12.1. Introduction

Since Babcock (1953) and Kiepenheuer (1953) invented the photoelectric magnetograph, which maps the distribution of longitudinal magnetic flux across the solar surface by recording the circular polarization in a Zeeman-sensitive spectral line, the spatial resolution of the magnetic-field observations has been gradually increased from about one arcmin to less than one arcsec. Even the seeing limit determined by the earth’s atmosphere can now be overcome by image reconstruction techniques like speckle polarimetry (see next chapter). Each new advance in spatial resolution has revealed new, smaller flux structures not seen before, and we have reasons to believe that this structuring continues well beyond the presently achievable spatial resolution.

If the observations were really able to resolve the fields, then we could apply the line-formation theory developed in the preceding chapters directly to diagnose the field, and the present chapter would not be needed. As it is, however, all magnetic fields outside sunspots are spatially unresolved. A new set of diagnostic methods therefore has to be developed to allow us to extract information about the spatially unresolved field.

This situation is typical in stellar astrophysics. Although the stars are spatially unresolved point objects, it has been possible to determine the stratification of temperatures, densities, velocity fields, and element abundances by extracting the information stored in different parts of the stellar spectrum. Similarly, the spatially unresolved magnetic flux elements on the sun have left their “finger prints” in the polarized profiles of many different types of spectral lines. The height variation of the intrinsic thermodynamic and magnetic structure of the unresolved flux elements may be derived through the combined analysis of carefully selected sets of polarized spectral lines. As the diagnostic methods that we develop for the sun concern spatially unresolved structures, they can be applied to other stars as well.

Since all the magnetic flux elements outside sunspots are spatially unresolved, the diagnostic methods presented in the present chapter are fundamental not just to exploring small-scale fields but to solar magnetism in general. The lack of spatial information is compensated for by using additional information in the spectral domain, e.g. combinations of two or more simultaneously observed spectral lines, spectrally resolved polarized line profiles, or resolved Zeeman splittings in the infrared. It may seem that this indirect approach could not take us very far, but a fortunate circumstance provides exceptional help: A two-component model, with one magnetic and one non-magnetic component, turns out to be a good approxi-
mation in most practical situations. This circumstance greatly reduces the number of free parameters needed to fit the data in comparison with the case of a multi-component atmosphere or a continuous distribution of the unresolved physical parameters. After solutions for the two-component model have been established as new references for further refinements, one may build ever more sophisticated models with additional components or continuous distributions added, thereby introducing more physical realism.

Applications of the spectral techniques outlined in the present chapter have revealed that the circular-polarization signals of which magnetograms are made up have their sources in spatially unresolved, discrete patches of magnetic flux that only occupy a small fraction, typically 1% (varying between a fraction of a percent in very “quiet” regions to tens of percent in active-region plages), with no discernible flux in between (except for the “hidden”, turbulent field that was discussed in Chapter 10). We call these flux patches flux tubes, although they could sometimes be more slab-like than tube-like. While the unpolarized spectrum (Stokes $I$) receives contributions from both the flux tubes and their “non-magnetic” surroundings, with the surroundings dominating due to their larger relative area, the polarized spectrum has its sources exclusively inside the flux tubes.

The Stokes $Q$, $U$, and $V$ spectra thus provide diagnostic information on the flux tubes that is uncontaminated by the surrounding radiation, stray light, or limited spatial resolution, in contrast to the unpolarized spectra. This is a huge advantage for polarimetric observations. It is as if we would observe stars at daylight and had the means of completely filtering out the background radiation from the bright sky. In polarized light the flux tubes are similar to point-like objects against an entirely “black” background.

In the present chapter we will show how this polarimetric information can be exploited to build increasingly sophisticated empirical models of the magnetic and thermodynamic structure of the flux tubes and their immediate surroundings. The free parameters of these models have to satisfy not only the observational but also the theoretical constraints, since the requirements of magnetohydrodynamic (MHD) force balance (between gravitational and dynamical forces, gas pressure, magnetic pressure and tension) in all spatial directions have to be satisfied. The resulting empirically determined thermodynamic structure can be compared with the structure that emerges from numerical MHD simulations of collapsing flux tubes. The discrepancies between the empirical and theoretical models may result from simplifying assumptions made when deriving the empirical models but may also indicate what physics is missing in the theoretical description (e.g. the wave dissipation processes that contribute to the heating of the flux tube interiors, and which may be of critical importance for the overall energy balance of the outer solar atmosphere).
12.2. Diagnostic Contents of Stokes V Spectra: An Overview

The most powerful existing instrument in terms of spectral resolution, wavelength coverage, and straylight level is the Fourier Transform Spectrometer (FTS) at the McMath-Pierce facility of the National Solar Observatory (Kitt Peak). It was converted into a polarimeter in the end of the 1970’s, which allowed fully resolved Stokes spectra with high signal-to-noise ratio to be recorded over 1000 Å portions of the solar spectrum. This has made it possible to use many hundreds of simultaneously recorded polarized line profiles for magnetic-field diagnostics.

Figure Eq.12.1, from Stenflo et al. (1984), gives an example of a 6 Å wide segment of one of the first Stokes V recordings with the FTS polarimeter, as an illustration of the diagnostic contents of such spectra. Stokes V is given in units of the continuum intensity $I_c$. To achieve such high signal-to-noise ratio as in this recording a trade-off with spatial and time resolution had to be made. Thus a
circular aperture of 10 arcsec diameter and an integration time of about 35 min were used.

To aid the understanding of the meaning of the Stokes \( V \) spectrum, let us consider the special case of a height-independent longitudinal magnetic field that covers the fraction \( \alpha_m \) (the magnetic filling factor) of the spatial resolution element. Then a Taylor expansion gives according to Eqs. (11.97) and (11.98)

\[
\begin{align*}
I &= \alpha_m I_m + (1 - \alpha_m) I_{nm}, \\
I_m &\approx I_{m,0} + \frac{1}{2} (\Delta \lambda_H)^2 \frac{\partial^2 I_{m,0}}{\partial \lambda^2} + \cdots, \\
V &\approx -\alpha_m \Delta \lambda_H \left[ \frac{\partial I_{m,0}}{\partial \lambda} + \frac{1}{6} (\Delta \lambda_H)^2 \frac{\partial^3 I_{m,0}}{\partial \lambda^3} + \cdots \right].
\end{align*}
\]

Index \( m \) refers to the magnetic component, \( nm \) to the non-magnetic component, while index 0 refers to the case of zero Landé factor.
\[ \Delta \lambda_H = z g \lambda^2 B, \]  
(12.2)

where \( z \) is a fundamental, numerical constant with a value given by Eq. (10.62).

For most lines in the visible part of the solar spectrum the higher-order terms in the Taylor expansion are unimportant. In this case Stokes \( \tilde{V} \) is simply proportional to the Landé factor \( g \), the field strength \( B \), and the intensity gradient \( \partial I_{m,0}/\partial \lambda \) of the magnetic component. To properly calibrate a magnetograph such that polarization \( \tilde{V} \) may be converted to field strength \( B \) one would need to know \( I_{m,0} \) and \( \alpha_m \), which however are not observable quantities. The observables are Stokes \( I \) and \( V \), and Stokes \( I \) is dominated by the contribution \( I_{nm} \) from the non-magnetic component. The anti-symmetric shape of the Stokes \( V \) profiles is due to \( \partial I_{m,0}/\partial \lambda \), since all \( \partial I/\partial \lambda \) profiles have qualitatively this shape, as was illustrated in Figs. 11.4 and 11.5.

The average field strength over the resolution element is

\[ \langle B \rangle = \alpha_m B, \]  
(12.3)

which would be obtained from the observed Stokes \( V \) if the calibration problem could be solved (i.e., if \( I_m \) were known). The value of \( \langle B \rangle \) however does not tell us what the intrinsic field strength \( B \) may be as long as the filling factor \( \alpha_m \) is unknown. We need additional information to untangle \( \alpha_m \) and \( B \) from each other.

This additional information can be obtained by using a combination of two lines, which we refer to by indices 1 and 2. Let us assume that the two lines have \( \lambda_1 \approx \lambda_2 \) and \( I_{m,1} \approx I_{m,2} \), which is generally the case when they belong to the same atomic multiplet and have approximately the same oscillator strengths. In this case the value of the field strength \( B \) to be used in Eq. (12.2) is the same for the two lines even when \( B \) is strongly height dependent, since the lines are formed in the same atmospheric layers. The Stokes \( V \) ratio then becomes

\[ \frac{V_1}{V_2} = \frac{g_1}{g_2} \left( \frac{\partial I_{m,0}}{\partial \lambda} + \frac{z^2 \lambda^4}{6} g_1^2 B^2 \frac{\partial^3 I_{m,0}}{\partial \lambda^3} + \ldots \right) \left( \frac{\partial I_{m,0}}{\partial \lambda} + \frac{z^2 \lambda^4}{6} g_2^2 B^2 \frac{\partial^3 I_{m,0}}{\partial \lambda^3} + \ldots \right)^{-1}. \]  
(12.4)

Note that by forming this line ratio we have managed to eliminate the magnetic filling factor \( \alpha_m \). In the weak-field limit Eq. (12.4) reduces to

\[ V_1/V_2 \approx g_1/g_2. \]  
(12.5)

Let us now check how the Stokes \( V \) line ratios of Eqs. (12.4) and (12.5) compare with what we actually see in Fig. 12.1. The two lines Fe I 5247.06 and 5250.22 \AA represent an ideal combination for this purpose, since they belong to the same multiplet (no. 1) of iron, and have practically the same line strengths, excitation potentials, and wavelengths. As they are thus formed in the same way in the atmosphere, the conditions \( I_{m,1} \approx I_{m,2} \) and \( \lambda_1 \approx \lambda_2 \) for Eq. (12.4) are fulfilled. The only difference of importance is due to the effective Landé factors, 2.0 and 3.0, respectively, for the two lines. If the magnetic fields were intrinsically weak (\( \lesssim 0.5 \text{kG} \)), then the Stokes \( V \) amplitudes should be in the proportion 2 : 3 according
to Eq. (12.5), whereas Fig. 12.1 shows that the ratio is much closer to 1 : 1. The reason for this is “differential Zeeman saturation”, meaning that the higher order terms in the Taylor expansion are important for the line with the larger Landé factor while being rather small for the other line. This measured deviation from the weak-field relation (12.5) is a function of field strength $B$ and therefore allows the value of $B$ to be determined, regardless of what the value of the filling factor $\alpha_m$ is.

The conversion of the ratio $V_1/V_2$ to field strength $B$ however depends on $I_m$ according to Eq. (12.4), and as this is not an observable, some model dependence of the derived field strengths is introduced. This model dependence is however quite weak, since only differential line broadening but not line weakening may have an effect. If there is only differential line weakening by a factor $w$ but no differential line broadening, then the shapes of the line profiles inside and outside the flux element are the same. When the filling factor is small $\ll 1$, the rest intensities may be written

$$1 - I_{m,0}/I_{m,0(c)} \approx w (1 - I/I_c).$$  \hspace{1cm} (12.6)

Index $c$ refers to the continuous spectrum and $I$ is the observed Stokes $I$ of Eq. (12.1). In this case the unobserved $I_{m,0}$ in Eq. (12.4) may be replaced by the observable $I$. Then the field strength $B$ is the only unknown that is left in the equation and may be solved for in an unambiguous way. As the field strength varies with height we have to remember that the derived value of $B$ refers to the height of formation of the lines used. As we will see later the geometrical height of formation is a sensitive function of the opacity structure and thus also of the temperature of the flux tube interior.

Let us next compare the two iron lines 5247.06 and 5250.65 Å in Fig. 12.1. Although the latter line has a smaller Landé factor it has a larger Stokes $V$ amplitude. This behavior cannot be explained in terms of Zeeman saturation, which is small for both lines, but is the result of different temperature sensitivities. The line weakening $w$ is more pronounced for the 5247.06 Å line because of its lower excitation potential (0.09 eV) as compared with that (2.20 eV) of the 5250.65 Å line.

For two lines with differential line weakening factors $w_1$ and $w_2$ but with no differential line broadening (between flux tube interior and exterior — the widths of the two lines may be different from each other) we have in the absence of Zeeman saturation according to Eqs. (12.4) and (12.6)

$$\frac{V_1}{V_2} = \frac{w_1}{w_2} \frac{g_1}{g_2} \frac{\partial I_1}{\partial \lambda} / \frac{\partial I_2}{\partial \lambda}. $$  \hspace{1cm} (12.7)

Comparison of the Stokes $V$ amplitudes of lines of different temperature sensitivities thus provides information on the temperature difference between flux tube interior and exterior.

The reason why the line weakening is larger for lines with low excitation potential is due to the temperature dependence of the level populations. If $n_{0,s}$ is the population number density for level $s$ of the neutral atom, and $\chi_i$ and $\chi_{0,s}$ are the ionization and excitation potentials, then in the limit of small relative temperature
changes $\Delta T/T$ we obtain the relative population change by Taylor expansion of the Saha-Boltzmann law:

$$\frac{\Delta n_{0,s}}{n_{0,s}} \approx - \left( \frac{3}{2} + \frac{\chi_i - \chi_{0,s}}{kT} \right) \frac{\Delta T}{T}. \tag{12.8}$$

Here we have ignored the temperature dependence of the electron density $n_e$ and made use of the fact that most metal atoms in the solar photosphere are singly ionized. Eq. (12.8) shows that lines with smaller excitation potentials $\chi_{0,s}$ of their lower levels are more sensitive in their level populations than lines with higher excitation potential. The temperature dependence outside sunspots is always such that the population decreases when the temperature increases. Note however that this situation can change in the cool sunspots, where neutrals become the dominant species.

Another temperature effect that has influence on the line weakening is the difference in the temperature gradient $dT/d\tau$ between the inside and the outside of the flux tube. As can be seen from Eq. (11.53) the line depth scales with $dB/\tau$, where $B_\nu$ is the Planck function. Due to the general differential cooling of the lower layers and heating of the upper layers of magnetic elements, $dT/\tau$ is shallower in flux tube interiors, which leads to weakened spectral lines.

The discussion so far can be summarized as follows. If we use the observed intensity profile for calibration to obtain an “observed” longitudinal field strength $B_{\text{obs}}$, assuming the weak-field limit and a homogeneous magnetic field, then we may write

$$B_{\text{obs}} = w_B w_T \alpha_m B, \tag{12.9}$$

where $w_B$ is the Zeeman saturation factor that is a function of field strength $B$ and equals unity for $B = 0$, while $w_T$ is the temperature weakening factor (same as the $w$ and $w_{1,2}$ factors used in Eqs. (12.6) and (12.7)), which is a function of $\Delta T$ (temperature difference between flux tube interior and exterior) and is unity for $\Delta T = 0$. In the weak-field limit in the absence of temperature effects $B_{\text{obs}}$ would equal the average field strength $\langle B \rangle$ of Eq. (12.3) (net flux divided by the area of the resolution element). A magnetic line ratio would give us the value of $w_B$ and thus also of $B$, a thermal line ratio would give us $w_T$ and thus $\Delta T$-information, while the Stokes $V$ amplitude gives us $B_{\text{obs}}$ and thus, in combination with the other parameters, the magnetic filling factor $\alpha_m$. Note that the values of $w_B$ and $w_T$ depend on the wavelength within the line profile that is used for the determination of $B_{\text{obs}}$.

Another property that clearly stands out in Fig. 12.1 is the difference between the Stokes $V$ amplitudes in the blue and red line wings. Both the amplitude and area of Stokes $V$ in the blue line wing are systematically larger than their values in the red wing. This contradicts Eq. (12.1), since the area asymmetry of $\partial I_{m,0}/\partial \lambda$ is always zero. If it did not vanish, then the levels of the continuum intensity $I_{m,c}$ on the blue and red sides of the line would be different. Such Stokes $V$ asymmetries can only arise if there are correlations between magnetic and velocity gradients in the atmosphere. For a non-zero area asymmetry we need such correlated gradients along the line of sight. The asymmetries thus contain information on the dynamics
of the spatially unresolved flux tubes. As we will see later, they even inform us about the dynamics of the immediate, non-magnetic surroundings of the flux tubes.

Additional dynamical information is obtained from the wavelength position of the Stokes $V$ zero crossing with respect to the position of the Stokes $I$ profile, as well as from the widths of the Stokes $V$ profiles. While the zero crossing tells us about the systematic mass flows inside the flux tubes, the Stokes $V$ line width tells us about the non-thermal Doppler broadening inside the flux elements. The directional properties of the motions may be diagnosed from the observed center-to-limb variations of these effects.

### 12.3. Evidence for the Validity of a Two-component Model

The interpretation of the observable effects that we have reviewed in the preceding section is enormously simplified by the possibility of applying a simple two-component approach with a magnetic filling factor $\alpha_m$. It was one of the pleasant surprises in solar exploration that such an approach can be used at all. The prime evidence for the validity of the two-component approach has come from line-ratio observations with magnetographs being able to sample a large number of points on the sun. If one makes a scatter-plot of the fields $B_{\text{obs},1}$ and $B_{\text{obs},2}$ observed simultaneously in two spectral lines, then the points in the diagram would fall along a 45° line if $w_{B,1} w_{T,1}$ were equal to $w_{B,2} w_{T,2}$. Instead the points fall beside the 45° line because the various $w$ factors deviate differently from unity.

The line ratio may according to Eq. (12.9) be written in the form

$$\frac{B_{\text{obs},1}}{B_{\text{obs},2}} = \frac{w_{B,1} w_{T,1}}{w_{B,2} w_{T,2}} = f(B, \Delta T), \quad (12.10)$$

since the flux $\alpha_m B$ is the same for both spectral lines (which is true even when the lines are formed at different heights, if the flux within the spatial resolution element is conserved). Since $w_B$ and $w_T$ are functions of $B$ and $\Delta T$, the line ratio is a function $f$ of these two parameters.

The line ratio is sampled at many different solar locations with different filling factors $\alpha_m$ (we recall that $B_{\text{obs}}$ is proportional to $\alpha_m$). If $B$ and $\Delta T$ would vary from place to place on the sun, then we would observe a considerable spread in the line-ratio values. This is not the case, as indicated by the scatter plot in Fig. 12.2 of $B_{\text{obs}}$ for the line pair at 5250.22 (g = 3.0) and 5247.06 Å ($g_{\text{eff}} = 2.0$) that was discussed in the preceding section. These two lines have the same temperature sensitivities and are formed at nearly the same heights, so that $w_{T,1}/w_{T,2} = 1$. The line ratio $B_{\text{obs},1}/B_{\text{obs},2}$ is therefore exclusively a function of the field strength $B$. The points in the diagram fall along a well-defined regression line with a slope that is less than unity, as expected from differential Zeeman saturation. A field strength of 1 kG follows from the value of the slope (assuming a rectangular cross section, see below). The scatter around the regression line does not exceed the instrumental scatter, which indicates that the scatter of the intrinsic $B$ values is small. There is a slight tendency for the value of the slope to decrease with
increasing flux, but to a first approximation the regression relation is a straight line (as drawn in the figure).

If there were a distribution of intrinsic field strengths between zero and kG values, then the region between the regression line and the 45° line would be populated with points, which is not the case. This has led to the conclusion that it is a good approximation to consider the flux tube properties as being “unique” (Frazier and Stenflo, 1972, 1978). It means that whatever solar region we look at, the difference between apparently strong and weak “observed” fields $B_{\text{obs}}$ is not due to variations of the intrinsic properties of these flux regions, but is instead due to the filling factor or number density of flux elements.

To identify the true spread of the intrinsic properties we have to reduce the instrumental contribution to the observed spread. This has been done through Stokesmeter recordings with long integration times using the vertical grating spectrometer and the FTS at the McMath-Pierce facility of the National Solar Observatory. As a trade-off we are not able to sample so many points on the sun when pushing for extremely high polarimetric accuracy. Figure 12.3, from Stenflo and Harvey (1985), gives examples of the results obtained.
As seen in the left diagram, the spread of the points around the regression line is indeed very small. In the right diagram the Stokes $V$ ratio for the points with a $V/I_c$ amplitude in excess of 0.4% has been plotted. The difference between the data from the grating spectrometer (filled symbols and solid line) and from the FTS (the two crosses and dashed line) is largely due to the difference in spectral resolution. The FTS data point with the larger $V$ amplitude has been extracted from the recording that was shown in Fig. 12.1.

Figure 12.3 shows that there is a small trend for the field strength to increase with increasing filling factor (increasing deviation of the line ratio from unity with increasing Stokes $V$ amplitude). The most striking feature of the diagram is however the complete absence of any points in the upper 2/3 of the diagram to the right. Although the statistical sample is limited, it suggests an absence of weak fields.

As we will see later, observations in the near infrared have confirmed and extended these conclusions. In the infrared the Zeeman splitting may be measured directly, which reduces the model dependence and increases the sensitivity. This has allowed field strengths for still smaller filling factors than those of Fig. 12.3 to be measured and has made more sophisticated modelling possible, including 3-component models. As we will see, the intrinsic spread of field strengths in the kG range is rather small, but weaker fields (below kG) become more common when we go to very small observed fluxes (or $B_{\text{obs}}$ values).
12.4. Line Ratios

The observed polarization at any given wavelength in a Zeeman-sensitive spectral line depends on the detailed line-of-sight variation of the magnetic field, temperature, density, turbulent velocities, etc. The contributions from these various physical parameters are coupled to each other in a highly non-linear way. The basic philosophy of the line-ratio technique is to observe the polarization simultaneously in two spectral lines that respond as similarly as possible to all the various physical parameters except one. The ratio of the $B_{\text{obs}}$ values in the two lines is then primarily a function of this parameter alone, which allows it to be untangled from the other parameters when interpreting the data.

A generalization of the line-ratio approach is to use a set of observables (which may represent a combination of amplitudes, asymmetries, widths, and shifts of $Q$, $U$, and $V$ polarized line profiles) carefully selected such that each observable responds as differently as possible to the atmospheric parameters as compared with the other observables. We may then apply an inversion approach, as done by Keller et al. (1990), to determine from the set of observables the free physical parameters of the model flux tube, as outlined in Sect. 11.13. As however this is a complex and computer intensive undertaking, the much simpler line-ratio approach is usually the preferred method that has advanced our understanding of the nature of the small-scale magnetic field.

It is important that the line pair is chosen with great care so that a certain effect or parameter may be isolated as cleanly as possible. In the present section we will give examples of three types of line ratios, one that isolates the field strength (magnetic line ratio), one that is dominated by the temperature effects (thermal line ratio), and one (for the infrared) that isolates Zeeman line broadening from Doppler line broadening.

The “cleanest” line ratio that has been found so far is the already mentioned magnetic line ratio based on the line pair Fe I 5247.06 and 5250.22 Å (Stenflo, 1973). These lines, which belong to the same multiplet and have the same line strength, are almost identical in all respects except for the Landé factor and the Zeeman splitting pattern. A $B_{\text{obs},1}/B_{\text{obs},2}$ ratio different from unity can therefore only be produced by Zeeman saturation, which is a function of the intrinsic field strength alone.

In early work in the 1970’s this line ratio was used as a $B_{\text{obs}}$ ratio derived from scans of the solar disk with a Babcock-type magnetograph (cf. next chapter), whereby the Stokes $V$ signal was recorded within fixed spectral windows in the two spectral lines. With the advent of the FTS polarimeter it became possible to determine the $V_1(\Delta \lambda)/V_2(\Delta \lambda)$ ratio as a function of wavelength position $\Delta \lambda$ with respect to the centers of lines 1 and 2. Such “ratio profiles” provide a verification of the consistency of the interpretation in terms of Zeeman saturation and also introduce additional observational constraints on the field distribution. An example is given in Fig. 12.4, from Stenflo and Harvey (1985), showing in the left diagram Stokes $V$ vs. $-\Delta \lambda$ for the blue wings of the 5250.22 and 5247.06 Å lines. The $V$ signal of the latter line has been multiplied by 1.5 to scale it to the 1.5 times larger Landé factor of the other line. Two cases are shown, labeled strong and
weak plage, which differ in $V$ amplitude by a factor of six. They correspond to the two FTS points in Fig. 12.3. The strong plage spectrum was shown in Fig. 12.1.

There are two main effects caused by the higher-order non-linear terms in the Taylor expansion (12.1): Suppression of the Stokes $V$ amplitude (which we have called Zeeman saturation), and broadening of the $V$ profile. Due to the broadening effect the $V$ ratio becomes larger than unity in the far line wings, but as the signal gets small in that portion of the line, the ratio between the integrated $V$ profiles is always less than unity. It is particularly remarkable that the line-ratio curves (right diagram) are so similar for the two cases that differ in amplitude by as much as a factor of six. This demonstrates that the $B$-dependence on filling factor is quite minor.

To convert the line ratio to field strength $B$ we have to solve the radiative-transfer problem for $V$ in the two lines. To reproduce the full line-ratio profile one needs a realistic, numerical model atmosphere. An example of such a calculation is given in Fig. 12.5, from Emonet (1992), showing a nearly perfect fit between the synthetic and observed line ratio profiles in the case of the strong plage data of Figs. 12.1 and 12.4. The model uses a magnetic field that varies linearly with log $\tau_{5000}$, where $\tau_{5000}$ is the continuum optical depth at 5000 Å inside the flux tube. The data are reproduced for a model that has $B = 1432$ G at log $\tau_{5000} = -1.46$, a typical height of formation for the wings of the 5250.22 and 5247.06 Å lines. The field decreases with height according to $dB/d(\log \tau_{5000}) = 273$ G in this model.

In the discussion so far we have implicitly assumed that the magnetic cross section is rectangular, i.e., that the field is single-valued at each given height. If we have a distribution of field strengths within the resolution element, then we have to spatially integrate the Stokes $V$ signals to obtain the $V$ line ratio. The
peak field strength of the distribution then has to be higher than the field strength of the single-valued case to fit a given observed line ratio. Thus, when the magnetic line ratio was first applied to the quiet sun (Stenflo, 1973), the field strength was found to be 1.1 kG if a rectangular magnetic cross section was assumed, while for a Gaussian cross section the field amplitude had to be 2.3 kG to fit the same line ratio. We will later see that infrared data favor a rectangular cross section.

A “thermal” line ratio cannot be as “clean” as the magnetic line ratio, since the lines need to come from different multiplets to have significantly different excitation potentials, and when the thermal response is different, the heights of formation will also be different. Since an ideal thermal line pair thus cannot be found, one may relax the requirement that the Landé factors and line strengths should be equal. The main requirement should simply be that the line ratio should be dominated by thermal effects (cf. Landi Degl’Innocenti and Landolfi, 1982).

In the discussion of Fig. 12.1 we noticed that the Fe I 5250.65 Å line was much less weakened by temperature effects than the other lines, due to its higher excitation potential. We may therefore form a “thermal line ratio” with this line, but we should not use the neighboring 5250.22 Å line as the other partner of the pair, since it is affected by Zeeman saturation. Figure 12.6 gives an example of the observed center-to-limb variation of the Stokes V amplitude ($V_{\text{max}}$) ratios for the line pair Fe I 5247.06 and 5250.65 Å (filled symbols and thick curve) and the pair Cr I 5247.57 and Fe I 5250.65 Å (crosses and thin curve). $\mu$ is the cosine of the heliocentric angle (the angle between the line of sight and the vertical direction).

**Fig. 12.5.** Comparison between observed (solid) and theoretical (dashed) profiles of the magnetic line ratio for the strong plage of Figs. 12.1 and 12.4. The theoretical model has $B = 1432$ G at $\log \tau_{5000} = -1.46$ and a gradient of $dB/d(\log \tau_{5000}) = 273$ G, where $\tau_{5000}$ represents the continuum optical depth inside the flux tube. From Emonet (1992).
Fig. 12.6. Center-to-limb variation of the “thermal” line ratio \( g_2 V_{\text{max,1}} / g_1 V_{\text{max,2}} \). The filled squares and thick curve represent the line pair Fe I 5247.06 (line 1) and 5250.65 Å (line 2), the crosses and thin curve the line pair Cr I 5247.57 and Fe I 5250.65 Å. The curves are cubic spline fits to the points. From Stenflo et al. (1987a).

As the value of the Zeeman saturation factor \( w_B \) (cf. Eqs. (12.9) and (12.10)) is almost unity for all of these three lines, the line ratios plotted represent primarily \( w_T : g_1 V_{\text{max,2}} / g_2 V_{\text{max,1}} \) (although the differential effect of the line profile shapes may also play a role). The line ratios are considerably smaller than unity as expected from the smaller temperature weakening of the 5250.65 Å line. The conversion of such line-ratio data to information on the height variation of \( \Delta T \) however requires careful numerical radiative-transfer calculations with realistic flux tube models, for instance as done by Zayer et al. (1990).

Since the Zeeman splitting varies with \( \lambda^2 \) according to Eq. (12.2), the infrared region of the spectrum contains lines for which the Zeeman splitting in the kG flux tubes is complete, something that does not occur for any line in the visible part of the spectrum. For such lines the field strength can be measured directly from the wavelength separation of the sigma components, instead of, as in the visible, having to translate polarization ratios to field strength. Still, even in the complete splitting case, a line ratio is of great advantage, since it allows us to untangle the distribution of field strengths from the other atmospheric parameters.

Field strength distributions cause significant broadening of the Stokes \( V \) profiles when the Zeeman splitting is large. Such Zeeman line broadening, which is correlated with the Landé factors of the lines, is insignificant for lines in the visible part of the spectrum (cf. our search for a turbulent magnetic field in Sect. 10.14). A field-strength distribution will always be present even when the magnetic cross section of an element is single-valued, since the field strength always varies along the line of sight within the line-forming layers due to the height divergence of the
field lines. To separate the contributions from the Zeeman broadening and the non-thermal Doppler broadening we need to use a combination of two similar lines with different Landé factors (Zayer et al., 1989).

An example of such an infrared line pair is shown in Fig. 12.7, from Rüedi et al. (1992). The two lines are Fe I 1.5649 μm (the “Zeeman line”) with a Landé factor $g$ of 3.0, and Fe I 1.5653 μm with $g = 1.53$. While the solid curves represent the observed $V/I_c$ profiles, the dashed curves have been derived from the two-component model that fits the observations. In the upper diagram the field strength at $z = 0$ (where the continuum optical depth in the external atmosphere $\tau_{5000} = 1$) is 1.52 kG, and the Zeeman splitting is complete for both lines (with the consequence that the Stokes $V$ amplitudes of the two lines are of similar magnitude). The $g = 3.0$ line is however significantly broader than the $g = 1.53$ line. In the lower diagram the field strength is only half as large (0.75 kG), leading to much less Zeeman broadening and a relative suppression of the Stokes $V$ amplitude of the $g = 1.53$ line, since for these weaker fields the Zeeman splitting is not complete for this line.

The lower diagram of Fig. 12.7 in fact represents the first identification of intrinsically weak (below kG) fields in the lower photosphere. The infrared region thus has a great deal of diagnostic potential, but to properly exploit it for Stokes inversion and flux tube model building we need to be able to sufficiently constrain the free physical parameters like the macroturbulent velocities and the height gradients from the observations. This is only feasible by using line combinations like the example of Fig. 12.7.

### 12.5. Fraction of Magnetic Flux in kG Form

The first determinations of the kG field strengths of the spatially unresolved flux elements were done with the line-ratio method via regression-line analysis of scatter-plot diagrams like that of Fig. 12.2. The observable that contains the information on the intrinsic field strength is the value of the slope of the regression line. Using Stokesmeter recordings of polarized line profiles with very high signal-to-noise ratio one can abandon the statistical approach and apply the line ratio to each individual point, as was shown in Fig. 12.3. The results of the statistical approach are thereby confirmed. Still, the observational uncertainties in the line ratio become excessive for small fluxes or filling factors, corresponding to the points near origo in the scatter-plot diagram. This necessitated the introduction of a polarization cut-off of 0.4% in the right panel of Fig. 12.3, below which the line ratio could not be reliably measured.

In the regression analysis it is the points with the largest filling factors that carry the most weight when the slope is determined, so neither the regression analysis of Fig. 12.2 nor the individual line ratios of Fig. 12.3 provide information on whether the smallest observed fluxes are due to flux elements with intrinsically strong or weak fields.

Still it is these small flux values that are the most numerous when we make a raster scan across an arbitrary area of the sun, as shown by the heavy clustering of the measured points around origo in Fig. 12.2. Since these small fluxes are
representative of most of the area of the sun, their accumulated flux may amount to a substantial fraction of the total flux through the solar surface. On the other hand we know from the application of the line ratio to the larger fluxes that at least some fraction of the total flux through the solar surface, both in active and quiet regions, has its sources in kG flux elements. The question that we now want to address is how large this fraction really is.

Let $\Phi_i$ be the true vertical magnetic flux through the sampling area $A$ (the resolution element) at position $i$ on the solar disk. The absolute value of the total flux through the scanned region is then

$$\Phi_{tot} = \sum_i |\Phi_i|.$$  \hfill (12.11)

Let us further assume that there are two types of flux within each resolution element: strong-field flux $\Phi_{s,i}$ and weak-field flux $\Phi_{w,i}$. In the strong-field elements
the contribution to the magnetograph signal is according to Eq. (12.9) reduced by
the factor
\[ \delta = w_B w_T \] (12.12)
due to Zeeman saturation and temperature line weakening. In the weak-field flux
elements, on the other hand, we assume that these effects are absent (which must
certainly be true for Zeeman saturation at least), which means that \( \delta = 1 \) there.
Then, according to Eqs. (12.3), (12.9), and (12.12), the observed apparent field at
solar position \( i \) is
\[ B_{\text{obs},i} = (\delta \Phi_{s,i} + \Phi_{w,i})/A . \] (12.13)
The fraction \( R_s \) of the total flux that occurs in strong-field form is defined by
\[ R_s = \sum_i |\Phi_{s,i}|/\Phi_{\text{tot}} . \] (12.14)

We now want to determine the value of \( R_s \) from observational data. This was
done by Howard and Stenflo (1972) and Frazier and Stenflo (1972) through the
use of the line pair Fe I 5250.22 (line 1) and Fe I 5232.95 Å (line 2). While line
1, as we have seen, has a small value of \( \delta \) due to both Zeeman saturation and
temperature line weakening, these effects are rather insignificant for line 2, which
is much broader, has a smaller Landé factor (1.3), and is much less affected by
temperature effects in the line wings. We may therefore assume that \( \delta \approx 1 \) for line
2.

To deduce the value of \( R_s \) we need another observable (besides \( \delta \), which is
determined from the slope of the regression line) that is sensitive to the effect of
the weak-field fluxes \( \Phi_w \). Such an observable can be obtained from the data in the
scatter-plot diagram by forming the ratio \( \rho \) between the total apparent fluxes in
the two lines:
\[ \rho = \sum_i |B_{\text{obs},1}|_i / \sum_i |B_{\text{obs},2}|_i , \] (12.15)
or, using Eq. (12.13),
\[ \rho = \sum_i |\delta \Phi_{s,i} + \Phi_{w,i}|/\Phi_{\text{tot}} , \] (12.16)
since according to Eqs. (12.11) and (12.13)
\[ \Phi_{\text{tot}} = \sum_i |\Phi_{s,i} + \Phi_{w,i}| . \] (12.17)

Let us now make the reasonable assumption that as soon as there are any
strong-field contributions at all for a given solar position \( i \) (i.e., when \( \Phi_{s,i} \neq 0 \)),
then that contribution dominates over the weak-field one within the same
resolution element, which implies that \( |\Phi_{w,i}| < \delta |\Phi_{s,i}| \). This assumption allows us
to write
\[ |\delta \Phi_{s,i} + \Phi_{w,i}| = \delta |\Phi_{s,i}| + \epsilon_i |\Phi_{w,i}| , \] (12.18)
where \( \epsilon_i = +1 \) or \(-1 \), depending on whether \( \Phi_{s,i} \) and \( \Phi_{w,i} \) are of equal or opposite
signs. For the special case that \( \Phi_{s,i} = 0 \), Eq. (12.18) is always valid with \( \epsilon_i = +1 \).
Using Eqs. (12.14), (12.16), and (12.18),
\[ \rho = \delta R_s + \sum_i \epsilon_i |\Phi_{w,i}|/\Phi_{\text{tot}}. \] (12.19)

If \( \delta \) were equal to unity, then \( \rho \) would also equal unity according to Eqs. (12.16) and (12.17). Eq. (12.19) then becomes
\[ 1 = R_s + \sum_i \epsilon_i |\Phi_{w,i}|/\Phi_{\text{tot}}. \] (12.20)

Subtraction of Eq. (12.19) from Eq. (12.20) finally gives
\[ R_s = \frac{1 - \rho}{1 - \delta}. \] (12.21)

The determinations of \( \rho \) and \( \delta \) from the FeI 5250.22 and 5232.95 Å line pair by Howard and Stenflo (1972) and Frazier and Stenflo (1972) resulted in a value for \( R_s \) that was \( \approx 1 \) within the uncertainties of the observational data. These uncertainties could be translated to a 10% uncertainty in \( R_s \). The conclusion then follows that at least 90% of the total magnetic flux that is recorded by solar magnetographs is in strong, kG form. This figure of 90% has not yet needed revision, but new diagnostic techniques in the near infrared have now provided us with the tools to make more accurate determinations not only of \( R_s \), but of the full distribution of field strengths in different regions on the sun.

In our statement about the 90% limit, the qualification “recorded by solar magnetographs” is important. It means that when the true field distribution across the solar disk is smeared by a spatial window that has the same size as the effective spatial resolution element of the magnetograph observations (which is about 4" × 4" in the case of the observations of Frazier and Stenflo (1972) if we include the seeing effects in addition to the 2.4" × 2.4" sampling aperture), then at least 90% of that flux is due to kG fields. Note that this is a resolution-dependent statement. Thus the properties of a hypothetical small-scale turbulent magnetic field of the kind discussed in Chapter 10 are not constrained. For such a field the magnetic polarities are mixed on a small scale so that the flux contributions cancel out when averaged with a spatial window of a few arcsec. The turbulent field is therefore not “seen” by the magnetograph and is accordingly unconstrained by the method discussed in the present section, although it may carry total fluxes much in excess of the 10% weak-field flux limit. For a detailed discussion of how to diagnose this turbulent field we refer to Chapter 10.

12.6. Integrated Stokes \( V \) Profiles

In connection with the Taylor expansion (12.1) we noted that the Stokes \( I \) profile \( I_m \) of the flux tube is not directly an observable, since the actually observed Stokes \( I \) profile has its main contributions from the surroundings. In the case of weak magnetic fields, however, when the higher-order terms in the expansion can be
neglected, \( I_m \) may be retrieved directly through integration, since \( I_m \sim \int V \, d\lambda \), although the scale factor \( \alpha_m \Delta \lambda_H \) remains an unknown.

For the purpose of a statistical analysis of the Stokes \( V \) spectrum it is therefore useful to define the integrated Stokes \( V \) profile \( I_V \) by

\[
\frac{I_{V_c} - I_V}{I_c} = -\frac{1}{\alpha_m \Delta \lambda_H} \int_{\lambda_1}^{\lambda} \frac{V(\lambda')}{I_c} \, d\lambda',
\]

(12.22)

where \( I_{V_c} \) is the continuum intensity in the magnetic element on the blue side of the line (see below), and \( I_c \) is as before (cf. Eq. (12.6)) the continuum intensity averaged over the resolution element. The lower integration limit should be chosen to be sufficiently far to the blue side of the line so that there is no significant Stokes \( V \) signal for \( \lambda < \lambda_1 \) (cf. Solanki and Stenflo, 1984).

Under the assumptions used for Eq. (12.1) (height-independent field) \( I_V \) is clearly identical to \( I_m \) in the weak-field limit, and this limit represents a very good approximation for almost all visible spectral lines except narrow lines with a large Landé factor, like the Fe I 5250.22 Å line. It is nevertheless important to make a distinction between \( I_m \) and \( I_V \), since real, observed Stokes \( V \) profiles are asymmetric due to correlated gradients of the magnetic and velocity fields, which is not accounted for in Eq. (12.1).

The situation is illustrated in Fig. 12.8, from Solanki and Stenflo (1985). The diagram to the left shows an observed Stokes \( V \) profile with definitions of the amplitudes \( a_b, r \) and areas \( A_b, r \) of the blue and red line wings. Not only is there an amplitude asymmetry (\( a_b \neq a_r \)), but an area asymmetry (\( A_b \neq A_r \)) as well. An area asymmetry of the observed magnitude can only arise if there are correlated magnetic and velocity field gradients along the line of sight (cf. Auer and Heasley, 1978). Since \( A_b - A_r = \int V \, d\lambda \neq 0 \), the continuum levels of the integrated \( V \) profile \( I_V \) will be different on the blue and red sides of the line, as indicated by the middle diagram of Fig. 12.8. Thus, if as defined by Eq. (12.22), \( I_V = I_{m_c} \) on the blue side, then \( I_V < I_{m_c} \) on the red side. Such an \( I_V \) profile cannot represent a Stokes \( I_m \) profile of a real magnetic element.

For a statistical regression analysis of the \( I_V \) profiles on an equal basis with the ordinary Stokes \( I \) profiles, \( I_V \) needs to be differentially renormalized to force the continuum levels on both sides of the line to be equal. This is done through multiplication of the blue Stokes \( V \) wing by \( \sqrt{A_r/A_b} \), the red wing by \( \sqrt{A_b/A_r} \), before \( V \) is inserted for integration in Eq. (12.22). With this procedure the \( I_V \) profile emerges with a single-level continuum, as illustrated by the diagram to the right in Fig. 12.8. These differentially renormalized \( I_V \) profiles lend themselves to the same parametrization as was used for the Stokes \( I \) profiles in Fig. 10.6.

### 12.7. Regression Analysis of the Stokes \( V \) Profiles

We now parametrize the differentially renormalized \( I_V \) profiles and the simultaneously recorded Stokes \( I \) profiles, using for both the parametrization described by Fig. 10.6. The main parameters are the ones described in Sect. 10.14: the relative line depth \( d \), the area \( S \) of the profile below the half-level chord (the shaded area
Fig. 12.8. Illustration of the conversion of a Stokes $V$ profile to an $I_V$ profile. Left panel: Observed Stokes $V$ profile of the Fe I 5250.22 Å line. Middle panel: Relative line depth of the $I_V$ profile derived from Stokes $V$ according to Eq. (12.22). Right panel: The $I_V$ profile after differential renormalization, to make the continuum levels on the red and blue sides of the line equal. From Solanki and Stenflo (1985).

of Fig. 10.6), and the width $v_D$ of the half-level chord expressed in velocity units as the Doppler width of a Gaussian with the same half width. The parameters that have been extracted from the $I_V$ profiles are marked by index $V$, those extracted from the $I$ profiles by index $I$. Additional parameters that will be used are the central wavelengths $\lambda_I$ and $\lambda_V$, and for the Stokes $V$ profiles the amplitude and area asymmetries $\Delta a = a_b - a_r$ and $\Delta A = A_b - A_r$. $\lambda_V$ is the wavelength of the zero crossing of the Stokes $V$ profile. The corresponding quantity for the Stokes $I$ profile would be the wavelength of its minimum, but in practice we use for $\lambda_I$ the center of gravity of the lower half of the Stokes $I$ profile, since this quantity is less affected by noise.

The power of this type of statistical approach does not lie in the analysis of the Stokes $V$ parameters $d_V$, $v_{D_V}$, etc., by themselves, but in the analysis of the difference between these Stokes $V$ parameters and the corresponding Stokes $I$ parameters. While it is hard to achieve high accuracy in the absolute values of the derived physical parameters of the atmosphere, the difference between the interior and exterior of the flux tube can be obtained with a high degree of confidence by employing such a differential approach.

For the differential line width $v_{D_V} - v_{D_I}$ we may use a regression equation of the same form as the one of Eq. (10.65) for $v_{D_I}$ alone, i.e.,

$$v_{D_V} - v_{D_I} = x_1 + x_2 S_I + x_3 S_I^2 + x_4 v_0 x_5 + x_5 g_{\text{eff}}^2 \lambda^2 / v_0,$$

where $v_0$ is a second-order polynomial fit of $v_{D_I}$ to $S_I$. In comparison with Eq. (10.65) we have omitted the $x_6$ term, which was only essential when searching for the minute effects of a turbulent magnetic field.

Like the turbulent case the $x_5$ coefficient may be translated to a field strength, but in the present case the derivation is different, since we assume a longitudinal magnetic field (for observations at the center of the solar disk) and a two-component model with filling factor $\alpha_m$. We further assume that the line profile is
Gaussian and that the Zeeman splitting is small in comparison with the line width. Let us denote by \( w_c \) the relative change \( I_m / I_{nm} \) of the continuum level in the flux tubes, and let as before \( w_T \) represent the thermal line weakening (cf. Eq. (12.9)), assuming that only the line depth but not the line width is affected by the differential thermodynamics. With the numerical constant \( z \) given by Eq. (10.62) we can then from Eqs. (12.1) and (12.22) determine the field strength that corresponds to the \( x_5 \) coefficient:

\[
B_{vD} = \frac{\sqrt{3x_5}}{zc} \left(1 - 3w_c w_T \alpha_m \right)^{-1/2}.
\]  

(12.24)

Here we have used index \( vD \) for the field strength \( B \) to indicate that it has been derived from the differential \( vD \) parameter. The source of the first term (unity) in the square root is the \( I_V \) profile, while the second term with the \( w \) factors is due to the field-dependent contribution to Stokes \( I \) in Eq. (12.1).

For the differential line-depth effects \( (dV/dI) \) a similar but slightly modified regression equation has been found to work well:

\[
\ln(dV/dI) = x_1 + x_2 S_I + x_3 S_I^2 + x_4 \chi_e + x_5 \chi_e S_I + x_6 g_{\text{eff}} \chi^2 / v_0^2.
\]  

(12.25)

An interpretation of the magnetic-field dependent coefficient \( x_6 \) with the same assumptions as were used for Eq. (12.24) results in the same expression for the corresponding field strength \( B_d \) derived from the differential line depth, if we simply replace \( x_5 \) in Eq. (12.24) by \( x_6 \).

Regression analysis with Eqs. (12.23) and (12.25) has been applied to FTS polarimetric observations of the 402 Fe I lines that were used by Stenflo and Lindgren (1977) for the turbulent magnetic field. The derived field strengths fall in the range 1.4–1.7 kG for both \( B_{vD} \) and \( B_d \) (Solanki and Stenflo, 1984), which is consistent with the results of the line-ratio method when one accounts for the circumstance that field strengths of the regression analysis refer to somewhat larger depths in the atmosphere than the line-ratio method. The reason for this is that by accounting for the \( S_I \) dependence in the regression equation one effectively reduces the differential line widths or depths to the case of \( S_I = 0 \), i.e., to the case of the weakest lines, which are the ones that are formed the deepest, where the field strengths are the largest. In contrast the values derived from the line ratio refer to the height of formation of the wings of medium-strong lines.

Figure 12.9 gives an example of a scatter plot of \( \ln(dV/dI) \) vs. \( S_I \) for a network region at the center of the quiet solar disk. The diagram to the left shows a plot of the original data, where Fe I lines with \( \chi_e < 3 \text{ eV} \) are represented by stars, those with \( \chi_e \geq 3 \text{ eV} \) by open circles, Fe II lines by filled squares. We may generalize Eq. (12.25) to include the Fe II lines by replacing \( \chi_e \) by \( \chi^* = \chi_e + \chi_I \), where \( \chi_I \) is the ionization potential for neutral iron (and is set = 0 for the Fe I lines, 7.87 eV for the Fe II lines). The \( \chi^* \)-dependent terms in the regression equations then have to be slightly modified as described in Solanki and Stenflo (1985). If we make the regression fit and then subtract from the \( \ln(dV/dI) \) values the terms that depend on \( \chi^* \) and \( g_{\text{eff}} \), then we obtain the diagram to the right of Fig. 12.9, which represents \( \ln(dV/dI) \) for the case of zero \( \chi^* \) and Landé factor. The greatly reduced scatter in the right diagram is evidence that the regression model used is good.
Since the magnetic filling factor is unknown, the apparent values of \( d_V/d_I \) directly evaluated from the observed Stokes \( I \) and \( V \) spectra deviate from the “true” values by a certain factor. In terms of \( \ln(d_V/d_I) \) the filling factor represents an offset, which however is common to all the points of Fig. 12.9 (since all the spectral lines have been recorded strictly simultaneously with the FTS). It is the shape of the dependence on \( S_I \) and \( \chi^* \) that contains the information on the temperature structure, regardless of where the zero point of the \( \ln(d_V/d_I) \) diagram is. The zero point in Fig. 12.9 has been chosen such that it corresponds to \( \alpha_m B = 1 \) G in Eq. (12.22). By choosing this small value for the average field strength, the values of \( \ln(d_V/d_I) \) become correspondingly large.

When we compare with model calculations, which are done for a filling factor of unity, we may shift the zero point of the diagrams such that the results for the Fe\( \Pi \) lines, which have little temperature sensitivity, overlap. Figure 12.10, from Solanki (1984), shows such a comparison, using the same observational data as in the left diagram of Fig. 12.9. The region where the Fe\( \Pi \) data fall is marked by dark shading, that of the Fe\( \I \) data by light shading. The various plotted curves represent predictions by a number of previously published facular models (which are now outdated). The label ‘1’ to the left of the curves refer to the predictions for Fe\( \I \), the label ‘2’ to the predictions for Fe\( \Pi \). It is obvious from Fig. 12.10 that many of the previous models can immediately be ruled out by such a diagnostic diagram. The \( d_V/d_I \) ratio accordingly places quite stringent constraints on the allowed temperature structure of the flux tubes. These constraints have served as a starting point for a flurry of new empirical models of flux tube thermodynamics (reviewed by Solanki, 1993). This model building activity has later been augmented by
the introduction of Stokes inversion methods, which may use the $d_V/d_I$ models as an “initial guess” in a perturbation approach to solve the non-linear inversion problem.

12.8. Diagnostics of Flux Tube Dynamics

Since the dynamical state of the flux tube interior greatly differs from that of the external atmosphere, we have to use Stokes $V$ data to isolate the flux tube contribution from its surroundings and apply differential methods as we did in the preceding section for the temperature diagnostics. Only then can the difference between the interior and exterior be accurately determined.

The main diagnostic parameters for the dynamics are: (1) The Stokes $V$ zero crossing wavelength $\lambda_V$, which contains information on the systematic mass flows inside the flux tubes. (2) The Stokes $V$ amplitude and area asymmetries, which provide information on the correlations between the magnetic and velocity field gradients. (3) The line width difference $v_{D_V} - v_{D_I}$ between the $I_V$ and $I_I$ line profiles, which gives information on the difference of the non-thermal line broadening between flux tube interior and exterior. (4) The center-to-limb variation of the above parameters, which tells us about the directional properties of the motions and gives some height information as well.
In Fig. 12.11, from Stenflo et al. (1987a), we illustrate the center-to-limb variation of the Stokes $V$ zero crossing wavelength $\lambda_V$ relative to the position $\lambda_I$ of the Stokes $I$ profile, expressed in velocity units by plotting $c(\lambda_V - \lambda_I)/\lambda_I$. The observational data, obtained with the FTS polarimeter, are given for the Fe I 5250.22 (filled squares) and 5247.06 Å (crosses) lines. It is important to note that $\lambda_I$ does not constitute an absolute wavelength reference, since the average Stokes $I$ profiles are Doppler shifted due to the velocity-brightness correlations in the solar granulation. Balthasar (1984) has determined the absolute value of $\lambda_I$ for the undisturbed, non-magnetic atmosphere with respect to laboratory wavelengths (after subtraction of the gravitational redshift and the relative motion between the observer and the solar region observed). The solid line represents the negative value of the $\lambda_I$ determined this way. This curve can accordingly be regarded as the true zero level to which $\lambda_V - \lambda_I$ should be referred. It is well supported by absolute wavelength calibrations of Stokes $V$ profiles near disk center (Solanki, 1986).

**Fig. 12.11.** Center-to-limb variation of the position of the Stokes $V$ zero crossing wavelength with respect to the position of the Stokes $I$ profile, $c(\lambda_V - \lambda_I)/\lambda_I$, in velocity units. The filled squares represent the Fe I 5250.22 Å, the crosses the Fe I 5247.06 Å line. The solid curve represents $-c(\lambda_I - \lambda_0)/\lambda_V$, where $\lambda_0$ is an absolute line center wavelength for the respective spectral line (the laboratory wavelength, adjusted to include the effects of the gravitational redshift and the motion of the observer). This curve serves as the corrected zero level for the Stokes $V$ Doppler shifts. From Stenflo et al. (1987a).

If we disregard the point at $\mu = 0.1$, which is very close to the solar limb, the observational points hardly deviate from the solid curve by more than their error bars. This sets an upper limit of about 0.2 km s$^{-1}$ to the possible net mass flows inside the flux tube, which is a strong constraint in view of the much larger average downdraft velocities, typically 0.5 km s$^{-1}$ or more, observed at the location of flux concentrations (Simon and Leighton, 1964; Tanenbaum et al., 1969; Frazier, 1970), and the very large velocities of several km s$^{-1}$ needed to explain...
the Stokes $V$ asymmetries and line widths (see below). This result of essentially zero downdrafts inside the flux tubes is gratifying from a theoretical point of view, since even a small downdraft velocity in the photosphere would as a consequence of mass conservation lead to drainage of the thin corona in a matter of minutes (because diffusion across the field lines from the external atmosphere in the denser layers is too slow to be able to replenish the mass loss).

The left diagram of Fig. 12.12, from Solanki and Stenflo (1985), shows the variation of the absolute amplitude asymmetry $a_b - a_r$ with line strength $S_I$ for the same set of 400 Fe I and 50 Fe II lines that were used for the temperature and magnetic field diagnostics in the preceding section. The portions of the diagram where the lines of different excitation potential or ionization stage fall are marked with different shadings. To theoretically reproduce such Stokes $V$ asymmetries large mass motions with correlated magnetic and velocity field gradients are required. This general interpretation in terms of mass motions is strongly supported by the diagram to the right in Fig. 12.12, from Solanki (1985), which gives the magnitude of the macroturbulent velocity $\xi_{\text{mac}}$ that is needed to reproduce the observed widths of the $I_V$ profiles and the $v_{D_V} - v_{D_I}$ data. The qualitative similarity between the asymmetry $a_b - a_r$ and the macroturbulent broadening parameter $\xi_{\text{mac}}$ in their dependence on $S_I$ and $\chi^*$ is evidence that mass motions are a common source of both diagrams.

![Fig. 12.12.](image.png)

The high values of several km s$^{-1}$ for $\xi_{\text{mac}}$ show that we are dealing with quite large non-thermal velocities inside the flux tubes. Still the Stokes $V$ zero-crossing data of Fig. 12.11 tell us that these large-amplitude motions do not carry any significant net mass flow. One type of motion that could reconcile these apparently conflicting requirements is flux tube oscillations, which lead to Doppler line broad-
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enning when the FTS polarimeter performs temporal and spatial averaging over flux tubes in different phases of their oscillations (Solanki, 1989; Grossmann-Doerth et al., 1991).

The area asymmetries $A_b - A_r$ respond very differently to the flux tube dynamics as compared with the amplitude asymmetries $a_b - a_r$. While the amplitude asymmetry can be due to correlated magnetic and velocity field gradients both parallel and perpendicular to the line of sight in the layer where the Stokes $V$ maxima are formed, the observed area asymmetry can only be caused by correlated line-of-sight gradients, as first noted by Illing et al. (1975). Furthermore, as the area asymmetry is an integral quantity, it is formed over a larger range of heights.

A unique feature of the area asymmetry is its center-to-limb variation, which exhibits a sign reversal at $\mu \approx 0.4$. This is illustrated in Fig. 12.13, from Bünte et al. (1993b), for the Fe I 5250.22 Å line. Thus the red Stokes $V$ area dominates over the blue one on the limbward side of $\mu \approx 0.4$. In contrast, the amplitude asymmetry does not exhibit any such sign change.

![Fig. 12.13. Center-to-limb variation of the relative area asymmetry $\delta A = (A_b - A_r)/(A_b + A_r)$ for the Fe I 5250.22 Å line. The data, from Stenflo et al. (1987a), are represented by the filled squares. The theoretical curves, from Bünte et al. (1993b), differ in the assumed values for the flow speed in the external convection cell responsible for the area asymmetry (cf. Sect. 12.10 and Fig. 12.18).](image)

Models that try to reproduce the area asymmetry of course also have to satisfy the constraint of no zero-crossing shifts ($\lambda_V - \lambda_I \approx 0$). Using previous ideas of van Ballegooijen (1985), Grossmann-Doerth et al. (1988b, 1989a) have found a way to accomplish this. The condition of no zero-crossing shift of the Stokes $V$ profile is
satisfied when there are no mass flows inside the magnetic flux tube. Still a flow in the non-magnetic region immediately outside the flux tube can affect the $V$ profile and produce an area asymmetry when the line of sight crosses the boundary between the magnetic and non-magnetic regions. Although only unpolarized radiation enters the flux tube from the non-magnetic region, it affects the formation of the polarized profiles and produces an area asymmetry when the incident unpolarized absorption profile is Doppler shifted to an asymmetric position with respect to the two Zeeman-shifted sigma components of the absorption coefficient in the magnetic region. In Sect. 12.10 we will provide an illustration of a model that can reproduce the observed center-to-limb variation of the area asymmetry. The three curves in Fig. 12.13 have been obtained from such a model, which has an external convection cell with a cool downflow near the flux tube boundary and a hot upflow further away. The curves differ in the flow velocities assumed.

The observational data thus lead us to the following general picture of the flux tube dynamics. The field that is of kG strength in the lower photosphere diverges with height, forming “canopies” that overlie the non-magnetic atmosphere. Inside the magnetic region large-amplitude oscillations cause substantial non-thermal line broadening and may be a source of waves that could contribute to the heating of the upper layers of the flux tube. In the external, non-magnetic atmosphere below the canopies a flow pattern develops, with inflow and downdrafts immediately outside the flux tube. The situation is reminiscent of the drain in a bathtub, accompanied by the development of a whirl flow (Stenflo, 1975), which may help to stabilize the flux tube against the fluting or interchange instability (Schüssler, 1984; Bünte et al., 1993a,c). At the same time this picture explains the observed correlations between flux concentrations and downdrafts (Simon and Leighton, 1964; Tanenbaum et al., 1969; Frazier, 1970), without the need for systematic downdrafts inside the flux tubes.

12.9. Infrared Diagnostics

While the widths of spectral lines are to a first approximation proportional to wavelength, the Zeeman splitting increases with the square of the wavelength. Therefore the incomplete Zeeman splitting in the visible part of the spectrum can become complete for certain lines in the infrared. When the splitting is complete, the field strength can be read off directly from the wavelength separation of the sigma components, regardless of the magnitude of the magnetic filling factor. Harvey and Hall (1975) first demonstrated this for the $\text{Fe I}$ 1.5649 $\mu$m line, which has a Landé factor of 3.0.

The near infrared region around 1.6 $\mu$m is of special interest, since the solar opacity has a minimum at these wavelengths, which means that the radiation emanates from deeper layers in the atmosphere than in any other part of the solar spectrum. This enhances the Zeeman splitting still further, since the field strength increases with depth.
12.9.1. Weak and Strong Field Regimes

The amount of circular polarization (Stokes $V$) that we measure in a spectral line is primarily a measure of the magnetic flux through the resolution element (i.e., of the magnetic filling factor), but it does not by itself provide any information on the field strength. It is the shape of the Stokes $V$ profile that carries the field-strength information. The situation may be clarified by distinguishing between three splitting regimes (outside sunspot umbrae): (1) The weak-field regime, which applies to the great majority of lines in the visible part of the solar spectrum, and for which $V \sim \partial I / \partial \lambda$ (cf. Eq. (12.1)). (2) The partial splitting regime, an example of which is the Fe I 5250.22 Å line. (3) The complete splitting regime, represented for kG flux tubes by the Fe I 1.5649 μm line.

Figure 12.14, from Stenflo et al. (1987b), illustrates how these various regimes are connected to each other. The left panel shows half the separation between the centers of gravity $\lambda_r$ and $\lambda_b$ of the red and blue Stokes $I$ peaks, $0.5(\lambda_r - \lambda_b)$, expressed in units of $0.425\Delta \lambda_f$, where $\Delta \lambda_f$ is the total half width of the Stokes $I$ profile. On the abscissa the Zeeman splitting $\Delta \lambda_H$ is expressed in the same units. The significance of $0.425\Delta \lambda_f$ is that for a Gaussian profile it represents the distance from the line center to the inflexion point in the wing (where $\partial I / \partial \lambda$ has a maximum). According to Eq. (12.1) the Stokes $V$ peak should be located at this inflexion point in the case of weak fields, implying that half the Stokes $V$ peak separation should be unity in terms of such a normalization if the Stokes $I$ profile were Gaussian. Even for Gaussian profiles there will however be slight deviations from unity in the diagram since the position of the $V$ maximum and the center of gravity of the upper half of the $V$ peak are not entirely identical quantities. The solid and dashed curves correspond to different Voigt profiles used to represent Stokes $I$.

The complete splitting regime is reached when half the $V$ peak separation directly gives the Zeeman splitting $\Delta \lambda_H$ and thus the field strength $B$. It corresponds to the dotted 45° line in the left diagram of Fig. 12.14. The weak-field regime on the other hand represents the case when the $V$ peak separation is merely determined by the inflexion point separation of the Stokes $I$ profile, a quantity that is devoid of field-strength information. The transition between these two regimes, the partial splitting case, occurs over the range $0.5 \leq \Delta \lambda_H/(0.425\Delta \lambda_f) \leq 1.5$.

The two vertical lines mark the splittings in the case of the disk-center profiles of the Fe I 5250.22 Å and 1.5649 μm lines for a field strength of 1 kG. While the infrared line is located well within the complete splitting regime, the 5250.22 Å line lies in the transition zone of partial splitting. The Fe I 5247.06 Å line has a splitting that is 2/3 that of the 5250.22 Å line and is therefore substantially closer to the weak-field level of unity in the diagram. This difference between these two lines contains field-strength information.

The right panel of Fig. 12.14 shows the Stokes $V$ amplitude $V_{\text{max}}$ normalized to its weak-field value $\Delta \lambda_H (\partial I / \partial \lambda)_{\text{max}}$. In the complete splitting regime $V_{\text{max}}$ becomes a constant, independent of the field strength. The functional dependence of the normalized amplitude therefore becomes $\sim 1/\Delta \lambda_H$ in this regime, i.e., a hyperbola (dotted curve). The solid and dashed curves illustrate the transition between the weak-field and complete splitting regimes.
Fig. 12.14. Illustration of the transition from the weak-field to the complete splitting regime, from Stenflo et al. (1987b). The left panel gives half the wavelength separation between the Stokes V peaks vs. Zeeman splitting $\Delta \lambda_H$, both normalized by 0.425$\Delta \lambda_I$, the distance from the line center to the inflexion point of a Gaussian Stokes I profile. The right panel gives the Stokes V polarization amplitude $V_{\text{max}}$ in units of $\Delta \lambda_H (\partial I/\partial \lambda)_{\text{max}}$, its value in the weak-field limit. The dotted curves represent the asymptotic complete splitting limits, extrapolated to small values of the Zeeman splitting. The vertical lines mark the splitting values for the disk-center profiles of the Fe I 5250.22 Å and Fe I 1.5649 μm lines in the case of a 1 kG magnetic field.

If we disregard temperature line weakenings, then the solid and dashed curves can be seen as representing the ratio between the apparent and true magnetic fluxes, or $B_{\text{obs}}/(B)$, when the “observed” field $B_{\text{obs}}$ is derived from the observed $V_{\text{max}}$. This ratio deviates from unity with increasing splitting $\Delta \lambda_H$ due to what we have previously called Zeeman saturation. Thermodynamic effects in the flux tubes (line weakenings) reduce $B_{\text{obs}}$ by an additional factor, but since this factor is common to the Fe I 5250.22 and 5247.06 Å lines, the $V_{\text{max}}$ ratio, i.e., $B_{5250}/B_{5247}$, provides direct information on the differential Zeeman saturation and thus on the field strength.

12.9.2. ZEEMAN BROADENING

Due to the large Zeeman splitting in the infrared, a distribution of field strengths within the resolution element will induce significant Zeeman line broadening. In Fig. 12.7 we showed that the separation of the Zeeman broadening from the thermal and non-thermal Doppler broadening requires the use of an “infrared line ratio”, for example with the “Zeeman line” Fe I 1.5649 μm and the “Doppler line” Fe I 1.5653 μm that were illustrated in Fig. 12.7.

The Zeeman broadening determined this way may be due to different kinds of field-strength distributions: (1) A lateral variation of the field strength across an unresolved flux element. (2) The existence of different flux elements with different
field strengths within the spatial resolution element. (3) A variation of the field strength along the line of sight.

The previously unknown magnetic cross section of a flux element has led to some ambiguity in the determined field strengths, since the field strength amplitude of a flux element determined by any method (line ratio or infrared splittings) depends on the assumed shape of the cross section. Usually a rectangular cross section (single-valued field) has been assumed, which then removes the ambiguity, but empirical support for this assumption has only recently become available.

The field-strength variation along the line of sight can be determined by requiring that both the observed infrared line profiles and the 5250-5247 line ratio (which refers to higher atmospheric layers) should be satisfied simultaneously with a self-consistent magnetohydrostatic flux tube model that is used to derive the synthetic observations that fit the real data. The Zeeman line broadening follows from this line-of-sight variation as an automatic byproduct. In this way Zayer et al. (1989) found that the line-of-sight variation alone produced the observed amount of Zeeman broadening of the Fe I 1.5649 μm line. This speaks in favor of unique flux tubes with rectangular magnetic cross sections within the spatial resolution element, since additional lateral field-strength variations would make the Zeeman broadening too large.

12.9.3. Multi-component modelling

The majority of the Stokes V infrared spectra look similar to the two cases shown in Fig. 12.7 and can be interpreted in terms of our standard two-component model with no horizontal variations of the field strength within the flux elements or from one flux tube to the next within the same spatial resolution element. A number of spectra however show strange looking Stokes V profiles. The most “pathological” of them that has been analysed so far is shown in Fig. 12.15, from Rüedi et al. (1992). No such profiles have ever been seen in the visible due to the smaller Zeeman splitting there. Obviously two-component models are unable to reproduce spectra like that of Fig. 12.15.

It is clear that we have to add at least one additional magnetic component to be able to model such spectra. With each new component one gets more free parameters, so one might fear that the problem could get out of hand with too many free parameters. Fortunately, for all the 27 spectra that have been analysed so far (Rüedi et al., 1992), not a single case has been encountered where more than two magnetic components (plus a third, non-magnetic component) with rectangular cross sections were needed. This intriguing result provides additional support to the concept of discrete flux tubes.

It is important to stress that this model-fitting success is far from being a trivial result. When first attempts to fit the observed spectra were made with a height independent magnetic field (which is unphysical), then a whole distribution of magnetic components were needed, otherwise the Zeeman line broadening could not be fitted. As soon as self-consistent, magnetohydrostatic flux tube models with a diverging field geometry were introduced, however, the Zeeman broadening was
automatically taken care of, and no more than two magnetic components were ever needed.

The model fit to the observed "pathological" spectrum in the left panel of Fig. 12.15 is shown by the dashed curve. The contributions to the Stokes V profile of the Fe I 1.5649 μm line from each of the two magnetic components are shown in the right panel. The corresponding field strengths at geometrical height \( z = 0 \) (\( \tau_{5000} = 1 \), level of continuum formation in the external atmosphere at 5000 Å) are 1.70 and -1.05 kG, respectively. This implies that we have two discrete flux elements of opposite polarities and significantly different field strengths within a few arcsec from each other. A number of other cases requiring two magnetic components have been found, both with the same and with opposite polarities.

### 12.9.4. FIELD-STRENGTH DISTRIBUTIONS

We have seen how we, from two-component modelling of infrared Stokes V spectra like those of Fig. 12.7 and three-component modelling of spectra like that of Fig. 12.15, can derive the field strengths \( B \) at geometrical height \( z = 0 \) for each magnetic component. The results for all the 27 spectra that have been analysed so far are summarized in Fig. 12.16. The left diagram shows \( B(z = 0) \) vs. \( \langle B \rangle \), where \( \langle B \rangle \) is the field strength of the component when averaged over the spatial resolution element. The open circles represent spectra for which only a single mag-
nentic component was needed for the fit, the pluses the cases where two magnetic components were needed (each such spectrum is thus represented by two pluses). A regression line has been drawn through the points above 1.3kG (which have a plasma \( \beta \) below unity, see below), to indicate the slight filling-factor dependence of \( B \) for the “normal” kG flux elements. This weak dependence is consistent with the line-ratio results of Fig. 12.3.

Fig. 12.16. Left panel: \( B(z = 0) \) vs. \( \langle B \rangle \), the flux of a magnetic component divided by the area of the spatial resolution element. The open circles represent spectra that could be fit with a single magnetic component, the pluses the spectra requiring two magnetic components (there are thus two pluses for each such spectrum). Right panel: Histogram of the magnetic flux (proportional to \( \langle B \rangle \)) contributed by all the magnetic components vs. the plasma \( \beta(z = 0) \) within each magnetic component. The vertical dashed line at \( \beta = 1.8 \) marks the boundary for stability against convective collapse according to the linear stability theory of Spruit and Zweibel (1979). From Rüedi et al. (1992).

We see from this diagram that for large filling factors (\( \langle B \rangle > 140 \) G) all field strengths \( B(z = 0) \) are > 1.4 kG, while for smaller filling factors the incidence of weaker fields become increasingly abundant, although these weak fields generally correspond to the weaker component in models with two magnetic components within the resolution element, except for two cases (the two open circles that fall significantly below the regression line). As a matter of fact this analysis (Rüedi et al., 1992), independently complemented by the results of Rabin (1992a,b, see below), constituted the first real identification of intrinsically weak flux elements. The lowest field strength found so far is about 0.4 kG for the three-component case (two magnetic and one non-magnetic components) according to Fig. 12.16, 0.75 kG for the two-component case. The 0.75 kG case is the one that was shown in the lower panel of Fig. 12.7, where the Stokes \( V \) amplitude of the “Doppler line” became suppressed due to the incomplete Zeeman splitting. In contrast the upper panel of Fig. 12.7 represents the case when the Zeeman splitting is complete for both the Zeeman and Doppler lines.

Each of the determined magnetic components can instead of its field strength \( B(z = 0) \) also be represented by its plasma \( \beta = 2\mu_0 P/B^2 \) at height \( z = 0 \), where
$P$ is the gas pressure inside the flux tube. $\beta$ is the ratio between the gas pressure and the magnetic pressure. The right panel of Fig. 12.16 shows a histogram that gives the relative contributions to the magnetic flux from the various magnetic components per unit interval in plasma $\beta$. We see that the distribution is peaked around $\beta \approx 0.3$, but it has a low tail that extends to high values of $\beta$. The dashed vertical line in the diagram, at $\beta = 1.8$, is the theoretical limit below which flux tubes are stable against convective collapse according to the linear stability analysis of Spruit and Zweibel (1979). According to this theory flux elements with $\beta > 1.8$ spontaneously collapse to a strong-field state. Our empirical values of $\beta$ are however about a factor of six smaller than this theoretical limit, which implies a very low internal gas pressure and thus highly evacuated flux tubes.

Although weaker intrinsic fields are more abundant when the observed flux values are small, their accumulated contribution to the total magnetic flux still remains small. Thus, if we add up all the absolute flux values $|\langle B \rangle|$ in Fig. 12.16, then only about 10% of this total flux is due to fields with $B(z = 0) < 1.25$ kG. Although the statistical sample is limited, this result is consistent with the old line-ratio result of Howard and Stenflo (1972) and Frazier and Stenflo (1972) that more than 90% of the total flux is in strong-field form. The diagnostic advantages in the infrared will however allow us to explore this 90% statement in much greater quantitative detail.

Much improved field-strength statistics has become possible with the Near-Infrared Magnetograph (NIM) of Rabin (1992a,b), with which the polarized spectrum around the Fe I 1.5649 $\mu$m line is recorded with a two-dimensional InSb detector array. Thereby it becomes possible to produce 2-D maps of the intrinsic field strength across the solar surface. Initial results with NIM are shown in Fig. 12.17, from Rabin (1992a). The histogram to the right shows the observed distribution of intrinsic field strengths with NIM, while the histogram to the left represents the apparent field strengths $B_{\text{obs}}$ (cf. Eq. (12.9)) recorded across the same area of the solar disk with the solar magnetograph of the National Solar Observatory. We see that $B_{\text{obs}}$, which is primarily a measure of the magnetic flux through the spatial resolution element, has little to do with the intrinsic field strengths but is more a measure of the filling factor. While all the values of $B_{\text{obs}}$ are below 0.5 kG, most values of $B$ occur in the range 1.1–1.5 kG, with only a weak tail of the distribution extending below 1 kG. This also supports the conclusion that most of the magnetic flux in the solar photosphere occurs in kG form.

The field-strength values of Fig. 12.17 have been obtained directly from the Zeeman splitting without any model fitting. They therefore represent the values of $B$ at the height of formation of the sigma components, which is unknown without model fitting. In contrast the $B$ values of Fig. 12.16 refer to geometrical height $z = 0$ and have been obtained with a magnetohydrostatic model that is able to reproduce the full observed Stokes $V$ profiles. As $z = 0$ lies below the level where the sigma components are formed, the $B$ values of Fig. 12.16 are systematically larger than those of Fig. 12.17. The two sets of results are thus fully consistent with each other.
12.9.5. The 12 μm Region

The spectral region around a wavelength of 12 μm contains some emission lines that have been identified as due to Mg I and that exhibit strikingly large Zeeman splittings, as first noted by Brault and Noyes (1983). Of particular interest is the Mg I 12.32 μm line, which has been explored polarimetrically by Hewagama et al. (1993). Most 12 μm work has been done in sunspots, but some recordings in active plage regions indicate Zeeman splittings corresponding to intrinsically weak fields there, about 300–500 G.

At a first glance these small values may seem to contradict the kG field strengths that have been found with other methods, but they are in fact consistent with the empirical flux tube models when one takes into account the difference in height of formation of the spectral lines used. According to Carlsson et al. (1992) the Mg I 12.32 μm emission line is formed at a height in the upper photosphere where the field strength in a magnetohydrostatic flux tube with $B(z = 0) \approx 1.6 \text{kG}$ has fallen to a value of 300–500 G due to the expansion of the flux tube as the external gas pressure exponentially decreases. The 12 μm data are thus consistent with the results of other diagnostic methods, as shown by detailed non-LTE radiative-transfer calculations of Mg I line formation in a flux tube atmosphere (Bruls and Solanki, 1994). Although this region of the spectrum is technologically difficult to use for high-resolution observations, in particular due to the diffraction limit of normal-size solar telescopes, it has considerable future potential for the diagnostics of small-scale magnetic fields.

12.10. Model Building

Early attempts to derive the temperature-density structure of magnetic field concentrations, so-called facular models, were based on the use of unpolarized spectra and the assumption of a plane-parallel stratification of the atmosphere. We now know that such an approach is entirely inadequate, since the observed Stokes $I$
spectra are dominated by the contributions from the non-magnetic atmosphere, and the magnetic field elements have a geometry that is vastly different from that of a plane-parallel stratification. Only through the use of polarized spectra can the contributions from the flux elements be separated from the surroundings.

If we neglect magnetic tension forces, which are only important where the curvature of the magnetic field lines is large, then horizontal force balance requires equality between the external gas pressure \( P_{nm} \) of the non-magnetic surroundings and the total pressure inside the flux element, due to the gas pressure \( P_m \) and the magnetic pressure \( B^2/(2\mu_0) \):

\[
P_m + \frac{B^2}{2\mu_0} = P_{nm}.
\]  

(12.26)

Although this means that \( P_m < P_{nm} \), it does not necessarily imply that the density \( \rho_m < \rho_{nm} \), since this depends on the temperature difference \( \Delta T = T_m - T_{nm} \), which is determined by the energy transport mechanisms. Nevertheless it turns out that the flux tubes are in fact highly evacuated, as we have seen from the low empirical values of the plasma \( \beta \) in the preceding section (Fig. 12.16). \( \beta \approx 0.3 \) means that \( P_m \) is about 30% of the value of \( B^2/(2\mu_0) \) and thus only 20–25% of \( P_{nm} \) according to Eq. (12.26). Such a large reduction of the internal gas pressure can only be achieved if the density is also substantially reduced, which implies lower opacities in the flux tubes. The surfaces of equal optical depth are thus depressed in the flux elements, which constitutes the so-called Wilson depression that had first been discovered for sunspots.

Eq. (12.26) implies that the directly observed magnetic field structure of a flux element is coupled to its thermodynamic and opacity structure. A stronger magnetic field generally requires a deeper Wilson depression, which means that the height of line formation is lower. The diagnostic information then emanates from a region with larger field strength (since the field strength increases with depth). A change of the internal temperature modifies the opacity and thus shifts the formation heights, with the result that the apparent, observed field strength changes, although the field strength at a given geometrical height may not change. What we see is related to the optical depth scale, and it is the opacity that dictates the relation between the optical and geometrical depth scales.

Another consequence of Eq. (12.26) is that \( B \) decreases with height, since \( P_{nm} \) drops off almost exponentially with height. Due to flux conservation a decreasing value of \( B \) implies an increasing cross section of the flux element. The rate of divergence is determined by \( P_{nm} - P_m \). In the lower layers the expansion rate is moderate and the curvature of the field lines is small. In the upper photosphere and lower chromosphere, however, the field lines flare out to become almost horizontal. In this region with large field-line curvature the tension forces become important and need to be accounted for in the force balance equation. The assumption (12.26) that the tension forces can be neglected is called the thin tube approximation (cf. Roberts and Webb, 1978). It is a good approximation throughout most of the line-forming regions in the photosphere.

To model the flux tube interior we first have to adopt a model for the non-magnetic external atmosphere. A natural first choice is to use some standard,
reference atmosphere model that is representative of the average quiet sun, like the Harvard-Smithsonian Reference Atmosphere HSRA (Gingerich et al., 1971). Due to the Wilson depression the flux tube atmosphere that is accessible to observations however extends to larger depths than those covered by such reference atmospheres, which therefore need to be extended into the convection zone. Such an extension has been done for HSRA by combining it with the convection zone model of Spruit (1974). The resulting extended reference model atmosphere is called HSRASP (Chapman, 1979).

The initial flux tube models may be stepwise improved to increasing levels of sophistication. Thus the thin-tube approximation may be abandoned and replaced by the solution of the complete magnetohydrostatic equations for an axially symmetric flux tube with an arbitrarily prescribed temperature structure, as done by Steiner et al. (1986). The flux tube temperature at different heights may then be varied as free parameters, the values of which get determined by fitting synthetic Stokes spectra to the observed ones. Each choice of internal temperature stratification and field strength at a reference level automatically determines the magnetic field structure via the constraints imposed by the magnetohydrostatic equations.

The next step in the hierarchy of models is to include convective flow fields in the surrounding, non-magnetic atmosphere, as illustrated in Fig. 12.18, from Bünte et al. (1993b). In these models the type of flow and the flow amplitudes are again specified as free parameters to be determined by the observations, but the flow pattern and its modifying influence on the field structure are calculated self-consistently from these free parameters.

Models such as that of Fig. 12.18 are used to calculate synthetic polarized spectra through numerical solutions of the LTE polarized radiative transfer equations, e.g. using the DELO method of Sect. 11.11. When the flux tubes are being viewed obliquely, which is the case when we approach the solar limb, we need to take into account the contributions from the neighboring flux tubes as well. Since the flux tubes are optically rather thin in the horizontal direction, the contribution function along the line of sight may sample more than one flux tube. Therefore the structure of Fig. 12.18 has been used as part of the periodic arrangement of flux tubes that was illustrated in Fig. 11.5. The right panel of Fig. 12.18 shows an example of 20 oblique lines of sight with the grid points for the radiative transfer computations marked. The synthetic spectra from all the lines of sight have to be averaged before they may be compared with the observations.

The model of Figs. 12.18 and 11.5 was specially designed to explain the observed center-to-limb variation of the Stokes $V$ area asymmetry that was shown in Fig. 12.13. The sign reversal at a center-to-limb distance of $\mu \approx 0.4$ could not be explained by an outside downdraft alone. A complete convection cell was required, which included an upflow and a temperature-velocity correlation of the kind that is observed in the solar granulation.

The calculation of synthetic Stokes spectra with such sophisticated models requires massive computations on supercomputers and have therefore so far only been used for direct modelling, i.e., for the computation of synthetic spectra for a few hand-picked choices of free model parameters with subsequent visual comparison with the observations. The desired, ultimate approach is however Stokes
Fig. 12.18. Example of an axially symmetric model of a magnetic flux tube with a surrounding convective velocity field. It has been used to successfully reproduce the observed center-to-limb variation of the Stokes $V$ area asymmetry of Fig. 12.13. Left panel: Magnetic field lines and flow lines of the velocity field. Right panel: 20 lines of sight at an angle of 70° with the vertical direction, used for the radiative transfer calculations of the synthetic Stokes $V$ spectra. The grid points for the computations (filled circles) have been selected with an adaptive step size method. The flux tube is part of the array of flux tubes illustrated in Fig. 11.5. A line of sight that exits the left boundary enters the model again at the same height at the right boundary, to simulate the effect of neighboring flux tubes. From Bünte et al. (1993b).

Inversion, which was discussed in Sect. 11.13. The free parameters of the physical model are then determined by an iterative least-squares fit to a carefully selected set of observables that respond to the model parameters in ways that are mutually as “orthogonal” as possible. As this approach requires the calculation of synthetic observables for a rather large number of models, it is not yet computationally feasible to use for models as sophisticated as that of Fig. 12.18.

For the application of Stokes inversion to determine the temperature-density stratification of the flux tube interiors it has been found adequate to use the thin tube approximation without any convective mass flows (apart from the line-
broadening turbulent motions). The free parameters of the model are then the
field strength $B$ at a given height, the temperature at several different vertical
grid points, and the macroturbulent broadening velocity, parametrized with a
regression equation such that its value for any line strength and excitation potential
can be obtained. The temperature between the selected grid points is obtained by
cubic spline interpolation. The small set of free parameters in combination with
the thin tube approximation, the requirement of hydrostatic balance, etc., fully
specifies the model and allows synthetic spectra to be computed. A suitable set
of observables is a combination of the magnetic and thermal line ratios, Stokes $V$
area ratios between weak and strong Fe I and II lines, Stokes $V$ line widths, and
the separation between the red and blue Stokes $V$ peaks.

Figure 12.19 illustrates some flux tube properties derived from such Stokes
inversions. The left diagram, from Zayer et al. (1990), shows the variation of
the field strength with geometrical height $z$. The various curves in the diagram
refer to different magnetic filling factors. Although the filling factors vary by an
order of magnitude, the different $B(z)$ curves are practically identical, within the
instrumental noise of the data and the uncertainties of the inversion procedure.
When the field strength is plotted as a function of optical depth $\tau$, however, the
spread of the various $B(\tau)$ curves is very large, since the flux tube temperature and
thus also the opacities vary significantly with the filling factor. As however the flux tubes are so highly evacuated, with a plasma $\beta \approx 0.3$ as shown by Fig. 12.16, the $B(z)$ curves are primarily determined by the gas pressure $P_{nm}(z)$ of the external atmosphere. All $B(z)$ curves therefore resemble the asymptotic case of completely evacuated flux tubes (with $P_m = 0$) and are thus insensitive to the thermodynamic state of the interior.

In the diagram to the right of Fig. 12.19 four flux tube temperature curves are compared with $T_{nm}(z)$ of the surroundings (solid curve). The two dashed curves that run almost parallel to each other have been obtained via Stokes inversion by Keller et al. (1990). While the upper, hotter curve represents a network element with a small filling factor, the lower, cooler curve represents a facular element with a larger filling factor. The two remaining curves represent purely theoretical models that we will discuss later. Common to all these flux tube curves is a temperature deficit in the lower layers with respect to the surroundings and a temperature excess in the upper layers.

What matters for the optical appearance of the flux tubes is not their $T(z)$ curves but their $T(\tau)$ curves. Although the network elements are cooler in the deeper layers at equal geometrical height, they are hotter than their surroundings at equal optical depth. They therefore appear as bright features in continuum radiation. This could be demonstrated directly with speckle polarimetry by Keller (1992), who spatially resolved a flux tube with a diameter of about 200 km and a continuum brightness exceeding the surroundings by 30% near the center of the solar disk. The facular elements with larger filling factors on the other hand are cooler and appear as dark features in disk-center continuum radiation. Using Stokes $V$ profiles of C I lines to diagnose the conditions at the level of continuum formation, Solanki and Brigljević (1992) were able to clarify this filling-factor dependence of the continuum brightness. Since the continuum radiation has no polarized Zeeman component that can be used to isolate the radiation from flux tubes with small filling factors, one has to use spectral diagnostics like deeply forming C I lines to diagnose the continuum layers.

In all the MHD models of flux tubes that we have discussed so far the thermodynamic properties have been left as free parameters to be determined by fitting synthetic Stokes spectra to observed ones. The thermal structure however no longer would remain free if an energy equation were to be included in the MHD problem. In its full generality the energy balance is a formidable problem, which has so far only been approached with incomplete and approximate energy equations.

The main energy transport processes are due to radiation, convection, and waves. While the 2-D radiative transfer problem for flux tube geometries has been taken care of in previous work, convection has been treated in an approximate and idealized way, and wave heating has hardly been included at all. Spruit (1976) did the pioneering work, while treating the radiative transport in the diffusion approximation and the convective transport with a mixing-length theory. The most comprehensive 2-D models of the flux concentrations have been developed for slab geometry by Deinzer et al. (1984a,b), Knölker et al. (1988), Knölker and Schüssler (1988), and Grossmann-Doerth et al. (1989b). Instead of seeking stationary solutions, the time-dependent MHD equations in a compressible medium are
solved. One lets the solution relax towards a stationary end state that describes the properties of the flux concentration. Convection is treated with the mixing-length
Fig. 12.20. Theoretical model of a magnetic element in slab geometry, from Grossmann-Doerth et al. (1989b). The horizontal coordinate scale has been stretched to enhance the visibility of the structure. The densities are in units of $3 \times 10^{-4}$ kg m$^{-3}$, the temperatures in K.

theory and the radiative transfer with a grey atmosphere (wavelength-independent opacity).
Figure 12.20 illustrates the result of such calculations for a slab of thickness 150 km at optical depth $\tau_{5000} = 1$ inside the slab. The final end state has been approached but not yet reached for the model results that are shown in Fig. 12.20. The Wilson depression is about 180 km. Cooling of the lower layers, seen in the figure as depressions of the temperature contours, is due to inhibition by the magnetic field of lateral convective energy transport. The heating of the upper, observable layers is due to radiation from the hot bottom and walls of the fairly transparent flux slab.

The most complete treatment of the 2-D radiative transfer with proper opacities has been done by Steiner and Stenflo (1990) for stationary, axially symmetric flux tubes. Their calculations also lead to radiative heating of the upper flux tube atmosphere. The $T(z)$ temperature curves that result from the models of Grossmann-Doerth et al. (1989b) and Steiner and Stenflo (1990) at the flux tube axis have been given in the right panel of Fig. 12.19 (as the dashed-dotted and the long-dashed curves, respectively), where they may be directly compared with the empirically determined curves from Stokes inversion. The theoretical and empirical curves share the same qualitative properties, namely a temperature deficit with respect to the surroundings in the lower layers, which changes to become a temperature excess higher up. The detailed shapes of the curves however differ, indicating that all the physics has not yet been properly modelled, or that some of the assumptions underlying the empirical models are inadequate (e.g. LTE, simple geometry, no cool surroundings).

The theoretical models may be made more realistic by going to 3-D, as done in the extensive numerical simulations of granular convection by Nordlund (1983, 1986). Although these models can also include magnetic fields, they have not yet been able to treat the detailed physics of flux tubes due to the inadequate spatial resolution of the computational grid. Much progress can be expected in the near future from the application of new numerical techniques with grid refinement procedures on ever more powerful computer systems.