

Chapter 3

Transits of planets: mean densities

Close-in (short period) planets have a relatively high chance to transit in front of the star. A transit introduces a small periodic dimming of the star which is for a given star proportional to the size of the planet. The photometric observation of transits provides therefore the orbital period and the radius ratio between planet and star. Because the radii of stars are quite well known, the planet radius can be derived and with the planet mass from the radial velocity method also the mean density. Planet transits have been detected for more than 1000 stars. Such a large sample yields important information, not only on planet frequency and orbit properties, but also on the composition and internal structure of different types of planets based on the mean density determinations.

In this chapter we discuss first the radius and mean density of solar system objects. They serve again as well studied test cases. Then we describe the transit technique for extra-solar planets and describe some important scientific results from transit observations.

3.1 The structure of solar-system planets

3.1.1 Radius, mass and mean density for planets

A rough characterization an extra-solar planet, comparable to observations and results for Solar System planets, can be based on the size, the mass, and the mean (or bulk) density.

The radii of solar system planets can be derived from angular diameter measurements. Because of their fast rotation the giant planets are flattened significantly ($> 1\%$). The difference between equatorial and polar radius R_e and R_p are given in Table 3.1. The flattening $f = (R_e - R_p)/R_e$ for the terrestrial planets is less than 1%. It is useful to express the size of a planet by its volumetric mean radius R_V defined as radius of a spherical body with the same volume.

Masses of solar system planets and large bodies can be determined from the orbits of moons using Kepler's third law.

From the radius R and the mass M one can calculate the mean density $\bar{\rho}$ of a planet:

$$\bar{\rho} = \frac{M}{V} = \frac{M}{(4/3)\pi R^3}. \quad (3.1)$$

Table 3.1: Giant planet data: rotation period P_{rot} , equatorial radius R_e , polar radius R_p , volumetric mean radius R_V , and flattening $f = (R_e - R_p)/R_e$.

Planet	P_{rot}	R_e [km]	R_p [km]	R_V [km]	f
Jupiter	9^h55^m	71 490	66 850	69 910	6.5 %
Saturn	10^h39^m	60 270	54 360	58 230	9.8 %
Uranus	17^h15^m	25 560	24 970	25 360	2.3 %
Neptune	16^h06^m	24 770	24 340	24 620	1.7 %

Table 3.2 gives mass M , radius R , and ρ for the large bodies in the solar system. A few points are notable from Table 3.2:

- the objects with the highest density are the terrestrial planets Mercury, Venus and Earth with $\bar{\rho} \approx 5.5 \text{ g cm}^{-3}$, followed by Mars with $\bar{\rho} \approx 4 \text{ g cm}^{-3}$,
- the Earth’s moon and the Galilean moons Io and Europa have intermediate densities of about $\bar{\rho} \approx 3.0$ to 3.5 g cm^{-3} ,
- all other large solid bodies have a density around $\bar{\rho} \approx 2.0 \text{ g cm}^{-3}$,
- the gas planets and the sun have a low mean density of $\bar{\rho} \approx 0.7 - 1.7 \text{ g cm}^{-3}$.

The main reason for the different mean densities are the different compositions, but also the different structure. As a very rough statement one can say, that high density objects $\bar{\rho} > 3.0 \text{ g cm}^{-3}$ are composed of rocks and iron, intermediate density objects $\bar{\rho} \approx 2.0 \text{ g cm}^{-3}$ contain in addition a substantial fraction of water ice, while for low density objects a substantial part of their volume is due to H and He.

3.1.2 Composition of solar system planets

It is not easy to derive the composition of planets because one cannot look deep into their interior. There are two very important sources of information about elemental abundances in the solar system:

- **The solar photosphere**, which is considered to show representative abundances for the convective envelope of the sun and the elemental abundances in the pre-solar nebula.
- **Carbonaceous chondrites**, a special type of primitive meteorite, which were not strongly altered by heating, melting, differentiation and other processes. Therefore, their abundances are considered to represent well the dust particles of the early solar system.

The abundances in the solar photosphere can be determined by the spectroscopic analysis of the solar spectrum. This method is accurate ($\pm 10\%$) for abundant elements with many absorption lines and well determined atomic data. Abundances of H, He, noble gases, and other volatile elements can be directly compared to heavy elements (e.g. Mg, Si, Fe). The abundance determination is difficult for rare elements with only few spectral lines in the solar spectrum.

The abundances of meteorites can be determined with very high accuracy with modern laboratory techniques. Abundances of very rare elements can be determined as well as

Table 3.2: Mean radius R , mass M and mean density $\bar{\rho}$ for large solar system objects with $R > 1000$ km. The mass M is given in Earth mass $M_E = 5.97 \cdot 10^{24}$ kg.

Planet / moon	R [km]	M [M_E]	$\bar{\rho}$ [g cm^{-3}]
sun	696'000	333'000	1.41
planets			
Mercury	2 440	0.0553	5.43
Venus	6 050	0.815	5.24
Earth	6 370	1.0	5.51
Mars	3 390	0.107	3.94
Jupiter	69 910	318.	1.33
Saturn	58 230	95.2	0.70
Uranus	25 360	14.5	1.30
Neptune	24 620	17.1	1.76
dwarf planets			
Pluto	1 160	$2.2 \cdot 10^{-3}$	2.0 (?)
Eris	1 160	$2.7 \cdot 10^{-3}$	2.5 (?)
moons			
Moon (E)	1 740	0.0123	3.35
Io (J)	1 820	0.0150	3.53
Europa (J)	1 560	$8.0 \cdot 10^{-3}$	3.01
Ganymede (J)	2 630	0.0248	1.94
Callisto (J)	2 410	0.0180	1.83
Titan (S)	2 580	0.0255	1.88
Triton (N)	1 350	$3.6 \cdot 10^{-3}$	2.06

values taken from en.wikipedia.org/wiki/list_of_Solar_System_objects_by_size.

the different isotopes for the elements. A major disadvantage is that volatile elements are strongly depleted, most notably H, He, C, N, O, S, and the noble gases. Especially for the elements H, C, O, S, there are strong differences between different chondritic meteorites, depending on the individual history of the sample. Abundances of many samples must be compared to exclude the possibility of abundances anomalies due to a special sample.

The solar and chondrite abundances form also the backbone for the determination of the universal “cosmic” abundances of the elements. Data from other stars and emission nebulae confirm that the abundances determined in the solar system are representative for the whole Universe.

The mass fraction f_{mass} of the most abundant elements is given in Table 3.3 for the solar photosphere, chondrites, the Earth and Jupiter. The sun, like the entire Universe, is essentially made of 98 % H and He plus about 2 % of “heavy” elements. Heavy elements include first O, C, Ne and N, then less abundant Fe, Si, Mg, and then the other elements. They can be explained by the stellar nucleo-synthesis theory (see also slide). The composition for chondritic meteorites is very different because of the strongly reduced abundance of the volatile elements or “gases”. For the abundances of Earth and Jupiter only the elements which contribute more than 1 % to the mass are given in Table 3.3 to emphasize the large differences in composition between these two planets. Of course also

Table 3.3: Abundances of the most frequent elements for the sun, chondrites, Earth and Jupiter expressed as mass fraction f_m and for the sun as atomic abundance relative to Si.

Z element	sun N(X)/N(Si)	sun f_m	Chondrite f_m	Earth f_m	Jupiter
1 H	27200	74 %	2.0 %	<	71 %
2 He	2180	24 %	<	<	26 %
6 C	12.1	0.40 %	3.5 %	<	1 %
7 N	2.5	0.10 %	0.3 %	<	<
8 O	20.1	0.86 %	46.6 %	30 %	2 %
10 Ne	3.8	0.20 %	<	<	<
11 Na	0.06	<	0.5 %	<	<
12 Mg	1.08	0.07 %	9.6 %	12 %	<
13 Al	0.08	<	0.9 %	1 %	<
14 Si	1	0.08 %	10.7 %	19 %	<
16 S	0.52	0.05 %	5.2 %	1 %	<
18 Ar	0.10	0.01 %	<	<	<
20 Ca	0.06	<	0.9 %	1 %	<
24 Cr	0.01	<	0.3 %	<	<
26 Fe	0.90	0.14 %	18.5 %	33 %	<
28 Ni	0.05	<	1.1 %	2 %	<
< is:		< 0.01 %	< 0.1 %	< 1 %	< 1 %

estimates for the abundances of the rare elements are available. However, these are often inferred from abundance ratios from the sun or chondrites, because some pairs of elements (e.g. Ni/Fe) are expected to behave very similar. Alternatively the abundance ratio Si/Fe follows from determinations of the size of the planet core based on models for the internal structure.

3.1.3 Differentiation: the example of Earth

Terrestrial planets were formed by the accumulation of solid bodies with elemental abundances similar to (chondritic) meteorites. There are four elements which dominate:

- O - oxygen: mostly bound in silicates $X\text{-SiO}_x$, but also other minerals like MgO,
- Fe - iron: bound in FeS, in silicates FeSiO_x and other minerals, but if melted then it accumulates as iron metal or melt,
- Si - silicon: main components of silicate rich rock, quartz (sand),
- Mg - magnesium: important constituent of silicate rich rock MgSiO_x .

The interior of Earth has a temperature of a few 1000 K because of internal heating by the decay of radioactive nuclei. This temperature is high enough that the rock/magma and iron in the Earth interior behave like fluids on geological time scales. This means that Earth had time for sedimentation into layers of progressively higher density towards the center. This explains the formation of an iron core and the silicate-rich mantle. This structure is well established based on the analysis of seismic waves. Due to the

differentiation the surface composition is not representative for the average composition of a planet, especially if predominant constituents have a high density and accumulate in the center.

Thus the material composition and rough structure of Earth is:

- 33 % of the mass are Fe or iron alloys, mainly in the core,
- 67 % of rock-like material mainly based on silicates $(\text{Mg,Fe})\text{SiO}_x$ in the mantle,
- the water H_2O in the oceans contributes only 0.024 % to M_E ,
- the N_2 and O_2 in the atmosphere is only 0.0001 %.

This very simple structure description for the planet Earth is of course far from a geological description of the inner structure of this planet. However, this is a useful starting point for the comparison of the structure of Earth with other solar system bodies and extra-solar planets.

3.2 Basic equations for the structure of planets

The radial model structure of a spherical planets can be derived from three basic equations:

The hydrostatic equation

$$\frac{dP(r)}{dr} = -g(r)\rho(r) = -\frac{GM_r(r)}{r^2}\rho(r), \quad (3.2)$$

where $P(r)$ is the pressure, $\rho(r)$ the density, $M_r(r)$ the mass inside radius r , and $g(r) = GM_r(r)/r^2$ the gravitational acceleration (G the gravitational constant).

The equation for mass conservation

$$\frac{dM_r(r)}{dr} = 4\pi r^2 \rho(r). \quad (3.3)$$

The equation of state (EOS) which describes the density as function of pressure, temperature and composition:

$$\rho(r) = \rho(P, T, \text{composition}). \quad (3.4)$$

The EOS cannot be self-consistently derived from the above equations because the temperature cannot be determined without equations for the energy generation and the energy transport. Therefore, assumptions or prescriptions of the temperature are required. Also the composition must be given as input parameter for the modelling.

The following boundary conditions are valid for self-gravitating sphere:

$$M_r(r) = 0 \quad \text{at} \quad r = 0, \quad \text{and} \quad P(r) = 0 \quad \text{at} \quad r = R,$$

where R is the radius of the planet. For gaseous planets there is not a well-defined outer boundary and often the boundary condition $P(R) = 1 \text{ bar}$ (10^5 Pa) is used.

The modelling can be substantially simplified if the equation of state depends only on the pressure and a predefined radius-dependent composition according to

$$\rho(r) = \rho(P(r), \text{composition}(r)).$$

In this case the three equations can be solved iteratively and one obtains density, pressure and composition as function of radius.

3.2.1 Central pressure for a homogeneous planet

First we consider the very simple model case of a homogeneous planet. The assumption $\rho(r) = \bar{\rho}$ is a very rough over-simplification because there are strong composition and therefore density changes in planets and the materials may be compressible. For small solar system bodies, the assumption of a homogeneous structure may be acceptable.

Nonetheless the central pressure $P_0 = P(r = 0)$ of a homogeneous sphere yields a first guess on the typical pressures in planets which must be considered for the equation of state.

The pressure profile and the central pressure follows from an integration of the equation for the hydrostatic equilibrium using $M_r(r) = (4/3)\pi r^3 \bar{\rho}$. The pressure profile is:

$$P(r) = - \int_r^R \frac{dP}{dr} dr = \int_r^R \frac{GM_r(r)}{r^2} \bar{\rho} dr = \frac{4\pi G \bar{\rho}^2}{3} \int_r^R r dr = \frac{2\pi G \bar{\rho}^2}{3} (R^2 - r^2).$$

This can be written in the simple form

$$P(r) = P_0 \left(1 - \frac{r^2}{R^2} \right), \quad (3.5)$$

with the central pressure

$$P_0 = \frac{2}{3}\pi G \bar{\rho}^2 R^2 \quad \text{or} \quad P_0 = \frac{3GM^2}{8\pi R^4}, \quad (3.6)$$

where for the second equation the $\bar{\rho}$ is replaced according to Eq. 3.1. For a given mean density $\bar{\rho}$ the central pressure increases like

$$P_0 \propto R^2 \propto M^{2/3},$$

and for a given mass like

$$P_0 \propto 1/R^4.$$

Table 3.4 gives P_0 for several solar system objects using Eq. 3.6 and R and M from Table 3.2. The central pressures derived with the homogeneous planet $\bar{\rho}$ approximation are already accurate within a factor of a few for the terrestrial planets. The discrepancy is larger for the giant planets because they have a low density envelope $< 1 \text{ g cm}^{-3}$ and a high density core $> 10 \text{ g cm}^{-3}$.

Table 3.4: Central pressures P_0 calculated for homogeneous planets and based on detailed model calculations

object	R/R_E	M/M_E	$P_0(\bar{\rho})$	P_0 model ¹	T_0 model ¹
Moon	0.273	0.0123	47 kbar	45 kbar	1 800 K
Mercury	0.383	0.0553	240 kbar	400 kbar	2 000 K
Earth	1	1	1.7 Mbar	3.6 Mbar	6 000 K
Jupiter	11.0	318.	12 Mbar	80 Mbar	20 000 K

1: central pressures and temperatures from detailed modelling taken from the compilation of de Pater and Lissauer

3.2.2 Phase diagrams

The equation of state relates the pressure, density and temperature for the different materials in a planet. For the structure equation, we need a relation for the density as function of radius $\rho(r)$ or as function of pressure $\rho(P)$. The density of a material depends on the phase, which can be described in phase diagrams (Fig. 3.1).

Figure 3.1: Generic $P - T$ phase diagram with the different dividing lines between phases.

The triple points, dividing lines, and the different phases in the phase diagram (Figure 3.1) can be characterized as follows:

- **The triple point** in the low T - low P domain defines the areas for the solid, liquid, and gas phases.
- **The critical point** defines the transition between the gas or liquid phase and the super-critical fluid phase.
- **Solid** materials are relatively cold or under high pressure. The density of solids varies between about $\rho \approx 1$ and 10 g cm^{-3} . Depending on the pressure the materials can have different solid state structures (crystalline structures), with more compact = higher density configurations under high pressure. The density changes typically less than a factor of 2 between the different solid state structures.
- **Liquid** materials have a temperature above the triple point and below the critical point. Often a liquid becomes a solid if the pressure is strongly enhanced and it evaporates for very low pressures. The density of a liquid changes only little ($< 30 \%$) if pressure or temperatures are changed.
- **Gas** occurs for material under relatively low pressures, below the critical point or below the triple point. If gas is further heated it dissociates and becomes an **atomic gas**, which may become an **ionized plasma** for even higher temperatures. Gas behaves like (ideal gas)

$$\rho(P) = \frac{P}{\mathcal{RT}} \mu,$$

where μ is the mean particle mass. Thus, a gas is compressible and the density increases for a given temperature like $\rho \propto P$ until a critical density is reached at which a cold gas condenses or a hot/warm gas becomes a super-critical fluid.

- **Super-critical fluids** are at temperatures and pressures which are both above the critical point. Under very high pressures a super-critical fluid goes into a solid state. A super-critical fluid has properties of the liquid (can solve other materials) and the gas phase. For example, a super-critical fluid fills the whole volume of a container homogeneously and does not form a transition between a gas in the upper part and a liquid in the lower part of the container.

Figure 3.2: Behavior of liquid/gas and a super-critical fluid in a container

In particular, super-critical fluids are **compressible** and their density increases if the pressure is enhanced. The dependence $\rho(P, T)$ can not be described by a simple relationship.

- **Dissociation of molecules** in the gas phase or the super-critical fluid phase occurs in the temperature regime of 1000 K or a few 1000 K typically. The bindings between the individual atoms is broken up and the gas contains only atoms.
- **Ionization** of atoms occurs at temperature above a few 1000 K and if the temperature is further enhanced then atoms can be progressively ionized.
- **Pressure ionization** of solids or super-critical fluids occurs in the density regime of the order Mbar (10^{11} Pa). These pressures are high enough to squeeze together the nuclei in the material despite the Coulomb forces. The nuclear potentials start to overlap and the electrons can move freely between the nuclei - the material is pressure ionized. Many properties are similar to metals (e.g. electric conductivity). Pressure-ionized materials are compressible with the approximative dependence

$$\rho(P) \propto P^{1/2}.$$

Figure 3.3: Schematic difference for solid and pressure ionized materials.

3.2.3 Phase diagrams for the solar system planets and moons

We discuss in this section strongly simplified (qualitative) phase diagrams for hydrogen, water and iron in order to get the general picture about the EOS for solar system objects (see slides). The diagrams include points for the central pressure and temperature of objects, as well as lines and a few values which illustrate the run of parameters from the center towards the surface of the highlighted objects given in red. All diagrams cover the identical parameter space for an easy comparison. Thermodynamic parameters for hydrogen, water and iron are given in Table 3.5.

Table 3.5: Thermodynamic quantities of important materials for planets; melting T_{melt} and boiling temperatures T_{boil} and density ρ at $P = 1$ bar and T_{CP} and P_{CP} for the critical point.

material	1 bar properties			critical point	
	T_{melt}	T_{boil}	ρ	T_{CP}	P_{CP}
H ₂	14 K	21 K	0.09 mg cm ⁻³	33 K	13 bar
H ₂ O	273 K	373 K	1.0 g cm ⁻³	647 K	220 bar
Fe	1810 K	3130 K	7.9 g cm ⁻³	9250 K	8.8 kbar

Hydrogen phase diagram. Hydrogen is an important constituent for the giant planets. Because it is a light element it will be located outside the planet core which is composed of heavier materials.

In Jupiter and Saturn hydrogen is the dominant constituent. Pressure-ionized (metallic) liquid hydrogen with a density of about 4 g cm⁻³ at a pressure of 10 Mbar is expected outside the central rocky/iron core. The transition to a super-critical fluid (around 1 Mbar) occurs for Jupiter at about 0.8 R_J and in Saturn at about 0.5 R_S . As super-critical fluid the hydrogen has a density of the order 0.7 g cm⁻³. H₂ gas is only present close to the surface.

Neptune and Uranus are also called ice-giants because a substantial fraction of their mass is made up by icy materials, mostly H₂O, but also CH₄ and NH₃ ice. Hydrogen as light super-critical fluid $\rho = 0.7$ g cm⁻³ occurs outside $> 0.7 R_N$ of the ice - rock core and extends up to the hydrogen gas atmosphere.

H₂O phase diagram. Water is an important constituent in the ice giants Neptune and Uranus and in TNOs and the moons of the giant planets. We take Europa as an example for a “solid” icy body.

In Neptune and Uranus a zone of liquid (metallic) pressure-ionized ice in the form of H₃O⁺, OH⁻ is predicted from about 0.2 to 0.7 R_N . The density is about 2.5 to 5 g cm⁻³. Inside of the ice zone is a rocky/iron core and outside the hydrogen envelope discussed above.

For Europa and other solid bodies in the outer solar system, water is the lightest material and it forms therefore the outermost layer. The internal temperature of Europa is probably high enough that there exists an ocean of liquid water below the surface crust of water ice. For smaller bodies than Europa the temperature might be too low for the presence of water in liquid form.

Fe phase diagram. Iron is the most abundant heavy material in planets and therefore it is a main constituent in the cores of the planets. The melting temperature of iron at low pressure is around 1500 - 2000 K and it depends a lot on the presence of other elements, like sulfur, nickel and others. At high pressure > 1 Mbar T_{melt} is enhanced to values above 4000 K.

For Earth, the largest terrestrial body in the solar system, the iron core is in a pressure-ionized solid (metallic) phase, surrounded by a layer of liquid iron. In smaller bodies the central pressure is not high enough for pressure ionization and the temperature is probably too low for liquid iron. Thus, one may assume that all other terrestrial planets and differentiated asteroids have a solid iron core. The presence or absence of liquid iron may also explain the presence or absence of strong magnetic fields for terrestrial planets.

The giant planets have pressure-ionized cores. It is not clear whether the iron and rocky materials in their cores are differentiated, or whether the core is too stiff for sedimentation.

Rocky materials. Much of the Fe phase diagram applies also for rocky materials. The melting temperatures for rocks depends in a very complex way on the rock composition. However, roughly T_{melt} of rocks are comparable to iron so that one can assume that rock is solid in terrestrial bodies. Exceptions are the largest terrestrial planets Earth (and perhaps Venus). Another exception is Jupiter's moon Io, for which the interior is heated up by tidal forces. When terrestrial planets were young their internal temperature was higher and therefore they had sufficient magma and melts for volcanic activity (e.g. Mars, Venus and the Moon). The rock in the center of giant planets is at such high pressures that it is in the solid pressure-ionized phase.

3.2.4 Simple approximation for the equation of state

Because the observational data for M and R of extra-solar planets are not very accurate, a simplified treatment of the equation of state (EOS) can be chosen for an assessment of the internal structure. According to the study of Seager et al. (2007) the EOS for different solid and liquid materials can be described with a modified ‘‘polytropic’’ equation:

$$\rho(P) = \rho_0 + cP^n, \quad (3.7)$$

where ρ_0 is the low pressure density of the considered material, and c and n are material dependent parameters. The $\rho(P)$ curves for some important materials are shown in one of the slides.

Equation 3.7 and the curves in the above mentioned slide indicate that the planetary material is incompressible for pressures significantly below the critical pressure $P < P_{\text{crit}}$ while for $P > P_{\text{crit}}$ the material becomes compressible or the density behaves like a polytrope $\rho(P) \approx cP^n$ with $n \approx 1/2$. The critical pressure is defined by $P_{\text{crit}} = (\rho_0/c)^{1/n}$. The curves for all materials are similar because the underlying physical cause for the density is the balance between gravitational forces and the Coulomb forces of the electrons and nuclei.

- At low pressures $P < P_{\text{crit}}$ the density is essentially constant $\rho(P) = \rho_0$, because iron, rock, water-ice, and carbon are solid or liquid in the planets and the density of the material is defined by atomic Coulomb forces (solid state forces).

Table 3.6: The parameters ρ_0 , c and n for the EOS $\rho(P) = \rho_0 + cP^n$ of important materials (Seager et al. 2007). P_{crit} indicates at which pressure the material changes from incompressible to compressible:

material	ρ_0 [kg m ⁻³]	c [kg m ⁻³ Pa ⁻ⁿ]	n	P_{crit}
H ₂ O	1460	$3.11 \cdot 10^{-3}$	0.513	114 GPa
C (graphite)	2250	$3.11 \cdot 10^{-3}$	0.514	251 GPa
SiC	3220	$1.72 \cdot 10^{-3}$	0.537	479 GPa
MgSiO ₃ (perovskite)	4100	$1.61 \cdot 10^{-3}$	0.541	693 GPa
(Mg,Fe)SiO ₃	4260	$1.27 \cdot 10^{-3}$	0.549	770 GPa
Fe(α)	8300	$3.49 \cdot 10^{-3}$	0.528	1192 GPa

- At higher pressure $P < P_{\text{crit}}$ the material becomes compressible because the gravitational forces are so large that the atoms are squeezed together to distances which are smaller than the typical size of atoms. This means that the material becomes “pressure ionized”. The electrons “float” like in a metal.
- The critical density for pressure ionization depends on the material but lies in the range 1 - 12 Mbar (100 - 1200 GPa).
- For very high pressures $P \gtrsim 300$ Mbar the electron gas becomes degenerate. In this state the electron density is defined by the Pauli exclusion principle. This principle states, that fermions of a given spin can only be stacked in a momentum-location cell of $\Delta p \Delta V = h^3$. This state is called “degenerate gas” and it becomes important for objects more massive than $\approx M_J$. We will discuss this equation of state in the next section.

For hydrogen this simple description is not applicable because the hydrogen density is not constant at pressures < 1 Mbar, because it is then in the phase of a compressible super-critical fluid.

3.2.5 Interior structure of solar system planets

The interior structure of the planets can now be summarized qualitatively.

Earth and the terrestrial planets. The reference model for Earth interior is plotted (see slide). The diagram gives the PREM density structure, and rough indications about the pressure and temperature within Earth. The other terrestrial planets have a similar internal structure, with an iron core and a rocky mantle. The size of the core is different for Venus, Mars, Mercury and the Moon. Important differences are that the smaller objects Mars, Mercury and Moon are not or much less-pressure ionized and have therefore flatter density profiles for the iron core or the mantle, because the material is not compressed. It is also not clear whether there is liquid iron in Venus and Mercury.

Important properties for the terrestrial planets and the Moon just based on their mean density parameter:

- The mean density of Earth and Venus is above 5 g cm^{-3} because their interior is compressed due to the high pressure.

- Mercury is too small for significant compression and its high density is due to a large overabundance (e.g. with respect to Earth) of iron and a correspondingly large iron core. This property of Mercury is explained by the strong irradiation by the sun at this small separation and the partial evaporation of silicate grains during the planet formation process.
- The Moon has a low mean density indicating that there is only a small iron core present. The low mean density is one important fact pointing to the formation of the Moon by mantle material from Earth after a large collision of Earth with a Mars-sized body.

Jupiter and Saturn. Figure 3.4 gives the structure of Jupiter and Saturn. Both planets have first an outer region of molecular hydrogen which translates at the pressure of about 1 Mbar into metallic hydrogen. There must be a core of high density material, presumably a layer of O, C, and N rich “ices” and then a central rocky core. From the molecular abundances measured at the surface it is impossible to estimate the abundance of heavy elements (Si and Fe) and the size of the central high density core.

Figure 3.4: Interior structure for Jupiter (top) and Saturn (bottom).

Neptune and Uranus For Saturn and Jupiter one can at least assume that H and He dominates for radii larger than $> 0.15 R$. The situation is more ambiguous for Uranus and Neptune because the molecular H, He, CH₄ envelope extends only down to $r \approx 0.7 R$. It is assumed, but not sure, that there is an extended region of O-, N- and C-rich materials (ices) and then presumably a rocky core within $r < 0.2 R$.

Figure 3.5: Interior structure for Neptune.

3.3 Mass - radius relation for planets

In the previous sections we have discussed in detail the internal structure of solar system planets and the equation of state for the different materials for high pressure. Here we repeat a few basic principles in the context of the mass - radius relationship for substellar objects.

3.3.1 Low mass planets

For a terrestrial low mass planet, the densities of the different radial layers is given by ρ_0 from Table 3.6. The radius of the planet increases then with mass just like $R \propto M^{1/3}$

$$R = \frac{M^{1/3}}{\frac{4}{3}\pi\bar{\rho}} . \quad (3.8)$$

The mean density $\bar{\rho}$ is given by the composition. For example, a small ($< M_E$) terrestrial planet with $\bar{\rho} > 4 \text{ g cm}^{-3}$ must contain a substantial fraction of iron and a planet with $\bar{\rho} < 2.5 \text{ g cm}^{-3}$ a lot of ices.

This general relationship is also a good approximation for low mass $< 0.1 M_J$ giant planets made of H and He, because the H and He material are not yet strongly compressed under these conditions.

The situation is a bit more complicated for Neptune like gaseous planets. like Neptune, with a solid core of a few M_E . The radius of such objects changes quite rapidly if the mass of the hydrogen envelope is changed. Because the gravitation is not so high, a relatively small amount of H in gaseous form increases strongly the radius by a factor of about 1.5, because the H-gas adds an extended low density $\rho < 1 \text{ g cm}^{-3}$ “envelope” around the core, which reduces significantly the mean density from a mean core value of $\bar{\rho} \approx 3 - 5 \text{ g cm}^{-3}$ to $1 - 2 \text{ g cm}^{-3}$.

3.3.2 Degenerate high mass planets and brown dwarfs

Under high pressure, like in the centers of high mass ($> M_J$) planets, the material is pressure ionized and therefore compressible. This state is also called electron degenerate matter. According to the Pauli-principle two spin-1/2 particle (Fermions) cannot exist at the same location (quantum cell dV_x) in the same quantum-mechanical state. Possible states for electrons are two spin orientations and different momenta $p = m_e v$, where the “cell size” in momentum space is $dV_p = h^3$ (h =Planck constant).

For a degenerate gas the density of electrons can be described by the number of electrons in momentum space

$$n_e = 2 \cdot \frac{4}{3\pi} \frac{1}{h^3} p_0^3, \quad (3.9)$$

where p_0 is the Fermi-momentum and $E_0 = p_0^2/2m_e$ the corresponding (kinetic) Fermi-energy. For a cold, fully degenerate electron gas all low energy (or low momentum) states are occupied up to the Fermi-energy or Fermi-momentum:

$$p_0 = \left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{1/3} \approx \frac{h}{2} n_e^{1/3}. \quad (3.10)$$

For an ideal gas the pressure P is related to the kinetic energy, and therefore also the momentum, of the gas particles, according to:

$$P = nkT = \frac{2}{3}n\langle E \rangle = \frac{2}{3}n\left\langle \frac{p^2}{2m} \right\rangle.$$

This relation is also valid for a degenerate electron gas where the mean electron energy is related to Fermi energy by $\langle E \rangle = 3/5 \cdot E_0$. This yields then an equation of state for a degenerate electron gas describing the relation for the electron gas pressure and the electron density

$$P_e = \frac{2}{5}n_e \cdot E_0 = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3}. \quad (3.11)$$

This is essentially equivalent to the description of the high pressure regime of Table 3.6 describing the compressibility of matter at $P \gg 100$ GPa (or \gg Mbar)

$$n_e \propto \rho \propto P^{3/5}.$$

What happens phenomenologically? An electron degenerate material parcel with a volume V_1 is squeezed under enhanced gravitational pressure to a smaller volume $V_2 < V_1$, so that the Fermi-momentum and the associated pressure of the electron “gas” is enhanced in order to reach a new pressure equilibrium. In this state the gas pressure does not depend on temperature as long as the kinetic momentum of the electrons due to the temperature is smaller than the Fermi momentum. This is the case for all “cold” objects.

Figure 3.6: Relation between volume and Fermi momentum of electron degenerate matter under low and high pressure P_1 and P_2 .

The equation of state for degenerate matter has a most important effect on the mass-radius relationship of substellar objects. In a hydrostatic equilibrium the gravitational pressure P_G is equal to the electron pressure P_e :

$$P_G \propto \frac{GM^2}{R^4} \propto P_e \propto n_e^{5/3} \propto \left(\frac{M}{R^3}\right)^{5/3} = \frac{M^{5/3}}{R^5}.$$

This yields the mass-radius relation for degenerate matter

$$R \propto M^{-1/3}. \quad (3.12)$$

The exact relationship requires the consideration of the density profile through the planet, instead of a simple consideration of a mean density. However, the relation given above

describes roughly the functional dependence. For example a white dwarf star is an electron degenerate “cold” objects with $M_{\text{wd}} \approx 10^3 M_J$ and $R_{\text{wd}} \approx 10^{-1} R_J$.

The most important message is, that cold substellar object in the regime $M > 3 M_J$ become smaller in radius if more mass is added. The maximum radius of cold substellar object is therefore

$$R_{\text{max}} \approx R_J,$$

and brown dwarfs are slightly smaller than Jupiter.

Slide 3.7 and 3.8 illustrate the mass-radius relationship. Slide 3.7 is based on an analytic result for spheres with different (homogeneous) composition. This shows that the radius of an object depends on its composition. The curve $X=0.75$ is the result for a model with 75 % of H and 25 % of He. White dwarf stars are composed of C and O and the curve for C reaches for $1 M_\odot$ the expected radius of $0.01 R_\odot$ which is the same radius like for a C rich planet-mass object with $5 \cdot 10^{-5} M_\odot$. The mean density of such an electron degenerate white dwarfs is therefore very extreme with $\bar{\rho} \approx 1000 \text{ kg cm}^{-3}$. Slide 3.8 illustrate the mass-radius relation calculated for stars and substellar gaseous objects. This diagram shows a strong break in the relation at $0.08 M_\odot$, where the nuclear hydrogen burning of stars sets in. For stars on the main sequence the kinetic momentum of the electron p_e due to the temperature is larger than the Fermi-momentum. These stars behaves like ideal gas spheres and the mass-radius relationship is roughly

$$\frac{R}{R_\odot} \approx \frac{M}{M_\odot}. \quad (3.13)$$

In addition the mass-radius relationship in Slide 3.8 is less peaked in the substellar regime when compared to the analytic solution. Detailed calculations take more accurately into account the different phases and the internal thermal energy which is still generated due to the ongoing contraction of brown dwarfs and giant planets.