Chapter 3

Galactic dynamics

Galactic dynamics can be divided into different regimes. First, there is the motion of the gas and the stars in the overall galactic potential. On these large scales the stars behave in a first approximation like test masses in a smooth potential for which collisions, near encounters with other stars, can be neglected. The stellar dynamics and the motion of the gas can be used to constrain the potential and the corresponding density distribution of the different galactic components: the stars in the bulge and the disk, the gas located in disk, and the extended dark matter halo.

Second, On small scales the motion of a star is determined by the gravitational potential of many stars in a smooth "background" potential. Depending on the case, it must be distinguished whether the dynamics of a star is strongly affected by individual encounters, collisions, with other stars or not. In this context it is important to consider the difference between collisionless systems and systems with collisions.

In this chapter we describe first simple models for smooth gravitational potentials, the associated density distributions and the expected motion parameters and time scales. Then we consider relaxation (collision) time scales and discuss the impact of collisions on the dynamics.

3.1 Potential theory

In this section we describe the force field for a smooth distribution of mass. There exist simple but powerful analytic formula with give a lot of insight on the motion of test particles in a smooth potential. In particular, we will discuss how the density structure of the Milky Way can be modelled. The description of this section follows the corresponding chapter in the book "Galactic Dynamics" from Binney and Tremaine.

3.1.1 Basic equations for the potential theory

The force $\vec{F}(\vec{x})$ at position $\vec{x}$ on a star with mass m_S is generated by the space distribution of mass $\rho(\vec{x}')$:

$$\vec{F}(\vec{x}) = m_S\,\vec{g}(\vec{x}) = m_S\,G \int \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|}\,\rho(\vec{x}')\,\mathrm{d}^3\vec{x}'\,. \tag{3.1}$$

$\vec{g}(\vec{x})$ is the vector gravitational field, the force per unit mass or the gravitational acceleration.

The gravitational potential $\Phi(\vec{x})$ is defined by

$$\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} \, \mathrm{d}^3\vec{x}' \,, \tag{3.2}$$

which is the integral of the mass distribution weighted by the inverse distance to the point $\vec{x}$. The gradient for the inverse distance is

$$\vec{\mathrm{grad}}_x \Big(\frac{1}{|\vec{x}' - \vec{x}|} \Big) = \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|}$$

and therefore the gravitational field $\vec{g}(\vec{x})$ can be expressed by the gravitational potential according to

$$\vec{g}(\vec{x}) = -\vec{\mathrm{grad}}_x \Phi(\vec{x}) = \vec{\mathrm{grad}}_x \Big(G \int \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|} \, \mathrm{d}^3\vec{x}' \Big) \,. \tag{3.3}$$

The potential $\Phi(\vec{x})$ is very useful because it is a scalar field which can be described and analyzed based on equipotential surfaces. Φ contains the same information as the vector gravitational field $\vec{g}(\vec{x})$ and the acceleration $\vec{g}(\vec{x}$ can follows from the gradient of the potential.

The potential energy of a system follows from an estimate of expected change in potential energy if a small additional mass is added to the system with potential $\Phi(\vec{x})$. If a small increment of density $\delta\rho(\vec{x})$ is added then the change in potential energy is:

$$\delta E_{\mathrm{pot}} = \int \delta\rho(\vec{x}) \, \Phi(\vec{x}) \mathrm{d}^3\vec{x} \,. \tag{3.4}$$

3.1.2 Newton's theorems

Let's start with the simple case of spherical systems to get familiar with the mathematical procedures. Spherical systems are particularly simple because of Newton's theorems.

First theorem of Newton. A body inside a spherical shell experiences no net gravitational force from that shell.

Second theorem of Newton. A body outside a spherical shell experiences a gravitational force equal to the force of a mass point in the center of the shell with the mass of the shell.

Figure 3.1 illustrates the proof of the first theorem. A point P inside the shell is attracted equally strong by opposite shell section "seen" under the same solid angle $\mathrm{d}\Omega$. This is obvious for radial sight lines through the center of the shell because the areas (with surface mass $m_{1,2}$) of the opposite regions are proportional to the distances squared $d_{1,2}^2$ from point P. Thus there is $F_1 = m_1/d_1^2 = m_2/d_2^2 = F_2$

This is also valid for an arbitrary "sight" line (full line) because the tilt angles θ_1 and θ_2 between the tangential surfaces and the cone center lines are equal on both sides. Therefore the surface area defined by the solid angle cones are proportional to $d_1^2/\cos\theta_1$ and $d_2^2/\cos\theta_2$ and the attraction from the opposite sides is also equal.

.

Figure 3.1: Figures for the proof of Newton's first theorem (left) and Newton's second theorem (right).

Inside the shell the gravitational potential is constant because the gravitational force is zero

$$\vec{\text{grad}}_x \Phi = -g = 0\,.$$

The gravitational potential in the shell can be easily calculated for the central point from a radial form of Eq. 3.2 (see also Eq. 3.9)

$$\Phi(0) = -\frac{GM}{R}\,, \tag{3.5}$$

where $M = 4\pi\rho(r)\mathrm{d}r$ is the total mass of a shell with thickness dr and R is the shell radius.

For the proof of Newton's second theorem a trick according to Fig. 3.1 with a special configuration of points p_1, p_2, q_1 and q_2 is needed. We consider two concentric shells with radius R_1 and R_2 and equal mass $M_1 = M_2$. Then one can write the potential for a point p_2 on the outer shell by a surface area region $\delta\Omega$ at point q_1 of the inner shell

$$\delta\Phi(\vec{p}_2) = -\frac{GM}{|\vec{p}_2 - \vec{q}_1|}\frac{\delta\Omega}{4\pi}\,.$$

This potential is equal to the potential for the point p_1 on the inner shell by a surface area region of the outer shell with the same angular dimensions $\delta\Omega$ at point q_2.

$$\delta\Phi(\vec{p}_1) = -\frac{GM}{|\vec{p}_1 - \vec{q}_2|}\frac{\delta\Omega}{4\pi}\,.$$

Thus, there is $\delta\Phi(\vec{p}_2) = \delta\Phi(\vec{p}_1)$ because $|\vec{p}_2 - \vec{q}_1| = |\vec{p}_1 - \vec{q}_2|$ (symmetry) and the summation yields then that the potential due to the entire inner and outer shells are equal

$$\Phi_{\text{shell\_1}}(\vec{p}_2) = \Phi_{\text{shell\_2}}(\vec{p}_1)\,.$$

We know $\Phi_{\text{shell\_2}}(\vec{p}_1) = -GM/R_2$ from Eq. 3.5) and therefore this is also the result for $\Phi_{\text{shell\_1}}(\vec{p}_2)$ for the potential of a point at a radius $R = R_2$ outside a shell with $R_1 < R$ and mass M

$$\Phi_{\text{shell\_1}}(R) = -\frac{GM}{R}\,. \tag{3.6}$$

This outside potential of a spherical shell is equal to the potential of a point with the same mass located at the center.

3.1.3 Equations for spherical systems

Simple equations can be derived for spherical systems using Newton's theorems.

The gravitational force of a spherical density distribution $\rho(r')$ on a star m_S at radius r is determined by the mass $M(r)$ interior to r

$$\vec{F}(r) = m_S\,\vec{g}(\vec{x}) = -\frac{GM(r)}{r^2}\,\vec{e}_r\,, \tag{3.7}$$

where

$$M(r) = 4\pi \int_0^r \rho(r')\,r'^2\,\mathrm{d}r\,. \tag{3.8}$$

The total gravitational potential of a spherical system is the sum of the potentials of spherical mass shells $\mathrm{d}M(r) = 4\pi\rho(r)r^2\mathrm{d}r$ with $r' < r$ (located inside r):

$$\Phi_{r'<r}(r) = -\frac{G}{r}\int_0^r dM(r')$$

and the mass shells at $r' > r$ (located outside r):

$$\Phi_{r'>r}(r) = -G\int_r^\infty \frac{dM(r')}{r'}\,,$$

or written alternatively

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')\,r'^2\,\mathrm{d}r + \int_r^\infty \rho(r')\,r'\,\mathrm{d}r\right]. \tag{3.9}$$

The circular speed $v_c(r)$, which is the speed of a test particle with negligible mass m_S in a circular orbit at radius r, is an important parameter for the characterization of the gravitational potential. The circular speed follows from the equilibrium $F_g(r) = -F_c(r)$ of gravitational force and centrifugal force $F_c = m_S v_c^2/r$:

$$v_c^2(r) = r\,g(r) = r\frac{d\Phi}{\mathrm{d}r} = \frac{GM(r)}{r}\,. \tag{3.10}$$

This can also be expressed with angular velocity

$$\Omega(r) = \frac{v_c(r)}{r} = \sqrt{\frac{GM(r)}{r^3}}\,.$$

The potential energy of a spherical system can be calculated from the incremental potential energy formula 3.4. For a spherical system this can be expressed as a change in potential energy due to the small addition of density in a shell at radius r:

$$\delta E_{\mathrm{pot}}(r) = 4\pi r^2 \delta\rho(r)\,\Phi(r)\,,$$

If we build up a whole spherical mass distribution from inside out by such small spherical mass (density) shell increments then the final potential energy is obtained by integration:

$$E_{\mathrm{pot}} = -\int_0^\infty 4\pi r^2\,\rho(r)\,\frac{GM(r)}{r}\mathrm{d}r = -4\pi\,G\int_0^\infty r\,\rho(r)M(r)\mathrm{d}r\,. \tag{3.11}$$

3.1.4 Simple spherical cases and characteristic parameters

Potential of a point mass. This is a very simple case which is often referred as a Keplerian potential. For a point mass there is

$$\Phi(r) = -\frac{GM}{r}, \quad \text{and} \quad v_c(r) = \sqrt{\frac{GM}{r}}\,. \tag{3.12}$$

The potential energy of a point is $-\infty$ (or not defined).

Potential of a homogeneous sphere. Inside a homogeneous sphere with constant ρ there is:

$$M(r) = \frac{4}{3}\pi\, r^3 \rho\,. \tag{3.13}$$

The circular velocity increases linearly with radius

$$v_c(r) = \sqrt{\frac{GM(r)}{r}} = \sqrt{\frac{4\pi G\rho}{3}}\, r\,. \tag{3.14}$$

The orbital period is then defined by the density ρ

$$T = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G\rho}} \tag{3.15}$$

The potential energy of a homogeneous sphere with radius a, density ρ and total mass $M = (4/3)\pi G\rho a^3$ follows from Eq. 3.11:

$$E_{\rm pot} = -4\pi\, G\rho \int_0^a r\, M(r) {\rm d}r = -\frac{16\pi^2\, G\rho^2}{3} \int_0^a r^4 {\rm d}r = -\frac{16}{15}\pi^2 G\rho^2\, a^5 = -\frac{3}{5}\,\frac{GM^2}{a}\,. \tag{3.16}$$

The gravitational potential of homogeneous sphere with radius a is

$$\Phi(r) = -2\pi G\rho(a^2 - \frac{1}{3}r^2) \qquad \text{for} \qquad r < a\,, \tag{3.17}$$

$$\Phi(r) = -\frac{GM}{r} \qquad \text{for} \qquad r > a\,. \tag{3.18}$$

Gravitational radius. The size of a system is sometimes characterized by the gravitational radius which is defined as ratio between mass squared divided by the total gravitational energy:

$$r_g = \frac{GM^2}{|W|}\,. \tag{3.19}$$

For a homogeneous sphere, where $W = -(3/5)GM^2/a$ the corresponding gravitational radius is $r_g = (5/3)a$. The gravitational radius can be a convenient quantity for the definition of the size of systems which have no sharp boundary (e.g. stellar cluster).

The dynamical time scale. The homogeneous sphere is a useful model for an estimate of the dynamical time scale of a system.

If a mass is released from rest in a gravitational field of a homogeneous sphere then its equation of motion is given by the gravitational acceleration

$$g(r) = \frac{d^2r}{dt^2} = -\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3} r\,.$$

This is the equation of motion for a harmonic oscillator ($\ddot{x} = -\omega^2 x$) with oscillation period $T = (2\pi/\omega) = \sqrt{3\pi/G\rho}$. This is the same time as is required for a full circular orbit (Eq. 3.15).

Thus, there is for a homogeneous sphere not only an unique circular orbital period but also an unique free fall time $t_{\rm ff}$, which is the time it takes for any particle released at rest to fall into the center. This time is

$$t_{\rm ff} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}} = 0.767\,(G\rho)^{-1/2}$$

The dynamical time scale is defined as

$$t_{\rm dyn} = (G\rho)^{-1/2}\,. \tag{3.20}$$

This quantity is of the same order as the free-fall time, the crossing time or the orbital time for a particle. According to our definition there is:

$$t_{\rm dyn} = 1.3\,t_{\rm ff} = 0.65\,t_{\rm cross} = 0.33\,t_{\rm orbit}$$

The dynamical time scale is also a good parameter for the characterization of systems with not to extreme density gradients, as long as ρ is replaced by the mean density $\bar{\rho}$ inside the location of the particle.

$$t_{\rm dyn} \approx (G\bar{\rho})^{-1/2}\,.$$

This relation is therefore used for the characterization of systems like open clusters, globular clusters, bulges of galaxies, or clusters of galaxies.

The Plummer model. Plummer proposed in 1911 a spherical density model with a "soft edge" which can be described by a simple gravitational potential

$$\Phi(r) = -\frac{\rm GM}{\sqrt{\rm r^2 + b^2}}\,. \tag{3.21}$$

The corresponding density can be described by

$$\rho(r) = \frac{3M}{4\pi}\,\frac{b^2}{(r^2+b^2)^{5/2}}\,. \tag{3.22}$$

Thus, there is a density distribution like for a homogeneous sphere for $r < b$ without a sharp edge but with a steep density fall off like $\propto r^{-5}$.

3.1.5 Spherical power law density models

Many galaxies have luminosity profiles which can be fitted with power law profiles. Therefore it seems useful to investigate spherical potentials for density distributions which can be described by a power law of the form

$$\rho(r) = \rho_0 \Big(\frac{r_0}{r}\Big)^{\alpha} . \tag{3.23}$$

The mass inside r is then

$$M(r) = 4\pi \int_0^r \rho(r')\, r'^2 \,\mathrm{d}r' = 4\pi\rho_0 r_0^{\alpha} \int_0^r r^{2-\alpha} \mathrm{d}r' = 4\pi\rho_0 r_0^{\alpha} \frac{r^{3-\alpha}}{3-\alpha} .$$

We consider only $\alpha < 3$, because only for such cases the mass interior to r is finite. On the other side the mass $M(r)$ diverges for $r \to \infty$ at large radii if $\alpha < 3$. The models are still useful because according to Newton's first theorem the spherical mass shells outside r do not affect the gravitational forces and dynamics inside r.

Thus, we can derive the circular velocity v_c for the power law models and obtain

$$v_c^2(r) = \frac{GM(r)}{r} = 4\pi G\rho_0 r_0^{\alpha} \frac{r^{2-\alpha}}{3-\alpha} . \tag{3.24}$$

This is a very interesting formula which can be used for the interpretation of the flat rotation curves observed in disk galaxies out to very large radii. The circular velocity $v_c(r)$ is constant if $\alpha \approx 2$ or for a dark matter density distribution which behaves at radii $\gtrsim 10$ kpc like

$$\rho_{\mathrm{dm}}(r) \propto \Big(\frac{1}{r}\Big)^2 .$$

Two-power density models. A spherical density model combining two power laws, one approximating the flatter central region and one approximating a steeper density fall-off at larger radius provides more modelling possibilities. Well studied is a analytical parameterization for which the density is described by

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}(1+r/a)^{\beta-\alpha}} = \frac{\rho_0}{(r/a)^{\alpha} + (r/a)^{\beta}} \tag{3.25}$$

where a is a scaling radius. The α parameter is $\alpha < 3$ to avoid that the mass at small radius goes to infinity and $\beta \geq 3$ so that the total mass remains finite for large radius.

The following cases are simple and popular solutions:

– Hernquist model with $\alpha = 1$ and $\beta = 4$; this yields

$$\rho(r) \propto \frac{1}{(r/a)(1+r/a)^3} , \quad \Phi(r) \propto \frac{GM}{a+r} \quad v_c(r) = \frac{\sqrt{GMr}}{b+r} .$$

– Jaffe model with $\alpha = 2$ and $\beta = 4$,

$$\rho(r) \propto \frac{1}{(r/a)^2(1+r/a)^2} , \quad \Phi(r) \propto \frac{GM}{a} \ln(1+a/r) , \quad v_c(r) = \sqrt{\frac{GM}{b+r}} .$$

– Navarro, Frenk and White or NFW model with $\alpha = 1$ and $\beta = 3$.

$$\rho(r) \propto \frac{1}{(r/a)(1+r/a)^2} , \quad \Phi(r) \propto GM \frac{ln(1+r/a)}{r/a} .$$

3.1.6 Potentials for flattened systems

Potential of a "Toomre" disk. A simple potential for a disk was introduced by Kuzmin in 1956 and independently by Toomre in 1963. The disk potential can be described by

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + (|a| + |z|)}} \,. \tag{3.26}$$

According to Fig. 3.2 the potential at point $(R, -z)$ is equal to a potential generated by a mass M located at the point (0,a) or for points above the disk by a mass located at (0,-a).

Such a potential can be generated by a razor-thin disk with the surface density distribution

$$\Sigma(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}} \,. \tag{3.27}$$

The central surface density at $R = 0$ is $M/2\pi a^2$ while the surface density drops for large R like $\Sigma(R) \approx aM/R^3$. The constant a is just a scale parameter which indicates where the surface density changes from constant to a step gradient.

.

Figure 3.2: Illustration of the parameters for Toomre's disk.

A hybrid model between Toomre's disk and the Plummer sphere We can now generalize the disk model to include also a matter distribution in z-direction. This can be achieved with a parameterization of the potential according to

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}} \,. \tag{3.28}$$

This potential has two extreme cases:

– for $b = 0$ the potential of a thin Toomre's disk is obtained,

– for $a = 0$ and using $R^2 + z^2 = r^2$ yields the spherical Plummer potential.

Depending on the selection of the parameters a and b one can create a family of potentials covering density distributions from a thin disk to a sphere. The corresponding mass distributions for this types of potentials are described by Miyamoto and Nagai)

$$\rho(R,z) = \Big(\frac{b^2 M}{4\pi}\Big) \frac{aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2}{[R^2 + (a + \sqrt{z^2 + b^2})^2]^{5/2} (z^2 + b^2)^{3/2}} \tag{3.29}$$

Slide 3–1 shows contour plots of this density distribution for a few parameter cases. The case $b/a \approx 0.2$ is at least qualitatively a quite good representation for a disk galaxy, while $b/a \approx 1$ resembles a S0 galaxy (e.g. Sombrero galaxy).

Potential of spheroids. Many astronomical systems are spheroidals, flattened spheres, because of the presence of angular momentum. The evaluation of potentials for spheroids in general is very difficult, because we have to consider the 2D or 3D density distribution of the system.

An important simplification is possible if we consider, homoeoids, thin concentrically nested spheroidal shells. These shells are similar to the spherical shell used for spherical systems.

One homoeid shell is bound by an inner surface and an outer surface described by

$$\frac{R^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{and} \quad \frac{R^2}{a^2} + \frac{z^2}{b^2} = (1 + \delta\beta)^2 \, ,$$

respectively. The perpendicular distance between the two surfaces varies with position. This happens in such a way that Newton's first theorem can be generalized to spheroidal (ellipsoidal) shells.

Newton's third theorem. A mass that is inside a homoeid experiences no net gravitational force from the homoeoids.

The potential theory of spheroids was further developed in order to model with high precision the potential of the Milky Way and other galaxies. Important for these models is Newton's third theorem and theory of multipole expansions for the gravitational potential. This theory is not discussed in this lecture. Some of the important results are:

- many potentials for flattened (oblate) spheroid and triaxial ellipsoids have been derived and applied to galaxy bulges, bars, and elliptical galaxies,
- potentials of exponential galactic disks are successfully described by strongly flattened spheroid using Newton's third theorem,
- potentials for non-axisymmetric disks can be calculated using Bessel functions, and special potential functions are used for the description of logarithmic spiral structures.

3.1.7 The potential of the Milky Way

In this subsection the potential of the Milky Way is described. In particular the density distributions of the main mass components are given: the bulge, the disk with different distributions for the stars and the interstellar gas, and the dark halo. The described model is only partly derived from studies of the dynamical properties of the Milky Way. A lot of information on the mass distribution is also derived from photometric studies. In this description the Milky Way is an axisymmetric system given in cylindrical coordinates R and z. The model picked for this description has the parameters of Model I in the book of Binney & Tremaine. This is a Milky Way model with a relatively small disk but all parameters are compatible with the available observations. Slide 3–2 shows the equipotentials for this model as well as the different components and Slide 3–3 illustrates the corresponding circular velocities $v_c(r)$.

The central bulge. The bulge can be described by a oblate, spheroidal power law model which is truncated at an outer radius r_b:

$$\rho_b(R,z) = \rho_{b0}\left(\frac{m}{a_b}\right)^{\alpha_b} e^{-m^2/r_b^2}\,, \tag{3.30}$$

with

$$m = \sqrt{R^2 + z^2/q_b^2}\,.$$

The parameters describe:

- $\rho_{b0} = 0.43\,\mathrm{M}_\odot/\mathrm{pc}^3$ is the density normalization for the bulge
- $a_b = 1$ kpc is the size normalization of the bulge,
- $q_b = 0.6$ describes the bulge flattening,
- $\alpha_b = -1.8$ is the power law index for the density distribution,
- r_b is the cut-off radius for the bulge.

The galactic disk. The Milky way disk consists of the stellar disk and a gas disks.

The stellar disk is described by an exponential fall near radius R_S and two exponential laws for the vertical direction, one for the thin disk and one for the thick disks. The used formula is

$$\rho_s(R,z) = \Sigma_S\, e^{-R/R_S}\left(\frac{a_0}{2z_0}e^{-|z|/z_0} + \frac{\alpha_1}{2z_1}e^{-|z|/z_0}\right). \tag{3.31}$$

The parameters describe:

- $\Sigma_S \approx 1500\ \mathrm{M}_\odot/\mathrm{pc}^2$ is the central surface density of the stellar disk which is not well known except for the solar radius R_0. At R_0 the surface density of the stars is about 35 $\mathrm{M}_\odot\mathrm{pc}^2$, while the thick disk contributes about 3 $\mathrm{M}_\odot\mathrm{pc}^2$.
- $R_S = 2.5$ kpc is the disk scale length,
- $\alpha_0 = 0.9$ and $\alpha_1 = 0.1$ are the relative normalizations of the thin and thick disk,
- $z_0 = 0.3$ kpc is the scale hight of the thin disk,
- $z_0 = 1$ kpc is the vertical scale hight of the thick disk.

The radial distribution of the distribution of the interstellar disk is also described with an exponential law with a much larger scale length than for the star. However there is a hole with a radius of about 4 kpc in the center which is considered with an exponential cut-off. The vertical density distribution of the gas is much narrower than for the stars:

$$\rho_g(R,z) = \Sigma_g\, e^{-R/R_g}\, e^{-R_m/R}\,\frac{1}{2z_g}e^{-|z|/z_g}\,. \tag{3.32}$$

where the parameters are:

- $\Sigma_S \approx 500\ \mathrm{M}_\odot\mathrm{pc}^2$, the surface density of the gas in the disk is not well known except for R_0 where the surface density is about $\Sigma_g R_0 \approx 12\mathrm{M}_\odot\mathrm{pc}^2$
- $R_g = 4$ kpc is the disk scale length for the gas (twice the value for the stellar disk),
- $R_m = 4$ kpc is the radius of the inner hole,
- $z_g = 80$ pc is the scale hight of the gas disk.

The dark halo. The dark halo can be described by a extension of the spherical two-power-law model to an oblate geometry.

$$\rho_h(R,z) = \frac{\rho_{h0}}{(m/a_h)^\alpha (1+m/a_h)^{\beta-\alpha}} \tag{3.33}$$

where the flattening is described like for the bulge

$$m = \sqrt{R^2 + z^2/q_h^2}\,.$$

The parameters describe:

- $\rho_{h0} = 0.71\mathrm{M}_\odot/\mathrm{pc}^3$ is the density normalization for the bulge,
- $a_h = 3.8$ kpc is the size normalization for the halo,
- $q_h = 0.8$ is a guess for the possible flattening of the dark halo,
- $\alpha_h = 2.0$ and $\beta_h = 3$ are the power law indices for the halo density distribution.

3.2 The motion of stars in spherical potentials

This section discusses the orbits of individual stars in a static, spherical potential. Spherical potentials serve again as simple cases for the description of general principles.

3.2.1 Orbits in a static spherical potential

Spherical potentials describe very well globular cluster but less well flattened or triaxial systems like galaxies. Nonetheless the solutions for spherical potentials serve as very important guide for more complicated gravitational fields.

In a centrally directed gravitational field the position vector of a star is

$$\vec{r} = r\vec{e}_r$$

The motion of a star with a mass m_S in spherical potential is defined by the radially directed gravitational force

$$\vec{F}(r) = m_S \frac{d^2\vec{r}}{dt^2} = m_S g(r)\vec{e}_r\,.$$

Further we use that the angular momentum in a static spherical system is constant

$$\vec{L} = m_s \vec{r} \times \frac{d\vec{r}}{dt} = \mathrm{const.}\,.$$

This implies that the stars moves in a plane. For this reason we can use plane polar coordinates.

Lagrange function. We introduce the Lagrange-function, which is a general formulation for the equations of motions. The Lagrange-function for a star in free space is in Cartesian coordinates

$$\mathcal{L} = \frac{m_S}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,,$$

and in polar coordinates

$$\mathcal{L} = \frac{m_S}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)\,.$$

The Lagrange-function for a mass m_S in a spherical potential $\Phi(r)$ can then be written as

$$\mathcal{L} = \frac{m_S}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \Phi(r)\,. \tag{3.34}$$

as in polar coordinates because we can align the spherical coordinate system always with the orbital plane (where $\theta = 0$).

Equation of motion. The equations of motions follow from the derivatives of the Lagrange equation

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = m_S\ddot{r} - m_S\dot{\phi}^2 - m_S\frac{d\Phi}{dr}\,, \tag{3.35}$$

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}(m_s r^2 dot\phi)\,. \tag{3.36}$$

The second equation is the formulation of the angular momentum conservation in polar coordinates

$$L = m_S r^2\dot{\phi} = \text{const}\,.$$

With the angular momentum equation we can substitute the time derivative by the angle derivative

$$\frac{d}{dt} = \frac{L^2}{r^2}\frac{d}{d\phi}\,,$$

and this yields the equation of motion in the following form:

$$\frac{L^2}{r^2}\frac{d}{d\phi}\Big(\frac{1}{r^2}\frac{dr}{d\phi}\Big) = -\frac{d\Phi}{dr}\,. \tag{3.37}$$

With the substitution $\frac{u=1}{r}$ a simplied form for the equation of motion is obtained:

$$\frac{du^2}{d\phi^2} + u = \frac{1}{L^2u^2}\frac{d\Phi}{dr}(1/u)\,. \tag{3.38}$$

Energy equation. We can write for a mass in a central potential the following energy equation

$$E_{\text{tot}} = \frac{m_S\dot{r}^2}{2} + \frac{L^2}{2m_S r^2} + \Phi(r)\,. \tag{3.39}$$

This provides very convenient formula for the motion of particles in a centrally symmetric gravitational field.
Further we can use for a stationary gravitational potential the **virial theorem**

$$2E_{\text{kin}} + E_{\text{pot}} = 0\,,$$

where $\Phi(r) = E_{\text{pot}}$ for the star in the central potential.

Effective Potential. The energy equation (3.39) shows that the radial motion can be described as 1-dimensional motion in an effective radial potential of the form

$$\Phi_{\rm eff}(r) = \Phi(r) + \frac{L^2}{2m_S r^2}\,. \tag{3.40}$$

This potential goes, except for the case $L = 0$, for $r \to 0$ to infinity and for $r \to \infty$ from negative values to zero.

This potential has for small radii a centrifugal barrier if $L \neq 0$. The r-values where the total energy is equal to the effective potential energy defines the radial range of motion:

$$\frac{m\dot{r}^2}{2} = E - \Phi_{\rm eff}\,. \tag{3.41}$$

The borders of this range are defined by the radius where the radial kinetic energy is zero or where $\dot{r} = 0$. At these points the total energy is equal to the effective potential energy. For bound orbits and $L \neq 0$ this equation has two roots r_1 and r_2 which are called the pericenter and apocenter distances, respectively.

Figure 3.3: Radial dependence of the effective potential energy for potentials with different angular momentum.

The different curves in Fig. 3.3 illustrate what happens if the total energy or the angular momentum is changed in the system. A change in angular momentum is equal to a jump to a different effective potential energy curve and a change in energy enhances or lowers the eccentricity. A dynamical interaction between two stars changes typically both, the total energy and the angular momentum.

The radial dependence of the effective potential energy is similar for essentially all gravitating systems. For small separation there is the centrifugal force barrier, in the intermediate range is the minimum of the potential energy, and for large separations the effective potential energy towards zero.

3.2.2 Radial and azimuthal velocity component.

The motion in r and ϕ can be derived from the energy equation. The equation for the radial velocity component is

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m_S}[E - \Phi(r)] - \frac{L^2}{m_S^2 r^2}}\,, \tag{3.42}$$

with the time dependence

$$t(r) = \int \frac{dt}{dr}\mathrm{d}r = \int \frac{dr}{\sqrt{\frac{2}{m_S}[E - \Phi(r)] - \frac{L^2}{m_S^2 r^2}}} + \text{const.}\,, \tag{3.43}$$

and using the definition for the angular momentum $L = m_S r^2 \dot{\phi}$ or $\mathrm{d}\phi = L/m_S r^2 \mathrm{d}t$ yields the equation for the azimuthal velocity component

$$\phi(r) = \int \frac{\phi}{dr}\mathrm{d}r = \int \frac{\frac{L}{r^2}\mathrm{d}r}{\sqrt{2m_S[E - \Phi(r)] - \frac{L^2}{r^2}}} + \text{const.}. \tag{3.44}$$

Figure 3.4: Typical orbit of a star in a spherical potential.

The radial period T_r is the time required for the star m_S to travel from apocenter to pericenter and back. This is:

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m_S}[E - \Phi_{\text{eff}}(r)]}}\,. \tag{3.45}$$

Similarly one can derive the azimuthal angle increase $\Delta\phi$ from pericenter to apocenter and back, which is

$$\Delta\phi = 2L\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^2\sqrt{\frac{2}{m_S}[E - \Phi_{\text{eff}}(r)]}}\,.$$

The azimuthal period is then

$$T_\phi = \frac{2\pi}{\Delta\phi}\,T_r\,, \tag{3.46}$$

or the mean azimuthal speed is equal to $2\pi/T_\phi$. The orbit will only be closed if $2\pi/\Delta\phi$ is a rational number, what is typically not the case except for the potential of a point source and a homogeneous sphere. The star moves therefore in general on a rosette around the center of the spherical potential (Fig. 3.4).

3.2.3 Motion in a Kepler potential

Effective potential. The effective potential energy for a point source is

$$\Phi_{\rm eff}(r) = -\frac{GM}{r} + \frac{L^2}{2m_s r^2}\,. \tag{3.47}$$

The equation

$$\frac{\mathrm{d}\Phi_{\rm eff}(r)}{\mathrm{d}r} = \frac{GM}{r^2} - \frac{L^2}{m_s r^3} = 0$$

provides the radius of the minimum

$$r_{\rm min} = \frac{L^2}{GMm_S} \tag{3.48}$$

and the corresponding minimum effective potential energy

$$\min(\Phi_{\rm eff}(r)) = -\frac{G^2M^2m_S}{2L^2}\,. \tag{3.49}$$

The total energy is for a given angular momentum equal or larger to

$$E_{\rm tot} \geq \frac{L^2}{2m_S r_{\rm min}^2} + \Phi(r_{\rm min}) = \frac{G^2M^2m_S}{2L^2} - \frac{G^2M^2m_S}{L^2} = -\frac{G^2M^2m_S}{2L^2}\,.$$

For $E_{\rm tot} = \min(\Phi_{\rm eff}(r))$ we have a circular orbit with no radial motion component. In this case the angular momentum energy term is half the potential energy term. This is as predicted by the virial theorem for a system in gravitational equilibrium:

$$2E_{\rm kin} + E_{\rm pot} = 0\,.$$

The circular orbit is a minimum energy orbit for an object with a given angular momentum moving in a spherical potential.

Orbital periodicities in a Kepler potential. The motion in a Kepler potential can be derived from the equation of motion described in Equation 3.38.

We know from the first and third Kepler law that the orbits are closed:

$$T_r = T_\phi$$

and that the orbital period or radial oscillation period is

$$T_r^2 = 2\pi\frac{a^3}{GM}\,.$$

The Keplerian motion has the following properties:
- the mass m_S moves on closed ellipses with the point source in one focal point,
- according to the angular momentum conservation, the azimuthal velocity during an orbit behaves like

$$\frac{d\phi(r)}{dt} \propto 1/r\,.$$

3.2.4 Motion in the potential of a homogeneous sphere

According to Section the potential at $r < a$ inside a sphere with radius a is

$$\Phi(r) = -2\pi G\rho a^2 + \frac{2\pi G\rho}{3}r^2 = \frac{\omega^2}{2}r^2 + \text{const.}\,,$$

with $\omega^2 = 4\pi G\rho/3$. The equation of motion $m_S\ddot{r} = m_S(\mathrm{d}\Phi/\mathrm{dr})$ can be written in Cartesian coordinates $x = r\,\cos\phi$ and $y = r\,\sin\phi$:

$$\ddot{x} = -\omega^2 x\,, \quad \ddot{x} = -\omega^2 y\,, \tag{3.50}$$

with the solutions:

$$x = a_x \cos(\omega t + \delta_x)\,, \quad y = a_y \cos(\omega t + \delta_y)\,. \tag{3.51}$$

where a_x, a_y, δ_x and δ_y are arbitrary constants. The motion has the following properties:

- x and y have the the oscillation period $T_r = 2\pi/\omega$,
- the oscillation phase in the x and y directions are independent,
- the mass m_S moves on closed ellipses which are centered on the center of the sphere $r = 0$,
- the radial period is half the orbital period, or an object completes two in-and-out oscillations during an orbital period:

$$T_r = \frac{1}{2}T_\phi\,. \tag{3.52}$$

If the x- and y-oscillations are in phase, then the motion corresponds to a swing from one side of the center to the other side and back along a straight line with a full oscillation period identical to the orbital period. However, for a radial coordinate system this corresponds to two full oscillation $r_{\max} - 0 - r_{\max} - 0 - r_{\max}$.

Figure 3.5: Qualitative illustration of the ellipse shape of a mass in a Kepler potential and a mass inside a homogeneous sphere.

Figure 3.5 illustrates the fundamental difference between the orbits in a homogeneous sphere and around a point source. All smooths potentials create orbits which have typically about two radial oscillation per azimuthal period.

3.3 Motion in axisymmetric potentials

Stars moving in the equatorial plane of an axisymmetric potential behave like stars in a spherical potential, because one can always find a spherical gravitational potential which induces the same gravitational force on the stars in the equatorial plane as the axisymmetric potential. For this reason the orbits discussed in the previous chapter for spherical potentials apply also for the stars in the equatorial plane of an axisymmetric potential.

The motion of the stars located in or near the equatorial plane is an important problem for the investigation of disk galaxies.

3.3.1 Motion in the meridional plane

We assume that the potential is symmetric with respect to the plane $z = 0$. Then we can write the Lagrange equation with the following terms

$$\mathcal{L} = \frac{m_S}{2}(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2) - \Phi(R, z)$$

The 3-dimensional motion of a star in an axisymmetric potential can be reduced to a 2-dimensional motion of a star in the R-z-plane, the meridional plane.

The equation of motion in this co-rotating plane are:

$$m_S\ddot{R} = -\frac{\partial\Phi_{\text{eff}}(R, z)}{\partial R}\,, \quad m_S\ddot{z} = -\frac{\partial\Phi_{\text{eff}}(R, z)}{\partial z}\,, \tag{3.53}$$

where the effective potential is

$$\Phi_{\text{eff}}(R, z) = \Phi(R, z) + \frac{L_z^2}{2m_S R^2} \tag{3.54}$$

Similar to the spherical case we can write the total energy equation, but now with an R and a z term for the kinetic energy

$$E_{\text{tot}} = \frac{1}{2m_S}(p_R^2 + p_z^2) + \Phi_{\text{eff}}(R, z)\,. \tag{3.55}$$

The kinetic energy of motion in the R-z-plane is

$$\frac{1}{2m_S}p(^2_R + p_z^2) = E_{\text{tot}} - \Phi_{\text{eff}}(R, z)\,.$$

Orbits in the meridional plane are restricted to the area $E_{\text{tot}} > \Phi_{\text{eff}}(R, z)$ and one can define contour lines or the zero velocity curves in the meridional plane where the kinetic energy term is instantaneously zero

$$\Phi_{\text{eff}}(R, z) = E_{\text{tot}}\,.$$

The minimum of Φ_{eff} is in the equatorial plane $z = 0$ and the radial value follows from

$$0 = \frac{\partial\Phi_{\text{eff}}}{\partial R} = \frac{\partial\Phi}{\partial R} - \frac{L_z^2}{m_S R^3}$$

This yields the radius for a circular orbit with angular speed $\dot{\phi}$ which is identical to the radius of the minimum of the effective potential. At this radial point, which is called the **guiding-center radius** R_g, there is

$$\Big(\frac{\partial \Phi}{\partial R}\Big)_{(R_g,0)} = \frac{L_z^2}{m_S R^3} = m_S R_g \dot{\phi}^2 \,,$$

($L_z = m_S R^2 \dot{\phi}$). This is the condition for a circular orbit with angular speed $\dot{\phi}$ for a mass located at the radius R_g, which is at the minimum of the effective potential.

Example. Slide 3-4 shows as example the contour plot and orbits for the effective potential

$$\Phi_{\rm eff}(R,z) = \frac{v_0}{2m_S} \ln\Big(\frac{R^2+z^2}{q^2}\Big) + \frac{L_z^2}{2m_S R^2} \,,$$

for $v_0 = 1$, $L_z = 0.2$ and axial ratio $q = 0.9$ and 0.5. This represents the effective potential for an oblate, spheroidal mass distribution like a central bulge of a disk galaxy, an elliptical galaxy, or a dark matter halo with a constant circular velocity speed $v_c = $ const. The effective potential energy rises strongly near $R = 0$ because of the "centrifugal barrier" for the given angular momentum L_z.

The equations (3.53) for the relative motion in a co-rotating frame must be integrated numerically. Slide 3-4 shows the calculated motion. The given results are for stars in the same potential, same energy and same angular momentum but they still differ significantly. This problem is called the third integral problem and it is linked in this case to the precession of the angular momentum vector in a flattened potential.

3.3.2 Nearly circular orbits: epicycle approximation

In disk galaxies many stars are on nearly circular orbits. For this case we can simplify the equation of motion in the co-rotating system (Eq. 3.53)

$$m_S \ddot{R} = -\frac{\partial \Phi_{\rm eff}(R,z)}{\partial R}\,, \quad m_S \ddot{z} = -\frac{\partial \Phi_{\rm eff}(R,z)}{\partial z}\,, \tag{3.56}$$

with a linearization of the corresponding effective potential at $R = R_g$ and $z = 0$. We introduce x as new variable in the radial direction

$$x = R - R_g$$

The effective potential in Eq. 3.54 can then be written as Taylor expansion:

$$\Phi_{\rm eff} = \Phi_{\rm eff}(R_g,0) + \frac{1}{2}\Big(\frac{\partial^2 \Phi_{\rm eff}}{\partial^2 R}\Big)_{(R_g,0)} x^2 + \frac{1}{2}\Big(\frac{\partial^2 \Phi_{\rm eff}}{\partial^2 z}\Big)_{(R_g,0)} z^2 + O(xz^2) + ... \tag{3.57}$$

The first order terms are zero because $\Phi_{\rm eff}(R_g, 0)$ is at a minimum. One can introduce abbreviations for the second derivatives (curvature in the effective potential):

$$\kappa^2(R_g) = \Big(\frac{\partial^2 \Phi_{\rm eff}}{\partial^2 R}\Big)_{(R_g,0)}\,, \quad \text{and} \quad \nu^2(R_g) = \Big(\frac{\partial^2 \Phi_{\rm eff}}{\partial^2 z}\Big)_{(R_g,0)}\,.$$

This approximation, which is called the **epicycle approximation**, yields very simple, harmonic, equations of motions for the radial x and vertical z directions:

$$\ddot{x} = -\kappa^2 x\,, \quad \ddot{z} = -\nu^2\,. \tag{3.58}$$

The two time scales are called:

- the epicycle or radial frequency κ,
- the vertical frequency ν.

These frequencies can be evaluated using Eq. 3.54 for the effective potential in a co-rotating system

$$\Phi_{\rm eff}(R,z) = \Phi(R,z) + \frac{L_z^2}{2m_S R^2}\,.$$

This yields for the vertical frequency the simple relation

$$\nu^2(R_g) = \Big(\frac{\partial^2\Phi}{\partial^2 z}\Big)_{(R_g,0)} \qquad (3.59)$$

Solution for the epicycle frequency. There are two terms for the epicycle frequency κ, a potential energy term and an angular momentum term

$$\kappa^2(R_g) = \Big(\frac{\partial^2\Phi}{\partial^2 R}\Big)_{(R_g,0)} + \frac{3L_z^2}{m_S^2 R^4}\,. \qquad (3.60)$$

We can now use the "global" angular velocity dependence for the circular motion at R_g which is (using also $L_z = m_S R\, v_c$)

$$\Omega^2(R) = \frac{v_c^2(R)}{R^2} = \frac{1}{R}\Big(\frac{\partial\Phi}{\partial R}\Big)_{(R_g,0)} = \frac{L_z^2}{m_S^2 R^4}$$

With this relation we can rewrite the equation for the epicycle frequency in terms of global, "galactic", quantities:

$$\kappa^2(R_g) = \Big(R\frac{d\Omega^2}{dR} + 4\Omega^2\Big)_{R_g} \qquad (3.61)$$

using $d^2\Phi/d^2R = d/dR(R\Omega^2) = \Omega^2 + R(d\Omega^2/dR$. This relates the epicycle frequency to the radial dependence of the angular velocity $d\Omega^2(R)/dR$.

Comparison of epicycle period and orbital period. We can now compare the epicycle period T_r with the azimuthal orbital period $T_{\rm orb}$ which are simply:

$$T_r = \frac{2\pi}{\kappa} \quad \text{and} \quad T_\phi = \frac{2\pi}{\Omega}\,.$$

There are three useful approximate cases for a comparison between orbital frequency and epicycle frequency:

- Near the center of galaxies the circular speed v_c increases linearly and $\Omega(R)$ is essentially constant and therefore $d\Omega^2/dR \approx 0$. In this case there is

  $$\kappa^2(R_g) \approx 4\,\Omega^2 \quad \text{or} \quad \kappa \approx 2\,\Omega\,,$$

  This corresponds to the case of a homogeneous sphere where the epicycle frequency is twice the orbital frequency, or the radial period is half the orbital period $T_r = T_\phi/2$. This is again the limiting case for a homogeneous sphere.

– At large radii from the center the circular velocity falls off like (but usually less rapid) the Kepler law. For a Kepler law there is $\Omega \approx R^{-3/2}$ (using $R(d\Omega^2(R)/dR) = -3\Omega^2$). This limit implies

$$\kappa^2(R_g) \gtrsim \Omega^2 \quad \text{or} \quad \kappa \gtrsim \Omega \,.$$

Thus we have the case where the radial period and orbital periods are equal or $T_r = T_\phi$. This is as expected for a closed Keplerian orbit.

– At most points in a typical disk galaxy the circular velocity is constant or $\Omega \propto R^{-1/2}$. For this case the formula for the epicycle frequency is

$$\kappa^2 = 3\,\Omega^2 \quad \text{or} \quad \kappa \approx 1.7\,\Omega \,.$$

This indicates that in a disk the stars oscillate with a frequency of roughly 1.5 times the orbital frequency.

Application for the solar neighborhood. The third case, for intermediate separations, can be evaluated in detail for the solar neighborhood. As described in Chapter 2, we know quite well the Oort's constants A and B from the measurement of the radial and tangential velocities of stars in the solar neighborhood. We used the following formula for the Oort's constants:

$$A = \frac{1}{2}\Big[\frac{\Theta_0}{R_0} - \Big(\frac{d\Theta}{dR}\Big)_{R0}\Big] \quad \text{and} \quad B = -\frac{1}{2}\Big[\frac{\Theta_0}{R_0} + \Big(\frac{d\Theta}{dR}\Big)_{R0}\Big] \,.$$

With $v_c = \Theta$ and $R\Omega = v_c$ we can write:

$$A = -\frac{1}{2}R\frac{d\Omega}{dR} \quad \text{and} \quad B = -\Big(\Omega + \frac{1}{2}R\frac{d\Omega}{dR}\Big)$$

The circular angular velocity is $\Omega = A + B$ while the epicycle frequency is

$$\kappa^2 = -4B(A-B) = -4B\Omega$$

which yield the ratio between epicycle frequency and orbital period for the solar neighborhood

$$\frac{\kappa_0}{\Omega_0} = 2\sqrt{\frac{-B}{A-B}} \approx 1.3 \pm 0.1 \,. \tag{3.62}$$

The result is obtained for the typical values for the Oort's constants $A \approx +15$ km/(s kpc) and $B \approx -12$ km/(s kpc). This means that the sun makes about 1.3 oscillations in radial directions within one orbit around the galactic center.

3.3.3 Density waves and resonances in disks

In the previous subsection we have introduced the following quantities for stars with almost circular orbits in disk galaxies:

- T_r: the epicycle or radial period for the in-and-out motion in radial direction,
- $\Delta\phi$: the azimuthal angle increase during the epicycle period,
- $\Omega_r = 2\pi/T_r$: the radial oscillation frequency,
- $\Omega_\phi = \Delta\phi/T_r$: the corresponding azimuthal oscillation frequency,
- $\Omega = 2\pi/T$: the orbital frequency or orbital angular velocity where T is the time for a full orbit around the galaxy center.

We now describe the motion of the stars in a frame which is rotating with some special angular velocity. The following quantities are defined in this system:

- Ω_P: angular velocity (or pattern speed) for the selected rotating frame,
- $\phi_p = \phi - \Omega_p t$: the azimuthal angle in the rotating reference system which changes with time,
- $\Delta\phi_p = \Delta\phi - \Omega_p T_r$: the azimuthal angle increase in the rotating system for one epicycle period.

On can always choose an angular velocity Ω_p for a rotating coordinate system in which the orbits are closed or $\Delta\phi_P/T_r = (n/m)\Omega_r$. This follows from the definition of $\Delta\phi_p$

$$\Omega_p = \frac{\Delta\phi}{T_r} - \frac{n}{m}\Omega_r\,. \tag{3.63}$$

For orbits close to circular orbits we can approximate $\Delta\phi/T_r = \Omega_\phi \approx \Omega$ and $\kappa \approx \Omega_r$ and write

$$\Omega_p = \Omega - \frac{n}{m}\kappa\,. \tag{3.64}$$

Figure 3.6 illustrates the appearance of an orbit with $\kappa/\Omega_r \approx 1.5$ in rotating frames with different m and n.

Figure 3.6: Closed orbits with different n and m in a rotating system.

In general, $\Omega - n\kappa/m$ is a function of radius, and no unique pattern speed Ω_p can be defined to close the orbits at all radii. Slide 3-5 shows curves for $\Omega - n\kappa/m$ for the Milky Way (model 1).

However, it was first noticed by Lindblad that the curve for $\Omega - \kappa/2$ is relatively constant for a wide range of galactic radii. A constant curve $\Omega - \kappa/2$ would mean that in a frame rotating at Ω_p all orbits would be ellipses, which are nested for a broad range of R. They would look like the ellipses shown in Slide 3-6. If stars move predominantly along these ellipses then they would create a **bar-like pattern**, which is stationary in a rotating frame. In a fixed frame this would then look like a **density wave** rotating with a pattern speed Ω_p

In a real galaxy $\Omega - n\kappa/m$ depends on radius. Therefore, independent of the selected Ω_p, most orbits will not be exactly closed. The orientations at different radii will drift at slightly different speeds, and the pattern will twist, and might look like a spiral pattern (see Slide 3-6). This type of kinematic density waves, produce a non-axisymmetric disk pattern and an exact calculation of stellar orbits needs to take this into account.

3.4 Two-body interactions and system relaxation

Up to now we have assumed that collisions, ie the interaction between individual stars, can be neglected. This is a reasonable assumption for galactic dynamics. We discuss now cases where such collisions between stars play an important role.

A star within a more or less homogeneous distribution of other stars "feels" the gravitational force of all these stars. The force $F = \Sigma F_i$ of all stars i in a given solid angle (see Fig. 3.7) behaves as follows:

- the force induced by an individual star is $F_i \sim 1/r_i^2$ and decreases with distance,
- the volume and therefore the number of stars in a fractional distance interval, e.g. $[r - r/2, r + r/2]$ increases like $\sim r^3$,
- the total force on the sample star is dominated by the more distant stars.

Figure 3.7: On the force induced by near and distant stars in a homogeneous distribution.

Therefore it is reasonable to assume that stars are smoothly accelerated by the force field that is generated by the Milky Way as a whole. In the following we investigate more quantitatively this simpflication and consider cases where this approximation is no more valid.

3.4.1 Two-body interaction

We consider an individual star, called the **the subject star**, and investigate how much its velocity is disturbed by encounters with other stars, called two-body interactions, during the crossing through a system like a galaxy, or a star cluster. Thus we calculate the expected deflection of the trajectory of the subject star from the path it would have in the smooth overall potential. For our estimate we assume that all stars have the same mass m_S.

The velocity deflection $\delta\vec{v}$ induced by a two-body interaction can be simplified, if we consider only weak (distant) encounters which introduce small velocity deflections $|\delta\vec{v}|/v \ll 1$. Further it is assumed that the field star is stationary during the encounter. The velocity deflection follows then the perpendicular force $F_\perp$ of the field star induced onto the subject star which is moving with velocity v along an essentially straight line with impact parameter b (Fig. 3.8).

.

Figure 3.8: Geometry for an estimate of the deflection by a star-star interaction.

If both stars have the same mass, then the perpendicular force induced on the subject star is:

$$F_\perp \approx \frac{G m_S^2}{b^2 + x^2} \cos\theta = \frac{G m_S^2 b}{(b^2 + x^2)^{3/2}} \,, \tag{3.65}$$

using the trigonometric relation $\cos\theta = b/r = b/\sqrt{b^2 + r^2}$. The coordinate along the trajectory x can be expressed by the time and the velocity of the subject star $x = v \cdot t$ so that

$$F_\perp \approx \frac{G m_S^2}{b} \left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2} .$$

According to Newton's law the acceleration or change in velocity $\dot{\vec{v}} = \vec{F}/m_S$ is the time integral of the acting force, or

$$\delta v \approx \int_{-\infty}^{+\infty} F_\perp \,\mathrm{d}t = \frac{G m_S}{b^2} \int_{-\infty}^{+\infty} \frac{1}{[1 + (vt/b)^2]^{-3/2}} \mathrm{d}t \tag{3.66}$$

The integral is equal to $2b/v$ and the deflection is

$$\delta v \approx \frac{2 G m_S}{b v} \,. \tag{3.67}$$

This equation can be interpreted as follows:

– δv is proportional to the acceleration at closest approach $G m_S / b^2$ times a characteristic duration of the acceleration $2b/v$,

– the derived approximative value is only valid for $\delta v \ll v$ or for an impact parameter larger than

$$b \gg G m_S / v^2 = 900\mathrm{AU} \frac{(m_S/\mathrm{M}_\odot)}{(v/(\mathrm{km/s}))^2} \,.$$

As next step we estimate the number of encounters of the subject star in a stellar system for the impact parameter range $[b, b + \mathrm{d}b]$. We ue an estimate for the surface density of field stars

$$\Sigma_{\mathrm{stars}} \approx \frac{N}{\pi R^2} \,,$$

where N is the total number of stars and R the radius of the considered system, e.g. the stellar cluster or galaxy. The subject star will have during one crossing of the system the following number of encounters

$$\delta n = \frac{N}{\pi R^2}\, 2\pi b\, \mathrm{d}b = \frac{2N}{R^2}\, b\, \mathrm{d}b\,. \tag{3.68}$$

with impact parameter between b and $b + \mathrm{d}b$. Each such encounter produces a deflection $\delta\vec{v}$ to the subject star, but these deflections are randomly oriented and their mean will be zero. But the mean-square change will not be zero and after one crossing. The squared velocity deflection (change) for an impact parameter intervall $\mathrm{d}b$ will be:

$$\Sigma\, \delta v^2\, \mathrm{d}b \approx \delta v^2 \delta n\, \mathrm{d}b = \Big(\frac{2Gm_S}{bv}\Big)^2 \frac{2N}{R^2}\, b\, \mathrm{d}b\,. \tag{3.69}$$

Now, we have to take into account all impact parameters by integrating from b_{min} to b_{max}

$$\Delta v^2 = \int_{b_{\mathrm{min}}}^{b_{\mathrm{max}}} \Sigma\, \delta v^2\, \mathrm{d}b = 8N\Big(\frac{Gm_S}{Rv}\Big)^2 \ln b\Big|_{b_{\mathrm{min}}}^{b_{\mathrm{max}}}\,. \tag{3.70}$$

The logarithm term can be written as

$$\ln\Lambda = \ln b_{\mathrm{max}} - \ln b_{\mathrm{min}}\,.$$

The maximum impact parameter is of the order $b_{\mathrm{max}} \approx R$, the smallest, where the small deflection approximation is still valid, is $b_{\mathrm{min}} \approx 2Gm_S/v^2$. These are only approximate values with an uncertainty of a factor of a few. For this reason we can write

$$\ln\Lambda = \ln\Big(\frac{R}{b_{\mathrm{min}}}\Big) + \ln(\epsilon_1/\epsilon_2)\,.$$

In most systems $R \gg b_{\mathrm{min}}$ and the typical ratio is $R/b_{\mathrm{min}} \gg 10^4$ while the uncertainty term ϵ_1/ϵ_2 is much smaller, of the order of a few. This term can therefore be neglected with respect to the first term. Thus, the parameter Λ is approximately

$$\Lambda \approx \frac{Rv^2}{2Gm_S} \approx N\,,$$

where we already used the next approximation for the typical stellar velocity v.

The encounters between the subject star and the field stars produce a diffusion of the star's velocity which is different from an acceleration induced by a smooth, large scale potential. This velocity diffusion is called **two-body relaxation**, because it is the result of a large number of mostly weak two-body interactions.

The typical speed v of the a field star can be approximated by the circular velocity of a star at radius R (at the edge) of the system

$$v^2 \approx \frac{GNm_S}{R}\,.$$

Equation 3.70 can be simplified with this velocity to

$$\frac{\Delta v^2}{v^2} \approx \frac{8\ln\Lambda}{N}\,, \tag{3.71}$$

The subject stars makes typically many crossings until the velocity $\vec{v}$ changes by roughly Δv^2. The number of crossings $n_{\rm relax}$ required for a change of the velocity by a value comparable to v is then

$$n_{\rm relax} \approx \frac{N}{8\ln\Lambda} \approx \frac{N}{8\ln N}\,.$$

3.4.2 Relaxation time

The relaxation time is defined as $t_{\rm relax} = n_{\rm relax} t_{\rm cross}$, where the crossing time is $t_{\rm cross} = R/v$. Using all the approximations from above we can express the relaxation time by the number of stars and the crossing time

$$t_{\rm relax} \approx \frac{0.1N}{\ln N}\, t_{\rm cross}\,. \qquad (3.72)$$

Thus the relaxation time exceeds the crossing time in a self-gravitating system for $N \gtrsim 40$.

After the relaxation time the orbit of a (subject) star is changed significantly by all the small kicks induced by other stars, so that its velocity is now different than from the what one would expect in a smooth potential.

Table 3.1: Typical characteristic parameters for stellar systems

system	R	N	$t_{\rm cross}$	$t_{\rm relax}$	$t_{\rm lifetime}$
clusters of galaxies	1 Mpc	1000	1 Gyr	14 Gyr	10 Gyr
galaxies	10 kpc	10^{11}	100 Myr	$\gg$ 100 Gyr	10 Gyr
central pc of galaxies	1 pc	10^6	10^4 yr	100 Myr	10 Gyr
globular clusters	10 pc	10^5	10^5 yr	100 Myr	10 Gyr
open clusters	10 pc	100	1 Myr	10 Myr	100 Myr

Table 3.1 gives typical numbers for the different stellar systems using these approximations. The numbers show that the relaxation times are extremely long for galaxies. Therefore they can be treated as collision-less systems. Important to notice is, that the dynamics of stars in galaxies preserve at least partly information from past eventes.

On the other side there are the stellar clusters. Globular clusters relax in about 100 Myr and open clusters on a very short timescale of about 1 Myr. Their dynamics is dominated by relaxation and the star motions are rapidely randomized. For this reason it is often not possible to extract the past history of clusters from dynamical studies.

3.4.3 The dynamical evolution of stellar clusters

Galactic stellar clusters have a very short relaxation time. A disturbance of their dynamics is therefore rapidely randomized. In addition, galactic stellar clusters are also quite fragile and disolve rapidely, typically within about 300 Myr. On the other side there are the globular clusters which have survived more than 10 Gyr. For these systems we have only little information about their formation history and all signatures from the formation process in the stellar dynamics has been washed out.

Some important dynamical processes for the evolution of stellar cluster can be infered from their current properties:

For globular clusters, we know that
- they have a long life time of 10 Gyr or more,
- they have typically $N \approx 10^5$ to 10^6 stars,
- many globular clusters show a dense core and a low density halo,
- there are often "hard binary systems" in the center.

For galactic open cluster, we know that
- they have typically 100 - 1000 star members,
- they disolve in about 300 Myr,
- they show often a mass segregation with more massive stars in the center and lower mass stars further out.

In the following we discuss a few processes in stellar dynamics which influence the evolution of stellar clusters.

Cluster formation. We discuss the formation of a stellar cluster, considering a very young population of N stars which is still embedded in the gas cloud out of which the stars were formed. We define the total embedded stellar mass of the cluster $M_{\rm ecl}$ and use m_S as mean stellar mass

$$m_S = \frac{M_{\rm ecl}}{N}\,.$$

Further we can define a fractional star-formation efficiency ϵ, the fraction of the total mass of the initial gas cloud $M_{\rm cloud}$ which ends up in newly formed stars

$$\epsilon = \frac{M_{\rm ecl}}{M_{\rm ecl} + M_{\rm gas}}\,.$$

Here $M_{\rm gas}$ is the gas left over from the star-formation process ($M_{\rm cloud} = M_{\rm ecl} + M_{\rm gas}$). Usually it is very difficult to determine observationally the mass of the remaining gas after the end of the star formation process. For this reason the existing "typical" fractional star formation efficiency parameter is very uncertain. A value in the range

$$0.2 \lesssim \epsilon \lesssim 0.4$$

is often quoted. This means that less than half of the mass of a collapsing cloud ends up in stars. This is a strong hint that the star formation in a collapsing cloud is terminated by the newly formed stars: this is called **feedback effect** in star formation. Energetic processes are responsible for the termination of the star formation:
- the production of turbulence by the outflows from circumstellar disk around newly forming stars ,
- photoionization and heating by the energetic radiation produced by the gas accretion processes of protostellar sources or the UV radiation from the hot photosphere of young, massive stars,
- shocks created by the stellar winds of young stars,
- shocks from supernova explosions of very massive, short lived stars.

Feedback: instantaneous gas removal. We can estimate what happens if there is an embedded cluster of protostars where the gas is removed in a short time by energetic stellar processes.

We assume that the embedded, proto-stellar cluster is in a dynamical equilibrium state what is a reasonable assumption for a 10 Myr young cluster (see Table 3.1). Then the total energy (or binding energy) is

$$E_{\rm ecl} = -\frac{GM_{\rm init}^2}{r_{\rm init}} + \frac{1}{2} M_{\rm init} \sigma_{\rm init}^2 \tag{3.73}$$

where $\sigma_{\rm init}$ is the velocity dispersion which can be written for a virialized system $E_{\rm pot} + 2E_{\rm kin} = 0$ as

$$\sigma^2 = \frac{GM}{r} \,. \tag{3.74}$$

A virialized systems relates also the binding energy and the potential energy

$$E = -\frac{1}{2} E_{\rm pot} \,.$$

The initial mass is $M_{\rm init} = M_{\rm ecl} + M_{\rm gas}$ and this quantity can be the same as the total mass of the collapsing cloud $M_{\rm init} = M_{\rm cloud}$. The formalism is also valid for later stages where already some gas is lost, so that $M_{\rm init} < M_{\rm cloud}$ and $M_{\rm gas}(t) < M_{\rm gas}(t=0)$.

If energetic processes remove instantaneously the gas then the total mass of the cluster is changed from $M_{\rm init}$ to $M_{\rm after} = M_{\rm ecl}$ which includes only the total mass of the stars. The instantaneous gas removal does not change instantaneously the radial distribution $r_{\rm init}$ and the kinetic motion $\sigma_{\rm init}$ of the stars. The total binding energy of the cluster immediately after the gas removal is then

$$E_{\rm after} = -\frac{GM_{\rm after}^2}{r_{\rm init}} + \frac{1}{2} M_{\rm after} \sigma_{\rm init}^2 = -\frac{1}{2}\frac{GM_{\rm after}^2}{r_{\rm init}} \,. \tag{3.75}$$

The cluster evolves now with a timescale of the order of the relaxation time scale to a new equilibrium state. If we assume that the mass $M_{\rm cl} = M_{\rm after}$ and energy $E_{\rm cl} = E_{\rm after}$ are conserved during this phase then the new equilibrium state can be described by a new radius $r_{\rm cl}$ and a new velocity dispersion $\sigma_{\rm cl}$

$$E_{\rm cl} = -\frac{GM_{\rm cl}^2}{r_{\rm cl}} + \frac{1}{2} M_{\rm cl} \sigma_{\rm cl}^2 \,. \tag{3.76}$$

The resulting cluster radius follows from $E_{\rm cl} = E_{\rm after}$ where

$$E_{\rm after} = -\frac{GM_{\rm after}}{r_{\rm init}} \Big(M_{\rm after} - \frac{1}{2} M_{\rm init} \Big)$$

and with the relations from above $M_{\rm init} = M_{\rm cl} + M_{\rm gas}$ we obtain

$$\frac{r_{\rm cl}}{r_{\rm init}} = \frac{1}{2}\frac{M_{\rm cl}}{M_{\rm cl} - M_{\rm init}/2} = \frac{M_{\rm cl}}{M_{\rm cl} - M_{\rm gas}} \,. \tag{3.77}$$

This equation implies that the cluster radius goes to ∞, or becomes unbound

– for $M_{\rm gas} \to M_{\rm cl}$, or if the removed gas contains equal or more mass than the stellar mass of the cluster,
– this is potentially the case for inefficient star formation when the fractional star formation efficiency $\epsilon \leq 0.5$ is low and a lot of gas is still present in in the newly formed clusters.

Since, the inferred fractional star formation rate is low $\epsilon < 0.5$, and there are many gas-less clusters observed there must be alternatives to the instantaneous gas removal model. Instantaneous gas removal would lead to the destruction of many galactic clusters, but this did not happen for all the known stellar clusters in our Milky Way.

Feedback: continuous removal of gas. One can talk of a continuous mass loss if the time scale for gas removable is much longer than the relaxation time or the cluster crossing time:

$$\tau_{\rm gas} \gg \tau_{\rm cross} .$$

In this case the cluster adjusts its dynamics continously according to the virial equilibrium. The increase of the cluster radius can then be described as a result of small (infinitesimal) mass removals:

$$\frac{r_{\rm init} + \delta r}{r_{\rm init}} = \frac{M_{\rm init} - \delta M_{\rm gas}}{M_{\rm init} - \delta M_{\rm gas} - \delta M_{\rm gas}} . \tag{3.78}$$

which can be written as

$$\frac{r + dr}{r} = \frac{M - dM}{M - 2dM} ,$$

We search now for the formula for the relative radius increase of the cluster because of a small mass loss. Useful formulae are obtained by rearranging

$$\Big(\frac{dr}{r} + 1\Big)(M - 2dM) = M - dM \quad \text{or} \quad \frac{dr}{r}(M - 2dM) = -(M - 2dM) + M - dM = dM .$$

Since the radius increases for a reduced mass we can write

$$\frac{dr}{r}\frac{M - 2dM}{M} = -\frac{dM}{M} \quad \text{or} \quad \frac{dr}{r} = -\frac{dM}{M}\Big(\frac{M}{M - 2dM}\Big)$$

For slow mass loss, there is $|dM| \ll M$. and we can approximate

$$\frac{dr}{r} \approx -\frac{dM}{M} . \tag{3.79}$$

Integration yields $\ln(r_{\rm cl}/r_{\rm init}) = -\ln(M_{\rm cl}/M_{\rm init})$ or

$$\frac{r_{\rm cl}}{r_{\rm init}} = \frac{M_{\rm init}}{M_{\rm cl}} = \frac{M_{\rm ecl} + M_{\rm gas}}{M_{\rm cl}} = \frac{1}{\epsilon} . \tag{3.80}$$

If the mass-loss is slow, then one can have a low fractional star formation efficiency (say 0.2) and loose a lot of gas (80 %) from the initial cloud mass and still end up with a bound cluster. The radius of the cluster expands like $1/\epsilon$. For example, if 80 % of the mass is lost by a continous gas removal then the initial radius of the cluster expands by a factor of 5.

The conclusion is that with a slow mass loss, which allows a continuous re-virialization of the cluster dynamics, the mass loss causes less expansion and a more likely survival of a cluster compared to an instantaneous mass loss.

Mass segregation and core-formation. A cluster contains stars with a range of masses. The interactions of stars in a cluster induces, like in the kinetic gas theory, an evolution towards **equipartition**:

- in two-body interactions, the more massive stars transfer a significant amount of their large kinetic energy $m_1 v_1^2/2$ to less massive stars i, until $m_1 v_1^2 \approx m_i v_i^2$,
- this leads in a self-gravitating star clusters to a mass segregation, the more massive stars have less specific (per unit mass) kinetic energy and sink towards the cluster center, while less massive stars gain kinetic energy and diffuse outwards to larger radii.

The concentration of massive stars towards the center would just continue and lead to a core collapse. A relatively small number of massive stars concentrate in a very compact cluster core while the halo expands. This evolution would lead to a singularity if hard binaries would not counteract to this process.

Compact binary stars. Binary stars can transfer a lot of energy to a dense stellar system by dynamic interactions. We consider here only a simple energy argument.

A virialized system has a binding energy of

$$E_{\rm cl} \approx -\frac{GM^2}{R_{\rm cl}} \approx -\frac{G\,N^2 m_S^2}{R_{\rm cl}}\,.$$

We can compare this to the binding energy of a binary star which is

$$E_{\rm bin} \approx -\frac{G\,m_S^2}{a}$$

where a is the orbital separation (semi-major axis).

If the binary is sufficiently compact then its binding energy (negative total energy) is equal to the total binding energy of the entire cluster. The corresponding binary separation is

$$a_{\rm eq} \approx \frac{R_{\rm cl}}{N^2}\,. \tag{3.81}$$

This separation corresponds to

- $a_{\rm eq} \approx 2$ AU for a open cluster with 1000 stars and a radius of 10 pc,
- $a_{\rm eq} \approx 10^{-4}$ AU (or 0.1 $R_\odot$) for a globular cluster with 10^5 stars and 10 pc radius.

This comparison shows that compact binaries, also called hard binaries, can stabilize a stellar cluster against collapse. Interaction of a hard binary star with a single star can transfer orbital energy of the binary to the third star, which gains then kinetic energy and moves outward in the cluster. This interaction reduces of course the separation and the total energy of the binary. However, a compact binary can have more binding energy than an entire open cluster. Such binary star interactions act against the cluster core collapse due to two-body interactions and equipartition. Of course, the binaries become more and more compact with time and they may even merge. This scenario can also explain the presence of the blue stragglers in the HR-diagram of clusters.

In globular cluster several hard binaries are required to stabilize the system. With X-ray observations such hard binaries were indeed found in several globular clusters. There are cases with about 10 or even more such binaries in one globular cluster. These X-ray binaries have characteristics of low mass X-ray binaries, which are composed of a neutron star and a companion, often a white dwarfs, in a very compact orbit with an orbital period of about an hour. Thus the orbital separation is indeed very small, of the order 10^{-3} AU, or even less. Several such binaries are capable to stabilize a globular cluster against collapse of the compact core.

Evaporation. The stars in the cluster halo can escape from a cluster if the encounters with other stars transfer enough energy so that they can escape from the system. For this a star must reach a velocity above the **escape speed** $v_e(r)$ or its total energy must become positive:

$$E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2} m_S v^2(r) + m_S \Phi(r) > 0\,.$$

or

$$v(r) > v_e(r) = \sqrt{2\Phi(r)}\,.$$

This can be generalized to an expression $v_e^2(\vec{x}) = -2\Phi(\vec{x})$ so that we can write a general mean-squared escape velocity for a system with a density $\rho(\vec{x})$ according to

$$\langle v_e^2 \rangle = \frac{\int \rho(\vec{x}) v_e^2(\vec{x})\,\mathrm{d}\vec{x}}{\int \rho(\vec{x})\,\mathrm{d}\vec{x}} = -2\frac{\int \rho(\vec{x})\Phi(\vec{x})\,\mathrm{d}\vec{x}}{M} = -4\frac{E_{\text{pot}}}{M}$$

According to the virial theorem $2E_{\text{kin}} + E_{\text{pot}} = 0$, where E_{kin} is the total kinetic energy $M\langle v^2 \rangle/2$, the root mean squared (rms) escape speed is just twice the rms speed:

$$\langle v_e^2 \rangle = 2\langle v^2 \rangle\,.$$

We may assume that the velocity distribution behaves in a collisionally dominated system ($t > t_{\text{relax}}$ like a Maxwellian distribution, where a fraction of about $\gamma = 0.7$ % of particles have a velocity which is $v > 2\langle v \rangle$. Thus we can assume that the two-body interaction removes about a fraction γ of stars by evaporation every relaxation time:

$$\frac{dN}{dt} = -\frac{\gamma N}{t_{\text{relax}}} = -\frac{N}{t_{\text{evap}}}\,.$$

Thus the **evaporation time** is of the order

$$t_{\text{evap}} = \frac{t_{\text{relax}}}{\gamma} \approx 140\,t_{\text{relax}}\,.$$

Thus any system with an age comparable to $\tau \approx t_{\text{relax}}$ will have lost a substantial fraction of its stars. If we use the characteristic relaxation time scale for open cluster $t_{\text{relax}} \approx 10$ Myr then we obtain an evaporation time scale of the order 1.5 Gyr. This is of the same order of magnitude, although a bit higher, than the estimated typical age of stellar cluster of about $t \approx 0.3$ Gyr.

Most likely, there exist additional processes, which accelerates the evaporation of open clusters in the galactic disk. A possible process it the gravitational interaction of clusters with molecular clouds which enhances the stellar velocity dispersion in the cluster and shortens the evaporation time scale.