^{\infty} a_k y^k\right) \text{ and} \\ & w_2(y) = y^{r_2}\left(1 + \sum_{k=1}^{\infty} b_k y^k\right) + C w_1(y)\ln(y)\mathbb{1}\{r_1 \in r_2 + \mathbb{N}\}. \end{aligned} \tag{29}$$

Now we look at two separate cases.

- For $p_0 = q_0 = 0$, (28) shows that $r_1 = 0, r_2 = -1$. Then (29) implies that $\lim_{n\to\infty} n/w(1/n) > 0$.
- For $q_0 > 0$, (28) shows that $r_1 > 0$, $r_2 = -1$ for $p_0 = q_0$ and $r_2 < -1$ for $p_0 < q_0$. As $r_1 > 0$, using (26) and $w_1(0) = 0$ we get that $\beta > 0$. Now (29) shows that $\lim_{n\to\infty} n/w(1/n) = 0$ if and only if $p_0 < q_0$.

In both cases, we have that $\lim_{n\to\infty} n/v(n) = \lim_{n\to\infty} n/w(1/n) = 0$ if and only if $p_0 < q_0$. Now we can conclude with referring to (24) above. The proof of Lemma 1 is thus completed. ■

Theorem 1. *Assume the setting and the assumptions in section 3; then it holds that*

$$\mathbb{E}\left[B_{\tau_J} - \frac{b(B_{\tau_J})}{h(B_{\tau_J})}\right] = B_0 \Longleftrightarrow \left(\lim_{x\to\infty} \frac{b(x)}{h(x)x} < 1\right) \text{ and } \left(\lim_{x\to\infty} \frac{h(x)x^2}{\sigma^2(x)} > 0\right). \tag{30}$$

Proof. Recall the $(\mathcal{F}_t)_{t\in[0,\infty)}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ given by $\tau_n = \inf\{t \geqslant 0 \colon B_t \geqslant n\}$ for $n \in \mathbb{N}$ and $\tau = \lim_{n\to\infty} \tau_n$. Let $t \in [0,\infty)$, fix $n \in \mathbb{N} \cap [B_0,\infty)$, and let $T = t \wedge \tau_n$. Then integration by parts and Itô's formula show that

$$\begin{aligned} B_T e^{-\int_0^T h(B_s)ds} = B_0 &- \int_0^T B_s h(B_s) e^{-\int_0^s h(B_u)du} ds \\ &+ \int_0^T e^{-\int_0^s h(B_u)du} b(B_s)ds + \int_0^T e^{-\int_0^s h(B_u)du} \sigma(B_s)dW_s. \end{aligned} \tag{31}$$

For fixed n, the integrand in the stochastic integral is bounded. Taking expectations, we get

$$\mathbb{E}\left[B_T e^{-\int_0^T h(B_s)ds}\right] = B_0 - \mathbb{E}\left[\int_0^T \left(1 - \frac{b(B_s)}{h(B_s)B_s}\right) h(B_s)B_s e^{-\int_0^s h(B_u)du} ds\right]. \tag{32}$$

Using $\mathbb{P}[\tau_n < \infty] = 1$, bounded convergence, and monotone convergence, respectively, we get with $t \to \infty$ that

$$\mathbb{E}\left[B_{\tau_n} e^{-\int_0^{\tau_n} h(B_s)ds}\right] = B_0 - \mathbb{E}\left[\int_0^{\tau_n} \left(1 - \frac{b(B_s)}{h(B_s)B_s}\right) h(B_s)B_s e^{-\int_0^s h(B_u)du} ds\right]. \tag{33}$$

Equation (33), $B_{\tau_n} = n$, and monotone convergence imply that

$$\begin{aligned} \mathbb{P}[\tau_J < \tau] &= 1 - \mathbb{E}\left[e^{-\int_0^{\tau-} h(B_s)ds}\right] \\ &= 1 - \mathbb{E}\left[e^{-\int_0^{\tau} h(B_s)ds}\right] \\ &= 1 - \lim_{n\to\infty} \mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds}\right] \\ &\geqslant 1 - \lim_{n\to\infty} \frac{B_0}{n} \\ &= 1. \end{aligned} \tag{34}$$

In particular, (34) shows that the jump happens with probability 1 on $[[0,\tau))$. Using again monotone convergence and Proposition 1 we let $n \to \infty$ in (33) to arrive at

$$\lim_{n\to\infty} n\mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds}\right] = B_0 - \mathbb{E}\left[\left(1 - \frac{b(B_{\tau_J})}{h(B_{\tau_J})B_{\tau_J}}\right) B_{\tau_J}\right]. \tag{35}$$

Let $v_n : (0,n) \to [0,1]$ be the function with the property that for $x \in (0,n)$ it holds that

$$v_n(x) = \mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds} \,\middle|\, B_0 = x\right], \tag{36}$$

and let $D_n = (0,n)$ be an open domain with boundary $\delta D_n = \{0\} \cup \{n\}$. Note that the setting in section 3 ensures that b, σ are continuous and h is Hölder continuous on $[0,n]$. Using the

Feynman–Kac formula for a degenerate second order differential operator on D_n and attainable boundary n,[7] we get that v_n is twice differentiable, satisfies the ordinary differential equation

$$\frac{1}{2}\sigma(x)^2\frac{\partial^2 v_n}{\partial x^2}(x) + b(x)\frac{\partial v_n}{\partial x}(x) = h(x)v_n(x), \quad x \in (0,n), \tag{37}$$

and is uniquely determined by the boundary condition

$$v_n(n) = \mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds}\,\middle|\, B_0 = n\right] = 1. \tag{38}$$

For all $n \in \mathbb{N}$, $x \in (0,n)$, (33) implies that

$$v_n(x) = \mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds}\,\middle|\, B_0 = x\right] \leqslant \frac{x}{n} \tag{39}$$

and thus $\lim_{n\to\infty} v_n(x) = 0$ for all $x \in (0,\infty)$, which is used in (44) below. Now, let $v : (0,\infty) \to [0,\infty)$ be the function given by

$$v(x) = \frac{v_n(x)}{v_2(1)\cdots v_n(n-1)} \quad \text{for} \quad x \in (0,n]. \tag{40}$$

The boundary condition $v_n(n) = 1$ implies the equality

$$\frac{v_{n+1}(n)}{v_2(1)\cdots v_n(n-1)v_{n+1}(n)} = \frac{v_n(n)}{v_2(1)\cdots v_n(n-1)}, \quad n \in \mathbb{N}, \tag{41}$$

and thus, together with uniqueness of v_n, we can conclude that v is well defined, independent of n. Moreover, v is a twice differentiable function with the property that

$$v(n) > 0 \text{ and } \frac{v(x)}{v(n)} = v_n(x) \quad \text{for all } n \in \mathbb{N}, x \in (0,n]. \tag{42}$$

By definition, v satisfies the ordinary differential equation

$$\frac{1}{2}\sigma(x)^2\frac{\partial^2 v}{\partial x^2}(x) + b(x)\frac{\partial v}{\partial x}(x) = h(x)v(x), \quad x \in (0,\infty), \tag{43}$$

and, using (39) and (42) for arbitrary x, the boundary condition

$$\lim_{n\to\infty} v(n) = \lim_{n\to\infty} \frac{v(x)}{v_n(x)} = \infty. \tag{44}$$

Using (35), (36), and (42) we get that

$$\begin{aligned} B_0 - \mathbb{E}\left[\left(1 - \frac{b(B_{\tau_J})}{h(B_{\tau_J})B_{\tau_J}}\right) B_{\tau_J}\right] &= \lim_{n\to\infty} n\mathbb{E}\left[e^{-\int_0^{\tau_n} h(B_s)ds}\right] \\ &= v(B_0)\lim_{n\to\infty}\frac{n}{v(n)}. \end{aligned} \tag{45}$$

[7] See, e.g., Theorem 1.1 in Chapter 13 of Friedman (1976).

Finally, Lemma 1 implies that

$$B_0 - \mathbb{E}\left[\left(1 - \frac{b(B_{\tau_J})}{h(B_{\tau_J})B_{\tau_J}}\right) B_{\tau_J}\right] = 0 \tag{46}$$

if and only if $\lim_{x\to\infty} b(x)/(h(x)x) < 1$ and $\lim_{x\to\infty} h(x)x^2/\sigma^2(x) > 0$. The proof of Theorem 1 is thus completed. ∎

Corollary 1. *Assume the setting and the assumptions in section 3. Then the process S is a uniformly integrable martingale if only if $\lim_{x\to\infty} b(x)/(h(x)x) < 1$ and $\lim_{x\to\infty} h(x)x^2/\sigma^2(x) > 0$.*

Proof. The single jump process S is a positive local martingale and, thus, by Doob's martingale convergence theorem, a supermartingale with

$$\mathbb{E}\left[S_\infty\right] \leqslant B_0. \tag{47}$$

Thus S is a uniformly integrable martingale if and only if

$$\mathbb{E}\left[\left(1 - \frac{b(B_{\tau_J})}{h(B_{\tau_J})B_{\tau_J}}\right) B_{\tau_J}\right] = B_0.$$

Theorem 1 completes the proof of Corollary 1. ∎

The following corollary covers a special case that is very common in the literature; see, e.g., the examples in sections 5.1.1 and 5.1.2 below.

Corollary 2. *Assume the setting and the assumptions in section 3 and let $\lim_{x\to\infty} b(x)x/\sigma^2(x) > 0$. Then the process S is a uniformly integrable martingale if and only if*

$$\lim_{x\to\infty} \frac{b(x)}{h(x)x} < 1. \tag{48}$$

Proof. From $\lim_{x\to\infty} b(x)x/\sigma^2(x) > 0$ and for all x: $b(x)/(h(x)x) \in [0,1]$ we know that $\lim_{x\to\infty} h(x)x^2/\sigma^2(x) > 0$. Then the claim follows from Corollary 1. ∎

5. Applications and discussion.

5.1. Examples.

5.1.1. Geometric Brownian motion. First we discuss the situation where the underlying process B is a geometric Brownian motion. In contrast to processes with deterministic jump intensity given by (5) and the example in section 5.1.2 below, this allows us to construct a process that is not a uniformly integrable martingale (and thus a mathematical bubble), while the underlying diffusion is not explosive.

To see this, let $\sigma_0, c, \epsilon \in (0,\infty)$, $\mu_0 \in [\sigma_0^2/2, \infty)$, and let $b(x) = \mu_0 x$, $\sigma(x) = \sigma_0 x$ and

$$h(x) = \left(\mu_0\left(1 + \frac{c}{\epsilon}\right)\right)\mathbb{1}\{x \leqslant \epsilon\} + \left(\mu_0\left(1 + \frac{c}{x}\right)\right)\mathbb{1}\{\epsilon < x\} \tag{49}$$

for $x \in (0, \infty)$. Then the process B given by (12) is transient geometric Brownian motion with explosion time $\tau \equiv \infty$. Let us check the assumptions of section 3.2. From the discussion on p. 197 in Karatzas and Shreve (1988) we know that $\mathbb{P}[\tau_n < \infty] = 1$ for all $n \geqslant B_0$. Moreover, it holds that

$$\frac{h(x)x^2}{\sigma^2(x)} = \frac{\mu_0}{\sigma_0^2}\left(1 + \frac{c}{x}\right), \quad x \in (\epsilon, \infty). \tag{50}$$

Thus assumptions (A) and (B) are satisfied and Corollary 2 shows that the resulting single jump process S is *not* a uniformly integrable martingale. S follows a geometric Brownian motion until the time of the jump that is distributed according to the hazard rate $(h(B_t))_{t\in[0,\infty)}$. Note that for $\mu_0 < \sigma_0^2/2$ it holds that $\mathbb{P}[\tau_n = \infty] > 0$ for $n > B_0$ and assumption (A) is not satisfied.

5.1.2. Andersen–Sornette model. In Sornette and Andersen (2002) and Andersen and Sornette (2004) a model of bubbles has been introduced that is based on superexponential diffusive growth and a crash represented by a single jump. The process satisfies the above assumptions and can thus be shown to be a mathematical bubble for a suitable jump intensity, highlighting a possible link between the two approaches of

1. bubbles driven by positive feedback mechanisms, superexponential growth, and a failure of market efficiency as in Sornette and Andersen (2002) and
2. mathematical bubbles as discussed in the introduction and section 5.2 below.

A similar link for processes based on deterministic jump intensity has been discussed in Herdegen and Herrmann (2019). To replicate the setting from Sornette and Andersen (2002), assume the setting in section 3, let $m \in (1, \infty)$, $\mu_0, \sigma_0 \in (0, \infty)$, let $b\colon [0, \infty) \to [0, \infty)$ and $\sigma\colon [0, \infty) \to [0, \infty)$ be given by $b(x) = (m\sigma_0^2/2)x^{2m-1} + \mu_0 x^m$ and $\sigma(x) = \sigma_0 x^m$ for $x \in [0, \infty)$, and let $T_c \in (0, \infty)$. In Sornette and Andersen (2002) it has been shown that for $\alpha = \frac{1}{m-1}$ the process B of (12) is given by

$$B_t = \alpha^\alpha \frac{1}{(\mu_0(T_c - t) - \sigma_0 W_t)^\alpha} \qquad \text{for } (\omega, t) \in [[0, \tau)), \tag{51}$$

with explosion time $\tau : \Omega \to [0, \infty)$ given by $\tau(\omega) := \inf\{t \in (0, \infty) : \mu_0 t + \sigma_0 W_t(\omega) = \mu_0 T_c\}$. As described in Sornette and Andersen (2002), the very form of μ and σ can be deduced from the Stratonovich formulation of a nonlinear SDE

$$dB_t = \mu_0 B_t^m dt + \sigma_0 B_t^m \circ dW_t, \tag{52}$$

where $\circ$ denotes the Stratonovich integral.[8] This has been introduced as a straightforward extension of a nonlinear differential equation $dx = x^m dt$ for $m > 1$ as a means of describing self-reinforcing behavior, leading to superexponential growth. From (51) we can deduce that assumption (A) is fulfilled. Let $\kappa : [0, \infty) \to [0, 1]$, the *relative jump size*, be any measurable function such that

[8]Introduced in Stratonovich (1966).

1. $\lim_{x\to\infty} \kappa(x) = 1$ and
2. $h(x) = \frac{b(x)}{x\kappa(x)}$ is locally Hölder continuous and fulfils assumption (B).

Then Corollary 2 implies that S is *not* a uniformly integrable martingale. In Sornette and Andersen (2002) it has been assumed that the relative jump size $\kappa(\cdot) \equiv \kappa \in (0,1)$ is constant, and Corollary 2 shows that in this case S is a uniformly integrable martingale.

5.1.3. Jump (to default) extended constant elasticity of variance model. Extending diffusive models with jumps can enhance their ability to capture defaults, crashes, or market anomalies. An example of this is the jump to default extended constant elasticity of variance (CEV) model, introduced in Carr and Linetsky (2006). Let us discuss a similar jump extended model and its classification as a mathematical bubble based on the results above. For this, assume the setting in section 3, and let $\mu_0 \in [0,\infty)$, $\sigma_0 \in (0,\infty)$, $\alpha, \beta \in \mathbb{R}$, $b(x) = \mu_0 x^\alpha$, and $\sigma(x) = \sigma_0 x^\beta$. For $\alpha = 1$ the process B (usually defined with absorption at 0) given by (12) is the CEV model, introduced by Cox (1996) and Emmanuel and MacBeth (1982). In Carr and Linetsky (2006) they allow for the parameter range $\beta \in (-\infty, 1)$ and $\alpha = 2\beta - 1$ and include a possible default (a jump to 0) to arrive at a local martingale in the form of (15), which they called the *jump to default extended CEV model*. Due to the fixed relative jump size of 1, the resulting process is a nonuniformly integrable martingale. Note that the latter parameter range fails to be included in section 3.1, as the resulting diffusion coefficient is not Lipschitz continuous at 0.

Instead consider the parameter range $\beta \in (1,\infty)$, $\alpha \in (1, 2\beta-1]$. Then Example 4.4 in Mijatović and Urusov (2012) and its preceding remark show that the discounted process

$$B_t e^{-\int_0^{\tau\wedge t} \frac{b(B_s)}{B_s} ds}, \quad t \in [0,\infty), \tag{53}$$

is a strict local martingale and thus not a uniformly integrable martingale. Similar to Carr and Linetsky (2006), instead of discounting with the drift $\mu_0 x^{\alpha-1}$, let us add a jump to the model with $h(x) = \frac{\mu_0}{\kappa} x^{\alpha-1}$ for some $\kappa \in (0,1)$. Then Corollary 1 shows that the local martingale S, given by (16), is

1. not a uniformly integrable martingale for $\alpha \in (1, 2\beta-1)$ and
2. a uniformly integrable martingale for $\alpha = 2\beta - 1$ and $\mu_0 \geqslant \sigma_0^2/2$.

Note that assumption (A) for the CEV model with $\alpha = 2\beta - 1 \geqslant 1$ is fulfilled if and only if $\mu_0 \geqslant \sigma_0^2/2$; see Theorem 5.1 in Cherny and Engelbert (2005). In particular, Corollary 1 does not apply to the case $\mu_0 = 0$ and $\beta = 2$, in which the underlying diffusion $(B_t)_{t\in[0,\infty)}$ is given by the inverse Bessel process

$$B_t = B_0 + \int_0^t B_s^2 dW_s, \quad t \in [0,\infty), \tag{54}$$

a classical example of a strict local martingale.

5.2. Comments on mathematical bubble models. Based on the seminal paper by Loewenstein and Willard (2000), Cox and Hobson (2005), Heston, Loewenstein, and Willard (2007), Jarrow, Protter, and Shimbo (2007), (2010), and Herdegen and Schweizer (2016) have developed several attempts to describe financial bubbles through a deviation of a continuous stock price from its fundamental value.

To describe the essential idea of these approaches (see, e.g., Protter (2013) for a comprehensive introduction to mathematical bubble modeling), assume that some process S on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ describes a stock market (accompanied by the usual bank account as numéraire) on some time interval $[0,\tau]$, where $\tau : \Omega \to [0,\infty)$ is a $\mathbb{P}$-a.s. finite but possibly unbounded stopping time. Requiring the absence of arbitrage opportunities,[9] we know that in a complete market there exists a unique equivalent measure $\mathbb{Q} \approx \mathbb{P}$ such that S is a local $\mathbb{Q}$-martingale. Further assuming a complete market and the absence of dividends, one can define the fundamental price of an asset at time 0 as the expected value of the final payoff S_τ, that is, $\mathbb{E}_{\mathbb{Q}}[S_\tau|\mathcal{F}_0]$. Then it becomes clear that Corollary 1 above gives necessary and sufficient conditions to classify single jump processes of the form (15) (with the choice $\tau = \tau_J$) as mathematical bubbles.

This reasoning is complicated by the fact that a market generated by single jump processes is, in general, incomplete. Therefore, it is not immediately clear how to define the fundamental value of an asset and there exist several competing approaches in the literature; see, e.g., Jarrow, Protter, and Shimbo (2010) and Herdegen and Schweizer (2016) or the discussion in section 2.1 of Schatz and Sornette (2019). For a definite classification as either of the two approaches it will be necessary to examine how the processes studied in this paper behave under an equivalent change of measure.

Models of mathematical bubbles to date have mostly been considered on a finite time horizon $[0,T]$ for some deterministic $T \in (0,\infty)$. While this is generally rationalized in many financial problems by a finite time investment horizon of the agent, this immediately excludes nonuniformly integrable martingales studied in Corollary 1. If we distinguish, however, *investment horizon*—constraining the trading activity of a market participant—and *model horizon*—the lifetime of a financial asset—then nonuniformly integrable martingales are readily conceivable as mathematical bubble models.

5.3. Discussion.

5.3.1. Relaxing assumptions on characteristics. Processes as defined by (15) in the setting in section 3 are well defined for very general coefficient functions b, σ, h. The local Lipschitz condition on b and σ can be significantly relaxed (see Theorem 4.53 in Engelbert and Schmidt (1991) or Proposition 2.2 in Cherny and Engelbert (2005)), yielding a so-called weak solution B for (12). Moreover, the Hölder condition on h can be relaxed to mere measurability; see, e.g., section 6.5 in Bielecki and Rutkowski (2002). The additional assumptions we make are in order to apply the Feynman–Kac formula and analyze the expectation of (34) in Theorem 1. Very recently there has been an effort by Feehan, Gong, and Song (2015) and Feehan and Pop (2015) to extend the stochastic representation of Dirichlet boundary problems for a degenerate differential operator to more general[10] diffusion processes. Using these results, one may be able to extend the analysis in the present paper to such processes.

[9]In the form of *No free lunch with vanishing risk*, developed by Delbaen and Schachermayer (1994), (1998a).

[10]They allowed for σ that is not Lipschitz continuous to cover, e.g., the *Heston stochastic volatility model*, the *SABR model*, and the *CEV* model.

5.3.2. Relaxing the assumptions in section 3.2. Assumption (B) in section 3.2 is necessary to guarantee that the ordinary differential equation appearing in Lemma 1 has at most regular singular points and can be analyzed using Frobenius series. There exist some results on the (asymptotic) behavior around *irregular singular points*; see, e.g., sections 3.4 and 3.5 in Bender and Orszag (1978) for a textbook treatment. However, there is no unified treatment of such equations and an extension may be fruitful only for particular examples, if at all.

Assumption (A) in section 3.2 ensures that

$$\mathbb{P}\left[\lim_{t\to\infty} B_t = 0\right] = 0 \tag{55}$$

for the process B given by (12). If we drop this assumption, we cannot conclude that the jump happens with probability 1 on $[0,\infty)$ and additional terms appear in the probabilistic representation of (37),[11] thus leading to a nontrivial extension of the approach in this paper.

5.3.3. Analysis of the martingale property. Let us close with a revisit to the discussion in the introduction. Assume we know that a process S is not a uniformly integrable martingale. Then the obvious next question to ask is whether the process is a strict local martingale or a martingale that is not uniformly integrable. To make this point clear, consider the following simple example. Assume the setting in section (3), let $\sigma_0, \mu_0 \in (0,\infty)$, and let $b(x) = \mu_0 x, \sigma(x) = \sigma_0 x$, and $h(x) = \mu_0$ for $x \in (0,\infty)$. Then the process B given by (12) is a geometric Brownian motion and the process S follows B up to the jump, where it jumps to 0 (we have a *relative jump size* $b(x)/(h(x)x) \equiv 1$). It is clear that S is not a uniformly integrable martingale. However, as h is not state-dependent it holds that

$$\mathbb{E}\left[S_t\right] = \mathbb{E}\left[B_t \mathbb{1}\{t < \tau_J\}\right] = \mathbb{E}\left[B_t\right]\mathbb{E}\left[\mathbb{1}\{t < \tau_J\}\right] = B_0, \tag{56}$$

which implies that the supermartingale S is a true martingale. It is crucial to have this simple, state-independent form of h to evaluate the expectation; in general one is confronted with a parabolic problem (cf. Chapter 15 in (Friedman, 1976)) as opposed to the elliptic problem we encountered in Theorem 1. Thus it is not immediately clear in general whether the process is a true martingale.

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[11] For details, see equation (1.17) and Theorem 1.1 on p. 311 of Friedman (1976).

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