HANLE-ZEEMAN REDISTRIBUTION MATRIX. III. SOLUTION OF THE POLARIZED LINE FORMATION PROBLEM

M. Sampoorna and K. N. Nagendra
Indian Institute of Astrophysics, Koramangala, Bangalore 560 034, India

AND

J. O. Stenflo
Institute of Astronomy, ETH Zürich, CH-8092 Zürich, Switzerland

Received 2007 October 8; accepted 2008 February 12

ABSTRACT

The polarized radiative transfer equation is solved numerically, taking into account both the Zeeman absorption matrix and the Hanle-Zeeman redistribution matrix, to obtain line profiles for arbitrary magnetic field strengths, partial frequency redistribution, and scattering-dominated line transitions. The limiting cases of weak-field Hanle scattering and strong-field Zeeman true absorption are retrieved. The intermediate regime, where both Zeeman absorption and scattering effects are important, is studied in some detail. The numerical method is applied to various test cases to illustrate aspects of partial frequency redistribution on line scattering in magnetic fields of arbitrary strength and direction.

Subject headings: line: formation — magnetic fields — methods: numerical — polarization — radiative transfer — scattering — Sun: atmosphere

1. INTRODUCTION

The linearly polarized spectrum of the Sun, known as the “second solar spectrum,” contains a wealth of information about the physics of light scattering on atoms. The recent series of “Solar Polarization Workshops” (and their proceedings: Stenflo & Nagendra 1996; Nagendra & Stenflo 1999; Trujillo Bueno & Sánchez Almeida 2003; Casini & Lites 2006; Berdyugina et al. 2008) and important polarimetry workshop proceedings (Sigwarth 2001; Trujillo Bueno et al. 2002) highlight the growing body of research directed toward a deeper understanding of the physical processes used to diagnose solar magnetic fields. The solution of the polarized line transfer equation is used to interpret the shapes of observed Stokes line profiles. The physics of anisotropic absorption and scattering has developed hand in hand with sophisticated numerical methods to solve these complex radiative transfer equations.

Reviews on the methods of solving the polarized transfer equation (Trujillo Bueno 2003; Nagendra 2003a, 2003b; Nagendra & Sampoorna 2008) list several exact and approximate numerical methods. Novel iterative schemes have been developed by Trujillo Bueno and coworkers (see Trujillo Bueno 2003 and references therein to their previous works) to solve the polarized nonlocal thermodynamic equilibrium (NLTE) radiative transfer equation together with the statistical equilibrium equation in multilevel atoms, including multidimensional geometries. Here we limit our attention to the vector version of the standard two-level atom NLTE line transfer equation (Mihalas 1978; Stenflo 1994). The new aspect of our treatment is that it allows us to solve the line transfer problem for the general case of arbitrary magnetic fields with a full account of partial frequency redistribution (PRD).

The solar atmosphere with its magnetically active regions harbors a whole range of field strengths (milligauss to kilogauss fields). Hence, a general form of the radiative transfer equation that is valid in this entire range has to be formulated and tested. Such a transfer equation has been formulated by Stenflo (1994; see also § 2 of the present paper). It is also briefly discussed in the review of Trujillo Bueno (2003).

Polarized PRD scattering in the presence of magnetic fields of arbitrary strength and direction is called Hanle-Zeeman scattering. We use a perturbation approach to solve the Hanle-Zeeman line transfer problem. Perturbation methods have often been applied in polarized line transfer (Rees 1978; Rees & Murphy 1987; Faurobert 1987; Nagendra et al. 2002). It is a practical approach which works for any type of scattering redistribution matrix in the polarized source vector.

Efficient numerical methods to solve the polarized radiative transfer equation in the presence of magnetic fields have been developed in the past. In the weak-field limit of the Hanle effect, techniques like polarized approximate lambda iteration (PALI) taking into account both complete frequency redistribution (Nagendra et al. 1998) and PRD (Nagendra et al. 1999; Fluri et al. 2003; Sampoorna et al. 2008a) have been developed. However, in the weak-field limit a scalar isotropic absorption is often assumed, even when the magnetic field is nonzero. At the other extreme, for the strong-field Zeeman effect, techniques like the Feautrier method (Auer et al. 1977), the diagonal element lambda operator (DELO) method (Rees et al. 1989), and DELOPAR (Trujillo Bueno 2003) have been developed. In this strong-field limit, one assumes local thermodynamic equilibrium (LTE), where scattering effects are either neglected or are taken into account through a complete redistribution (CRD) line source function, which is then solved using the polarization-free approximation of Trujillo Bueno & Landi Degl’Innocenti (1996). However, scattering included in this way does not account for coherent scattering in the presence of strong magnetic fields.

In a correct treatment of NLTE line formation in the presence of a magnetic field of arbitrary strength, one has to take into account both the Zeeman absorption matrix and PRD effects. Such a treatment requires a redistribution matrix which is valid in all field strength regimes. Clearly, the weak-field Hanle phase matrix...
(Landi Degl’Innocenti & Landi Degl’Innocenti 1988; Stenflo 1994) does not suit these requirements. Quantum electrodynamical (QED) and classical theories of PRD line scattering in the presence of magnetic fields of arbitrary strength and including collisions have been formulated, respectively, by Bommpier (1997a, 1997b) and Bommpier & Stenflo (1999). The classical theory describes the scattering process in a transparent way. In Bommpier & Stenflo (1999), the redistribution matrices have been derived in the atomic rest frame. The corresponding laboratory frame redistribution matrices are given by Sampoorna et al. (2007a, hereafter Paper I). The equivalence between the classical (Paper I) and QED approach of Bommpier (1997b) for the Zeeman triplet case has been established in Sampoorna et al. (2007b, hereafter Paper II; see also Sampoorna et al. 2008b). Following Stenflo (1998) we refer to this as the Hanle-Zeeman redistribution matrix, since it covers the entire field strength regime. In the present paper we incorporate the Hanle-Zeeman redistribution matrix of Paper I and Paper II into the polarized line transfer equation.

In § 2 we present the governing equations, a description of the general line radiative transfer equation, and its various limiting forms. In § 3 we introduce a numerical method to solve the polarized transfer equation including both the Zeeman absorption matrix and the Hanle-Zeeman redistribution matrix. Section 4 is devoted to results and discussions, and § 5 to the concluding remarks.

2. GOVERNING EQUATIONS

In the formulation that we are using here, the full Zeeman absorption matrix and the Hanle-Zeeman redistribution matrix (see Paper I, Paper II) are simultaneously and self-consistently incorporated (assuming an unpolarized ground state). We remark that the Hanle-Zeeman line transfer theory presented in this paper is valid for a normal Zeeman triplet ($J = 0 \rightarrow 1 \rightarrow 0$) case only.

2.1. The Radiative Transfer Equation

In the presence of a magnetic field, the vector radiative transfer equation for the Stokes vector $I(\tau, x, n) = (I, Q, U, V)^T$ may be written as (see Stenflo 1994)

$$\mu \frac{\partial}{\partial \tau} I(\tau, x, n) = K I(\tau, x, n) - S(\tau, x, n),$$

(1)

where the source vector $S(\tau, x, n) = j(\tau, x, n)/k_L$, with $\tau$ being the line-center optical depth and $x$ being the frequency separation from line center in Doppler width units. The vector $n(\phi, \varphi)$ is the propagation direction of the ray (where $\phi$ is the colatitude and $\varphi$ the azimuth), and $\mu = \cos \phi$. The quantity $k_L = (N_l B_{ul} - N_u B_{ul}) h \nu_0 / 4 \pi$ is the line-center opacity, where $N_l$ and $N_u$ are the populations of lower and upper states, respectively, $B_{ul}$ and $B_{ul}$ are the Einstein $B$ coefficients, $h$ is the Planck constant, and $\nu_0$ is the line-center frequency. Positive optical depth is measured in the direction opposite to the vertical $Z$-axis (see Fig. 1). We use the convention that the positive $Q$-direction corresponds to the linear polarization perpendicular to the solar limb.

In equation (1), $K$ is the total absorption matrix

$$K = \Phi + r \mathbf{E},$$

(2)

where $\Phi$ is the $4 \times 4$ Zeeman line absorption matrix, $\mathbf{E}$ is the $4 \times 4$ unity matrix, and $r$ is the ratio of continuous to line-center opacity. The total emission vector $j(\tau, x, n)$ has contributions from both line and continuum,

$$j(\tau, x, n) = j_L(\tau, x, n) + j_0(\tau, x, n).$$

(3)

Assuming unpolarized continuum with emission occurring in LTE, $j_0(\tau, x, n) = r k_L B_{ul} U$, where $U = (1, 0, 0, 0)^T$ and $B_{ul}$ is the Planck function. The line emission vector has contributions from coherent scattering (undisturbed by collisions), incoherent scattering (collisionally disturbed), and nonscattering emission (see Stenflo 1994 for details) and may be written as

$$j_L(\tau, x, n) = j_{\text{scat}}(\tau, x, n) + j_0(\tau, x, n).$$

(4)

2.2. Scattering Emission Vector

The scattering emission vector $j_{\text{scat}}$ has contributions from both coherent and incoherent scattering. The term coherent scattering refers to all scattering processes that are purely due to radiative excitation, followed by spontaneous emission which is undisturbed by collisions. The term incoherent scattering refers to all those scattering processes that are disturbed by collisions. The scattering emission vector for this case may be written as

$$j_{\text{scat}}(\tau, x, n) = k_L \int_0^{+\infty} d\nu' \mathbf{R}(x, n; x', n'; B) I(\tau, x', n').$$

(5)

The redistribution matrix $\mathbf{R}(x, n; x', n'; B)$ accounts for the correlations in frequency, angle, and polarization between the incident radiation field at frequency $x'$ and direction $n'$ and the reemitted radiation at frequency $x$ and direction $n$. The quantity $d\nu'$ is an element of solid angle around $n'$. For lines formed with PRD (see Paper I, Paper II)

$$\mathbf{R}(x, n; x', n'; B) = \mathbf{R}^H(x, n; x', n'; B) + \mathbf{R}^{III}(x, n; x', n'; B),$$

(6)

where

$$\mathbf{R}^H(x, n; x', n'; B) = A \mathbf{M}^{II},$$

(7)

$$\mathbf{R}^{III} = \mathbf{B}^{(0)} \mathbf{M}^{III} + \mathbf{B}^{(1)} \mathbf{M}^{III} + \mathbf{B}^{(2)} \mathbf{M}^{III}.$$  

(8)
The branching ratio $A$ for coherent scattering is

$$A = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + \Gamma_E},$$

(9)

where $\Gamma_R$ is the radiative rate, while $\Gamma_I$ and $\Gamma_E$ are inelastic and elastic collision rates, respectively. The branching ratio $B^{(k)}$ which represents the fraction of the scattering processes for which the atom is subject to elastic collisions that destroy the frequency coherence but not the atomic polarization (the $2K$-multipole) is given by

$$B^{(k)} = \frac{\Gamma_E - D^{(k)}}{\Gamma_R + \Gamma_I + \Gamma_E} \frac{\Gamma_R}{\Gamma_R + \Gamma_I + D^{(k)}},$$

(10)

where $D^{(k)}$ is the rate of destruction of the $2K$-multipole, with $K = 0, 1, 2$ (note that $D^{(0)} = 0$). The explicit forms of $M_{II}$ and $M_{III}^{(k)}$ in a frame where the $Z$-axis is along the magnetic field rather than along the normal to the atmosphere are presented in Paper I (see eq. [46]) and Paper II (see eqs. [17]–[19]), respectively. The required transformation to a frame in which the magnetic field has an arbitrary orientation with respect to the normal to the atmosphere is described in Appendix D of Paper II.

When the atom is in the excited state, collisions cause shifts of the emitted frequency, change of phase, destruction of alignment, orientation, etc., depending on the nature of the collisions (quenching, elastic, inelastic). All these processes are contained in a general redistribution matrix (see eqs. [6]–[10]), through a combination of collisional branching ratios and magnetically modified redistribution functions (type II and III of Hummer 1962), which we have called “magnetic redistribution functions” (see Paper I). The theoretical study and applications of such collisional redistributions for the nonmagnetic case have been described by Domke & Hubeny (1988), Faurobert-Scholl (1992, 1993), Nagendra (1994), and Stenflo (1994) and for the arbitrary field strength by Bomnier (1997a, 1997b) and Bomnier & Stenflo (1999).

2.3. Nonscattering Emission Vector

We refer to a process as nonscattering if the atom has no memory of how it has been excited when it emits the photon. In other words, the upper state has been populated by a process different from radiative absorption from the lower state. If the upper state is populated by isotropic collisions, then we get purely thermal emission. If radiative transitions are involved (other than from our initial atomic state) which contribute to populating the excited state, then since they are unspecified, we have assumed that the memory of the population process has been erased. In this case we also get thermal emission as in the collisionally excited case (see the discussion in § 5.3 of Stenflo 1998). Therefore, we can lump all these unspecified processes together, since they all contribute with the same form of the emission vector. The nonscattering emission vector is given by (see Stenflo 1994)

$$j_{\text{nc}}(\tau, x, n) = k_1 I_{\text{nc}} \Phi U,$$

(11)

where $\epsilon = \Gamma_I/(\Gamma_I + \Gamma_R)$ denotes the photon destruction probability per scattering.

2.4. The Hanle-Zeeman Line Transfer Equation

Combining equations (2)–(5) and (11), we can rewrite the polarized transfer equation (1) as

$$\mu \frac{\partial}{\partial \tau} I(\tau, x, n) = (\Phi + eE) I(\tau, x, n) - \left[ (rE + e\Phi) B_{\nu_0} U + S_{\text{scat}}(\tau, x, n) \right],$$

(12)

where

$$S_{\text{scat}}(\tau, x, n) = \int \frac{dn'}{4\pi} \int_0^+ dx' R(x, n; x', n'; B) I(\tau, x', n').$$

(13)

Note that the factor $(1 - \epsilon)$ is implicitly contained in $R$ through the branching ratios. We refer to equation (12) as the Hanle-Zeeman line transfer equation, as it includes both the Zeeman absorption matrix ($\Phi$) and the Hanle-Zeeman redistribution matrix ($R$).

Three limiting forms of equation (12) may be considered.

True absorption or LTE.—We may set $\epsilon = 1$ and thereby recover the LTE Zeeman line transfer equation, for which rapid solutions are available.

NLTE strong-field limit.—In the presence of strong fields, if we neglect the coherent scattering (PRD) contribution and retain only completely incoherent scattering, then equation (12) takes the form given in Trujillo Bueno & Landi Degl’Innocenti (1996). In this case, scattering effects are taken into account through a CRD-like line source function, and the radiative transfer equation is solved by using the polarization-free approximation of Trujillo Bueno & Landi Degl’Innocenti (1996).

NLTE weak-field limit.—In the weak-field limit, the line absorption matrix can be approximated by a diagonal matrix with a single absorption coefficient profile $\phi(x)$, and the redistribution matrix can be approximated by a scalar redistribution function (CRD or PRD) times a weak-field frequency-independent Hanle phase matrix. This is the conventionally employed Hanle scattering line transfer problem that has been solved in various papers in the past (e.g., Faurobert-Scholl 1991; Nagendra et al. 1998, 1999).

Bomnier (1997b) proposed three kinds of approximations (I, II, III, in order of decreasing complexity) for weak-field Hanle scattering. In approximation I, the redistribution functions are not factorized and retain their couplings to the angular phase matrix. Approximations II and III are based on asymptotic expansion of the generalized profiles (see eqs. [86] and [87] of Bomnier 1997b). These expansions in the line center and in the line wings lead to a decomposition of frequency space into several domains. In each domain the frequency redistribution function is decoupled from the angular phase matrix. Approximation II uses the angle-dependent (AD) redistribution functions, and approximation III uses the corresponding angle-averaged (AA) functions. Approximations II and III have been applied and tested in line transfer calculations by Nagendra et al. (2002) using a computationally expensive perturbation approach. Approximation III was later integrated into a high-speed PALI method by Fluri et al. (2003). Both these papers use a domain-based treatment of polarized PRD. Recently, Sampoorna et al. (2008a) have developed a PALI method based on a non-domain-based treatment of PRD, namely, adopting the AA version of approximation I of Bomnier (1997b). The present paper deals with the more general problem that fully includes the
Zeeman absorption matrix when solving the full Hanle-Zeeman scattering problem with PRD. We use a perturbation technique similar to the one proposed by Nagendra et al. (2002) with appropriate generalizations.

3. NUMERICAL METHOD OF SOLUTION

Here we present a perturbation method to solve the Hanle-Zeeman PRD line transfer problem. There are two stages in the perturbation method that we use. In stage 1 we solve the Hanle-Zeeman scattering problem neglecting Zeeman absorption (see § 3.1). In stage 2 the full problem including both Zeeman absorption and the Hanle-Zeeman redistribution matrix is solved (see § 3.2).

3.1. Stage 1: Initializing $S_{\text{scat}}(\tau, x, n)$

If we replace $\Phi$ by $\phi(x)$ in equation (12), we obtain the transfer equation traditionally used for solving the weak-field Hanle effect (e.g., Faurobert-Scholl 1991; Nagendra et al. 1998, 1999, 2002),

$$
\frac{\partial}{\partial \tau} I(\tau, x, n) = [\phi(x) + r] I(\tau, x, n) - \{[r + c(\phi(x))] B_{\nu} U + S_{\text{scat}}(\tau, x, n)\}. \tag{14}
$$

However, the difference is that we now use the newly derived Hanle-Zeeman redistribution matrix instead of the weak-field Hanle redistribution matrix.

Equation (14) is solved using a perturbation technique presented in Nagendra et al. (2002). We first calculate the scalar PRD intensity using a scalar approximate lambda iteration (ALI) method, which is used to compute a first estimate $S_{\text{scat}}(\tau, x, n) = [S(\tau, x), 0, 0, 0]^T$. In the second step, the polarized transfer equation is solved using a short characteristic formal solver of Auer & Paletou (1994) for which $S_{\text{scat}}$ is used as input. In this step we use the AA version of the magnetic redistribution functions (see § 3.5 for details). This procedure is repeated until the relative variations of the polarization rate at the surface between two successive perturbations is less than 1% (see § 3.4 for details). The converged source vector $S_{\text{pol, AA}}$ becomes the input to step 3. The third step is exactly the same as step 2, except for the use of AD magnetic redistribution functions (see eqs. [21]–[26] of Paper I) in the explicit computation of the scattering integral. In this way we obtain a good initial estimate of $S_{\text{scat}}$, which we call $S_{\text{pol, AD}}$.

3.2. Stage 2: Full Solution

The Hanle-Zeeman line transfer equation (12) is now solved in three steps described below.

1. The initial values $S_{\text{pol, AD}}$ obtained from stage 1 are used as input to a formal solver, which finds the solution of equation (12). For this purpose we use DELOPAR introduced by Trujillo Bueno (2003; see § 3.3 for details).
2. The Stokes vector obtained from the formal solver is used to compute the new value of $S_{\text{scat}}$ from equation (13).
3. The perturbation sequence is tested for convergence (see § 3.4) and returned to step 1 if the convergence is not yet reached.

3.3. Formal Solution with DELOPAR

A numerical method for the formal solution of NLTE Zeeman line transfer equations was first given by Auer et al. (1977). It is a Feautrier-type method which involves writing the equations in second-order form and solving them by a finite-difference technique. However, the Feautrier method involves handling large matrices, and hence, it is slow. A much faster method was developed by Rees et al. (1989) which they call the DELO method. Trujillo Bueno (2003) clearly showed that the DELO method, as a formal solver, is less accurate when compared with his new formal solver, which is a generalization of the scalar short characteristic method of Kunasz & Auer (1988) to the polarized case. The reason for the inaccuracies of the DELO method when solving hard problems of NLTE polarization transfer is that it is based on a linear interpolation approximation to the source vector in optical depth segments.

Trujillo Bueno (2003) overcame this problem by developing an improved DELO method, which is based on using a parabolic interpolation for the total source vector. This method is called DELOPAR.

In the present work, we apply the DELOPAR method. Let us define a total optical depth $d\tau_{\text{tot}} = d\tau(\varphi + r)/\mu$, where $\varphi$ is the diagonal element of the Zeeman absorption matrix $\Phi$. For notational simplification we will here call $d\tau_{\text{tot}} = d\tau$. Equation (12) can be rewritten as

$$
\frac{\partial}{\partial \tau} I(\tau, x, n) = I(\tau, x, n) - S_{\text{eff}}(\tau, x, n). \tag{15}
$$

Here, the effective source vector is

$$
S_{\text{eff}}(\tau, x, n) = S_{\text{tot}}(\tau, x, n) - K' I(\tau, x, n), \tag{16}
$$

where we have redefined the total absorption matrix of equation (2) as

$$
K' = \frac{K}{(\varphi + r)} - E. \tag{17}
$$

The total source vector is defined as

$$
S_{\text{tot}}(\tau, x, n) = \frac{1}{(\varphi + r)}[\phi(x) B_{\nu} U + S_{\text{scat}}(\tau, x, n)]. \tag{18}
$$

On the interval $(\tau_k, \tau_{k+1})$, the solution of equation (15) is

$$
I_k = E_k I_{k+1} + \int_{\tau_k}^{\tau_{k+1}} e^{\gamma - 1} |S(t) - K'(t) I(t)| dt, \tag{19}
$$

where $I_1 = I(\tau_2)$ and $E_k = e^{-\delta_k}$, with $\delta_k = \tau_{k+1} - \tau_k$. Note that we have dropped the other two arguments of $I$ and $S$ and also the subscript on $S_{\text{tot}}$. Changing the variable ($p = t - \tau_k$), equation (19) can be rewritten as

$$
I_k = E_k I_{k+1} + \int_0^{\delta_k} e^{-p} |S(p) - K'(p) I(p)| dp. \tag{20}
$$

We assume that $S$ varies parabolically,

$$
S(p) = S_k + C_1 p + C_2 p^2, \tag{21}
$$

where the coefficients are

$$
C_1 = \frac{(S_{k+1} - S_k) \delta_{k+1} - (S_k - S_{k-1}) \delta_k}{\delta_k (\delta_k + \delta_{k-1})}, \quad C_2 = \frac{(S_{k+1} - S_k) \delta_{k+1} - (S_k - S_{k-1}) \delta_k}{\delta_k (\delta_k + \delta_{k-1})}. \tag{22}
$$

Further, we assume that $K' I$ varies linearly,

$$
K'(p) I(p) = K'_1 I_k + (K'_{k+1} I_{k+1} - K'_k I_k) \frac{p}{\delta_k}. \tag{23}
$$
If we insert equations (21) and (23) into equation (20), it can be integrated analytically. After some algebra, we obtain

\[ I_k = P_k + Q_k I_{k+1}, \quad (24) \]

where

\[ P_k = \left[ E + \left( F_k - \frac{G_k}{\delta k} \right) K'_k \right]^{-1} (F_k S_k + G_k C_1 + H_k C_2), \]
\[ Q_k = \left[ E + \left( F_k - \frac{G_k}{\delta k} \right) K'_k \right]^{-1} \left( E_k E - \frac{G_k}{\delta k} K'_{k+1} \right), \quad (25) \]
\[ F_k = 1 - E_k, \]
\[ G_k = 1 - (1 + \delta k) E_k, \]
\[ H_k = 2G_k - \delta_k^2 E_k. \quad (26) \]

At the lower boundary we specify a boundary condition (either 0 or 1). Then equation (24) can be applied recursively to derive the emergent Stokes vector at the surface.

### 3.4. Convergence Criteria

The formal solver DELOPAR computes the Stokes vector at all depth, frequency, and angle points, provided a known source vector is given. To initialize the perturbation loop, the starting values \( S_{\text{pol}, \text{AD}} \) for \( S_{\text{scat}} \) (see §3.1) are given as input to DELOPAR, which delivers new values of the Stokes vector. These new values are then used to compute a new \( S_{\text{scat}} \) using the defining equation (13).

This procedure is repeated until the maximum relative variations of polarization rate at the surface between two successive iterations is less than 1%. The maximum relative variations of polarization rate, denoted by \( P_{\text{surf}} \), is defined as

\[ P_{\text{surf}} = \max_{\lambda, \mu, \varphi} \left[ \frac{|P^{(n)}(x, \mu, \varphi) - P^{(n-1)}(x, \mu, \varphi)|}{P^{(n)}(x, \mu, \varphi)} \right], \quad (27) \]

where \( P^{(n)} = (Q^2 + U^2 + V^2)^{1/2} \) is the degree of polarization at the surface for the \( n \)th iteration and \( P^{(n)} = |P^{(n)} + P^{(n-1)}|/2 \).

The same convergence criteria is used for steps 2 and 3 of stage 1 of the perturbation method.

### 3.5. Computational Details

The memory and CPU time requirements of the perturbation method is discussed in detail in Nagendra et al. (2002). In the Hanle-Zeeman scattering theory, the redistribution matrix contains the magnetic redistribution functions (see eqs. [21]–[26] of Paper I). The type II and III magnetic redistribution functions are denoted as \( R_{I;X}^n(x, x', \Theta) \) and \( R_{II;XY}^n(x, x', \Theta) \), respectively, where \( X \) and \( Y \) stand for the symbols \( H \) and/or \( F \), \( \Theta \) is the scattering angle, and \( q, q' = 0, \pm 1 \). Clearly, there are six type II and 36 type III magnetic redistribution functions. In contrast, in the limit of weak-field domain-based Hanle scattering theory, we need to compute only one each of the scalar type II and III redistribution functions of Hummer (1962). Therefore, there is a much larger demand on the CPU time and memory in the Hanle-Zeeman case. For this reason, we restrict ourselves to slabs of optical depth 20 and 200 in this paper, for which a smaller frequency bandwidth is sufficient.

It is well known that \((I, Q, U)\) are symmetric about \(x = 0\) and \(V\) is antisymmetric about \(x = 0\). Thus, the frequency integral in equation (13) can be written as

\[
\int_{-\infty}^{+\infty} dx' \sum_{i=1}^{4} \sum_{j=1}^{3} R_{ij}(x, n; x', n'; B) Z_{ij}(x', n')
\]

\[
= \int_{0}^{+\infty} dx' \sum_{i=1}^{4} \sum_{j=1}^{3} [R_{ij}(x, n; +x', n'; B) + R_{ij}(x, n; -x', n'; B)] Z_{ij}(x', n'),
\]

for \( Z_{ij}(x', n') = I, Q, U \), when \( j = 1, 2, 3 \). The corresponding equation for \( V(x, n') \) is

\[
\int_{-\infty}^{+\infty} dx' \sum_{i=1}^{4} R_{a0}(x, n; x', n'; B) V(x', n')
\]

\[
= \int_{0}^{+\infty} dx' \sum_{i=1}^{4} [R_{a0}(x, n; +x', n'; B) - R_{a0}(x, n; -x', n'; B)] V(x', n').
\]

Thus, the magnetic redistribution functions entering individual elements of redistribution matrix \( R \) are now computed in the range \((0, x_{\text{max}})\) and \((-x_{\text{max}}, x_{\text{max}})\), which are then folded as shown in the square bracket of equations (28) and (29). This reduces the computing time by a factor of 2. Further, the memory is also reduced, as we store the concerned quantities over the range \((0, x_{\text{max}})\) and \((0, x_{\text{max}})\) only.

We recall that in the second step of stage 1 of the perturbation method (see §3.1), we use the AA version of the individual AD magnetic redistribution functions appearing in the Hanle-Zeeman redistribution matrix. We call this the AA approximation. The AA functions can be computed from the AD functions by numerically integrating over the scattering angle \( \Theta \) (as see also Bommier 1997b)

\[ R_{I;X}^n(x, x', \Theta) = \frac{1}{2} \int_{0}^{\pi} R_{II;X}^n(x, x', \Theta) \sin \Theta \, d\Theta, \quad (30) \]

\[ R_{II;XY}^n(x, x', \Theta) = \frac{1}{2} \int_{0}^{\pi} R_{III;XY}^n(x, x', \Theta) \sin \Theta \, d\Theta. \quad (31) \]

The AA functions still contain the physics of frequency redistribution and are very economical to compute compared to corresponding AD functions. We use a Gauss-Legendre quadrature with 31 angle points to numerically compute equations (30) and (31). We remark that to obtain only the AA solution we skip step 3 of stage 1 and directly go to stage 2 of the perturbation method, when \( S_{\text{pol}, \text{AA}} \) becomes the initial guess for \( S_{\text{scat}} \).

Such a replacement of AD functions by their AA counterparts is not new (see, e.g., Rees & Saliba 1982). Although approximate, it gives very reasonable results. In line-formation computations, it is therefore reasonable to use AA functions in preliminary work. Faurobert (1987) has shown that in the nonmagnetic resonance scattering problem the use of AA type II redistributions instead of AD functions gives nearly the same overall behavior of the linear polarization profile, but it may lead to an overestimate in the near wing maximum by about 50% when the optical thickness is larger than 10. Nagendra et al. (2002) have shown for the case of the weak-field “domain-based Hanle redistribution matrix” (approximations II and III of Bommier 1997b) that there are large differences in particular for the \( U \) Stokes parameter between solutions.
that are obtained using AA and AD functions. We briefly examine this important question again, but now for the Hanle-Zeeman regime (see § 4.4).

4. RESULTS AND DISCUSSIONS

Line transfer computations involving the Hanle effect traditionally use only a scalar absorption coefficient. Here, we explicitly include the full Zeeman absorption matrix, as we consider intermediate and strong fields, where the Zeeman effect is important.

The atmospheric model parameters are represented by \((T, a, \epsilon, r, \Gamma_E/\Gamma_R)\), where \(T\) is the optical thickness of the slab and \(a\) is the damping parameter. The Planck function \(B_{\nu_0}\) is taken as unity for all the results presented here. The magnetic field parameters are \((\Gamma_B, \nu_B, \varphi_B)\), where \(\Gamma_B = g\omega_L/\Gamma_R\) with \(g\) being the Landé factor and \(\omega_L = 2\pi\nu_L\) being the Larmor frequency, \(\nu_B = g\nu_L/\Delta\nu_B\) is the magnetic splitting in Doppler units \((\Delta\nu_B)\), and angles \((\varphi_B, \varphi_B)\) define the field orientation with respect to the slab normal.

The magnetic splitting parameter \(\nu_B\) is related to the Hanle parameter \(\Gamma_B\) through the relation \(\nu_B = 2\Gamma_B\nu_0/\{1 + \Gamma_B + \Gamma_E\Gamma_R\}\), where \(\Gamma_B/\Gamma_R = \epsilon/(1 - \epsilon)\). The depolarizing elastic collision parameter is assumed to be \(D^{(1)} = D^{(2)} = 0.5\Delta E\), while \(D^{(0)} = 0\). The details of parameterizing the collisional rates are given in Nagendra et al. (2002; see also Paper II).

The grid resolution in the physical variables is given by the values of \((N_d, N_c, N_a, N_c)\). The quantity \(N_c\) represents the number of points per decade in a logarithmically spaced \(\tau\)-grid with the first depth point \(\tau_{min} = 10^{-2}\), unless stated otherwise. The frequency points are equally spaced in the line core, with a gradual switch over to logarithmic spacing in the wings, and satisfy the condition \(\phi(x_{max})/\nu_L < 1\) in the far wings (the atmosphere becomes so thin at these far wing frequencies that the radiative transfer effects gradually become insignificant). The quantity \(N_d\) represents the number of frequency points from 0 to \(x_{max}\). We use a Gauss-Legendre quadrature for colatitude \(\theta\) \((\mu \in \{0, 1\})\) with \(N_d\) points and trapezoidal quadrature for the azimuthal angles \(\varphi \in \{0, 2\pi\}\) with \(N_c\) points. Unless stated otherwise, we use \(N_d = 5\) points per decade on a logarithmic scale, \(N_c = 45\) (with \(x_{max} = 17\)), \(N_a = 7\), and \(N_c = 8\). Such a grid resolution requires about 3 GB of main memory and half an hour to one hour of computing time to obtain AA solutions. To obtain the corresponding AD solutions, we require about 8 GB of main memory and about 65–75 hr of computing time. Figures 2, 3, and 6 are computed using the AD magnetic redistribution functions, while Figures 4 and 6 are computed using the AA magnetic redistribution functions.

4.1. Scattering at Optically Thin Slabs

The classical theory for Hanle-Zeeman scattering developed by Stenflo (1998) considered only coherent scattering in the laboratory frame. Stenflo (1998, in Fig. 3) presents the scattered Stokes line profiles for Hanle-Zeeman coherent scattering in a 90° single scattering event. To mimic such a single scattering event from a Hanle-Zeeman line transfer problem with PRD (described in § 2 and solved by a perturbation method discussed in § 3), we consider an optically thin slab illuminated at the lower boundary by a unidirectional unpolarized beam of radiation, namely, \(I(\tau = T, x', \mu' = 0.95, \varphi' = 0°) = U\). Observation of radiation from the upper boundary along a nearly tangential direction \((\mu = 0.11, \varphi = 0°)\) mimics extreme limb observations. For our present studies we consider a solar chromosphere canopy–like magnetic field (fields parallel to the surface of the slab medium) with \(\varphi_B = 90°\) and \(\varphi_B = 45°\). The Zeeman splitting parameter \(\nu_B\) is chosen as a free parameter. The other parameters used are \((T, a, \epsilon, r, \Gamma_E/\Gamma_R) = (0.02, 4 \times 10^{-3}, 0, 10^{-9}, 0)\). The optical depth is chosen very small so that the emergent diffuse radiation field is “dominated” by single scattered photons. The first depth point is \(\tau_{min} = 10^{-4}\), \(N_d = 3\) points per decade, and \(N_c = 5\).

The scattering parameter \(\epsilon = 0\) and \(r = 10^{-8}\) together represent a highly scattering medium with negligible continuum absorption. The choice of \(\Gamma_E/\Gamma_R = 0\) corresponds to the pure \(\text{R}^\text{R}\) case and somewhat mimics Figure 3 of Stenflo (1998) which describes coherent scattering in the laboratory frame. A comparison with Figure 3 of Stenflo (1998) shows that the shape of these profiles are very similar to the “single scattered profiles” presented there, except for \(U/I\) (which slightly differs because we do not have perfect 90° scattering now and \(U\) is indeed very angle sensitive). In addition, an admixture of multiply scattered photons is also responsible for the differences with respect to the single scattering results.

For an optical line \(\lambda_0 = 5000\) Å with photospheric-like Doppler width \(\Delta\lambda_0 = 30\) mÅ and Landé factor of unity, the field strength \(\nu_B\) variation that is used here covers magnetic field strengths in the range of 2–6000 G, approximately. In the case of very weak fields (Fig. 2, solid curve), the \(-Q/I\) profile is nearly frequency independent with a value of \(\approx 80\%\) throughout the profile instead of 100% as in the single scattering case, because of multiple scattering contributions as mentioned above.

The \(-U/I\) profile shows Hanle rotation of the polarization plane (the generation of Stokes \(U\) in the line core \(|x| < 3.5\) (see the solid, dotted, and short-dashed curves in Fig. 2), but approaches zero in the line wings \(|x| > 3.5\). This is in agreement with the traditionally used assumption that, in the case of weak fields, it is sufficient to consider the combination of using the weak-field Hanle phase matrix (see e.g., Landi Degl’Innocenti & Landi Degl’Innocenti 1988; Stenflo 1994) in the line core and the Rayleigh phase matrix in the wings.

In the strong-field limit \((\nu_B = 0.5 [1285 G] \text{ and } 2.5 [6424 G]), Stokes \(Q/I, U/I, \text{ and } V/I\) appear very similar to the true Zeeman absorption profiles. For \(\nu_B = 2.5\) the intensity profile is clearly split into a Zeeman triplet.

It is well known that Stokes \(V\) gets completely decoupled from \((I, Q, U)\) \(T\) and transfers independently both in the nonmagnetic Rayleigh limit (Chandrasekhar 1960) and in the weak-field Hanle limit (Landi Degl’Innocenti & Landi Degl’Innocenti 1988). Thus, unless a Stokes \(V\) parameter is given as input or generated internally through a thermal emission coefficient, Stokes \(V\) is not generated at all. However, in the Hanle-Zeeman scattering problem, the \(V\) parameter is naturally coupled to \((I, Q, U)\) \(T\) and is generated even if no input \(V\) is given or the anisotropic Zeeman absorption matrix is not considered explicitly. The coupling of \(V\) to \((I, Q, U)\) \(T\) is weak in the weak-field case, and thus, the \(V/I\) signal is small (see the solid, dotted, and short-dashed curves in Fig. 2).

This coupling increases as the field strength increases, and hence, \(V/I\) also becomes large (see the dot-dashed, dot-dot-dashed, and long-dashed curves in Fig. 2).

4.2. Convergence Behavior of the Perturbation Method

A plot of \(p_{\text{pert}}\) (defined in eq. [27]) versus iteration number \(n\) is presented in Figure 3. The model parameters are \((T, a, \epsilon, r, \Gamma_E/\Gamma_R) = (20, 10^{-3}, 10^{-4}, 3 \times 10^{-9}, 1)\), and the field orientation is \((\varphi_B, \varphi_B) = (30°, 0°)\). The Hanle \(\Gamma_B\) parameter is chosen as a free parameter. Figures 3a and 3b correspond, respectively, to steps 2 and 3 of stage 1 of the perturbation method, while Figure 3c corresponds to stage 2 in the perturbation method. We note that for \(\Gamma_E/\Gamma_R < 0.01\), the convergence is reached within 10 iterations in both Figures 3a and 3b [except for \((\Gamma_E = 1, \nu_B = 0.001)\) in Fig. 3b]. Further, from Figure 3c we see that for \((\Gamma_E = 10, \nu_B \leq 0.01)\), the convergence is reached within five to six
iterations. This clearly shows that the contribution from Zeeman absorption is quite small, as expected.

For \((\Gamma_B > 10, v_B > 0.01)\), the increasing contribution from the Zeeman absorption matrix slows down the convergence (see dash-dotted, dot-dot-dashed, and long-dashed curves in Fig. 3c).

The numerical method fails to converge for \((\Gamma_B > 200, v_B > 0.2)\). This is because the perturbation method is based on the assumption that the source vectors corresponding to \(Q\), \(U\), and \(V\) serve as perturbations to that of \(I\). For \((\Gamma_B > 200, v_B > 0.2)\) this condition is no longer satisfied. In this paper we have applied the
perturbation method to solve hard problems and used it as far as it gives a stable and converged solution.

However, for optically thin slabs (e.g., see Fig. 2) where the single scattering dominates the line radiation field, the polarization at every frequency is produced in the very first scattering and has a small probability of undergoing modifications by subsequent scattering events (within the slab). For this reason, the perturbation convergence is rapid (three to five iterations), with hardly any difference from first iteration to the second, and so on, irrespective of the field strength. Therefore, for thin slabs the final solution is extremely close to the first iteration solution. For the same reason, in optically thin slabs, irrespective of whether one uses a scalar absorption coefficient or matrix absorption, the convergence rate is nearly the same.

4.3. Hanle Saturation of the Line Polarization

In Figure 4 we present the emergent Stokes parameters for a range of field strengths, varying from weak to the Hanle saturation regime. The model used is the same as that of Figure 3, but for $T = 200$. The Hanle $\Gamma_B$ parameter is chosen as a free parameter and varied from 0.1 to 100. The choice of $\Gamma_B/\Gamma_R = 1$ corresponds to a nearly equal mix of $R^R$ and $R^I$. The results in the left panels of Figure 4 are computed using the PALI method of Fluri et al. (2003), while the results in the right panels of Figure 4 are obtained by solving equation (12), using the perturbation technique described in the present paper with the AA Hanle-Zeeman redistribution matrix. We also show in Figure 5 the convergence behavior of the perturbation method for the model considered in this section. The convergence rate in the AA case is somewhat larger than the AD case (presented in Fig. 3). In addition, the convergence is more uniform in the AA case compared to the AD case.

Domain-based Hanle PRD theories show a saturation behavior as we go from weak fields to the intermediate-field regime ($\Gamma_B$ going from 0.1 to 100, see Fig. 4a). The $Q$ values initially decrease (depolarization) and then approach a saturation limit for all frequencies. The $U$ values initially increase (Hanle rotation of the polarization plane), then decrease and approach zero at all the line frequencies. Such a saturation behavior is not exhibited in the Hanle-Zeeman theory (see Fig. 4b). For $\Gamma_B > 10$, the Zeeman absorption matrix plays a significant role, and we get a smooth transition from the scattering-dominated weak-field Hanle effect to the Hanle-Zeeman effect for arbitrary fields. The contribution from the Zeeman absorption is responsible for larger values of Stokes $U$. For example, Stokes $U$ in the $\Gamma_B = 100$ case (see long-dashed curve in Fig. 4b) is 5 times larger when compared to the Stokes $U$ profiles computed using smaller values of $\Gamma_B$.

From Figure 4a we can define various field strength regimes for the Hanle effect. The Hanle effect becomes operative when the Zeeman splitting ($g\mu_B$) is comparable to the small damping width ($\Gamma_R$) of the line transition. Thus, the different field regimes for the Hanle effect can be defined in terms of the parameter $\Gamma_B$. The value of $\Gamma_B$ in the range $0 < \Gamma_B < 0.2$ may be considered as a weak-field regime for the Hanle effect, since the scattering polarization in this regime is close to the Rayleigh scattering polarization. The intermediate-field regime of the Hanle effect corresponds to the range $0.2 < \Gamma_B < 5$ (see also Stenflo 1994, p. 228), with $\Gamma_B = 1$ representing the optimum field strength for which the Hanle effect is most significant. The Hanle effect finally saturates for $\Gamma_B > 5$, and hence, $\Gamma_B > 5$ may be considered as the strong-field limit for the Hanle effect.

Similarly, we may define the weak-, intermediate-, and strong-field regimes for the Zeeman effect. The Zeeman effect becomes operative when the Zeeman splitting ($g\mu_B$) is comparable to the much larger Doppler width ($\Delta\nu_D$) of the line. Thus, we define the different field regimes for the Zeeman effect in terms of the parameter $\nu_B$. The weak-field regime of the Zeeman effect corresponds to $\nu_B \leq 0.3$. The intermediate- and strong-field regimes are, respectively, $0.3 < \nu_B < 1.2$ and $\nu_B > 1.2$ (see Stenflo 1994, p. 299). Clearly, the Hanle and Zeeman effects are two different physical effects. Further, it is important to note that the intermediate Hanle regime and the intermediate Zeeman regime are very different, with little overlap between the two.

The new aspect of the Hanle-Zeeman line transfer theory is that it covers all the field strength regimes defined above, hence the name Hanle-Zeeman regime. The polarized PRD scattering for all field strengths (from zero field to completely split lines) is described by the Hanle-Zeeman redistribution matrix, while the absorption effects are accounted for by the Zeeman absorption matrix. Both the parameters $\Gamma_B$ and $\nu_B$ enter the calculations, but in different ways. The Hanle parameter $\Gamma_B$ appears in the scattering part, while the parameter $\nu_B$ appears both in the absorption part (through $\Phi$) and in the scattering part (through the magnetic redistribution functions; see Paper I). The parameters $\Gamma_B$ and $\nu_B$ are related, but the relation is model dependent. For the model chosen here, the Zeeman absorption effects become significant
Fig. 4.—Comparison of (a) domain-based weak-field theory with the (b) full Hanle-Zeeman theory, for a range of field strengths varying from very weak to the Hanle saturation regime, for $\mu = 0.11$ and $\varphi = 0^\circ$. Model parameters are $(T, a, c, r, \Gamma_{\text{G}/\Gamma_{\text{B}}}) = (200, 10^{-3}, 10^{-4}, 1)$. Magnetic field orientations are $(\theta_B, \varphi_B) = (30^\circ, 0^\circ)$. The medium is assumed to be isothermal and self-emitting. The Hanle $\Gamma_B$ parameter is 0.1 (solid curves), 1 (dotted curves), 3 (dashed curves), 10 (dot-dashed curves), 50 (dot-dot-dashed curves), and 100 (long-dashed curves). We note that in the $U$ panel of (b) the long-dashed curve has been rescaled by the factor $2 \times 10^4$ and not by the $10^5$ that has been applied to the other curves in the panel.

Fig. 5.—Convergence behavior of the perturbation method for the model considered in Fig. 4. (a) Step 2 of stage 1 in the perturbation method (scalar absorption coefficient). (b) Stage 2 of the perturbation method (matrix absorption). We recall that to obtain the AA solution we skip step 3 (AD computation) of stage 1 in the perturbation method. Curve types have the same meaning as in Fig. 4.
for \((\Gamma_B > 10, v_B > 0.01)\). In this regime, the scattering effects can only be correctly included through the Hanle-Zeeman redistribution matrix. To obtain a self-consistent solution, one therefore has to solve the full Hanle-Zeeman line transfer problem.

4.4. Comparison of AD and AA Solutions

In Figure 6 we compare the AD (solid curves) and AA (dotted curves) solutions for the Hanle and Hanle-Zeeman regimes. The model used is the same as that of Figure 3. A detailed comparison of AA and AD solutions for the weak-field Hanle effect are presented in Nagendra et al. (2002). They show that for the effectively thin slabs \((\epsilon' \lesssim 1)\) the AA and AD solutions show quite a large difference in \(Q\) and \(U\) (which decreases for the effectively thick case). We reconfirm their result for the Hanle regime from the present computations. Our numerical experiments show that as we increase \(\Gamma_B\), the differences between the AA and AD solutions decrease and vanish for \(\Gamma_B = 200\). This behavior can be attributed to the progressive dominance of the Zeeman absorption effect (which is blind to the choice of AA or AD functions in the scattering part) over the scattering effects. It is interesting to note that the Stokes \(V\) parameter is also insensitive to the choice of AA or AD functions irrespective of the field strength as shown in Figure 6.

5. CONCLUDING REMARKS

A general form of the line radiative transfer equation applicable for a two-level atom \((J = 0 \rightarrow 1)\) in arbitrary field strengths is presented and solved. We use a standard perturbation method, extended appropriately to include both the Zeeman absorption matrix and the Hanle-Zeeman scattering redistribution matrix.

To reduce the complexity of the PRD treatment in the Hanle scattering problem, it has been common to use domain-based redistribution matrices. We show that such approaches are indeed valid for weak fields. For intermediate fields, it is necessary to apply the Hanle-Zeeman PRD line transfer problem presented in this paper.

The scattering as well as absorption mechanisms in intermediate fields are self-consistently included in the transfer equation. In the Hanle-Zeeman line transfer problem, the Stokes \(V\) parameter is coupled to \((I, Q, U)\) and is generated not only by Zeeman absorption but also by scattering. We have discussed the weak-, intermediate-, and strong-field regimes for both Hanle and
Zeeman effects independently. The Hanle-Zeeman line transfer theory covers all these regimes, from zero field to the completely split line. For the model used to present the numerical results in this paper, we conclude that one has to solve the Hanle-Zeeman line transfer theory to obtain self-consistent solutions for \((\Gamma_B > 10, v_B > 0.01)\).

In the solar atmosphere one encounters fields of all magnitudes (milligauss to kilogauss). The method developed here addresses this complex problem of Hanle-Zeeman line transfer in the presence of arbitrary field strengths. The weakness of the perturbation method is its great demand for computer memory and CPU time (because it solves the nonaxisymmetric Stokes vector transfer equation explicitly). We use the perturbation method, because the axisymmetric form (the so-called reduced form) of the concerned transfer equation is yet to be derived. A more elegant and powerful approach would have been to develop a polarized approximate lambda iteration (PALI) method, which uses the more simple axisymmetric form of the transfer equation.

M. S. is financially supported by the Council of Scientific and Industrial Research (CSIR) through an SRF [grant 9/890(01)/2004-EMR-I], which is gratefully acknowledged. The authors would like to thank M. Bianda for kind hospitality. A part of the work on this paper was completed when they were visiting IRSOL, Switzerland. M. S. would like to thank Hélène Frisch for useful discussions. The authors are grateful to an anonymous referee for constructive comments, which helped to improve the manuscript.

REFERENCES

Casini, R., & Lites, B. W., eds. 2006, Solar Polarization 4 (San Francisco: ASP)
Chandrasekhar, S. 1960, Radiative Transfer (New York: Dover)
Hummer, D. G. 1962, MNRRAS, 125, 21
Mihalas, D. 1978, Stellar Atmospheres (2nd ed.; San Francisco: Freeman)
Nagendra, K. N., & Stenflo, J. O., eds. 1999, Solar Polarization (Boston: Kluwer)
Rees, D. E. 1978, PASI, 30, 455