Huygens-Fresnel diffraction of Hermite-Gaussian modes by a circular aperture

Héctor Sainz Cruz
Trapped Ion Quantum Information. ETH Zurich. Supervisor: Klara Rhaissa Burlamaqui Theophilo
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We run numerical simulations to investigate the diffraction patterns of Hermite-Gaussian modes of a paraxial Gaussian laser beam, as truncated by a circular aperture. Diffractions are calculated via the Huygens-Fresnel-Kirchhoff integral. Previous research showed that if the aperture to beam waist ratio is $\frac{a}{w_0} > 2.3$, the fundamental mode $\text{TEM}_{00}$ presents intensity ripples which are $<1\%$. We find analogous inequalities for higher order modes $\text{TEM}_{lm}$ and based on these results, we conclude that diffraction losses will be negligible in the cavity for ion traps designed at ETH Zurich by the Trapped Ion Quantum Information group.

INTRODUCTION

Due to the omnipresence of lasers in modern scientific experiments, it has become crucial to comprehend the ways in which they interact with other optical elements, such as lenses or apertures. In this context, the existence of families of stable laser beam modes has attracted a lot of attention, especially since it was discovered that Fabry Perot resonators output beams with quasi Gaussian intensity profiles in the transverse direction [1][2]. These modes continue to be an object of study today, partly owing to the burgeoning popularity of optical systems such as cavities in the realm of quantum information technologies.

Several scholars have analysed the behaviour of the so-called fundamental Hermite-Gaussian mode, or $\text{TEM}_{00}$, when diffraction by elements such as apertures [3][4] or lenses [5], and in the Fresnel [6] and Fraunhofer regions [2]. However, studies about diffraction phenomena of higher order modes are scarce. The cases of single and multiple slits have been discussed [7][8] but, to the best of our knowledge, diffraction of Hermite Gaussian modes by a circular aperture has not yet been presented. In fact, Tanaka [9] proposed a method to treat this problem, in which diffraction integrals are converted into discrete sums, easing computational complexity at the cost of some approximations. Unfortunately, we found that the expression derived in that paper for the diffraction field associated to each mode does not match the plot presented therein. Thus, a different approach was taken, building on the integral formulation developed by Kraus [10].

The present study is motivated by a technical challenge faced by the Trapped Ion Quantum Information (TIQI) group at ETH Zurich. The subject experiment pursues building a Modular Ion Trap [11] integrated with an optical cavity, a system which could be regarded as a candidate universal node for a future quantum communications network. Indeed, a hybrid technology could unite some of the advantages of ion traps, such as high fidelity and long coherence times, with the information transmission velocity enabled by optical systems. That makes hybrid systems a promising research direction for those interested in scalable quantum networks [12].

As in most experiments in the field, our trap is realized via electrical confinement of the ion. This standard technique conflicts with the fact that the cavity mirrors are dielectric, since the field that traps the ion could also lead to accumulation of charge in the mirrors, which would in turn create fields that destabilize the ion. The problem is circumvented by using a trap that includes metallic shields in front of the mirrors, with small circular apertures to let the laser beam go through, see Figure 1. Consequently, we investigate the possibility of the beam suffering from diffractive phenomena at the aperture. In particular, we would like to rule out losses due to diffraction of higher order Hermite-Gaussian modes, which have larger divergence and spatial extension than the fundamental one [13], and could be diffracted by apertures large enough to leave $\text{TEM}_{00}$ intact.

This research addresses the issue by computing the diffraction fields of higher order Hermite-Gaussian modes $\text{TEM}_{lm}$, using the formalism developed by Kraus [10] for $\text{TEM}_{00}$. The report is organized as follows: firstly, we briefly discuss the theory behind Hermite Gaussian modes and the Huygens-Fresnel-Kirchhoff diffraction formulation, as well as its underlying assumptions. Next, we run numerical simulations in Python to study the diffracted fields of Hermite-Gaussian modes and we find the required aperture to beam waist ratio that induces weak
diffractive effects in each mode. Finally, in the light of the results, we comment on the possibility of diffraction losses affecting the Modular Ion Trap experiment currently being developed by the TIQI group.

THEORETICAL MODEL

Hermite-Gaussian laser modes

The spatial evolution of electromagnetic waves in isotropic media obeys the Helmholtz equation, also known as the scalar wave equation,

$$[\nabla^2 + k^2] E = 0,$$

(1)

where $k$ is the wavenumber. A laser beam $E(x, y, z) \equiv u(x, y, z)e^{-ikz}$ propagating along $z$ is said to be paraxial if its amplitude variations in the $z$ direction are slow compared to its wavelength and variations along the transverse direction. A laser operating in the optical range fulfils these criteria, unless its waist becomes comparable to its wavelength [14, 15]. The paraxiality conditions can be expressed as [13]:

$$|\frac{\partial^2 u}{\partial z^2}| < |2k \frac{\partial u}{\partial x}|, |\frac{\partial^2 u}{\partial x^2}|, |\frac{\partial^2 u}{\partial y^2}|. $$

(2)

If these inequalities hold, we can neglect the left hand side of (2), and (1) is reduced to the paraxial Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2ik \frac{\partial u}{\partial z} = 0. $$

(3)

An exact solution to this equation with the desirable physical properties of a laser beam is a gaussian-spherical wave [10][13], given by

$$E(x, y, z) = \frac{E_0 \omega_0}{\omega(z)} \exp\left(-ik\left[\frac{z^2 + y^2}{2R(z)}\right]\right) + itan^{-1}\frac{z}{z_r} - \frac{x^2 + y^2}{\omega^2(z)}$$

(4)

here,

$$\omega(z) = \omega_0 \left[1 + \left(\frac{z}{z_r}\right)^2\right]^{\frac{1}{2}}$$

$$R(z) = \frac{\pi \omega_0^2}{\lambda}$$

$$z_r = \frac{\pi \omega_0^2}{\lambda}.$$ 

The field in (4) is the fundamental Hermite-Gaussian mode, often denoted TEM$_{00}$, where TEM stand for ‘Transverse electromagnetic’. In the above, $E_0$ represents the wave’s amplitude at the beam waist location, and $\omega_0$ is defined as the radial distance at which the intensity at that location falls to $\frac{1}{e}$ of its central maximum. In this report, we will loosely refer to $\omega_0$ as beam waist. These two, together with the wavelength $\lambda$ completely define the beam. $\omega(z)$ is a function describing the growth of the beam radius as it distances from the waist location. The parameter $z_r$ is named Rayleigh range, and corresponds to the distance to the waist position at which the beam radius is $\sqrt{2}\omega_0$. $R(z)$ accounts for the transformations in the radius of curvature of the wave front, it is infinite at the waist location, and goes to zero as we look further away from the waist, where the plane wave progressively becomes spherical.

The second term, $\phi(z) = i tan^{-1}\frac{z}{z_r}$, is known as the Gouy phase shift, an anomaly which occurs when light travels from $-\infty$ to $+\infty$ and crosses its focus point, acquiring a $\pi$ phase shift in its propagation direction. In the case of spherical waves, $n = 2$ for a total phase increment of $\pi$ [13, 17]. Although this phenomenon was first observed in the XIX century [18], it was not until recently [17, 19] that physically insightful and mathematically elegant explanations were discovered for it. One well known argument is based on geometric optics [16]. The idea is illustrated by Figure 2, the optical path difference between wave fronts $AB$ and $DE$ is a straight line $BE$, while the properties of the laser make the light traverse path $BCD$, and this difference induces the phase shift. A more recent, quantum mechanical treatment [17] explains that when a laser is focused, the light is spatially confined in the transverse plane, which in virtue of Heisenberg’s uncertainty principle leads to increased transverse momenta, which produces the phase shift in the propagation direction. Finally, a more sophisticated theory [19] regards the Gouy phase shift as one of a larger class of phenomena, and goes on to prove that if a quantum system with Hamiltonian $\mathcal{H}$ is made to evolve adiabatically through path $C$, defined by varying parameters $R$, the system will gain a geometrical phase factor $exp(\gamma C)$, where $\gamma$ depends on the eigenvalues of $\mathcal{H}(R)$ along $C$.

Ideally we would like to sustain only TEM$_{00}$ in our

Figure 2: The Gouy phase shift understood geometrically.

The difference between the optical path $BE$ and the real path of light $BCD$ leads to a phase shift between wavefronts $AB$, $DE$. Source:[10].
Fabry Perot cavity. However its mirrors are not perfectly spherical, and these asymmetries are responsible for the mixing of TEM\textsubscript{00} with higher order Hermite Gaussian modes, at specific cavity lengths. When such mixing occurs, losses due to diffraction of higher order modes can reduce the finesse of the cavity. The challenge is exacerbated in microcavities, where small volumes lead to high resonance frequencies and strong intermodal coupling [20]. Indeed, the gaussian beam field (4) is just one of a whole family of solutions to the paraxial Helmholtz equation. To find them, we first note that (3) is separable in \(x \) and \(y\). Thus, one can seek to build other solutions with rectangular symmetry in the form of products \(u(x, y, z) = u_l(x, z)u_m(y, z)\). Exact solutions with cylindrical symmetry also exist: the popular Laguerre-Gaussian modes [21], states of light with orbital angular momentum and novel applications, such as optical tweezers able to trap metallic particles. We proceed by substituting a general ansatz for \(u_l\) of the form

\[ u_l(x, z) = A(q(z)) \times H_l \left( \frac{x}{p(z)} \right) \times \exp \left[-ik\frac{x^2}{2q(z)} \right], \quad (5) \]

where \(q(z)\) is a generalized beam radius that can take complex values. \(p(z)\) is a scaling factor, and the \(H_l\) are functions to be found. For the preliminaries to (5), the reader is referred to [13]. Naturally, and analogous ansatz can be used for \(u_m\). Substitution of (5) in the paraxial wave equation (3) reduces it to the differential equation giving rise to Hermite functions:

\[ \frac{d^2H_l}{dx^2} - 2 \left( \frac{\sqrt{2}x}{\omega(z)} \right) \frac{dH_l}{dx} + 2lH_l = 0, \quad (6) \]

where differentiation is done w.r.t. \(\tilde{x} = \frac{x}{p(z)}\). The solutions to this equation, expressed in terms of parameter \(\xi = \frac{\sqrt{2}x}{\omega(z)}\), are the physicists Hermite polynomials \(H_0 = 1, H_1 = 2\xi, H_2 = 4\xi^2 - 2, H_3 = 8\xi^3 - 12\xi, \) etc. As a consequence, higher order stable laser modes with rectangular symmetry differ from the fundamental one by Hermite polynomial modulations in the transverse coordinates,

\[ E_{l,m}(x, y, z) = \frac{E_0\omega_0}{\omega(z)} H_l \left( \frac{\sqrt{2}x}{\omega(z)} \right) H_m \left( \frac{\sqrt{2}y}{\omega(z)} \right) \exp \left[-ik\left[z + \frac{x^2 + y^2}{2R(z)} \right] \right. \\
\left. + i(l + m + 1) \tan^{-1} \frac{z}{2R(z)} - \frac{x^2 + y^2}{\omega^2(z)} \right], \quad (7) \]

Hermite-Gaussian laser modes of arbitrary order TEM\textsubscript{lm}, as given by (7) have rectangular symmetry by construction \(E_{l,m}(x, y, z) = E_{m,l}(y, x, z)\). These modes are stable in the sense that their intensity distributions scale uniformly while keeping their shape as one looks at different points in the propagation axis \(z'\). On that account, it is enlightening to appreciate that any given Hermite Gaussian mode will couple more strongly to certain cavity lengths, and that finding the optimal alignment, maximizing the intensity of TEM\textsubscript{00} (i.e. minimizing power losses due to high order couplings) is a remarkable feat in itself [20].

Owing to the analytical properties of Hermite polynomials, an undisturbed mode TEM\textsubscript{0m} will form a \((l+1) \times (m+1)\) grid of intensity maxima in the observation plane (see Figure 3). Besides the appearance of Hermite polynomials in \(x\) and \(y\), the reader might have realized that the Gouy phase shift is now magnified by a term that includes the sum of the mode indexes, \((l + m + 1)\). An intuitive reason for this increment is that higher order Gaussian modes occupy a larger volume both in real and in momentum space, thus larger phase shifts occur as mentioned above. Albeit subtle, the Gouy effect has a crucial consequence, due to mode dependent phase shifts, different Hermite-Gaussian modes (7) have dissimilar resonant frequencies inside optical resonators, like fibre-based Fabry Perot cavity used in the Modular Ion Trap experiment. Finally, higher order Hermite-Gaussian modes have a larger spatial extension than the fundamental mode. For a given mode TEM\textsubscript{lm}, the location \(r_{\text{max}}\) of the outermost peak goes roughly as the square root of the largest mode index, i.e. \(r_{\text{max}} \sim \omega_0 \sqrt{l}\), if \(l \geq m\) [13]. This feature is central to our problem because higher order modes, due to their larger extent, could be strongly diffracted even by apertures which are large enough to let the fundamental mode essentially unaffected. More on this issue will be debated in the Results and Discussion section.

![Figure 3: Simulated intensity profiles of Hermite Gaussian modes, up to TEM\textsubscript{55}. Note that maxima lie on the corners of each grid. Extracted from [22].](image-url)
Huygens-Fresnel diffraction theory

We have obtained the full expression describing the fields incident upon the circular aperture for each Hermite-Gaussian mode. Beyond the aperture, the fields will be distorted by diffractive effects, a phenomenon we intend to study in the framework of the Huygens-Fresnel principle. Thus, we regard every point in the wavefront as a source point of a spherical wave, which at a later time is also understood as a superposition of secondary spherical waves, and the envelope of the latter constitutes the actual wave [23]. Therefore, the diffraction integral takes the form

\[
E_p^{l,m}(r', \phi' ) = \int_A \left( \frac{-i}{2\lambda} \right) (1 + \cos \theta')
\times E_{l,m}(r, \phi, z) \times \frac{\exp (ikR')}{R'} dA .
\]

Here, the first term is Kirchhoff's obliquity factor, which correctly describes the fact that secondary spherical waves emanating from different points in the wavefront will contribute differently to integral (8). Its value depends on the angle between a) the line that joins the primary wave point source with a secondary wave point, and b) the line from that secondary point to the observation point [24]. The second term is the field created by an Hermite Gaussian mode at a point in the aperture as given in (7), expressed in cylindrical coordinates. The last term represents the spherical waves created at each point of the aperture. Integrated over the surface of the latter, these three terms give the diffracted field at a point in the observation plane, denoted by prime variables.

As noted in [10], usual simplifications to the Huygens-Fresnel integral include setting the obliquity factor equal to 1 and invoking Fresnel’s approximation [4-6, 25]. Fresnel’s approximation consists on retaining the first terms in the Taylor expansion of the spherical wave term in (8), transforming it into a "paraxial spherical wave". Its range of validity can be condensed in the inequality \(|z'| >> 2\omega_0\), i.e. that the diffracted fields must be evaluated at distances greater than the beam diameter (see Figure 4), for details refer to [13, p. 654]. It is noteworthy that the paraxial approximation given by (2) and (3) is in fact physically and mathematically analogous to Fresnel’s approximation, as they both stem from the same property of the beam [13].

Our approach includes Kirchhoff’s obliquity factor and the aperture plane as surface of integration, but does not require explicit use of Fresnel’s approximation. As a result, our diffraction integral reads

\[
E_p^{l,m}(r', \phi' ) = E_0 \omega_0 \int_A \left( \frac{-i}{2\lambda} \right) (1 + \cos \theta') \times 
\times H_l \left( \frac{\sqrt{2} x}{\omega(z)} \right) H_m \left( \frac{\sqrt{2} y}{\omega(z)} \right) \left( \frac{1}{\omega(z)} \right) \exp \left( -ik \left[ z + \frac{x^2 + y^2}{2R(z)} - \frac{x^2 + y^2}{w^2(z)} + i(l + m)\tan^{-1} \frac{z}{z_r} \right] \right) \times \frac{\exp (ikR')}{R'} dA ,
\]

which can be simplified [10] by switching to cylindrical coordinates, evaluating the field at the position of the aperture \(z = z_w\) and making use of the geometry depicted in Figure 4, to arrive at

\[
E_p^{l,m}(r', \phi' ) = C \int_0^{2\pi} \int_0^a \frac{(1 + \cos \theta')}{R'} H_l \left( \frac{\sqrt{2} r \cos \phi}{\omega(z_w)} \right) \times 
\times H_m \left( \frac{\sqrt{2} r \sin \phi}{\omega(z_w)} \right) \exp \left( i k \left[ R' - \frac{r^2}{2R(z_w)} \right] \right) - \frac{r^2}{\omega^2(z_w)} \right) rdrd\phi .
\]
where we defined

\[
C = \frac{E_0 \omega_0}{2 \lambda \omega(z_w)} \exp \left( i \left[ -k z_w + \tan^{-1} \frac{z_w}{z_r} \right] \right),
\]

\[
R' = \left[ r^2 - r''^2 - 2rr' \cos(\phi - \phi') + z'^2 \right]^{1/2},
\]

\[
\cos(\theta') = -\frac{a^2 + b^2 + c^2}{2bc},
\]

\[
a = [r''^2 + r''^2 - 2rr' \cos(\phi - \phi')]^{1/2},
\]

\[
b = [r''^2 - 2rr' + z'^2]^{1/2},
\]

\[
c = R',
\]

\[
r' = r \left( 1 + \frac{z'}{R(z_w)} \right).
\]

Once a field in the observation plane (10) is computed, its associated intensity is simply given by

\[
I_{l,m} = \frac{1}{2} |E_{l,m}^P|^2.
\]

Taking together, equations (10) and (11) constitute the result of this discussion, and are the ones that will be calculated numerically to study the diffraction patterns of the modes. Despite the apparent complexity of the field, the magnitude of diffractive effects will largely depend on two paramount quantities: the aperture to beam waist ratio \( \frac{a}{w} \), and the waist to aperture distance \( z_w \). For the former, large ratios will naturally lead to weak or null diffraction, whereas smaller ratios imply severe truncation of the mode beam, accompanied by strong diffractive effects. However, we can expect the aperture size necessary to avoid diffraction will strongly depend on the mode TEM_{lm} that impinges on it, since every mode has a different spatial extension. Regarding the distance \( z_w \), larger separation between waist and aperture results in larger beam radius at the aperture, as well as greater curvature of the incident wavefront, which opens the possibility for new diffractive effects, especially relevant in the far field [2, 4].

In the next sections we will often refer to the Fresnel and Fraunhofer regions, which correspond to the near far fields and occur when \( F_r \gg 1 \) or \( F_r \ll 1 \), respectively. Here \( F_r \) is the well-known Fresnel number. If the laser is focused at the aperture \( (z_w = 0) \), then it takes the simple form

\[
F_r = \frac{a^2}{\lambda^2}.
\]

A general expression in case \( z_w \neq 0 \) is derived in [10]. To conclude, we list the assumptions underlying expression (10), namely that

- (a) the characteristic size of the diffractive element is large compared to the wavelength \( a \gg \lambda \), which is a requisite to apply scalar diffraction theory,
- (b) our laser respects \( \omega_0 \gg \lambda \), to ensure paraxiality in (2),
- (c) the incident wavefront’s curvature \( R(z) \) is small at the aperture,
- (d) the incidence angle of the laser beam on the aperture is negligible.

If (a) was broken, we would need to employ a vector theory of diffraction [26]. If (b) was not respected, the wave could not be considered paraxial and the diffraction problem escapes our framework. Regarding (c), from the dependence of \( R(z) \) on the Rayleigh range \( z_0 \) and the waist to aperture distance \( z_w \), it follows that a small curvature can be achieved by either placing the aperture near the waist \( z_w \approx 0 \), or using a laser with long Rayleigh range. However, note that assumption (c) can be relaxed, and in fact our theoretical model considers the possibility of non-zero values of \( z_w \). However, if assumption (c) starts to break substantially, or if the angle in (d) goes beyond 0.05 milliradians, results improve notably in accuracy if a wavefront integration formulation is used instead of aperture integration, especially in the near field region [27][10]. The four assumptions described above concern the characteristics of the beam and the aperture. Now we introduce a fifth, inherent to the theory,

- (e) the on-axis distance to the aperture \( z' \) at which the field is calculated respects \( z' \geq 2a \).

Encapsulated in this inequality is the fact that Huygens-Fresnel integral formulation losses validity as we approach the aperture, because in doing so we escape the domain of scalar diffraction theory [27]. Note that condition (e) becomes more restrictive than the Fresnel approximation condition \( z' \gg 2\omega_0 \) when \( a > \omega_0 \), as is the case in the Modular Ion Trap experiment.

**METHODS**

**Python simulations**

The intensity of the diffraction fields of Hermite-Gaussian modes \((10,11)\) is calculated numerically with Python. The program, which can be found in the Appendix, allows to perform the integration and plot the results in three ways:

1. 2D plot. Intensity vs observation radius \( r' \), for fixed angle \( \phi' \) and on-axis distance \( z' \).
2. 2D plot. Intensity vs on axis distance \( z' \), for fixed \( \phi' \) and \( r' \).
3. Heat map, using the *pcolormesh* plotting tool. Color-coded intensity vs observation plane \((x'_p, y'_p)\).

A built-in integrator *dblquad* is employed. The program runs in about 5 minutes to generate a typical plot involving \(~1000\) points in the diffraction plane, and hence
an equal number of calculations of the integral written in (10). In the previous section we outlined the assumptions of our theoretical model, now we point at one computational restriction. For the integrals to converge fast enough, we need to enforce that

\[
(f) \text{ the maximum observation radius } r'_p \text{ in which the field is calculated respects } r'_p \leq 2a.
\]

This condition is meant as a guiding principle rather than a precise statement applicable to every case. Naturally, the diffraction problem in the near end of the Fresnel region can quickly become intractable at large observation radii due to the complexity of the fields that interfere beyond the aperture. Thus, the inequality becomes stricter in the Fresnel region (in some cases convergence demands \( r'_p \leq 1.5a \)), whereas it can be relaxed for Fraunhofer diffraction.

**Data analysis**

Our aim in the discussion is to gain knowledge about the joint beam-aperture parameters that lead to weak diffractive phenomena of Hermite Gaussian modes. The Modular Ion Trap experiment uses a laser beam described by \( \lambda = 866 \text{ nm}, \omega_0 = 6.0 \mu\text{m} \) and a spacious aperture \( a = 50.0 \mu\text{m} \). The beam is focused at the aperture location, hence \( z_w = 0 \). Because the trap is inside a microcavity, the distance from the aperture to the observation plane (the mirror) is only 200 \( \mu\text{m} \), while the mirror itself has a radius of \( r'_M = 40 \mu\text{m} \). In that situation, the wavefront is planar at the aperture; the relevant field is calculated in the near Fresnel region \( (Fr = 14.43) \), and one can foresee that the fundamental mode and others of low order will be free of any diffraction, owing to the size of the aperture \( a = 8.3 \omega_0 \). Nonetheless, due to their larger spatial extent, higher order modes could still suffer from diffraction, leading to photon losses in the cavity.

As a consequence, in the following we will focus our efforts in understanding the transition from null to weak diffraction in the Fresnel region, for higher order modes TEM\(_{lm} \). In previous investigations, scholars have concluded that if \( a / \omega_0 > 3 \), diffractive effects can be ignored altogether for the fundamental mode TEM\(_{00} \) [2]. More precisely, if \( a / \omega_0 > 2.3 \) the intensity ripples in the diffraction pattern are <1\% for TEM\(_{00} \) [10][13, p. 734]. We seek analogous conditions for a higher order modes. In other words, given a mode TEM\(_{lm} \) we would like to know the constant value \( \beta_{lm} \), such that if

\[
\frac{a}{\omega_0} > \beta_{lm},
\]

then diffraction effects are weak for that mode. The distinction between negligible and weak diffraction should be motivated by physical arguments, which will be unique to each experiment. Here we use a simple definition, under which which weak diffraction occurs if the intensity pattern presents <1\% ripples, relative to the bare (non-diffracted) case. In fact, such a criteria can be much more restrictive than one based on intensity truncation; for example, an aperture \( a = \frac{r_w}{\omega_0} \), for which the transmitted power is close to 99\%, will still lead to ripples of ±17\% [13, p. 667]. Our approach is the following: Given an Hermite Gaussian mode, firstly we compute its bare intensity profile (7), evaluated at the observation plane. Secondly, we make a guess for an aperture size that will induce diffraction ripples, and then we increase it in steps of 0.01 \( \text{mm} \) until the last 1\% ripple disappears. During this analysis, we shall keep the Fresnel number \( Fr \) constant, in order to remain in the same point of the Fresnel region. Therefore if we modify \( a \), we must also change the observation plane location \( z' \) accordingly. After recalling the expression for the evolving beam radius \( \omega(z) \) in (4), one could object that the beam experiences spreading at larger distances and therefore, by modifying \( z' \) we dilate or shrink the Gaussian profile in the radial direction, and such shift makes the comparison of profiles meaningless. However, such mismatch does not affect our aperture optimization problem, as long as the bare field used for comparison is always recalculated at the \( z' \) associated to the trial aperture in each 0.01 \( \text{mm} \) step.

For the analysis, we use a He-Ne laser, \( \lambda = 632.8 \text{ nm}, \omega_0 = 0.4 \text{ mm} \), with a larger waist to wavelength ratio, which eases the numerical computation of the fields. Naturally, once we understand the transition from null to weak diffraction in the Fresnel region for this laser beam, the conclusions will extrapolate to any other laser satisfying conditions (a)-(d).

**RESULTS AND DISCUSSION**

The fundamental mode

The fundamental mode is a good testbed for our program. Figure 5 presents two cases in stark contrast. The bottom image corresponds to TEM\(_{00} \), observed in the Fresnel region \( (Fr = 10) \) after surpassing an aperture \( a = 2.3 \omega_0 \). For this aperture size, diffraction is very weak and the beam conserves its transverse Gaussian profile (see Figure 5). The top simulation presents the mode in the same optical region \( (Fr = 10) \), but this time diffracted by aperture \( a = 1.2 \omega_0 \). The presence of diffraction manifests in the form of concentric rings, which arise from constructive and destructive interference of different parts of the truncated beam. These intensity rings owe to the symmetry of the fundamental mode, which result in intensity ripples in the radial direction, observed in Figure 6. It is clear that, even for an aperture bigger than the beam’s waist, diffraction effects are notable, and particularly strong near the propagation axis. In particular,
when a Gaussian beam encounters a circular aperture, its central intensity lobe suffers from destructive interference [13, p. 732], both in the near and far fields, which is in agreement with our simulation. Now we apply the protocol outlined in the Data Analysis section. We conjecture an aperture radius \( a = 0.90 \omega_0 = 2.25 \omega_0 \), which should still lead to \(>1\% \) ripples, a guess backed by the literature (see above). To find the desired value, we increase \( a \) in steps 0.01 mm. The result of this investigation is that the smallest aperture with \(<1\% \) ripples with respect to the bare intensity profile is

\[
a_{00} = (2.38 \pm 0.03) \omega_0 ,
\]

in good agreement with the literature [13, p. 734]. Figure 6 plots transversal intensity profiles of the fundamental mode for several aperture to waist ratios, along with the bare intensity profile. The Fresnel number is kept constant (\( Fr = 10 \)), and hence each curve is taken at a different on-axis distance \( z' \), as given by (12). As mentioned above, a direct consequence of this choice is the observation of beam radius spreading at larger \( z' \) distances, which naturally occurs to laser beam modes as they move away from the focus point, and results in a slight lateral displacement between curves in the Figure 6. Result (14) is a proof that the code works as it should. Withal, it is also obvious that the aperture in the Modular Ion Trap experiment (\( a = 8.3 \omega_0 \)) will leave TEM\(_{00}\) completely untouched, and so we need to expand the analysis to spatially larger modes TEM\(_{lm}\).

### Higher order modes

Now we proceed with the study of higher order modes TEM\(_{lm}\). Obtaining expressions like (14) for these modes is computationally more cumbersome. The reason is that these modes do not enjoy the angular symmetry of TEM\(_{00}\), which enabled us to look for intensity ripples reduction along a single transverse direction (i.e. \( \phi'_{p} = 0 \)), and obtain conclusions valid for the entire observation plane. Therefore, for higher order modes it is in principle necessary to scan through the whole surface of the heap map to search for intensity ripples. Because a map with \( \sim 1000 \) point takes between 5 and 10 minutes with our program and processor, such optimization process can be computationally exhausting, especially if our ini-
Figure 7: TEM_{12} bare intensity profile at \( z' = 132.901 \) mm. Other parameters: \( \lambda = 632.8 \) nm; \( \omega_0 = 0.4 \) mm; \( z_w = 0 \).

Figure 8: Mode TEM_{12}. Intensity profiles along a transversal cut \( \phi' = 1.002 \) for four different aperture to waist quotients. Other parameters: \( \lambda = 632.8 \) nm; \( \omega_0 = 0.4 \) mm; \( F_r = 10 \); \( z_w = 0 \).

A useful strategy for an arbitrary mode is to choose the privileged direction based on symmetry, the direction of choice is natural for modes with one null index; we shall take \( \phi_P = 0 \) for modes TEM_{0m}, and \( \phi_P = \frac{\pi}{2} \) for TEM_{0m}. However, it is not obvious how to approach the problem for other modes.

Other interesting locations are those of intensity maxima, because where the bare intensity is largest, ripples also have a good chance of being large. Based on these considerations, we choose a privileged direction and after finding the optimal aperture that avoids ripples along that axis, we test the aperture over the entire surface to check that the maximum intensity ripple we encountered, and which is \(< 1\%\), is truly a global maximum, and not a local one. Due to symmetry, the direction of choice is natural for modes with one null index; we shall take \( \phi_P = 0 \) for modes TEM_{0m}, and \( \phi_P = \frac{\pi}{2} \) for TEM_{0m}. However, it is not obvious how to approach the problem for other modes.

We see that low order modes do not demand much larger apertures than TEM_{00}, but as we go to larger orders, substantial size increments become necessary. Here we use a fixed maximum ripple of \( 1\%\); an open question is how the aperture spacing between modes would change as we went to weaker diffraction conditions, e.g. \( 0.01\%\). In any case, it is natural that a mode with equal indexes TEM_{l,m} requires a larger aperture than all modes with one (or both) smaller indexes. Thus, if an aperture diffracts, say TEM_{33} only weakly (with ripples \(< 1\%\)), the same will be true for TEM_{30}, TEM_{31}, TEM_{32} and of
UES in the third column lead to no ripples.

modes TEM\textsubscript{44}. In agreement with result (15), as expected, symmetry prevails and the largest ripples occur along the privileged directions of the mode (e.g. \( \phi' = 1.002 \) in the first quadrant).

Other parameters: \( z' = 132.901 \) mm; \( \lambda = 632.8 \) nm; \( \omega_0 = 0.4 \) mm; \( z_w = 0 \).

The Python program that performs the numerical simulation can be found in the Appendix and it is released for free use. All figures presented in this report can easily be reproduced with it. It should be noted that, since we intended to apply our findings to the Modular Ion Trap experiment, we focused on one aspect of the diffraction problem: varying the aperture to waist ratio, while keeping other variables controlled. Expansions of this research could employ or improve the program to analyse other features of the problem. For example, we could study an unfocused laser (\( z_w \neq 0 \)), which would lead to a curved wavefront at the aperture, enabling new diffractive phenomena, such as enhanced beam radius in the far field [2, 4]. Laser resonators and microcavities occupy a prominent position in the landscape of quantum information technologies, so understanding diffractive features of other families of stable modes, such as Laguerre-Gaussian or self-reconstructing Bessel beams [28], would also be worthwhile.

**CONCLUSIONS AND OUTLOOK**

We have explored the emergence of diffractive effects when Hermite-Gaussian modes impinge upon a circular aperture, as a function of the aperture radius to beam waist ratio \( \frac{a}{w_0} \). Diffracted fields were calculated via the Huygens-Fresnel integral, and the Gaussian beam was assumed to be paraxial. Defining weak diffraction as the regime that presents intensity ripples smaller than 1% with respect to the non-diffracted pattern, we find that \( a = (2.38 \pm 0.03) \omega_0 \) is the necessary aperture radius to avoid larger ripples in the fundamental mode TEM\textsubscript{00}, in harmony with the literature [10, 13]. We have presented analogous conditions for higher order modes TEM\textsubscript{lm}, which can be found in Table I. In virtue of these results we have concluded that, in the Modular Ion Trap experiment developed by the Trapped Ion Quantum Information group ETH, the aperture is wide enough to host modes TEM\textsubscript{lm} s.t. \( l, m \leq 5 \) without provoking diffraction ripples. Therefore, mode deformation and losses due to diffraction of higher order Hermite Gaussian modes should not be a concern in that setup.

![Figure 9: TEM\textsubscript{12} differential intensity profile. The weakly diffracting pattern caused by an aperture \( a = 2.90 \omega_0 \) is subtracted from the bare intensity pattern. All ripples are < 1%, in agreement with result (15). As expected, symmetry prevails and the largest ripples occur along the privileged directions of the mode (e.g. \( \phi' = 1.002 \) in the first quadrant). Other parameters: \( z' = 132.901 \) mm; \( \lambda = 632.8 \) nm; \( \omega_0 = 0.4 \) mm; \( z_w = 0 \).](image)

Table I: Hermite Gaussian modes, accompanied by their angular directions of high symmetry and intense peaks, and the associated minimum aperture to waist ratios needed to avoid > 1% ripples in their intensity diffraction patterns. The values in the third column lead to no ripples > 1% along at least 200 data points in the privileged directions, and 900 data points spanning across the whole surface.

<table>
<thead>
<tr>
<th>TEM</th>
<th>Privileged ( \phi'_p ) (rad)</th>
<th>( \frac{a}{w_0} ) (± 0.03)</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>None</td>
<td>2.38</td>
</tr>
<tr>
<td>10 &amp; 01</td>
<td>0 &amp; ( \pi/2 )</td>
<td>2.40</td>
</tr>
<tr>
<td>11</td>
<td>( \frac{\pi}{4} )</td>
<td>2.63</td>
</tr>
<tr>
<td>20 &amp; 02</td>
<td>0 &amp; ( \pi/2 )</td>
<td>2.80</td>
</tr>
<tr>
<td>30 &amp; 03</td>
<td>0 &amp; ( \pi/2 )</td>
<td>2.90</td>
</tr>
<tr>
<td>21 &amp; 12</td>
<td>0.568 &amp; 1.002</td>
<td>2.90</td>
</tr>
<tr>
<td>22</td>
<td>0 &amp; ( \pi/4 )</td>
<td>3.03</td>
</tr>
<tr>
<td>31 &amp; 13</td>
<td>0.458 &amp; 1.133</td>
<td>3.05</td>
</tr>
<tr>
<td>32 &amp; 23</td>
<td>0.663 &amp; 0.908</td>
<td>3.28</td>
</tr>
<tr>
<td>33</td>
<td>( \pi/4 )</td>
<td>3.55</td>
</tr>
<tr>
<td>44</td>
<td>0 &amp; ( \pi/4 )</td>
<td>3.88</td>
</tr>
<tr>
<td>55</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
</tbody>
</table>

In the light of the results available in Table I, which are valid for any laser fulfilling conditions (a)-(d), we can assure that the aperture that the laser needs to cross in the Modular Ion Trap experiment at ETH Zurich, which is \( a = 8.3 \omega_0 \), is more than large enough to host all modes TEM\textsubscript{lm} with indexes \( l, m \leq 5 \) without provoking diffraction losses in them. Thus, even if the microcavity has a length that mixes TEM\textsubscript{00} with some of those other modes, this will not lead to photon losses due to diffraction at the aperture.

<table>
<thead>
<tr>
<th>( a ) (( \mu \text{m} ))</th>
<th>( \frac{a}{w_0} )</th>
<th>( \phi'_p ) (rad)</th>
<th>( \frac{a}{w_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.3</td>
<td>0.01</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
<tr>
<td>8.3</td>
<td>0.013</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
<tr>
<td>10.4</td>
<td>0.015</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
<tr>
<td>12.5</td>
<td>0.017</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
<tr>
<td>14.6</td>
<td>0.019</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
<tr>
<td>16.7</td>
<td>0.021</td>
<td>( \pi/4 )</td>
<td>4.03</td>
</tr>
</tbody>
</table>
# Import mathematical libraries
from math import log, exp, sin, cos, atan, sqrt, pi, e, inf, floor
from numpy import array, copy, linspace, zeros, amax, amin, radians, meshgrid,
  arange, tile
from cmath import exp as cexp
from scipy.special import jn
from scipy import integrate
import operator

# Import plotting libraries
import matplotlib
import matplotlib.pyplot as plt

# Definition of constants:
wavelength=0.6328e6
k=2*pi/(wavelength) # Wavenumber
aperture=0.94e3 # Aperture radius
w0=0.4e3 # Beam waist
zw=0.0 # Position of the beam waist relative to the aperture
zr=pi*w0*w0/wavelength # Rayleigh range
Fresnel=10 # Fresnel number

# Definition of the Gaussian mode, e.g. for TEM10 we select ModeL=1 and ModeM=0.
ModeL=0
ModeM=0
N=ModeL+ModeM

# Now we define three functions, which we will use both outside and inside the diffraction integral.

# A function describing the evolution of the beam radius,
def beamradius(z):
wz=w0*sqrt(1+(z/zr)**2)
return(wz)

# Another function, describing the change in the radius of curvature,
def evolradius(z):
  if z==0:
    Rz=inf
  else:
    Rz=z*(1+(zr/z)**2)
return(Rz)

# A function for the Gouy phase,
def Gouy(z):
  Gouyarg=(N+1)*atan(z/zr)
  Gouyphase=complex(cos(Gouyarg),sin(Gouyarg))
  return(Gouyphase)
With these three we can already compute the constant outside the integral,
The Constant = \( \frac{w_0}{2 \times \text{wavelength} \times \text{beam radius}(zw)} \) * complex(0,1) * complex(\cos(kzw), \sin(kzw)) * Gouy(zw)

Next comes the main function, which takes as input the observation point and the aperture size and computes the value of the aperture plane integral.

```python
# The primes denote aperture plane variables, whereas r and phi refer to the aperture plane.
# We will define 6 functions as building blocks of the function we need to integrate.

def DiffractionIntegral(rprime, phi_prime, zprime, aperture):
    # The primes denote diffraction plane variables, whereas r and phi refer to the aperture plane.
    Rcapitalprime=(r**2 + rprime**2 + 2*r*rprime*cos(phi phi_prime) + zprime**2)**(1/2)
    return (Rcapitalprime)

def FunctionExpCos(r, phi):
    CosPieceA=r*exp((r/beamradius(zw)))**2
    CosPieceB=cos(k*(FunctionRcapital(r, phi) r**2/(2*evolradius(zw))))
    ExpCosvalue=CosPieceA*CosPieceB
    return(ExpCosvalue)

def FunctionExpSin(r, phi):
    SinPieceA=r*exp((r/(beamradius(zw)))**2)
    SinPieceB=sin(k*(FunctionRcapital(r, phi) r**2/(2*evolradius(zw))))
    ExpSinvalue=SinPieceA*SinPieceB
    return(ExpSinvalue)

def FunctionCos(r, phi):
    a_value=(rprime**2 + (r*(1+(zprime/evolradius(zw))))**2) 2*rprime*r*(1+(zprime/evolradius(zw)))**2 2*r*(1+(zprime/evolradius(zw)))**2
    b_value=(r**2 + (r*(1+(zprime/evolradius(zw))))**2) 2*r*(1+(zprime/evolradius(zw)))**2
    c_value=FunctionRcapital(r, phi)
cos_value=1+(a_value**2 + b_value**2 + c_value**2)/(2*b_value*c_value)
    return (cos_value)

# The next two functions are the Hermite polynomials determined by the chosen mode.
def FunctionHermiteX(r, phi):
    argX=sqrt(2)*r*cos(phi)/beamradius(zw)
    if ModeL==0:
        HermiteX=1
    elif ModeL==1:
        HermiteX=2*argX
    elif ModeL==2:
        HermiteX=4*argX**2 2
    elif ModeL==3:
        HermiteX=8*argX**2 12*argX
    elif ModeL==4:
        HermiteX=16*argX**2 48*argX**2+12
    elif ModeL==5:
        HermiteX=32*argX**2 160*argX**2+120*argX
    return (HermiteX)
```
def FunctionHermiteY(r, phi):
    argY=sqrt(2)*r*sin(phi)/beamradius(zw)
    if ModeM==0:
        Hermitey=1
    elif ModeM==1:
        Hermitey=2*argY
    elif ModeM==2:
        Hermitey=4*argY**2
    elif ModeM==3:
        Hermitey=8*argY**3
    elif ModeM==4:
        Hermitey=16*argY**4
    elif ModeM==5:
        Hermitey=32*argY**5
    return(Hermitey)

#At this point we break the integrand into two real pieces to avoid feeding complex
# numbers to the integrator.
def RealIntegrand(phi,r):
    #Note that the order of the variables needs to be
    # reversed here for dblquad to work
    Integrandvalue=FunctionCos(r,phi)*FunctionExpCos(r,phi)*FunctionHermiteX(r,phi)*
    FunctionHermiteY(r,phi)/FunctionRecapital(r,phi)
    return(Integrandvalue)

def ImagIntegrand(phi,r):
    Integrandvalue=FunctionCos(r,phi)*FunctionExpSin(r,phi)*FunctionHermiteX(r,phi)*
    FunctionHermiteY(r,phi)/FunctionRecapital(r,phi)
    return(Integrandvalue)

IntegralReal =integrate.dblquad(RealIntegrand,0,aperture,lambda phi:0,lambda phi:2*pi)[0]
IntegralImag =integrate.dblquad(ImagIntegrand,0,aperture,lambda phi:0,lambda phi:2*pi)[0]

Integralvalue=complex(IntegralReal, IntegralImag) #Here we join both real pieces in
# a complex number, the value of the integral.

DiffractionValue=TheConstant*Integralvalue #Total value of the diffraction field

DiffractionModulus=DiffractionValue*DiffractionValue.conjugate() #Intensity of the
# diffracted field

return(DiffractionModulus.real)

#The following function gives the undisturbed intensity (no aperture) at a point in
# the observation plane.
def Nakedfield(rprime, phiprime, zprime):

def NewExpCos(r, phi, z):
    NCosPieceA=exp((r/(beamradius(z)))**2)
    NCosPieceB=cos(k*(z+r**2/(2*evolradius(z))))
    NExpCosvalue=NCosPieceA*NCosPieceB
    return(NExpCosvalue)
```python
def NewExpSin(r, phi, z):
    NSinPieceA = \exp\left( \frac{r}{(\text{beamradius}(z))^{*2}} \right)
    NSinPieceB = \sin\left( k(z+r^{*2})/(2*\text{evolradius}(z)) \right)
    NExpSinValue = NSinPieceA*NSinPieceB
    return (NExpSinValue)

def ExpFinal(r, phi, z):
    ExpValue = \text{complex}(\text{NewExpCos}(r, phi, z), \text{NewExpSin}(r, phi, z))
    return (ExpValue)

def NewHermiteX(r, phi, z):
    argX = \sqrt{2} \cdot r \cdot \cos(\phi)/\text{beamradius}(z)
    if ModeL==0:
        HermiteX = 1
    elif ModeL==1:
        HermiteX = 2*argX
    elif ModeL==2:
        HermiteX = 4*argX**2
    elif ModeL==3:
        HermiteX = 8*argX**3 + 12*argX
    elif ModeL==4:
        HermiteX = 16*argX**4 + 48*argX**2 + 12
    elif ModeL==5:
        HermiteX = 32*argX**5 + 160*argX**3 + 120*argX
    return (HermiteX)

def NewHermiteY(r, phi, z):
    argY = \sqrt{2} \cdot r \cdot \sin(\phi)/\text{beamradius}(z)
    if ModeM==0:
        HermiteY = 1
    elif ModeM==1:
        HermiteY = 2*argY
    elif ModeM==2:
        HermiteY = 4*argY**2
    elif ModeM==3:
        HermiteY = 8*argY**3 + 12*argY
    elif ModeM==4:
        HermiteY = 16*argY**4 + 48*argY**2 + 12
    elif ModeM==5:
        HermiteY = 32*argY**5 + 160*argY**3 + 120*argY
    return (HermiteY)

NakedFieldValue = (w0/\text{beamradius}(zprime)) \cdot \text{NewHermiteX}(rprime, \phi prime, zprime) \cdot \text{NewHermiteY}(rprime, \phi prime, zprime) \cdot \text{Gouy}(zprime) \cdot \text{ExpFinal}(rprime, \phi prime, zprime)
NakedIntensity = NakedFieldValue \cdot \text{NakedFieldValue.conjugate()}
return (NakedIntensity.real)

# 2D plots (varying \( rprime \) and fixing the \( zprime \) axis distance) used for aperture optimization

# First we define some common parameters
```

---

# First we define some common parameters
Result = 0  # This only serves as a witness for the while loop, we initialize it to 0,
    # which means that there are still ripples >1%
rcom = 0.0  # A common radius, for fixed radius plots
phicom = 0.0  # A common angular direction \phi, for both fixed on axis distance (zprime)
    # and fixed observation radius (rprime) plots

# Definition of the radial interval we will inspect
r_initial = 0
range_r = w0 * 1.5
r_intervals = 200
rposition = linspace(r_initial, range_r, r_intervals)
lengthR = len(rposition)

# Additional apertures for comparison, to create multiple curve figures such as
    # Figure 6 (can be skipped).

# aperture2 = 0.96e3
# z2 = (aperture2 ** 2) / (Fresnel * wavelength)
# aperture3 = 0.80e3
# z3 = (aperture3 ** 2) / (Fresnel * wavelength)
# aperture4 = 0.60e3
# z4 = (aperture4 ** 2) / (Fresnel * wavelength)

# IntensityR2 = zeros(lengthR)
# IntensityR3 = zeros(lengthR)
# IntensityR4 = zeros(lengthR)
# I2norm = zeros(lengthR)
# I3norm = zeros(lengthR)
# I4norm = zeros(lengthR)

# for i in range(len(rposition)):
#     IntensityR2[i] = DiffractiveIntegral(rposition[i], phicom, z2, aperture2)
#     IntensityR3[i] = DiffractiveIntegral(rposition[i], phicom, z3, aperture3)
#     IntensityR4[i] = DiffractiveIntegral(rposition[i], phicom, z4, aperture4)
#     print(i)

# IRNormalization2 = amax(IntensityR2)
# IRNormalization3 = amax(IntensityR3)
# IRNormalization4 = amax(IntensityR4)

# for i in range(len(rposition)):
#     I2norm[i] = IntensityR2[i] / IRNormalization2
#     I3norm[i] = IntensityR3[i] / IRNormalization3
#     I4norm[i] = IntensityR4[i] / IRNormalization4

# In the following section we search for ripples > 0.01. If the search is successful
    # we plot the results, else we increase the aperture size by 1 mm.

while Result == 0:
    aperture = aperture + 0.01e3
    z1 = (aperture ** 2) / (Fresnel * wavelength)

    # The next lines calculate the non diffracted intensity pattern and normalize it
    IntensityN = zeros(lengthR)
INnorm=zeros(lenR)

for i in range(len(rposition)):
    IntensityN[i]=Nakedfield(rposition[i],phicom,z1)

INNormalization=amax(IntensityN)

for i in range(len(rposition)):
    INNorm[i]=IntensityN[i]/INNormalization

#And now we do the same for the diffracted intensity pattern
IntensityR1=zeros(lenR)
I1norm=zeros(lenR)
DeltaI=zeros(lenR)

for i in range(len(rposition)):
    IntensityR1[i]=DiffractionIntegral(rposition[i],phicom,z1,aperture)
    print(i)

I1Normalization1=amax(IntensityR1)

for i in range(len(rposition)):
    I1norm[i]=IntensityR1[i]/I1Normalization1
    DeltaI[i]=abs(INorm[i]/INNorm[i]) #Intensity difference between both cases

MaxDelta=amax(DeltaI) #Look for the largest ripple size
AverageDelta=sum(DeltaI)/len(DeltaI) #And the average ripple size
DeltaIndex, DeltaValue = max(enumerate(DeltaI), key=operator.itemgetter(1))
print(DeltaIndex, DeltaValue)

#Conditional part, the loop stops if the required precision of 0.01 is achieved
if MaxDelta>=0.01:
    Result=0
    print('The aperture')
    print(aperture)
    print('is too small, the maximum difference is:')
    print(MaxDelta)
    print('The average difference is:')
    print(AverageDeltaI)

elif MaxDelta<0.01:
    Result=1
    print('Success! The aperture is')
    print(aperture)
    print('The maximum difference is:')
    print(MaxDelta)
    print('The average difference is:')
    print(AverageDeltaI)

AptoWAist=aperture/w0
print('The required aperture to waist ratio is:')
print(AptoWAist)

plt.plot(rposition/w0, INnorm, linewidth=1, linestyle=" ", label="No aperture")
plt.plot(rposition/w0, I1norm, linewidth=1, linestyle=" ", label=r"$rac{a}{\omega_0}$=2.38")
```python
# Plot (rposition/w0, I2norm, linewidth=1, linestyle="", label=r"$\frac{a}{\omega_0}=$")
# Plot (rposition/w0, I3norm, linewidth=1, linestyle="", label=r"$\frac{a}{\omega_0}=$")
# Plot (rposition/w0, I4norm, linewidth=1, linestyle="", label=r"$\frac{a}{\omega_0}=$")
plt.legend(loc="upper right", prop={'size': 14})
plt.xlabel(r'$r^\prime_p \omega_0$ ', fontsize=12)
plt.ylabel('Normalized intensity', fontsize=13)
plt.show()

# Alternatively, one can plot the diffracted intensity versus the on axis distance (zprime) for fixed rprime and phiprime#
z_initial=200*aperture
range_z=500*aperture
z_intervals=100
zposition=linspace(z_initial,range_z,z_intervals)
lenghtZ=len(zposition)
IntensityZ1=zeros(lenghtZ)

# for i in range(len(zposition)):
# IntensityZ1[i]=DiffractionIntegral(r1,phi1,zposition[i])
# print(i)
# IZNormalization=max(IntensityZ1)

# Plot (zposition/w0, IntensityZ1/IZNormalization, color="blue", linewidth=1, linestyle="")
# Plot (zposition/w0, IntensityZ1/IZNormalization, color="blue", linewidth=1, linestyle="")
# Plot (zposition/w0, IntensityZ1/IZNormalization, color="blue", linewidth=1, linestyle="")
plt.legend(loc='upper left', borderpad=1.5)
plt.xlabel(r'$x^\prime_p \mathrm{/} \omega_0$ ', fontsize=14)
plt.ylabel('Normalized intensity ', fontsize=13)
plt.show()

# Heat map plots#
zfixed=(aperture**2)/(Fresnel*wavelength) #On axis distance
DIM=30 #Number of radii and angle partitions

# The tool we use for the plot (pcolormesh) only admits 2D arrays as inputs.
# In the next lines we simply create 1D arrays with entries in the correct order, and later we transform them to 2D arrays.

# First we generate an DIM**2 array from a NDIM array repeated (as a whole) NDIM times.
rlin=linspace(0,w0*1.5,DIM)
Rlin=tile(rlin,DIM)

# Here we generate an 100 array such that each entry is repeated 10 times.
Philin=zeros(DIM**2)
for i in range(DIM**2):
    Philin[i]=floor(i/DIM)*2*pi/(DIM 1)

# Now we need a few more empty arrays.
xvalues=zeros(DIM**2)
yvalues=zeros(DIM**2)
zvalues=zeros(DIM**2)
zvalues2=zeros(DIM**2)
```
zvalues2N=zeros(DIM**2)
zvaluesN=zeros(DIM**2)
Deltaz=zeros(DIM**2)

# And we fill them in with coordinates, the undiffracted intensity, and the
diffracted intensity.
for i in range (DIM**2):
xvalues[i]=Rlin[i]*cos(Philin[i])
yvalues[i]=Rlin[i]*sin(Philin[i])
zvalues[i]=Nakedfield(Rlin[i],Philin[i],zfixed)
zvalues2[i]=DiffractionIntegral(Rlin[i],Philin[i],zfixed,aperture)
print(i)

# Normalization constants
znormal=amax(zvalues)
znormal2=amax(zvalues2)

# Now we can choose to normalize both intensity arrays and compute their difference
Deltaz
for i in range (DIM**2):
zvaluesN[i]=zvalues[i]/znormal
zvalues2N[i]=zvalues2[i]/znormal2
Deltaz[i]=zvaluesN[i].zvalues2N[i]

# An we can also localize the maximum difference and its location
Deltaindex, Deltavalue = max(enumerate(Deltaz), key=operator.itemgetter(1))
Index=int(Deltaindex)
print(Deltaindex, Deltavalue)
print(xvalues[Index], yvalues[Index])

# As mentioned above, reshaping is necessary to use the heat map plotting tool.
Xval = xvalues.reshape(DIM, DIM).T
Yval = yvalues.reshape(DIM, DIM).T
Nval = zvaluesN.reshape(DIM, DIM).T
Zval = zvalues2N.reshape(DIM, DIM).T
Dval = Deltaz.reshape(DIM, DIM).T

# Axis labels
xminlabel=xvalues.min()/w0
xmaxlabel=xvalues.max()/w0
yminlabel=yvalues.min()/w0
ymaxlabel=yvalues.max()/w0

# Now we can plot Zval (diffracted), Nval (non diffracted) or Dval (difference),
depending on which pattern we are interested in.
plt.pcolormesh(Xval/w0, Yval/w0, Dval, cmap='coolwarm', shading='gouraud', vmin=amin
(Dval), vmax=amax(Dval))
plt.axis([xminlabel, xmaxlabel, yminlabel, ymaxlabel])
plt.xlabel(r'$x^\prime_\mathrm{P/\omega_0}$', fontsize=14)
plt.ylabel(r'$y^\prime_\mathrm{P/\omega_0}$', fontsize=14)
cbar = plt.colorbar()
cbar.ax.set_ylabel('Normalized intensity increment', fontsize=13)
plt.show()

# END