

# **High-Field Laser Physics**

(2,3)

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## DIPOLE APPROXIMATION (LONG-WAVELENGTH APPROXIMATION)

The dipole approximation [or electric-dipole approximation (EDA)] is used almost universally in Atomic, Molecular and Optical (AMO) Physics.

It is not generally recognized that its validity is limited by field intensity.

We start with the customary high-frequency limit on the EDA.

A plane wave field in general is a propagating field with trigonometric dependence on a phase factor  $\omega t - \mathbf{k} \cdot \mathbf{r}$ ,  $\mathbf{k} = \hat{\mathbf{k}}\omega/c = 2\pi\hat{\mathbf{k}}/\lambda$ , where  $\hat{\mathbf{k}}$  is the unit vector in the propagation direction.

The important range of  $|\mathbf{r}|$  is when the wave overlaps the atom, or  $|\mathbf{r}| = O(1 \text{ a.u.})$ .

Dependence of the phase on  $\mathbf{r}$  can be neglected when  $|\mathbf{k} \cdot \mathbf{r}| \ll 1$ , or  $\omega/c \ll 1$ , or  $\omega \ll 137$ .

(Fine-structure constant  $\alpha = e^2/\hbar c$  in Gaussian units; so  $c = 1/\alpha = 137.036 \text{ a.u.}$ )

$\omega t - \mathbf{k} \cdot \mathbf{r} \approx \omega t$  is called the *dipole approximation* or *electric dipole approximation (EDA)* or *long-wavelength approximation (LWA)*.

Dipole approximation, EDA, and LWA are not exactly the same thing, but that is usually ignored, and can be ignored here.

For more information, see:

HRR, PRA **22**, 770 (1980)

HRR, PRA **29**, 698 (1984)

**There is also a low-frequency, intensity-dependent limit on the dipole approximation, first pointed out long ago [HRR, Prog. Quant. Elex. **16**, 1 (1992), Sect. 1.5], but ignored until a more emphatic recent publication: HRR, PRL **101**, 043002 (2008); see also Erratum, PRL **101**, 159901 (E) (2008).**

## LOW FREQUENCY LIMIT ON THE DIPOLE APPROXIMATION

*The magnetic field  $\mathbf{B}$  is the key to the low frequency limit.*

$$\mathbf{B} = \nabla \times \mathbf{A}(t, \mathbf{r})$$

In the dipole approximation,  $\mathbf{A}(t, \mathbf{r}) \rightarrow \mathbf{A}(t) \Rightarrow \mathbf{B} = 0$ .

Usually,  $\mathbf{B} = 0$  is not considered a problem because of the Lorentz force expression:

$$\mathbf{F}_{\text{Lorentz}} = q \left( \mathbf{F} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

Even though  $|\mathbf{B}| = |\mathbf{F}|$  for a plane wave (Gaussian units), it is usually assumed that  $|\mathbf{v}|/c \ll 1$  for nonrelativistic conditions, so the neglect of the magnetic field doesn't matter.

RELATIVISTIC EFFECTS:

$$\gamma_{\text{rel}} = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2} \cdot \frac{v^2}{c^2} + \dots$$

MAGNETIC EFFECTS:

$$\frac{\text{magnetic force}}{\text{electric force}} = \frac{v}{c}$$

For  $v/c \ll 1$ :

$$1 \gg \frac{v}{c} \gg \frac{v^2}{c^2}$$

**MAGNETIC EFFECTS HAVE AN EARLIER ONSET THAN RELATIVISTIC EFFECTS.**

## EFFECT OF THE MAGNETIC FIELD OF A PLANE WAVE

For a plane-wave field of linear polarization, the electric field by itself causes (in the frame of reference in which the electron is at rest on average) an oscillation of amplitude  $\alpha_0$  along the direction of the electric field, with frequency  $\omega$ .

$$\alpha_0 = \frac{F_0}{\omega^2} = \frac{I^{1/2}}{\omega^2} = \frac{2}{\omega} U_p^{1/2} = 2 \left( \frac{z}{\omega} \right)^{1/2}$$

The coupling of the electric and magnetic fields of a linearly polarized plane wave produces a figure-8 motion with the long axis in the electric field direction  $\mathbf{F}$  and the short axis in the propagation direction  $\mathbf{k}$ . The component in the  $\mathbf{k}$  direction has amplitude  $\beta_0$  and frequency  $2\omega$ .

$$\beta_0 = \frac{I}{8c\omega^3} = \frac{U_p}{2c\omega} = \frac{z}{2c}$$

These expressions are nonrelativistic. The fully relativistic results are on the next slide.

Recall the definitions:  $z = U_p / \omega$ ,  $z_f = 2U_p / mc^2$

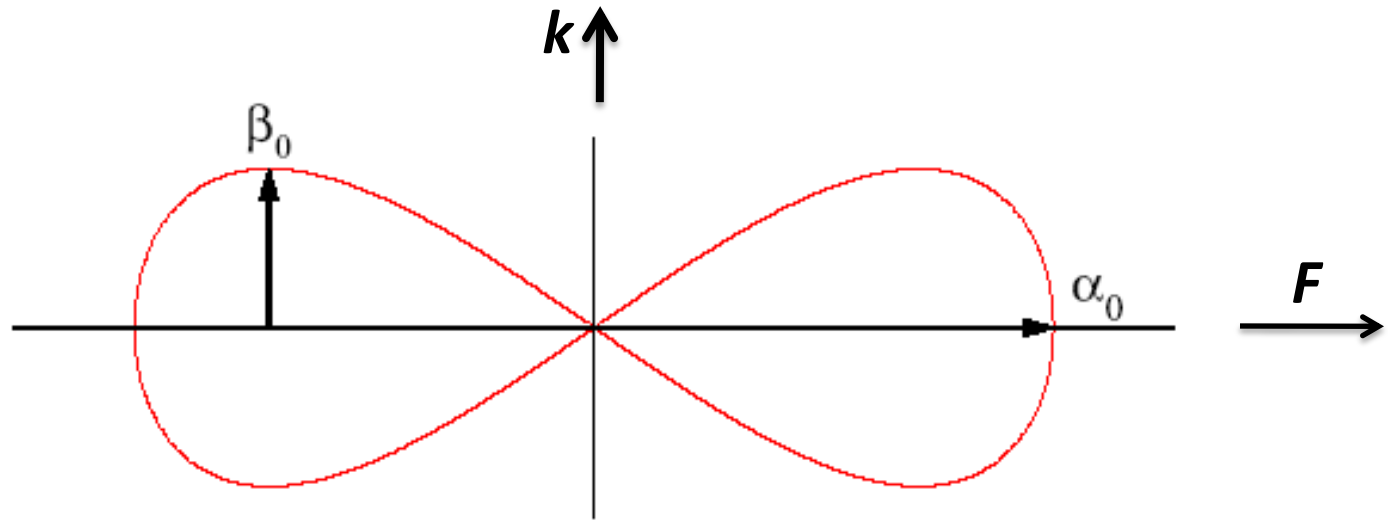
$$\frac{\beta_0}{\alpha_0} = \frac{1}{4} \sqrt{\frac{z_f}{2(1+z_f)}} \xrightarrow{z_f \rightarrow \infty} \frac{1}{4\sqrt{2}} = 0.177$$

Reference: Sarachik & Schappert, PRD **1**, 2738 (1970).

## CLASSICAL FREE ELECTRON IN A PLANE WAVE FIELD

Electron follows a figure-8 trajectory resulting from the combined action of the electric and magnetic fields.

$\alpha_0$  = amplitude along electric field direction  
 $\beta_0$  = amplitude along direction of propagation



$$\alpha_0 = 2\sqrt{\frac{z}{\omega(1+z_f)}}, \quad \beta_0 = \frac{z}{2c(1+z_f)}, \quad \frac{\beta_0}{\alpha_0} = \frac{1}{4}\sqrt{\frac{z_f}{2(1+z_f)}}$$

The figure-8 proportions shown correspond to the limiting case  $\beta_0 / \alpha_0 = 0.177$ .

Figure 6.8

## CONSEQUENCES OF MAGNETIC FIELDS FOR THE DIPOLE APPROXIMATION

If the magnetic-field-induced “thickness” of the figure-8 is of the order of the size of the atom, then overlaps of the field-induced electron motion on the atom will be altered, and matrix-element evaluations will thereby also change.

Require  $\beta_0 < 1$  a.u., or  $z/2c < 1$ ;  $z = U_p / \omega$ ;  $U_p = I / 4\omega^2 \Rightarrow \underline{I < 8c\omega^3}$

***This is now a lower bound on the frequency for which the dipole approximation is valid.***

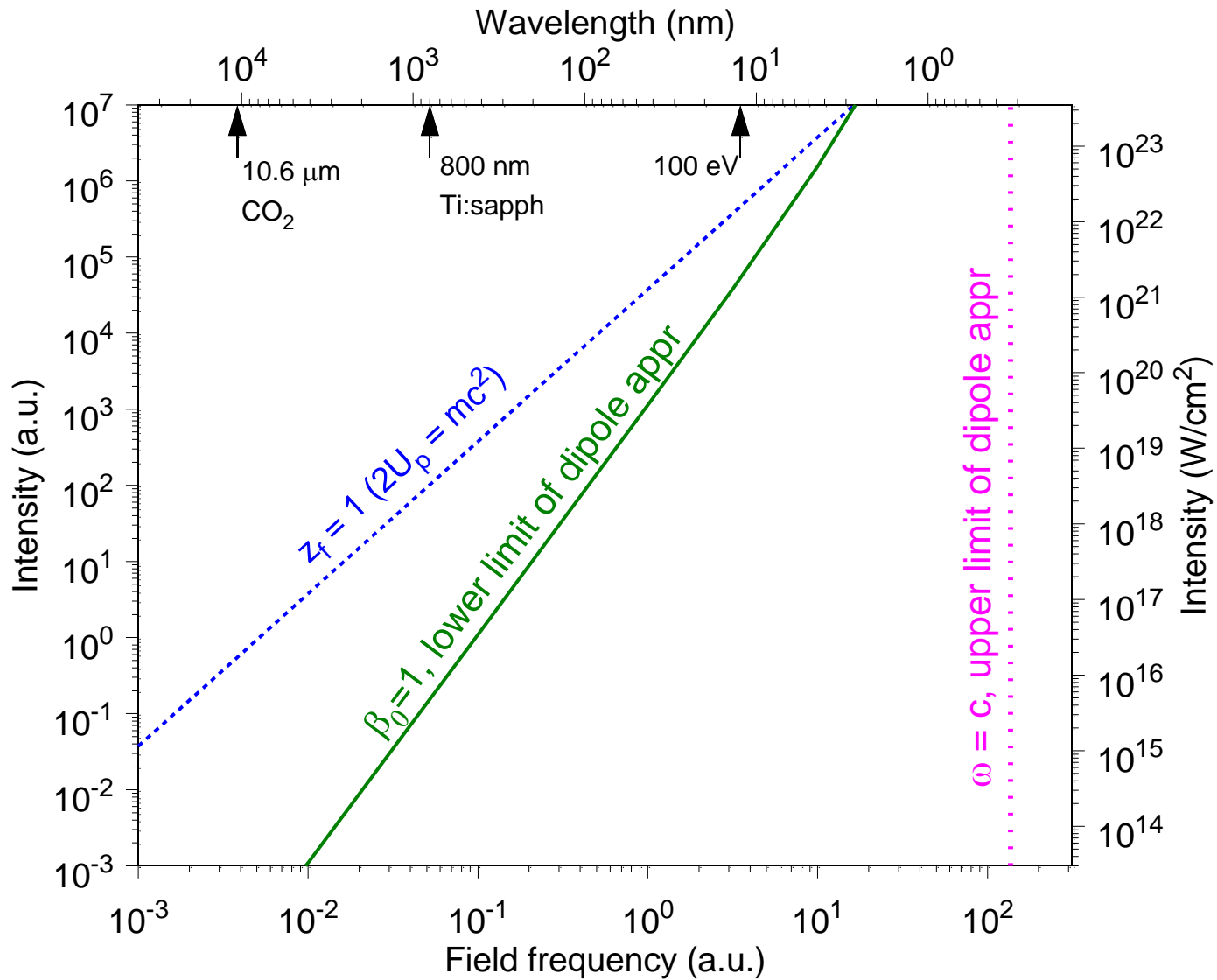
***Overall limits on the dipole approximation are shown on the next slide.***

***Taken together with the fact that a tunneling theory requires not only the dipole approximation, but also  $\omega \ll E_B$ , tunneling theories are limited as shown on the following slide.***

Reference: HRR, PRL **101**, 043002 (2008)

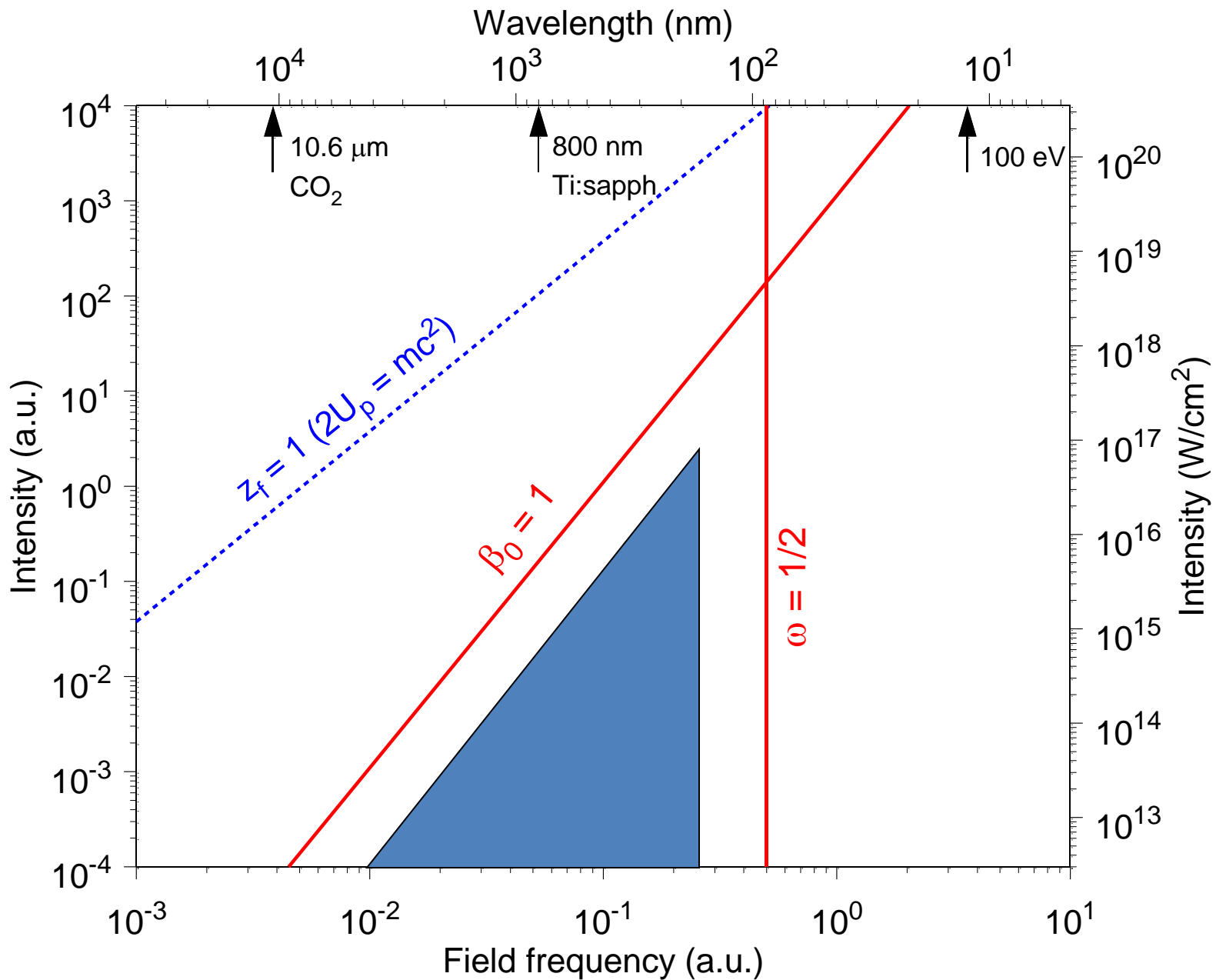
[Note: tunneling is inherently a static-field or very-low-frequency phenomenon; it is explicitly for QSE fields. For example, a high-frequency ( $\omega > E_B$ ) PW field can ionize with a single photon; it does not tunnel.]

# THE DIPOLE APPROXIMATION HAS UPPER AND LOWER FREQUENCY LIMITS





TUNNELING THEORIES FOR LASER-INDUCED PROCESSES ARE LIMITED TO THE SHADED AREA



## DERIVATION OF A GENERAL QUANTUM TRANSITION AMPLITUDE

- We develop a general quantum transition amplitude based only on measurements made in the laboratory.
- This allows us to avoid any description of *how* a state evolves in time. We need not even mention *adiabatic decoupling* (an artificial procedure used in some formal S-matrix theories to describe how fields stop interacting at large times).
- All time evolution is handled by incorporating the equations of motion (the Schrödinger equation) directly in the transition amplitude.
- What emerges is the standard transition amplitude, but with a *minimum of needless assumptions*.

## BRIEF HISTORY OF S MATRICES

Introduced for use in scattering problems; hence the letter **S**

J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).

W. Heisenberg, Z. Physik **120**, 513, 673 (1943).

E. C. G. Stückelberg, Helv. Phys. Acta **17**, 3 (1943); **18**, 21, 195 (1945).

In a search for a general nonperturbative formalism to use for strong-field problems, it was shown that the S-matrix formalism can be employed for *any* quantum process:

free-free (scattering)

bound-free (ionization, photodetachment)

free-bound (recombination)

bound-bound (excitation, de-excitation)

HRR, Phys. Rev. A **1**, 803 (1970).

## RIGOROUS DERIVATION OF AN S MATRIX

Basic requirement: The transition-causing interaction occurs only within a domain bounded in space and time.

(Example: transitions in the focus of a pulsed laser.)

Important properties of an S matrix as derived here:

- ***It can be formulated entirely in terms of quantities measurable in the laboratory.***
- ***It does not require that dynamics be tracked over time. Equations of motion are incorporated in the S matrix.***
- ***There is no need for “adiabatic decoupling”.***
- ***It gives unambiguous rules for gauge transformations.***
- ***It can be applied to any process as long as the space-time domain of the interaction region is bounded.***

The general problem is evoked of an atomic electron subjected to a pulsed, focused laser beam. This is a convenience, not a requirement.

There will be a complete set of states  $\{\Psi_n\}$  that satisfy the Schrödinger equation describing the atomic electron that may be undisturbed or in interaction with a laser beam:

$$i\partial_t\Psi = H\Psi$$

The outcome of any experiment will be measured by laboratory instruments that never experience a laser field. As far as the laboratory instruments are concerned, there is a complete set of states  $\{\Phi_n\}$  that satisfy the Schrödinger equation describing an atomic electron that does **NOT** experience the laser field:

$$i\partial_t\Phi = H_0\Phi$$

Algebraic notation will be used for Hilbert space quantities rather than Dirac bra-ket notation.

The correspondences and rules are:

$$\begin{aligned} |u\rangle &\rightarrow u) & \langle v| &\rightarrow (v, & \langle v|u\rangle &\rightarrow (v,u) \\ (v,cu) &= c(v,u) & (cv,u) &= c^*(v,u); & c &= \text{complex scalar} \end{aligned}$$

By hypothesis, the laser pulse is finite, so

$$\lim_{t \rightarrow \pm\infty} [H(t) - H_0] = 0$$

We can organize the two complete sets of  $\Phi$  states and  $\Psi$  states so that they correspond at  $t \rightarrow -\infty$  :

$$\lim_{t \rightarrow -\infty} [\Phi_n(t) - \Psi_n(t)] = 0$$

After the laser interaction has occurred, the only way for the laboratory instruments to discover what has happened is to form overlaps of all possible final  $\Phi_f$  states with the state that began as a particular  $\Psi_i$  state. This is the S matrix:

$$S_{fi} = \lim_{t \rightarrow +\infty} (\Phi_f, \Psi_i)$$

Subtract the amplitude that no transition has occurred:

$$M_{fi} \equiv (S - 1)_{fi} = \lim_{t \rightarrow +\infty} (\Phi_f, \Psi_i) - \lim_{t \rightarrow -\infty} (\Phi_f, \Psi_i)$$

We now have the form of a perfect differential:

$$M_{fi} = \int_{-\infty}^{+\infty} dt \frac{\partial}{\partial t} (\Phi_f, \Psi_i)$$

$$M_{fi} = \int_{-\infty}^{+\infty} dt [(\partial_t \Phi_f, \Psi_i) + (\Phi_f, \partial_t \Psi_i)]$$

$$i\partial_t \Phi = H_0 \Phi \Rightarrow \partial_t \Phi = -iH_0 \Phi$$

$$i\partial_t \Psi = (H_0 + H_I) \Psi \Rightarrow \partial_t \Psi = -i(H_0 + H_I) \Psi$$

$$M_{fi} = \int_{-\infty}^{+\infty} dt [(-iH_0 \Phi_f, \Psi_i) + (\Phi_f, -i(H_0 + H_I) \Psi_i)]$$

$$M_{fi} = \int_{-\infty}^{+\infty} dt [i(\Phi_f, H_0 \Psi_i) - i(\Phi_f, (H_0 + H_I) \Psi_i)]$$

$$M_{fi} = -i \int_{-\infty}^{+\infty} dt (\Phi_f, H_I \Psi_i)$$

An alternative form is especially useful for strong-field problems. Instead of making a one-to-one correspondence of  $\Phi$  and  $\Psi$  states at  $t \rightarrow -\infty$ , do it at  $t \rightarrow +\infty$  and then look for the probabilities that particular initial states could have led to this final result.



The end result of this procedure is the transition matrix element

$$M_{fi} = -i \int_{-\infty}^{\infty} dt (\Psi_f, H_I \Phi_i)$$

The first form above is called the direct-time S matrix, and the second form is the time-reversed S matrix.

In general,  $\Phi$  states are known exactly or can be simulated accurately (for example, by analytical Hartree-Fock procedures). It is the  $\Psi$  states that present the problem because they represent conditions for an electron simultaneously subjected to the Coulomb attraction and to the laser field. No exact solutions are known.

If the initial state  $\Psi_i$  is to be approximated, this is very difficult because the laser field (by hypothesis) is too strong to be treated perturbatively, and so is the Coulomb field because in an initial bound state the Coulomb field has a singularity at the origin.

For the final state  $\Psi_f$ , the electron is unbound, and the laser field can be assumed to be more important than the Coulomb field. This identifies the time-reversed S matrix as the preferred form, and it will be employed exclusively hereafter.

## GAUGE TRANSFORMATIONS

Electromagnetic fields can be represented in terms of derivatives of vector potentials  $\mathbf{A}(t, \mathbf{r})$  and scalar potentials  $\phi(t, \mathbf{r})$ .

These potentials are not unique; within specific constraints, one set of potentials can be exchanged for another that produce exactly the same electric and magnetic fields  $\mathbf{F}$ ,  $\mathbf{B}$ .

The alteration from one set of potentials to another describing the same fields is called a *gauge transformation*.

Within perturbative AMO physics, gauges are not important, and are usually (and safely) ignored. It is normally a matter of calculational convenience to decide what gauge to employ.

***In strong-field physics, gauges become very important; they are no longer a matter of indifference.***

***It will be necessary to explain why this occurs, and to identify the consequences of ignoring the effects of a change in gauge.***

*This matter has only recently been subjected to detailed study. Much of the AMO community is either unaware that gauge is important, or else reject the evidence.*

## CONCISE OVERVIEW OF GAUGES

Every intermediate or advanced textbook on classical electrodynamics gives a treatment of this subject. Only those matters of direct importance to these lectures will be reviewed.

The fields  $\mathbf{F}$  and  $\mathbf{B}$  are related to the potentials  $\mathbf{A}$  and  $\phi$  as

$$\mathbf{F} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

When substituted into the two source-free Maxwell equations, it is found that if the potentials satisfy the *Lorentz condition*

$$\nabla \times \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0,$$

and this then leads to two uncoupled wave equations for  $\mathbf{A}$  and  $\phi$  that follow from the remaining two Maxwell equations containing sources:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J},$$

where  $\rho$  is the density of electric charge and  $\mathbf{J}$  is the current density.

***If there are no sources:  $\rho = 0$  and  $J = 0$ , then these equations have plane-wave (PW) solutions. Once formed, plane waves can propagate indefinitely without sources.***

***A quasistatic electric (QSE) field cannot exist without sources:  $\rho \neq 0$ .***

***PW and QSE fields are fundamentally different types of fields.***

This difference will arise repeatedly in later work.

## GAUGE TRANSFORMATION

If  $\chi$  is a scalar function, then the gradient of  $\chi$  can be added to  $\mathbf{A}$  without any change in the magnetic field:

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A}, \quad \nabla \times (\nabla \chi) \equiv 0, \quad \text{set } \mathbf{A}' = \mathbf{A} + \nabla \chi \\ \Rightarrow \mathbf{B} &= \nabla \times \mathbf{A}'\end{aligned}$$

If the Lorentz condition is to remain valid, then it is necessary that

$$\phi \rightarrow \phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \quad \text{from which follows} \quad \mathbf{F} = -\nabla \chi' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t}$$

To maintain the wave equations for the vector and scalar potentials, it is necessary that

$$\nabla^2 \chi - \frac{1}{c} \frac{\partial^2 \chi}{\partial t^2} = 0$$

This is a gauge transformation. The new potentials  $\mathbf{A}'$ ,  $\chi'$  give exactly the same fields as the original  $\mathbf{A}$ ,  $\chi$ . The conclusion that any set of potentials meeting the constraints listed are as valid as any other set of potentials is called *gauge invariance*.

## EVIDENCE OF INCONSISTENCIES WITH STRICT GAUGE INVARIANCE

In support of gauge invariance, it is noted that Newton's equations depend directly on forces, and the Lorentz force associated with electromagnetic fields is given explicitly in terms of the fields themselves:

$$m\ddot{\mathbf{r}} = \mathbf{F}_{\text{Lorentz}} = q\left(\mathbf{F} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

It is generally shown in textbooks on classical mechanics that alternative formulations of mechanics infer Newton's equations.

*The inverse is **not true**. Newton's equations do **not** infer Hamiltonian, Lagrangian, etc. forms of classical mechanics. "System functions" like the Hamiltonian or the Lagrangian depend directly on potentials, and it has never been found possible to express those potentials in terms of the fields except in a "non-local" way.*

The inference, to be supported with much more evidence later, is that the potentials contain more information than the fields. That, in turn, suggests that the *potentials are more fundamental than the fields*.

This conclusion is in conflict with what (almost) all of the textbooks say.

Gauge selection is unimportant in perturbation theory, and the general attitude that “all gauges are created equal” causes no problems.

**In strong fields, gauge selection is fundamentally important.**

As already shown, in a gauge transformation:

$$\begin{aligned}\vec{A} &\rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi, & \phi &\rightarrow \phi' = \phi - \frac{1}{c}\partial_t\chi \\ \Rightarrow H &\rightarrow H', & H_I &\rightarrow H_I', & \Psi &\rightarrow \Psi' = U\Psi \\ \Phi, & H_0 & \text{unchanged}\end{aligned}$$

where  $\chi$  is a scalar *generating function* that satisfies the homogeneous wave equation.

It is clear and unambiguous that the states  $\{\Phi_n\}$  and Hamiltonian  $H_0$  for the world of the laboratory instruments have nothing to do with the laser field. A change of gauge of the laser field can have no possible effect.

**The transition amplitude changes as:**

$$\begin{aligned}M_{fi} &= -i\int_{-\infty}^{\infty} dt (\Psi_f, H_I \Phi_i) \rightarrow M_{fi}' = -i\int_{-\infty}^{\infty} dt (\Psi_f', H_I' \Phi_i) \\ \Rightarrow M_{fi} &\neq M_{fi}'\end{aligned}$$

## EVIDENCE OF MORE PROBLEMS IN THE “CONVENTIONAL UNDERSTANDING”

In numerous papers in the AMO community and in textbooks, the presumption is made that a gauge transformation is a unitary transformation; that is, a transformation generated by a unitary operator  $U$ , defined by the properties:  $UU^\dagger = U^\dagger U = 1$ , or  $U^\dagger = U^{-1}$ . Quantum states transform as  $\Psi' = U\Psi$ , and quantum operators as  $O' = UOU^\dagger = UOU^{-1}$ . The presumption is that transition matrix elements transform as

$$(\Phi_f, H_I \Psi_i) \rightarrow (\Phi_f', H_I' \Psi_i') = (U\Phi_f, (UH_I U^{-1})(U\Psi_i)) = (\Phi_f, H_I \Psi_i).$$

That is, it is presumed that all matrix elements are automatically gauge invariant.

**This is absolutely not true!**

The transformation just shown is a change of quantum picture. It is not a gauge transformation.

Were the AMO assumption correct, there would be no value or purpose in enforcing gauge invariance.

Note: in QED, gauge invariance is required, but it is not a universal gauge change; rather, it is a change to another transverse-field gauge.



## ANOTHER FUNDAMENTAL FEATURE OF STRONG FIELDS

Gauges become important in strong fields, but another basic concept from classical mechanics has to be altered.

In classical mechanics there is a simple theorem that any function of time alone:  $f(t)$  can be added to a Hamiltonian or Lagrangian function without altering the physical problem in any way.

An even simpler example is that a constant can be added to  $H(q,p,t)$  or to  $L(q, \dot{q}, t)$  without consequence. (For example, a constant shift in the zero of energy.)

On these grounds, Davidovich was able to show that the first correction term to the SFA should cancel the leading term (a mathematical impossibility: power series are unique and no term can cancel a preceding term).

Milonni concluded that the ponderomotive energy can be ignored because  $U_p$  depends on  $\mathbf{A}^2(t)$  and so it can be discarded. However, the ponderomotive energy is physically very real, and causes channel closings, among other effects.

*The simple and familiar principle about removal of  $f(t)$  from the Hamiltonian doesn't seem to be valid here. Why not??*

## CHANNEL CLOSING

There is a threshold order:  $n \geq E_B + U_p$  implies there is a smallest order  $n_0$  that satisfies this condition.

As  $U_p$  increases,  $n_0 \rightarrow n_0 + 1$ . This is a *channel closing*. Hence freely removing  $\mathbf{A}^2(t)$  changes the apparent physics.

*What is going on?*

In quantum mechanics, the  $\mathbf{A}^2(t)$  – dependent  $\Psi$  states are “pinned” to the  $\mathbf{A}^2(t)$  – independent  $\Phi$  states. Hence  $\mathbf{A}^2(t)$  can **NOT** be removed in quantum mechanics.

**This is a basic classical – quantum distinction.**

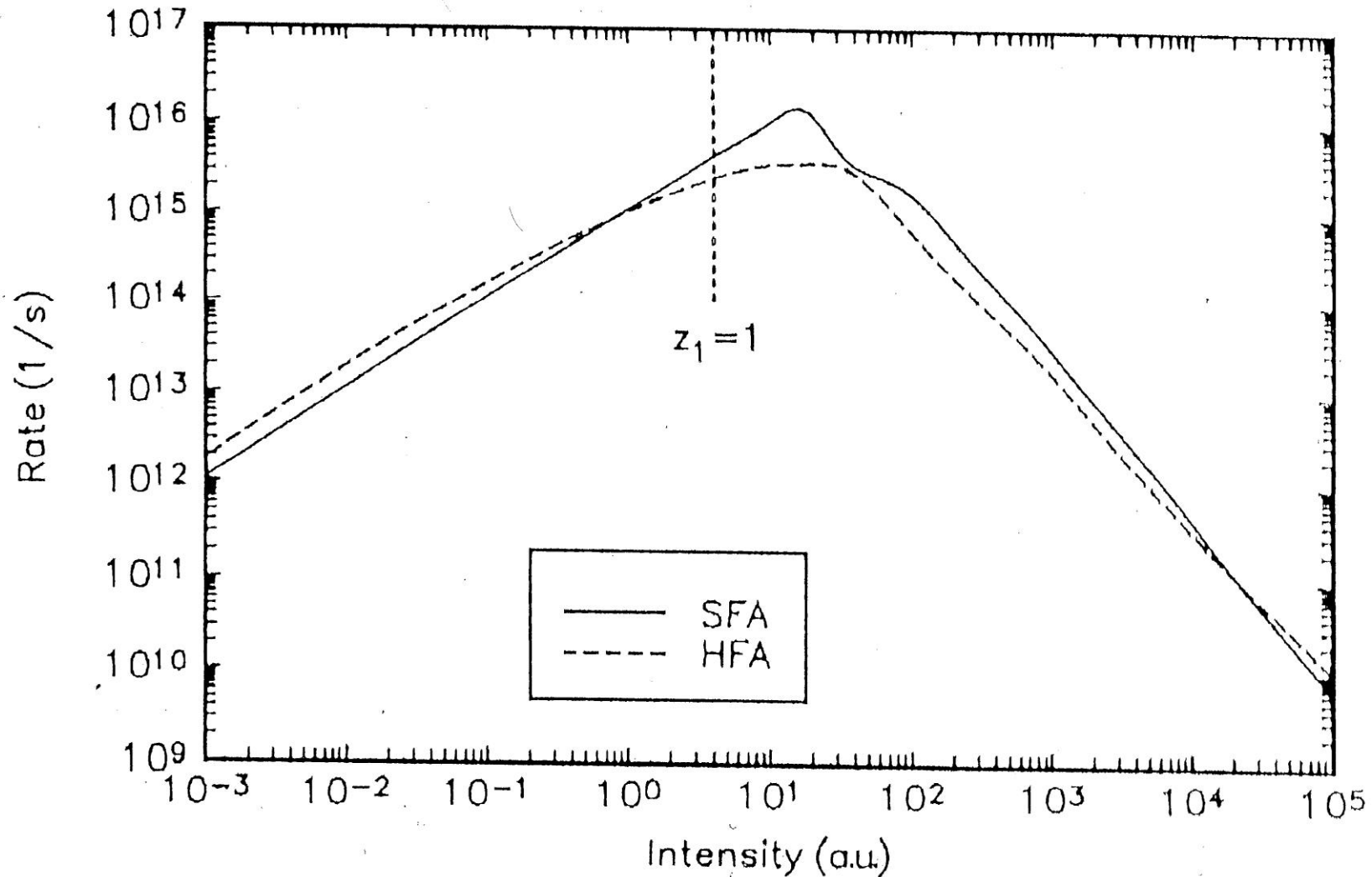
It is a fundamental result that is not in the literature.

The  $\mathbf{A}^2(t)$  term can also be removed by a gauge transformation. This is a different matter, since gauge invariance does “protect” basic measurables like total rates. The next slide shows two very similar-looking results from theories with and without removal of  $\mathbf{A}^2(t)$ .

*However, physical interpretations are completely different!*

# VELOCITY-GAUGE SFA AT HIGH FREQUENCY

Ionization of ground-state hydrogen,  $\omega = 2$  a.u.



SFA from: HRR, JOSA B **13**, 355 (1996)

HFA from Pont and Gavrila, PRL **65**, 2362 (1990)

The preceding slide shows many things that will be addressed in more detail later.

*SFA = Strong-Field Approximation*

*HFA = High-Frequency Approximation*

The fact that, as intensity increases, the transition probability eventually **decreases** is called *stabilization*. This appears to be a near-universal phenomenon theoretically, but it has never been conclusively demonstrated in the laboratory.

Because of the high frequency ( $\omega = 2$  a.u.),  $n_0 = 1$  at low intensities.

In the SFA, the maximum occurs when that channel closes, and  $n_0 \rightarrow 2$ .

In the HFA,  $A^2(t)$  is removed by a gauge transformation, and there is no such thing as a channel closing. Stabilization must be ascribed to some other cause.

This example establishes an *important basic concept*:

***PHYSICAL INTERPRETATIONS ARE GAUGE-DEPENDENT.***

*Not all aspects of a physical problem are preserved by a gauge transformation.*

This is another subject that will require more attention later.

## SOME TERMINOLOGY FOR GAUGES

**COULOMB GAUGE:** This is the most widely used gauge in all of physics. It has a formal mathematical definition, but for our purposes a qualitative description is more useful.

The *Coulomb gauge* is that gauge in which a longitudinal field is described by a scalar potential  $\phi$  alone, with no vector potential  $\mathbf{A}$ ; and a transverse field is described by a vector potential  $\mathbf{A}$  alone, with no scalar potential  $\phi$ .

A *longitudinal field* is one where the direction of propagation lies along the direction of the electric field. It is typified by a *quasistatic electric (QSE) field*, such as that which exists between a pair of parallel capacitor plates with a slowly varying potential difference across them. A static electric field (such as arises from a Coulomb center of attraction) is a longitudinal field.

A *transverse field* has the direction of propagation perpendicular to the direction of the electric field, so that  $\mathbf{k} \cdot \mathbf{F} = 0$ . A *plane-wave (PW) field* is a pure transverse field. It has the unique property that it can propagate infinite distances in vacuum with a loss of intensity arising only from solid-angle effects. Lasers produce transverse fields.

The *velocity gauge* is a Coulomb gauge in which the dipole approximation is employed. That is, where  $\mathbf{A}(t, \mathbf{r}) \rightarrow \mathbf{A}(t)$ . It is called the velocity gauge because the Hamiltonian contains the quantity  $\mathbf{p} - (1/c) \mathbf{A}$ , which is the *kinetic momentum* (proportional to *velocity*).

The *length gauge* (also called *Göppert-Mayer gauge*) is only used in the dipole approximation, and it expresses even transverse fields by a scalar potential. It is of the form  $\phi = \mathbf{r} \cdot \mathbf{F}$ . It is called the length gauge because of the presence of  $\mathbf{r}$  in the potential.

Reference: M. Göppert-Mayer, Ann. Phys. (Leipzig) **9**, 273 (1931).

The length gauge can also be described as that gauge where a PW field is treated as if it were a QSE field.

The *Lorentz gauge* is that broad class of gauges where vector and scalar potentials satisfy the relation

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0,$$

which decouples the wave equations for  $\phi$  and  $\mathbf{A}$ . Almost all gauges used in physics are Lorentz gauges, except for the length gauge.

When the scalar potential is either zero or independent of time, then a formal definition of the *Coulomb gauge* often encountered is

$$\nabla \cdot \mathbf{A} = 0$$

## STRONG-FIELD APPROXIMATION WITHIN THE DIPOLE APPROXIMATION

Start with the time-reversed S matrix:

$$M_{fi} = -i \int_{-\infty}^{\infty} dt (\Psi_f, H_I \Phi_i)$$

For an ionization problem,  $\Phi_i$  is an initial bound state. Two important facts:

1.  $\Phi_i$  can be assumed to be known accurately
2. Most of the binding potential dependence in the problem is in  $\Phi_i$

$\Psi_f$  is an ionized or photodetached state. For very strong fields, it can then be assumed that the dominant effect on the free particle will be the laser field.

As a first estimate, this requires  $U_p \gg E_B$

An exact solution is known for the motion of a free charged particle in a very strong plane-wave field: the **Volkov solution**.

When  $\Psi_f$  is replaced by  $\Psi_f^V$ , this gives the **Strong-Field Approximation**.

$$M_{fi}^{SFA} = -i \int_{-\infty}^{\infty} dt (\Psi_f^V, H_I \Phi_i)$$

References: HRR, PRA **22**, 1786 (1980); PRA **42**, 1476 (1990); Prog. Quant. Elect. **16**, 1 (1992).

## DIPOLE APPROXIMATION VOLKOV SOLUTION

Schrödinger equation:

$$i\partial_t\Psi = \frac{1}{2}\left(-i\nabla + \frac{1}{c}\mathbf{A}(t)\right)^2\Psi$$

The Volkov solution is explicit to the Coulomb gauge; within the Coulomb gauge, plane-wave fields are represented by vector potentials with zero scalar potential.

The Coulomb gauge within the dipole approximation is generally called the *velocity gauge*.

Another generic form for the Schrödinger equation:

$$i\partial_t\Psi = (H_0 + H_I)\Psi,$$
$$H_0 = -\frac{1}{2}\nabla^2, \quad H_I = \frac{1}{c}\mathbf{A}(t)\cdot(-i\nabla) + \frac{1}{2c^2}\mathbf{A}^2(t)$$

The dipole-approximation Volkov solution:

$$\Psi^V = V^{-1/2} \exp\left[i\left(\mathbf{p}\cdot\mathbf{r} - \frac{p^2}{2}t + \int_{-\infty}^t d\tau H_I(\mathbf{p},\tau)\right)\right]$$
$$H_I(t)\Psi^V = H_I(\mathbf{p},t)\Psi^V$$



With the last expression substituted:

$$M_{fi}^{SFA} = -i \int_{-\infty}^{\infty} dt H_I(\mathbf{p}, t) (\Psi_f^V, \Phi_i)$$

Express the initial state wave function as a stationary state:

$$\Phi_i(t, \mathbf{r}) = \phi_i(\mathbf{r}) \exp(-iE_i t)$$

The only spatial dependence in the Volkov function is  $\exp(i\mathbf{p} \cdot \mathbf{r})$ . The Hilbert-space inner product is then simply a Fourier transform, spatial dependence is replaced by momentum dependence, and the Hilbert-space inner product is no longer in evidence.

$$\hat{\phi}_i(\mathbf{p}) = (\exp(i\mathbf{p} \cdot \mathbf{r}), \phi_i(\mathbf{r}))$$

The matrix element is then

$$M_{fi}^{SFA} = -i \frac{1}{V^{1/2}} \hat{\phi}_i(\mathbf{p}) \int_{-\infty}^{\infty} dt \exp \left[ i \left( \frac{\mathbf{p}^2}{2} - E_i \right) t \right] H_I(\mathbf{p}, t) \exp \left[ i \int_t^{\infty} d\tau H_I(\mathbf{p}, \tau) \right].$$

## INTERACTION HAMILTONIAN AND POLARIZATION STATES

To proceed further requires a knowledge of the interaction Hamiltonian and that, in turn, requires a specification of the polarization state of the field. Only circular and linear polarization will be considered. Circular polarization is the simpler case.