II. Non-linear development

As Δ in a given place approaches unity, we can no longer use linear perturbations around a homogeneous Universe. We must use either

- Numerical simulations (easy due to only needing gravitational physics)
- Simple analytic approximations of idealized situations to understand the main physical effects

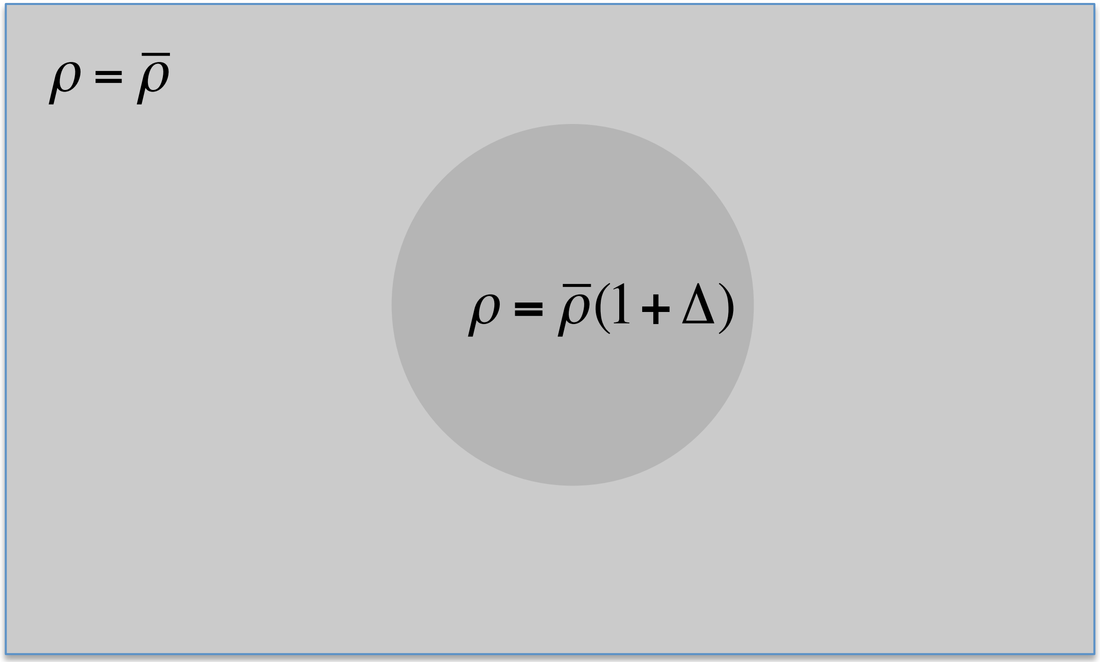
We will cover here two aspects of this:

- The analytic evolution of an idealized spherical "top-hat" over-dense region through to collapse.
- The analytic "Press-Schechter" calculation of the numbers of collapsed objects, i.e. the so-called "mass-function" $\phi(m,t)$: how many objects of mass *m* (in interval *dm*) are there at any given time

From Gauss' theorem, the evolution of a uniform over-dense sphere embedded in a uniform medium (assumed to have Ω_m = 1) will follow the solutions to the Friedmann equation of $\Omega > 1$ Universe.   This has a parameteric solution in terms of a parameter θ as follows:

$$r' = \frac{4\pi G\left[\rho' r'^3\right]}{3c^2}\left(1 - C_k(\theta)\right)$$

$$\tau = \frac{4\pi G\left[\rho' r'^3\right]}{3c^3}\left(\theta - S_k(\theta)\right)$$



Note that "turn-around" occurs at $\theta = \pi$

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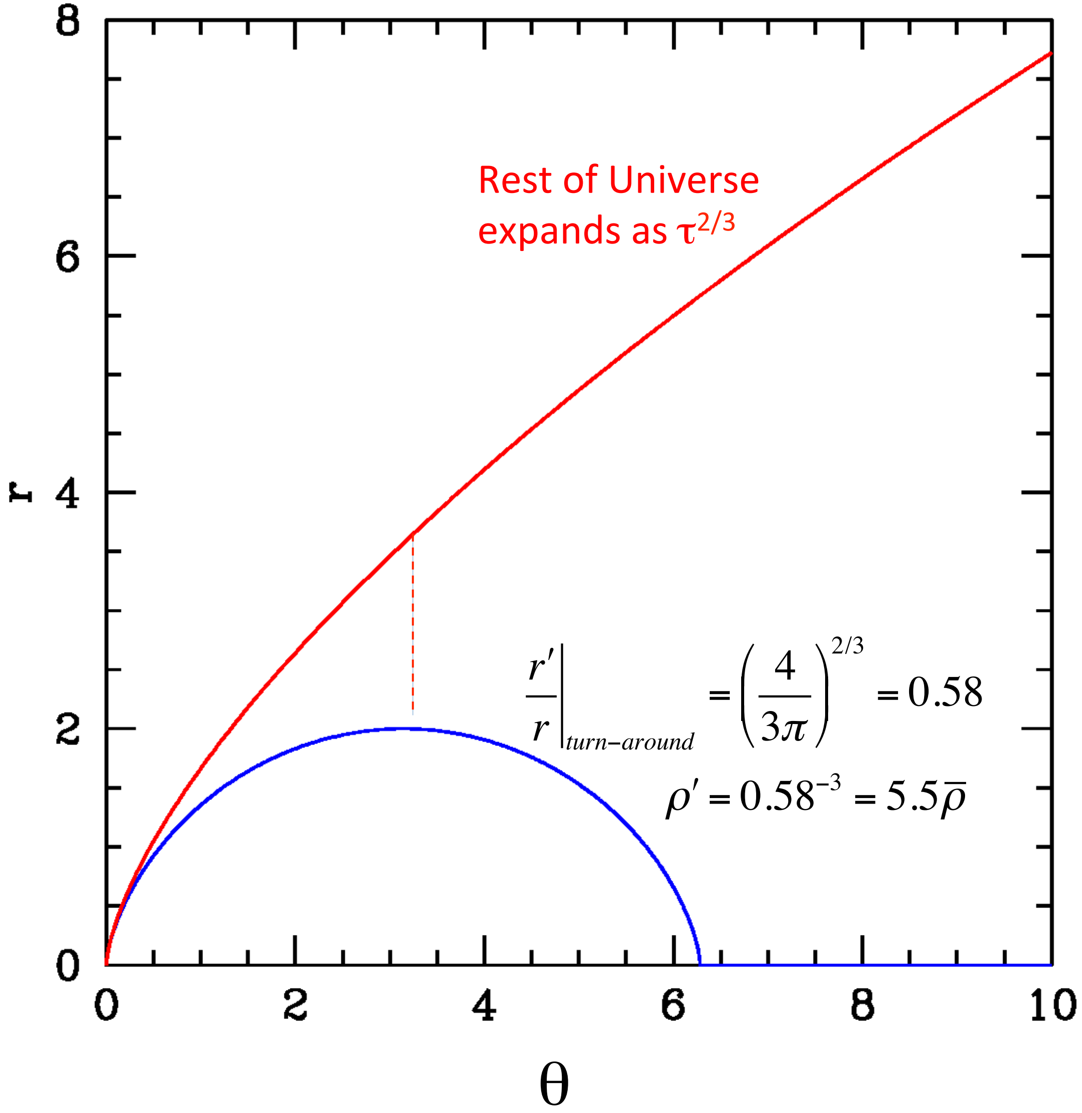
Turn-around at $\theta = \pi$



Note that already at turn-around, the density in the region is 5.5 times that of the surrounding Universe.

Virialization

All stable systems of particles moving under the influence of their own gravity (i.e. "self-gravitating") satisfy the virial condition between their total kinetic and total potential energies.

Virial condition: Kinetic Energy = −1/2 Potential Energy

For our density perturbation, collapse back to a point source will not happen in practice – structure will instead become stabilized ("virialized") through so-called "violent relaxation" that randomizes motions.

At turn-around:

$$\text{Kinetic Energy} = 0$$

$$\text{Potential Energy} = -\frac{GM^2}{r_{\max}}$$

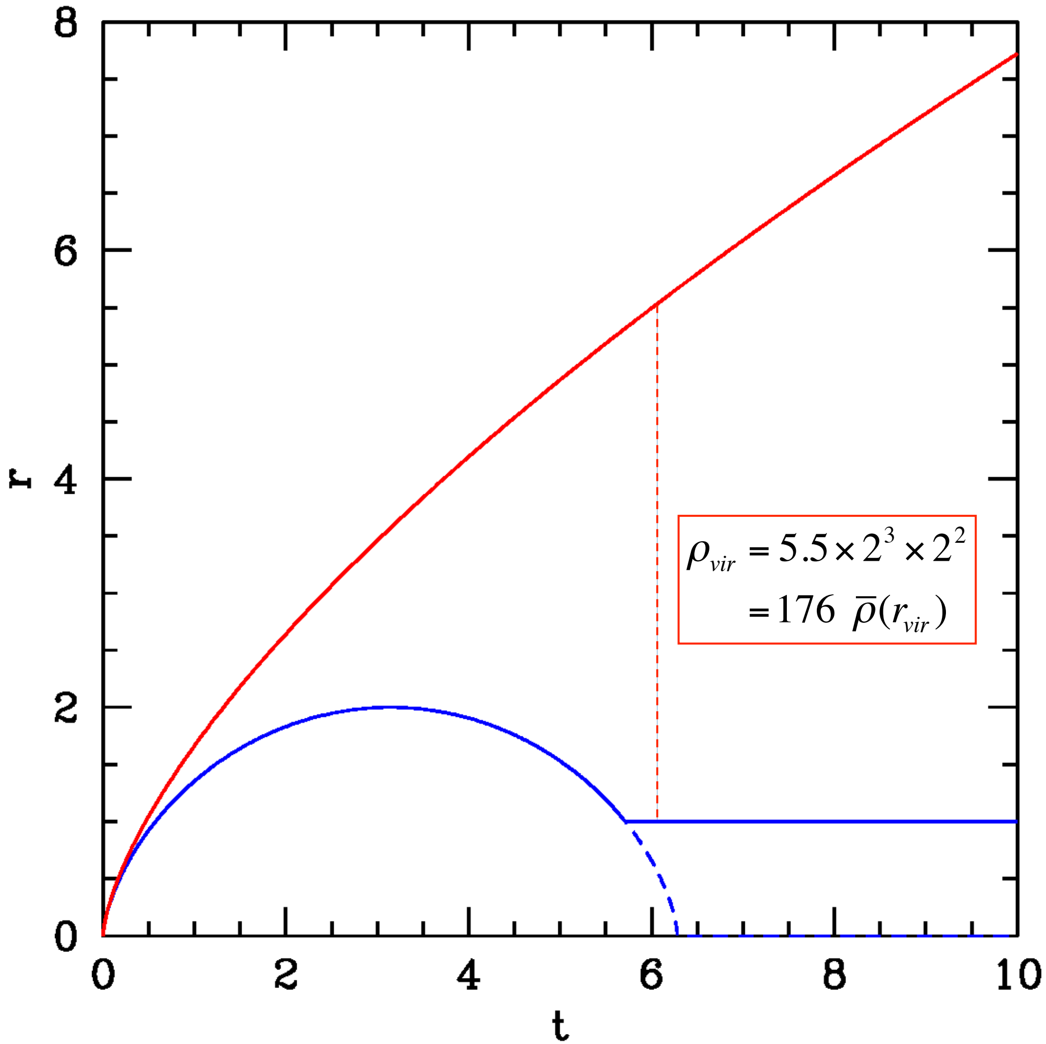
During subsequent collapse, converting P.E. to K.E.

$$\text{Potential Energy} = -\frac{GM^2}{r}$$

$$\text{Kinetic Energy} = GM^2\left(\frac{1}{r} - \frac{1}{r_{\max}}\right)$$

Virial condition is satisfied at $r_{vir} = 1/2\ r_{max}$, i.e. when structure has collapsed by factor of two from turnaround.

$$-\frac{1}{2}\frac{GM^2}{r} = GM^2\left(\frac{1}{r} - \frac{1}{r_{\max}}\right)$$



The density at virialization is eight times that at turn-around. By this time rest of Universe has decreased density by about $(2^{2/3})^3$

Density of the structure is therefore ~ 200 times that of rest of Universe when it virialises

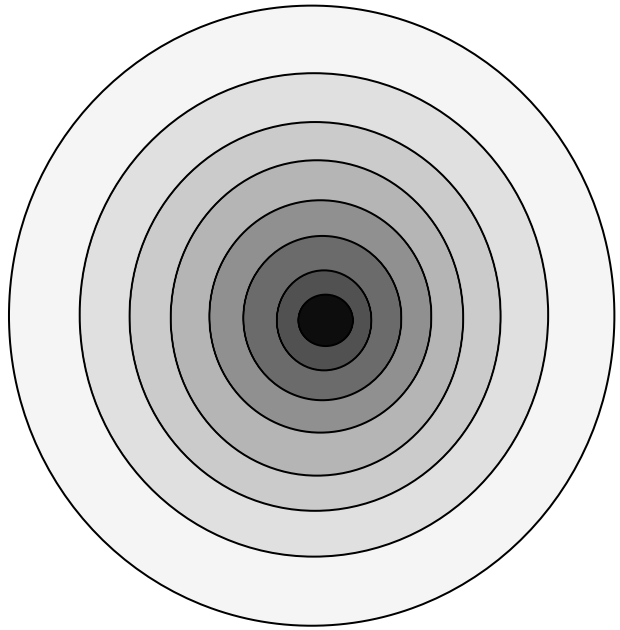
In principle, our idealized object would keep constant density after virialization, so the density of an object today tells us, <u>in principle</u>, when it formed

But, there are two effects that will cause continuing addition of material.

Even an idealized spherical over-density is unlikely to be uniform and will look more like an onion with over-density decreasing with radius.

Furthermore, for a spherically symmetric object, the idealized evolution of a spherical shell will not depend on the matter exterior to it, and will (only) depend on the mass (i.e. mean density) of the shells interior to it.

So, even a hypothetical point mass excess at the center will cause $\rho >$ average ρ around it.



The central virialized "object" will therefore grow in mass and the (virialization) radius of the object will also increase with time, so that the density within the virialization radius will always be ~ 200 times the average density of the Universe.

Important terminology: A <u>collapsed</u>, <u>self-gravitating</u> and <u>virialized</u> dark matter structure (or "object") is called a dark matter "halo".

The density distribution within a dark matter halo

The density distribution ρ(r) within a collapsed virialized dark matter object will reflect two things:

- The growth history of the halo: the earlier arriving material will have virialized at a higher density, because the density of the Universe was higher.
- The process of violent relaxation that redistributes PE and KE amongst the material subject to the virial condition.

Numerical *simulations* suggest that the density profile in dark matter haloes is well represented by the following analytic fitting formula, the so-called **Navarro-Frenk-White (NFW) profile** (there are other very similar choices)

NFW

$$\rho(r) = \frac{\rho_0}{\frac{r}{R_s}\left(1+\frac{r}{R_s}\right)^2}$$

The density varies as r^{-1} in the center and r^{-3} at large r
The enclosed mass in this representation actually logarithmically diverges, so we will assume the halo ends at r_{vir}

Aside: There is observational evidence for flatter profile at center "core" - Why?

The transition radius R_s is related to r_{vir} by a "concentration" parameter, c, which is about 10 for Milky Way, $4 < c < 40$ for other haloes

$$r_{vir} = c\, R_s$$

Total mass interior to r_{vir} given by

$$M = 4\pi\rho_0 R_s^3\left[\ln(1+c) - \frac{c}{1+c}\right]$$

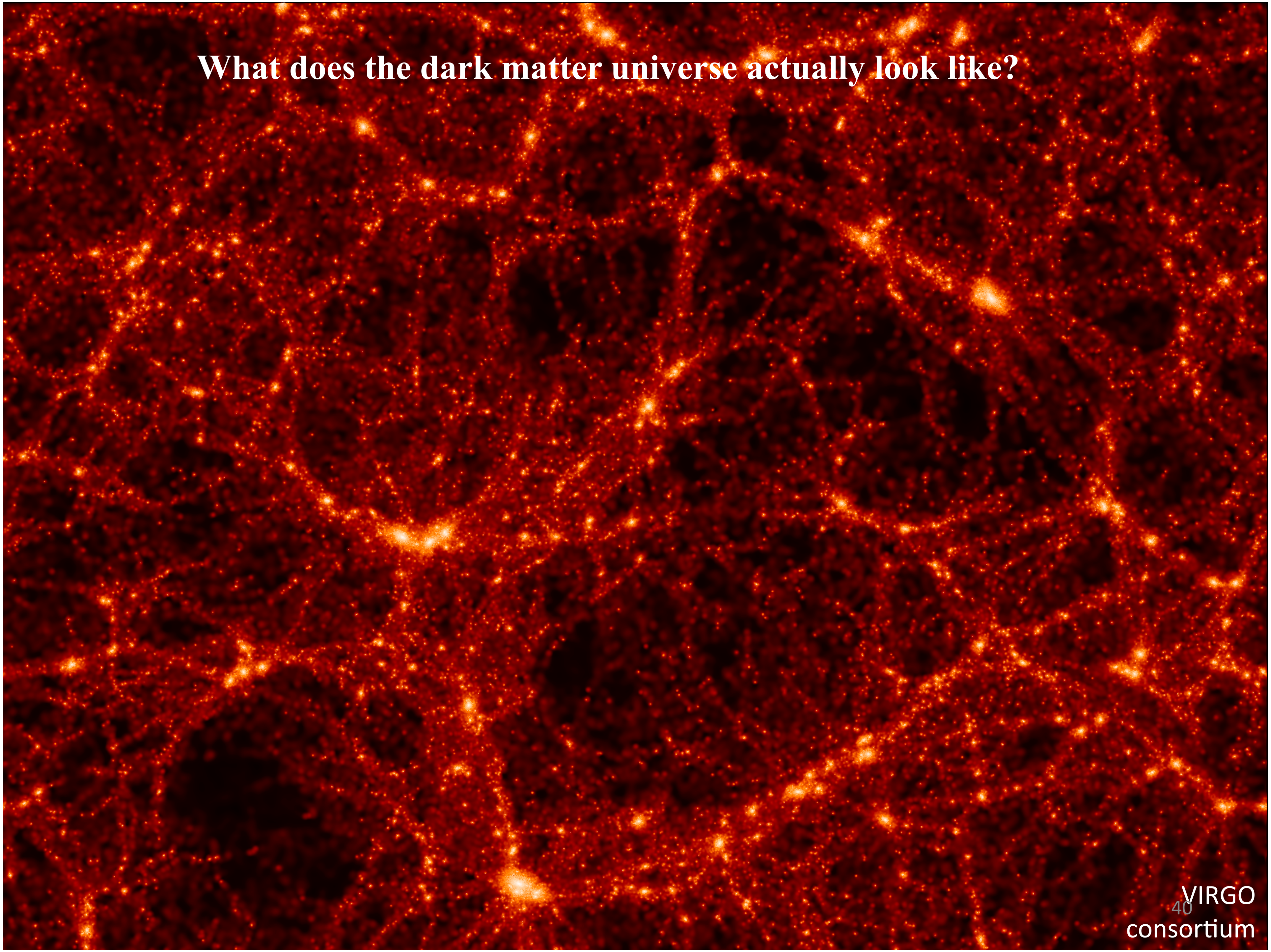
Important point: Assuming that Δ_M increases to small M, large haloes will have been built up by accreting smaller haloes which had already "formed" in the outer low density shells before accreting onto the main halo.

These pre-formed haloes may well survive virialization (at least partially) in the larger halo since their central densities will be much larger than the density in the outer parts of the larger halo. These surviving structures within haloes are called ***sub-haloes.***

At least to first order, the amount of sub-structure should be roughly self-similar, i.e. independent of halo mass.

Another aside: There is not much observational evidence in support of this much substructure - Why?

What does the dark matter universe actually look like?



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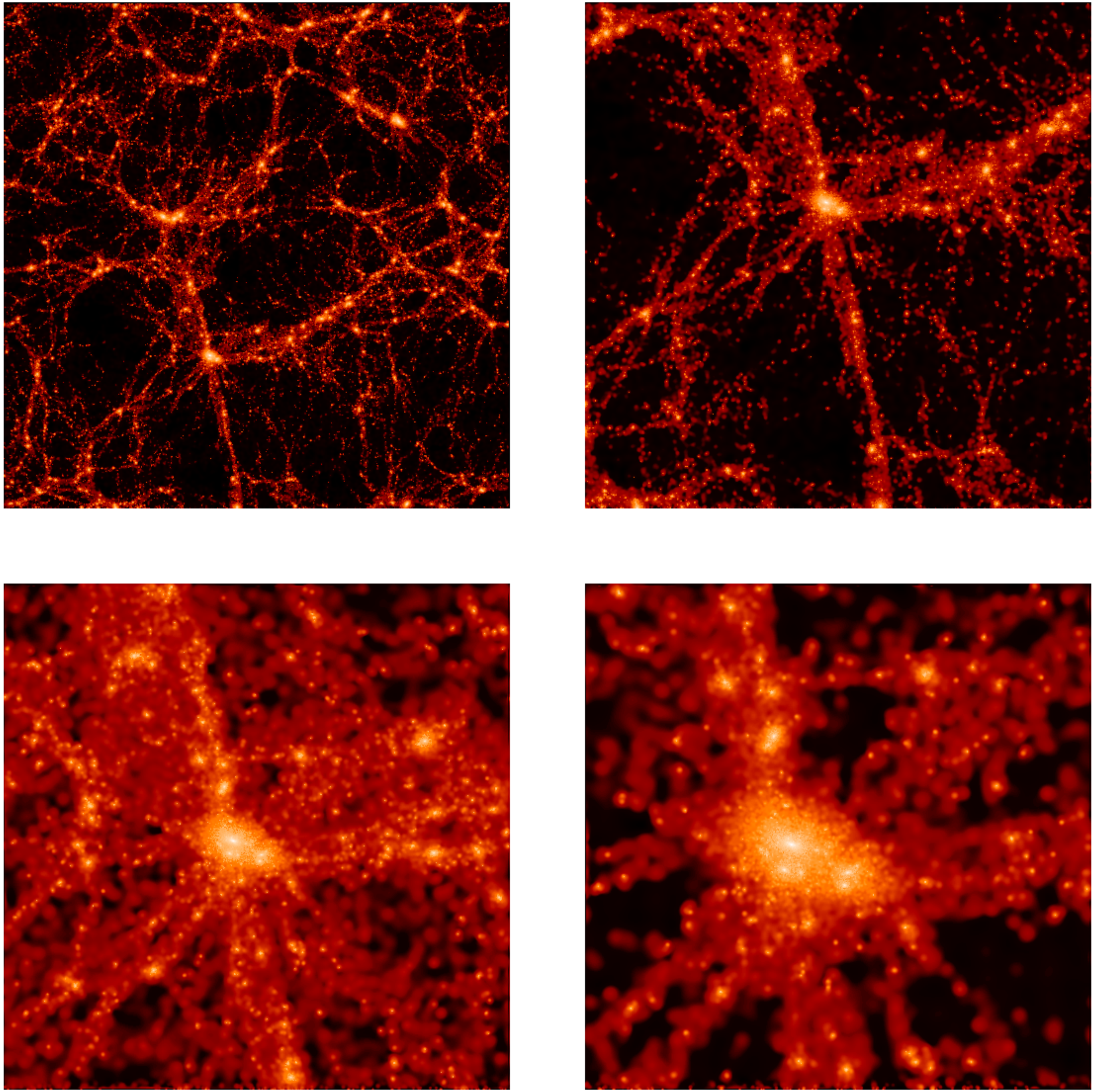
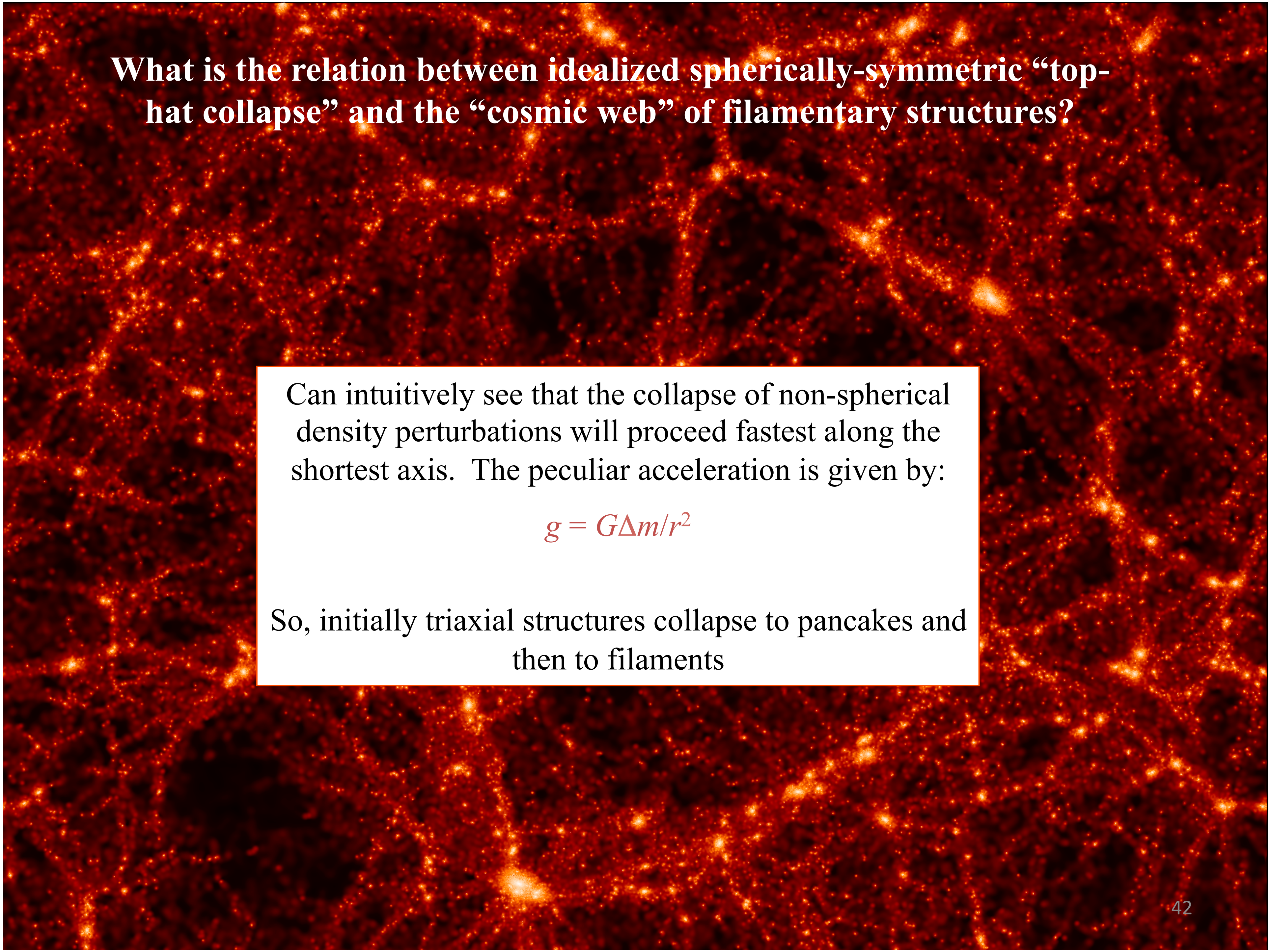
consortium





What is the relation between idealized spherically-symmetric "top-hat collapse" and the "cosmic web" of filamentary structures?

Can intuitively see that the collapse of non-spherical density perturbations will proceed fastest along the shortest axis. The peculiar acceleration is given by:

$$g = G\Delta m/r^2$$

So, initially triaxial structures collapse to pancakes and then to filaments

The number density of collapsed objects as *f*(mass,epoch):

First, I'd like to clarify the concept of the "mass function", $\phi(m)$. This is a conventional distribution function that describes the properties of a population:

In this case: "how many objects are there between mass m and $m+dm$ in unit volume"

$$dN = \phi(m)\, dm\, dV$$

The units are therefore per mass per volume, e.g. $M_\odot^{-1}$ Mpc^{-3}

We are dealing with a huge range of mass, e.g. at least 10^6-$10^{15} M_\odot$, so I will probably want to plot the mass and $\phi(m)$ on logarithmic axes (also makes it easier to see self-similarities etc.).

It then also makes sense to consider an increment in logarithmic mass. The units now become e.g. dex^{-1} Mpc^{-3}

$$dN = \phi'(\log m)\, d\log m\, dV$$

Since $dm = m\, d\ln m = 2.30\, d\log m$, it is easy to see that, if $\phi(m)$ is a power-law in m, then $\phi'(m)$ has an index differing by unity.

Whether the mass function is expressed per mass or per log mass interval is obvious from the units (and my pedantic use here of ϕ and ϕ' is not needed).

The Press-Schechter formalism

We had before: the probability distribution of density fluctuations when sampled on a mass scale *M* is gaussian

$$\boxed{P(\Delta)d\Delta = \frac{1}{\sqrt{2\pi}\,\sigma(M)} \exp\left(-\frac{\Delta^2}{2\,\sigma^2(M)}\right) \quad d\Delta}$$

Fraction of sampling spheres on this mass scale M (= fraction of mass in the Universe) that exceed a critical threshold for collapse is therefore:

$$F(M) = \int_{\Delta_c}^{\infty} P(\Delta)\,d\Delta$$

What is the appropriate threshold Δ_c? It is analytically convenient to choose the Δ that *would have been obtained* from purely linear growth at the time of (non-linear) virialization, so we can use the relations of linear growth.

$$\boxed{\Delta_c = 1.69}$$

F(M) is then given by a standard probability distribution function parameterized by t_c in terms of σ(M) and Δ_c.

$$F(M) = \frac{1}{2}(1 - \Phi(t_c))$$

with $t_c = \dfrac{\Delta_c}{\sqrt{2}\sigma(M)}$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\,dx$$

From earlier, we get the $\sigma(M)$ from the power-law fluctuation spectrum (valid for small range of k).

$$\sigma^2(M) = \left[k^3 |\Delta_k|^2\right] = AM^{-\frac{3+n}{3}}$$

The factor A (proportional to Δ^2) will incorporate the growth of density fluctuations with time. The growth of density fluctuations (squared) in an $\Omega_m = 1$ Universe is given by

$$A \propto R^2 \propto \tau^{4/3}$$

Now we introduce a characteristic mass, M*, such that at this mass M*, we have (see previous slide) $t_c = 1$, so that t_c at other masses is given by:

$$t_c = \frac{\Delta_c}{\sqrt{2}A^{1/2}} M^{(3+n)/6} = \left(\frac{M}{M^*}\right)^{(3+n)/6}$$

This characteristic mass M* is therefore given in terms of A (which increases with time as structure Δ in the Universe grows, depending on Ω_m) and also on the power law index n.

$$M^* = (2A/\Delta_c^2)^{3/(3+n)}$$
$$\propto \tau^{4/(3+n)}$$

Note for $n \sim -2$ relevant for galactic scales, M* grows as τ^4 (assuming $\Omega_m \sim 1$)

We actually want to know the number of objects of mass M.

This will be given by

- the <u>derivative</u> of $F(M)$ why?
- multiplied by ρ/M why?

$$N(M)dM = -\frac{\rho}{M}\frac{\partial F}{\partial M}dM$$

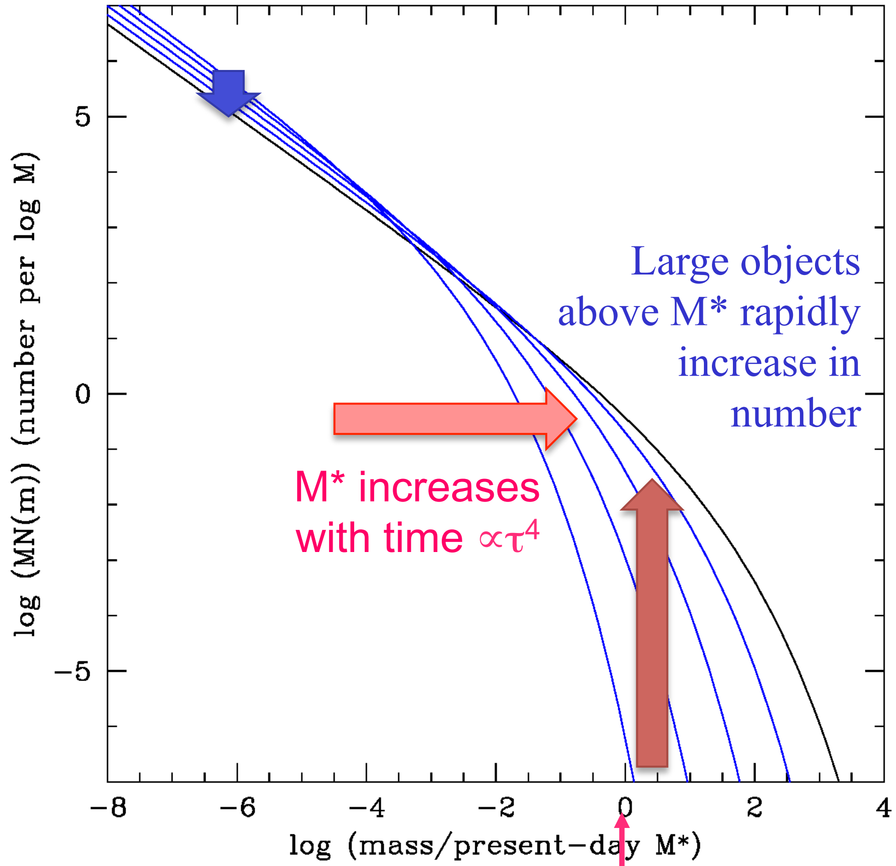
$$N(M)dM = \frac{1}{2\sqrt{\pi}}\left(1+\frac{n}{3}\right)\frac{\rho}{M^2}\left(\frac{M}{M*}\right)^{(3+n)/6}\exp\left[-\left(\frac{M}{M*}\right)^{(3+n)/3}\right]$$

For $n \sim -2$ (appropriate for the CDM spectrum on galactic scales) we then get:

$$\boxed{N(M)dM \propto M^{-11/6}M*^{-1/6}\exp\left[-\left(\frac{M}{M*}\right)^{1/3}\right] \quad \text{with } M* \propto \tau^4}$$

This is a power-law at low masses, with an exponential cut-off at high masses.

Small decrease below M* with time as some small objects are destroyed through merging



Present-day M* is about $10^{13.5}$ $M_\odot$, c.f. Milky Way halo ~ 10^{12} $M_\odot$

This power-law + exponential cut-off is called a Schechter function (at least by astronomers)

$$\boxed{N(M)dM \propto \phi^* \left(\frac{M}{M^*}\right)^{\alpha} \exp\left[-\left(\frac{M}{M^*}\right)\right]}$$

It is characterized by:

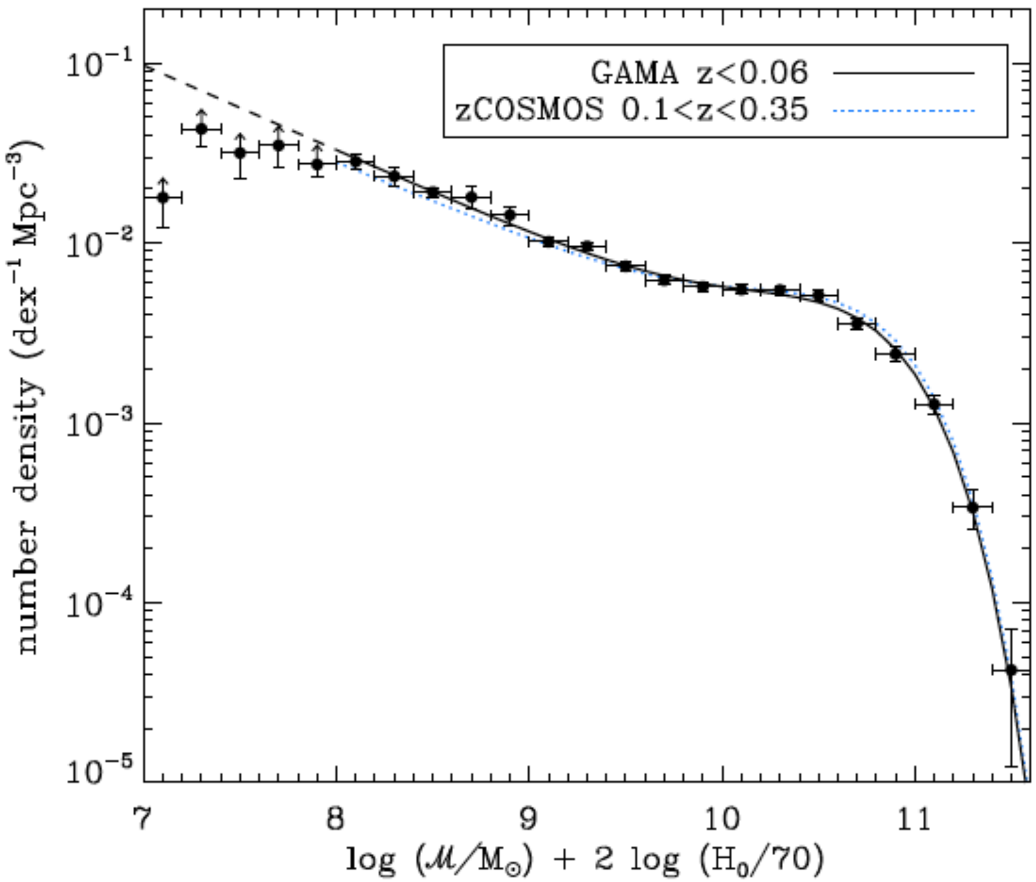
- M*, the mass of the exponential break from the power-law
- α the "faint end slope" of the power law
- φ* the density normalization

Intriguingly, it had been introduced by Schechter (1974) to empirically describe the galaxy luminosity function.

First thought: So, let's compare with the galaxian stellar mass function

This indeed looks quite like a Schechter function, but four things are in fact a bit wrong:

- It's got a bit of an extra bump around M_{star}*
- The faint end slope is a bit shallow, $\alpha \sim -1.5$, c.f. -11/6 for DM mass function
- M_{star}* is rather low: $M_{star}^* \sim 10^{11}\ M_\odot$ in stars, which is about 0.03 M_{DM}*, whereas we have a cosmic ratio of baryons to dark matter of $\Omega_B/\Omega_{CDM} \sim 0.2$.
- M_{star}* seems not to change significantly with time, whereas M_{DM}* increases rapidly.



There are no $10^{12}\ M_\odot$ super-galaxies in the center of $10^{13.5}\ M_\odot$ M* dark matter haloes.

Example: here we are plotting log {ϕ'(log m)}

Important points (Part III)

- Non-linear collapse produces self-gravitating, gravitationally bound objects, called dark matter haloes.

- The density after collapse (i.e. after virialization) is related to the average density of the Universe, such that $\rho > 200$ times the average density of the Universe.

- The density profile of a halo has a particular form (e.g. NFW) that reflects, at some level, the mass accretion history of the halo.

- Haloes likely grow by accreting other, previously formed, haloes, which subsequently become long-lived "sub-haloes" within the larger halo.

- The number density of haloes can be calculated using the Press-Schechter approach, and results in a so-called Schechter function (power-law plus exponential fall-off at high masses) whose characteristic mass M^* increases rapidly with time. This has some superficial similarities to the galaxy (stellar mass) mass function.