## Chapter 4

## Radiation from planets

We consider first basic, mostly photometric radiation parameters for solar system planets which can be easily compared with existing or future observations of extra-solar planets. In the next section we consider in more detail the physics of planetary atmospheres which is important for the interpretation of the thermal or reflected spectral radiation from planets.

### 4.1 Equilibrium temperature

The equilibrium temperature $T_{\text {eq }}$ of a planet is a theoretical parameter which assumes that the irradiated power coming from the flux $F_{\text {in }}$ from the star is equal to the thermal back-body emission luminosity of the planet $L_{\mathrm{P}}$. The following assumptions are made for the derivation of $T_{\text {eq }}$ :

- the irradiated radiation is either reflected or absorbed,
- the absorbed radiation energy is re-emitted as thermal radiation,
- there is no internal energy source,
- the planet is isothermal (same temperature on the day and night side!).

The irradiated power is:

$$
\begin{equation*}
P_{\mathrm{in}}=\frac{L_{\odot}}{4 \pi d_{P}^{2}} \pi R_{P}^{2}\left(1-A_{B}\right) \tag{4.1}
\end{equation*}
$$

where $L_{\odot}$ is the luminosity of the sun, $d_{P}$ the separation and $R_{P}$ the radius of the planet, and $A_{B}$ is the Bond albedo. $A_{B}$ is the fraction of the total irradiated energy which is reflected and which does not contribute to the heating of the planet. A Bond albedo $A_{B}=1$ means that all light is reflected, while $A_{B}=0$ indicates a perfectly absorbing (black) planet. Both cases are not realistic. Expected values for the Bond albedo are in the range $A_{B}=0.05$ to 0.95 .
The luminosity of the planet, which is assumed to radiate like a black body, is

$$
\begin{equation*}
L_{\mathrm{out}}=L_{P}=4 \pi R_{p}^{2} \sigma T_{\mathrm{eq}}^{4}, \tag{4.2}
\end{equation*}
$$

where $\sigma$ is the Stefan-Boltzmann constant and $T_{\text {eq }}$ the equilibrium temperature of the planet.

The equilibrium temperature $T_{\text {eq }}$ follows from $P_{\text {in }}=L_{\text {out }}$ :

$$
\begin{equation*}
T_{\mathrm{eq}}=\left(\frac{L_{\odot}\left(1-A_{B}\right)}{16 \pi \sigma}\right)^{1 / 4} \frac{1}{\sqrt{d_{P}}} \tag{4.3}
\end{equation*}
$$

This indicates that $T_{\text {eq }}$ decreases with distance from the sun for solar system objects or from the star for extra-solar planets. An important feature of this equations is, that it does not depend on the radius of the irradiated body which can be as small as a dust particle (mm-sized) or as large as a giant planet.

Temperatures for solar system planets. The equilibrium temperatures $T_{\text {eq }}$ for the solar system planets is given in Table 4.1 using the indicated Bond albedos $A_{B}$ and the planet separation $d_{P}=a$ from Table 2.1. The Table compares $T_{\text {eq }}$ also with the measured ground temperature $T_{\text {ground }}$ for terrestrial planets and the effective temperatures of the emitted thermal radiation $T_{\text {eff }}$. $T_{\text {eff }}$ is for Jupiter, Saturn and Neptune higher than the equilibrium temperature, because these planets have a substantial intrinsic energy source.

Mercury is a special case because this planet has no atmosphere and only a slow rotation. For this reason there are very large temperature differences between the irradiated ( 725 K ) and the non-irradiated ( 100 K ) hemisphere. For Mercury the assumption of an isothermal planet is not appropriate. However, averaged over all direction the effective temperature of the emitted thermal radiation agrees quite well with the equilibrium temperature.

Table 4.1: Radiation parameters for solar system planets: $A_{B}$ is the Bond albedo, $T_{\text {eq }}$, $T_{\text {ground }}, T_{\text {eff }}$ the equilibrium, ground and effective temperature, and $L_{P} / P_{\text {in }}$ the ratio between thermal emission and irradiation, $L_{p} / L_{\odot}$ the luminosity contrast, and $F_{p} / F_{\odot}$ (IR) the flux contrast at long wavelengths $\lambda \gg \lambda_{\max }$.

| Planet | $A_{B}$ | $T_{\text {eq }}$ | $T_{\text {ground }}$ | $T_{\text {eff }}$ | $L_{P} / P_{\text {in }}$ | $L_{p} / L_{\odot}$ <br> $10^{-10}$ | $\lambda_{\max }$ | $F_{p} / F_{\odot}$ <br> $10^{-6}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Mercury | 0.12 | 448 K | $725 / 100^{1} \mathrm{~K}$ | 448 K | 1 | 4.4 | $6.5 \mu \mathrm{~m}$ | 0.95 |
| Venus | 0.75 | 230 K | 730 K | 230 K | 1 | 7.7 | $8.8 \mu \mathrm{~m}$ | 4.3 |
| Earth | 0.31 | 253 K | 290 K | 279 K | 1 | 4.5 | $10.4 \mu \mathrm{~m}$ | 4.0 |
| Mars | 0.25 | 209 K | 225 K | 227 K | 1 | 0.56 | $12.8 \mu \mathrm{~m}$ | 0.93 |
| Jupiter | 0.34 | 110 K | - | 124 K | 1.6 | 21. | $23.4 \mu \mathrm{~m}$ | 220. |
| Saturn | 0.34 | 81 K | - | 95 K | 1.9 | 5.0 | $30.5 \mu \mathrm{~m}$ | 110. |
| Uranus | 0.30 | 59 K | - | 59 K | 1 | 0.14 | $49.2 \mu \mathrm{~m}$ | 13. |
| Neptune | 0.29 | 47 K | - | 59 K | 2.5 | 0.14 | $49.2 \mu \mathrm{~m}$ | 13. |

1: 725 K is for the irradiated hemisphere and 100 K for the "night" hemisphere. For the sun the adopted temperature is $T_{\text {eff }}=5800 \mathrm{~K}$.

Greenhouse effect for terrestrial planets. For the planets Earth and Venus the ground temperature $T_{\text {ground }}$ is a significantly higher than $T_{\text {eq }}$ due to the greenhouse effect.

Figure 4.1: Energy flow diagram for the greenhouse effect on Earth.
In the greenhouse effect (e.g. for Earth) the visual light from the sun penetrates through the atmosphere down to the surface and heats efficiently the ground. However, the thermal IR-radiation from the ground can only escape in certain spectral windows without strong molecular absorptions $\left(\mathrm{H}_{2} \mathrm{O}, \mathrm{CO}_{2}\right)$, while the rest is absorbed in the atmosphere. Energy transport from the warm/hot ground to higher cold layers occurs therefore through convection and radiation until the thermal radiation can escape to space. $T_{\text {eq }}$ represents the temperature of the atmospheric layers from which the thermal radiation can escape. Therefore, the ground temperature is higher than $T_{\text {eq }}$. The effect is stronger on Venus because of its much thicker atmosphere ( 90 bar) when compared to Earth (1 bar).

### 4.2 Thermal radiation from planets

Intrinsic energy for the giant planets. Table 4.1 gives the ratio between irradiated power $P_{\text {in }}$ and the total thermal emission $L_{P}$ which can be deduced from the equilibrium and effective temperatures according to

$$
\frac{L_{P}}{P_{\mathrm{in}}}=\left(\frac{T_{\mathrm{eff}}}{T_{\mathrm{eq}}}\right)^{4}
$$

A ratio $>1$ for Jupiter, Saturn, and Neptune indicates that these planets emit significantly more energy than they receive from the sun. This can be explained by the ongoing contraction, and differentiation, of these three planets. For Uranus, it is expected that there is also a small intrinsic flux but only at a level of about $5-10 \%$ of the irradiated flux. This effect is hard to measure due to uncertainties in the effective temperature determination. The presence of the internal energy source indicates that the central temperature of the giant planets is of the order $\approx 10^{\prime} 000 \mathrm{~K}$. Intrinsic energy sources can be neglected for the terrestrial planets in the solar system.

Black body radiation. The spectral intensity of the thermal radiation of an object at temperature $T$ can be described by the Planck or the black body intensity spectrum:

$$
\begin{equation*}
B(T, \lambda)=\frac{2 h c^{2}}{\lambda^{5}} \frac{1}{e^{h c / \lambda k T}-1} \tag{4.4}
\end{equation*}
$$

where $h, k$ and $c$ are Planck constant, Boltzmann constant and speed of light. The Planck intensity is given in unit of e.g. [ $J^{-2} \mathrm{sr}^{-1} \mathrm{~s}^{-1} \mu \mathrm{~m}^{-1}$ ] or [ $\left.\mathrm{erg} \mathrm{cm}^{-2} \mathrm{sr}^{-1} \mathrm{~s}^{-1} \AA^{-1}\right]$ ]. Black body radiation is isotropic so that the black body flux through a unit surface area is $\pi B(T, \lambda)$. It is assumed that the properties of the black body radiation are known and we remind here only some important facts:

- the black body spectrum can also be expressed as function of frequency

$$
B(T, \nu)=\frac{2 h \nu^{3}}{c^{2}} \frac{1}{e^{h \nu / k T}-1}
$$

- conversion between $B(T, \nu)$ and $B(T, \lambda)$ must use the factor $d \nu=-c / \lambda^{2} d \lambda$,
- the peak of the black body spectrum $B_{\max }(T, \lambda)$ is according to the Wien law at the wavelength:

$$
\begin{equation*}
\lambda_{\max }=\frac{2.9 \mathrm{~mm}}{T[\mathrm{~K}]} \tag{4.5}
\end{equation*}
$$

which is at $10 \mu \mathrm{~m}$ for a planet with $T=290 \mathrm{~K}$ ( $\approx$ Earth),

- for low frequency or long wavelengths the Planck radiation can be approximated by:

$$
\begin{equation*}
B(T, \nu)=\frac{2 \nu^{2}}{c^{2}} k T \quad \text { or } \quad B(T, \lambda)=\frac{2 c}{\lambda^{4}} k T \tag{4.6}
\end{equation*}
$$

- the total luminosity of the spherical black body (planet) with radius $R$ and effective temperature $T_{\text {eff }}$ is

$$
\begin{equation*}
L_{P}=4 \pi R_{P}^{2} \sigma T_{\mathrm{eff}}^{4} \tag{4.7}
\end{equation*}
$$

where $\sigma$ is the Stefan-Boltzmann constant (identical to Equation 4.2 except that $T_{\text {eff }}$ is used instead of $T_{\text {eq }}$ which does not account for intrinsic energy sources).

Thermal luminosity and flux contrast between planet and sun. The thermal luminosity $L_{P}$ of an irradiated planet without intrinsic energy source is given by Equations 4.1 or 4.2 . This can be expressed as thermal luminosity contrast $C_{\text {th }}$ between the planet and the sun

$$
\begin{equation*}
\frac{L_{P}}{L_{\odot}}=\frac{R_{P}^{2}}{R_{\odot}^{2}} \frac{T_{\mathrm{eq}}^{4}}{T_{\odot}^{4}}=\frac{R_{P}^{2}}{d_{P}^{2}} \frac{1}{4}\left(1-A_{B}\right) \tag{4.8}
\end{equation*}
$$

For solar system planets this ratio is very small, of the order $10^{-9}$ to $10^{-11}$ (see Table 4.1).
Equation 4.8 for the luminosity contrast is also valid for extra-solar systems. For hot Jupiters the ratio $L_{P} / L_{\odot}$ is much larger than for solar system planets.
The flux contrast as function of wavelength is important for observational studies. For long wavelengths, in the Rayleigh-Jeans part of the Planck function of the planet, one can use equation 4.6 which yields:

$$
\begin{equation*}
\frac{F_{P}\left(\lambda \gg \lambda_{\max }\right)}{F_{\odot}(\lambda)}=\frac{R_{P}^{2}}{R_{\odot}^{2}} \frac{T_{\mathrm{eq}}}{T_{\odot}} \tag{4.9}
\end{equation*}
$$

The factor for the temperature ratio between planet and sun (or star) $T_{\text {eq }} / T_{\odot}$ is of the order $\approx 10-100$. Thus the flux contrast at long wavelengths $\lambda \gg \lambda_{\max }$ is several orders of magnitudes $\left(10^{3}-10^{6}\right)$ larger than the total luminosity contrast (see Table 4.1). On the other hand, the planet to star flux contrast at short wavelengths $\lambda<\lambda_{\max }$ decreases rapidly to very small values because the thermal radiation of the planet drops-off exponentially. At short wavelengths the scattered light will therefore dominate.

### 4.3 Reflection from planets

Reflection by a Lambert surface. A Lambert surface is used as reference in many technical and scientific studies on reflectivities. A Lambert surface reflects all incident light and the surface brightness is the same for all viewing angles. However, for viewing directions with an angle $\theta$ with respect to the surface normal the apparent reflecting area and therefore also the reflected flux is reduced $\propto \cos \theta$. Thus the reflected intensity $I_{\text {Lam }}$ of a flat Lambert surface per unit solid angle is

$$
\begin{equation*}
I_{\mathrm{Lam}}(\theta)=F_{i} \frac{\cos \theta}{\pi} \quad \text { for } \quad 0^{\circ} \leq \theta<90^{\circ} \tag{4.10}
\end{equation*}
$$

where $F_{i}$ is the incident flux onto the considered surface. Thereby, the reflection from a Lambert surface does not depend on the direction of the irradiation. A sheet of white paper, a with screen or a white wall are close to a Lambert surface.

Figure 4.2: Reflection from a Lambert surface.
The factor $1 / \pi$ in Equation 4.10 is the normalization factor because energy conservation requires that the reflected intensity $I_{\text {Lam }}$ integrated over all direction is equal to $F_{i}$

$$
\int_{0}^{2 \pi} \int_{-\pi / 2}^{\pi / 2} I_{\mathrm{Lam}}(\theta) \sin \theta d \theta d \phi=\int_{0}^{2 \pi} \int_{0}^{\pi / 2} F_{i} \frac{\cos \theta}{\pi} \sin \theta d \theta d \phi=F_{i}
$$

An observer at a distance $D$ (much larger than the linear dimension of the surface area) measures a reflected flux $F_{\text {Lam }}$ per unit area of

$$
F_{\mathrm{Lam}}(\theta)=\frac{I_{\mathrm{Lam}}(\theta)}{D^{2}}=\frac{F_{i} \cos \theta}{\pi D^{2}}
$$

Normal retro-reflection of a Lambert disk irradiated by the sun. Based on the reflection law for a Lambert surface we can derive the normal retro-reflection (= normal irradiation and normal reflection) of solar light by a round Lambert disk with radius $R_{\text {disk }}$ at the distance $d_{\text {disk }}$ from the sun:

$$
F_{\mathrm{disk}}(\lambda, \theta=0)=\frac{F_{i}(\lambda)}{\pi D^{2}}=\frac{L_{\odot}(\lambda)}{4 \pi d_{\mathrm{disk}}^{2}} \pi R_{\text {disk }}^{2} \frac{1}{\pi D^{2}}
$$

where $F_{i}$ is replaced by the explicit formula for the sunlight intercepted by the Lambert disk.

Geometric albedo for solar system planets. The geometric albedo $A_{g}(\lambda)$ is the spectral reflectivity of a planet at zero phase angle $\alpha=0$ (full phase) relative to the reflectivity of a Lambert disk with the same cross section as the planet

$$
\begin{equation*}
A_{g}(\lambda)=\frac{F_{P}(\lambda)}{F_{\text {disk }}(\lambda)} . \tag{4.11}
\end{equation*}
$$

Thus, the geometric albedo of a planet can be determined by measuring the magnitude of that planet at opposition (normal retro-reflection), which is then compared to the calculated reflection of a Lambert disk with the same cross section.
It is convenient to express the theoretically reflected flux from a Lambert disk relative to the flux of the sun measured from Earth $F_{\odot}(\lambda)=L_{\odot}(\lambda) / 4 \pi d_{\mathrm{E}}^{2}$ (where $d_{\mathrm{E}}=1 \mathrm{AU}$ ), because this ratio is independent of wavelength:

$$
\begin{equation*}
R=\frac{F_{\text {disk }}}{F_{\odot}}=\frac{d_{\mathrm{E}}^{2}}{D^{2}} \frac{R_{\text {disk }}^{2}}{d_{\text {disk }}^{2}} . \tag{4.12}
\end{equation*}
$$

Opposition $\alpha=0^{\circ}$ occurs for the outer planets almost every year. Because the phase angles for the giant planets is never really large, $\alpha \lesssim 12^{\circ}$ for Jupiter, $\lesssim 5^{\circ}$, and less for Uranus and Neptune, one can correct for the small deviations for an "ideal" geometric albedo measurement.
Example Jupiter: As example we calculate with Equation 4.12 the case for Jupiter for which the distance to Earth at opposition is $D=5.2-1 \mathrm{AU}, R_{\text {disk }}=R_{J}=69910 \mathrm{~km}$ and $d_{\text {disk }}=d_{J}=5.2 \mathrm{AU}$ with $1 \mathrm{AU}=1.510^{8} \mathrm{~km}$. The ratio between the flux of a Lambert disk with a cross section equivalent to Jupiter and the solar flux is

$$
R=\frac{F_{\text {disk }}}{F_{\odot}}=4.55 \cdot 10^{-10} \quad \text { or } \quad m_{\text {disk }}-m_{\odot}=-2.5 \log R=23.36
$$

where the result is also given as magnitude difference. The apparent V-band magnitude for the sun is $m_{\odot}(\mathrm{V})=-26.74 \mathrm{mag}$ and for Jupiter at opposition about $m_{J}(\mathrm{~V})=-2.70$ mag. This yields an opposition contrast of $m_{J}-m_{\odot}=24.04 \mathrm{mag}$ or about $\Delta m=0.7 \mathrm{mag}$ more than expected for a Lambertian disk. The geometric albedo of Jupiter is this magnitude difference $\Delta m$ expressed as ratio $A_{g}=10^{-0.4 \cdot \Delta m}=0.52$ in good agreement with available literature values.

Figure 4.3: Typical constellation for the geometric albedo measurement of Jupiter or another outer planet.

Table 4.2: Reflection properties of solar system planets: geometric albedos for the V-band and the IR, phase integral $q$ and calculated spherical albedos. The last columns give the factor $R_{p}^{2} / d_{p}^{2}$ and the flux contrast for the scattered light at quadrature phase assuming $f\left(90^{\circ}\right)=0.3$.

| planet | $A_{g}(V)$ | $A_{g}(\mathrm{IR})$ | $q$ | $A_{s}(V)$ | $A_{s}(I R)$ | $A_{B}$ | $R_{p}^{2} / d_{p}^{2}$ <br> $10^{-10}$ | $F_{P} / F_{\odot}$ <br> $10^{-10}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Mercury | 0.142 |  | 0.48 | 0.07 |  | 0.12 | 18. | 0.77 |
| Venus | 0.67 |  |  |  |  | 0.75 | 31. | 6.2 |
| Earth | 0.367 |  |  |  |  | 0.31 | 18. | 2.0 |
| Mars | 0.170 |  |  |  |  | 0.25 | 2.2 | 0.11 |
| Jupiter | 0.52 | 0.27 | 1.25 | 0.65 | 0.34 | 0.34 | 77. | 12. |
| Saturn | 0.47 | 0.24 | 1.40 | 0.66 | 0.34 | 0.34 | 16. | 2.3 |
| Uranus | 0.51 | 0.21 | 1.40 | 0.71 | 0.29 | 0.30 | 0.78 | 0.12 |
| Neptune | 0.41 | 0.25 | 1.25 | 0.51 | 0.31 | 0.29 | 0.29 | 0.036 |

Geometric albedo of a Lambert sphere. It is important to note that a Lambert sphere has a geometric albedo of $A_{g}=2 / 3$. The surface brightness of a Lambert disk of normalized radius $R=1$ is constant over the whole disk and one can write for the normal retro-reflection $(\theta=0)$ :

$$
I_{\mathrm{disk}}(r)=\frac{F_{i}}{\pi} \quad \text { and } \quad \int_{0}^{1} I_{\text {disk }}(r) 2 \pi r d r=2 F_{i} \int_{0}^{1} r d r=F_{i}
$$

A sphere (not a disk) at zero phase angle has a surface brightness distribution with a limb darkening which behaves for the normalized radius $0 \leq r \leq 1$ like

$$
I_{\mathrm{sph}}(r)=F_{i} \frac{\cos \theta^{\prime}(r)}{\pi}=F_{i} \frac{\sqrt{1-r^{2}}}{\pi}
$$

Figure 4.4: Schematic difference of the geometric albedo of a Lambert disk and a Lambert sphere.

The angle $\theta^{\prime}$ is the angle of incidence with respect to the surface normal which depends on the radial distance $r=\sin \theta^{\prime}$ measured from the center of the illuminated hemisphere (apparent disk). The sub-solar point reflects like a disk (surface brightness $F_{i} / \pi$ ) but the irradiation of the more an more inclined surface towards the limb results in a reduced back-scattering because the strongest scattering occurs along the surface normal.
Integration for a fully illuminated Lambert sphere yields:

$$
\int_{0}^{1} I_{\mathrm{sph}}(r) 2 \pi r d r=2 F_{i} \int_{0}^{1} r \sqrt{1-r^{2}} d r=\left.2 F_{i}\left(-\frac{\left(1-r^{2}\right)^{3 / 2}}{3}\right)\right|_{0} ^{1}=\frac{2}{3} F_{i}
$$

A Lambert sphere reflects only $2 / 3$ of the light for phase angle $\alpha=0$ when compared to a Lambert disk because a substantial fraction of light is scattered into direction $\alpha>\pi / 2$ what does not occur for an illuminated disk. Lambert disk and Lambert sphere scatter both all light and have a Bond albedo (or spherical albedo) of $A_{B}=1$ but the angular distribution of the scattered light is different.

It is not surprising that the solar system planets have geometric albedos $A_{g} \lesssim 0.7$ when considering the case of the perfectly reflecting Lambert sphere. Averaged over all wavelengths the $A_{g}$ should be smaller (about $2 / 3$ ) than the Bond albedo $A_{B}$. This is roughly the case for Venus and Mars (see Table 4.2).

For Earth and the giant planets the situation is different. The geometric albedo in the visual is higher than the Bond albedo $A_{g}(\mathrm{~V})>A_{B}$. This indicates that the geometric albedo must be low at other wavelengths, what is the case for the IR wavelength regime because of molecular absorption by $\mathrm{H}_{2} \mathrm{O}$ for Earth and $\mathrm{CH}_{4}$ for the giant planets.

Spherical albedo and Bond albedo. The spherical albedo $A_{s}(\lambda)$ gives the reflection in all direction and not only the normal retro-reflection as measured for the geometric albedo $A_{g}(\lambda)$. The spherical albedo is required for an accurate derivation of the Bond albedo $A_{B}$. $A_{B}$, which is used for energy budget calculations, is the flux weighted wavelength average of the spherical albedo:

$$
\begin{equation*}
A_{B}=\frac{\int_{0}^{\infty} F_{i}(\lambda) A_{s}(\lambda) d \lambda}{\int_{0}^{\infty} F_{i}(\lambda) d \lambda} \tag{4.13}
\end{equation*}
$$

With a scattering model of a planet it is easy to calculate the geometric albedo and spherical albedo. Observationally, one needs to know the scattering in all direction, what is a very difficult to achieve. For example, the reflection $f(\alpha)$ of Earth for a phase angle $\alpha=90^{\circ}$ will be different if mainly the white polar regions are seen from a polar direction when compared to the dark oceans as seen from equatorial directions.

One simple way to address the problem of the reflection into different directions is the phase integral $q$ defined by

$$
q=2 \int_{0}^{\pi} \frac{F_{\mathrm{ref}}(\alpha)}{F_{\mathrm{ref}}(\alpha=0)} \sin \alpha d \alpha
$$

where $F_{\text {ref }}(\alpha)$ is a rotationally symmetric phase angle dependence of the reflected radiation normalized to the geometric albedo $F_{\text {ref }}(\alpha=0)=A_{g}$. With this definition the phase integral, geometric albedo, and spherical albedo are related by

$$
\begin{equation*}
A_{s}=A_{g} q \tag{4.14}
\end{equation*}
$$

It should be noted that this approach is only a first order approximation which is formally only correct for rotationally symmetric reflection from planets.
The phase integral $q$ for special cases is:

- $q=1$ for a Lambert disk,
- $q=3 / 2$ for a Lambert sphere,
$-q=4$ for (a theoretical) isotropically scattering body.
Some values of the phase integral for solar system planets are given in Table 4.2.
Reflectivity phase curves. The phase angle dependence of the reflected radiation from a planet is important for the analysis of observations. In general, planets are not observed at phase angle $\alpha=0^{\circ}$. For example, the inner planets, Mercury and Venus, are behind the sun for $\alpha=0^{\circ}$, and extra-solar planets are behind "their star". With direct imaging of extra-solar planets only data in the range $30^{\circ}<\alpha<150^{\circ}$ can probably be obtained in the near future. For this reason one needs to study the reflectivity phase curves $F_{\text {ref }}(\alpha)$ or the phase dependence of the reflection normalized to the geometric albedo:

$$
\begin{equation*}
f(\alpha)=\frac{F_{\mathrm{ref}}(\alpha)}{F_{\mathrm{ref}}(\alpha=0)} . \tag{4.15}
\end{equation*}
$$

Phase curve for a Lambert sphere. The phase curve for a Lambert sphere can be derived analytically by integrating the $\cos \theta$ reflection law of the visible part of the illuminated sphere as function of phase angle $\alpha$. The solution is:

$$
\begin{equation*}
f(\alpha)=\frac{1}{\pi}(\sin \alpha+(\pi-\alpha) \cos \alpha) . \tag{4.16}
\end{equation*}
$$

Flux contrast for reflecting extra-solar planets. Equation 4.12 is also valid for a very distant observer outside of the solar system or for the observations of extra-solar planets from Earth. In this case the distance of the observer to the central star $d_{\text {star }}$ (which was $d_{\mathrm{E}}$ for an Earth-based observer looking at a solar system planet) and the distance from the planet to the observer $D$ are equal and very large $d_{\text {star }}=D \gg 1 \mathrm{AU}$. Thus the contrast of a reflecting planet $C_{\text {ref }}$ with respect to its illuminating star is

$$
\begin{equation*}
C_{\mathrm{ref}}=\frac{F_{\mathrm{P}}}{F_{\mathrm{star}}}=A_{g}(\lambda) f(\alpha) \frac{R_{P}^{2}}{d_{P}^{2}}, \tag{4.17}
\end{equation*}
$$

where $A_{g}(\lambda)$ is the geometric albedo and $f(\alpha)$ a normalized phase function as described by Equation 4.15 which takes into account that the reflected light depends on the angle star - planet - observer. Table 4.2 gives the factors $R_{P}^{2} / d_{P}^{2}$ for the solar system planets and also estimates for the contrast of the reflected light for a phase angle $\alpha=90^{\circ}$.

