HIGH-FIELD LASER PHYSICS

FROM SFA TO TUNNELING

SFA & TUNNELING: COMPARISON AND CONTRAST

The SFA and Tunneling are the 2 basic analytical approximation methods in strong fields.
Both can be applied within the dipole approximation.

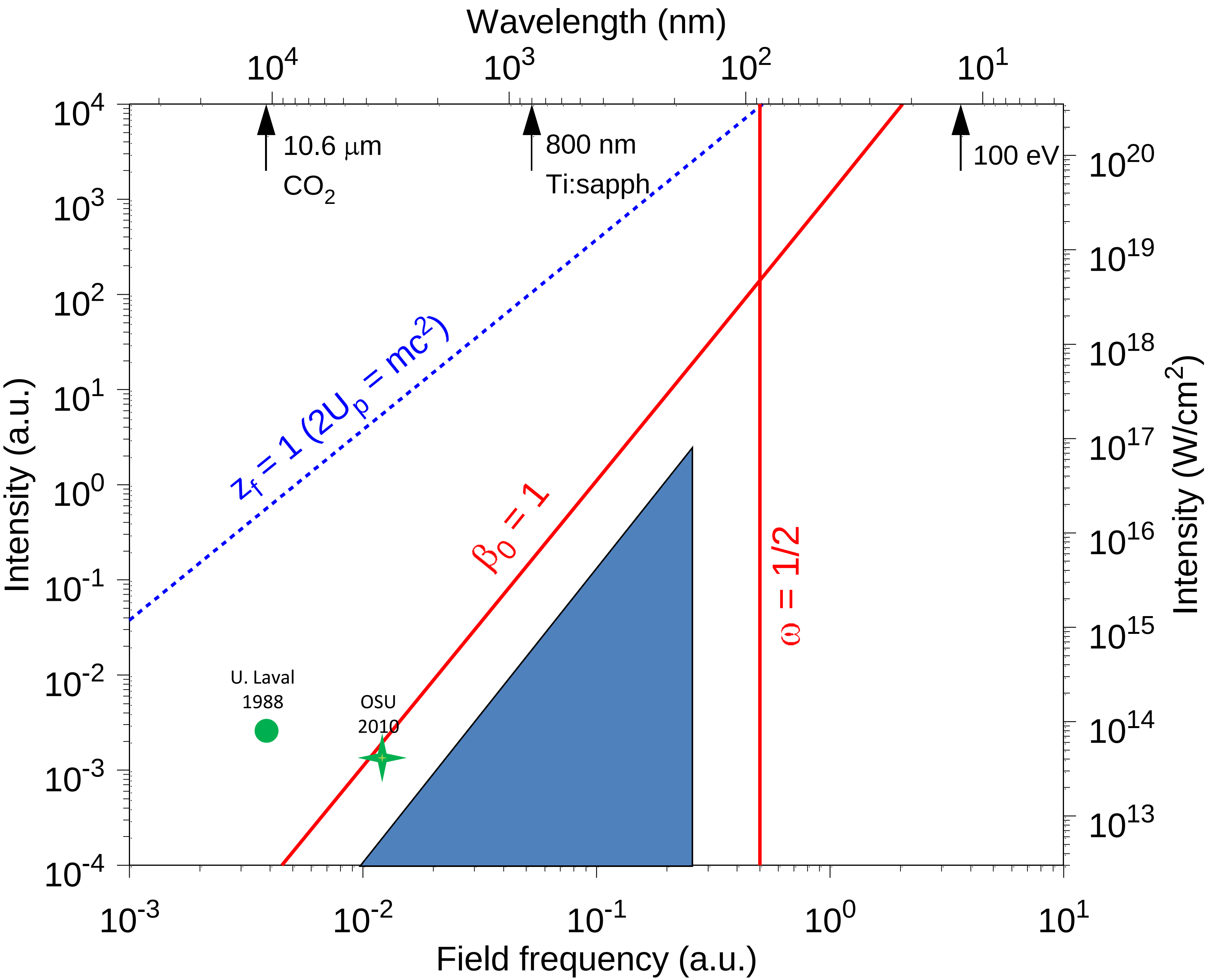
They are fundamentally different:

The SFA treats the laser field as a transverse (or vector) field. The laser field ***is*** a transverse field.	Tunneling treats the laser field as a longitudinal (or scalar) field. The laser field ***is not*** a longitudinal field.

A startling fact: Many of the leading theoreticians in the field don't know the difference.

The shaded area in the next figure shows the only region where the 2 methods overlap. Outside that area, only the SFA is applicable.

TUNNELING THEORIES FOR LASER-INDUCED PROCESSES ARE LIMITED TO THE SHADED AREA



BASIC PAPERS FOR TUNNELING IONIZATION

L. V. Keldysh, JETP **47**, 1945 (1964) [Sov. Phys. **20**, 1307 (1965)]

PPT1 = A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, JETP **50**, 1393 (1966) [Sov. Phys. **23**, 924 (1966)]

PPT2 = A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, JETP **51**, 309 (1966) [Sov. Phys. **24**, 207 (1967)]

PPT3 = A. M. Perelomov and V. S. Popov, JETP **52**, 514 (1967) [Sov. Phys. **25**, 336 (1967)]

ADK = M. V. Ammosov, N. B. Delone, and V. P. Krainov, JETP **91**, 2008 (1986) [Sov. Phys. **64**, 1191 (1986)]

These papers use Green's function techniques as well as Fourier transforms, so a brief introduction to both is given here.

GREEN'S FUNCTIONS

Green's functions form general solutions as superpositions of the weighted contributions of delta-function sources.

When used in operator form, they are also known as *propagators, time-development operators, time-evolution operators, …*

Rewrite the Schrödinger equation (SE):

$$i\partial_t \Psi = (H_0 + H_I)\Psi$$

$$(i\partial_t - H_0)\Psi = H_I \Psi$$

Atomic units are used. The right-hand side can be regarded as a *source term* for Ψ. The Green's function (GF) for this problem is

$$i\partial_t - H_0(\mathbf{r},t)\ \ G(\mathbf{r},t;\mathbf{r}',t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t') = \delta^4(x-x'),$$

where the last expression is a 4-dimensional form that is often useful as an abbreviation, as in:

$$i\partial_t - H_0(x)\ \ G(x;x') = \delta^4(x-x').$$

Before giving an example, introduce the Fourier transform.

FOURIER TRANSFORM

In one dimension:

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} f(p).$$

The function *f(p)* is the *Fourier transform* of *F(x)*.
To invert this:

$$\int_{-\infty}^{\infty} dx e^{ip'x} F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp e^{i(p'-p)x} f(p).$$

This expression contains an integral representation of the Dirac delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(p'-p)x} = \delta(p'-p) \quad \Rightarrow$$

$$\int_{-\infty}^{\infty} dx e^{ip'x} F(x) = \int_{-\infty}^{\infty} dp \delta(p'-p) f(p) = f(p')$$

If *f(p)* is the *Fourier transform* of *F(x)*, then *F(x)* is the *inverse transform* of *f(p).*

A symmetrized form is sometimes preferred:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{-ipx} f(p); \quad f(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ipx} F(x).$$

The variables *x* and *p* are called *conjugate variables,* where *x* and *p* are position and momentum.
Another pair of conjugate variables commonly found in physics problems is *t* and ω, time and frequency.

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} f(\omega); \quad f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} F(t).$$

In 3 dimensions:

$$F(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 p e^{i\boldsymbol{p}\cdot\boldsymbol{r}} f(\boldsymbol{p}); \quad f(\boldsymbol{p}) = \int d^3 r e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} F(\boldsymbol{r}).$$

The choice of sign in the exponent is arbitrary; it always reverses in the inverse transform. Signs will be selected here to achieve the form $exp[-i(\omega t - \boldsymbol{p}\cdot\boldsymbol{r})]$ that is appropriate for propagating waves.

EXAMPLE: FREE PARTICLE

$$\left(i\partial_t + \frac{1}{2}\nabla^2\right)G(\boldsymbol{r},t;\boldsymbol{r}',t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')$$

Since the source term depends on ***r,r'*** only as ***r* – *r'*,** and on *t,t'* only as *t* - *t'* , then *G(**r**,t;**r'**,t')* is of the form *G(**r**-**r'**;t-t')*.

This is called *translational invariance*; it doesn't matter where the origin of the space or time variable is selected.

Now insert a 4-dimensional Fourier transform for *G*.

$$G(\boldsymbol{r}-\boldsymbol{r}';t-t') = \frac{1}{2\pi^{4}}\int d^3p\int d\omega\, g(\boldsymbol{p},\omega)exp[-i\omega(t-t')+i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')]$$

When substituted into the equation of motion,

$$\left(i\partial_t + \frac{1}{2}\nabla^2\right)G = G(\boldsymbol{r}-\boldsymbol{r}';t-t') = \frac{1}{2\pi^{4}}\int d^3p\int d\omega\, g(\boldsymbol{p},\omega)exp[-i\omega(t-t')+i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\left(\omega-\frac{\boldsymbol{p}^2}{2}\right)$$

$$= \delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')$$

When integral representations are substituted for the delta functions,

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') = \frac{1}{(2\pi)^4}\int d^3p \int d\omega \exp[-i\omega(t-t') + i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')]$$

A comparison of these expressions yields directly

$$g(\boldsymbol{p},\omega) = \frac{1}{\omega - \boldsymbol{p}^2/2}.$$

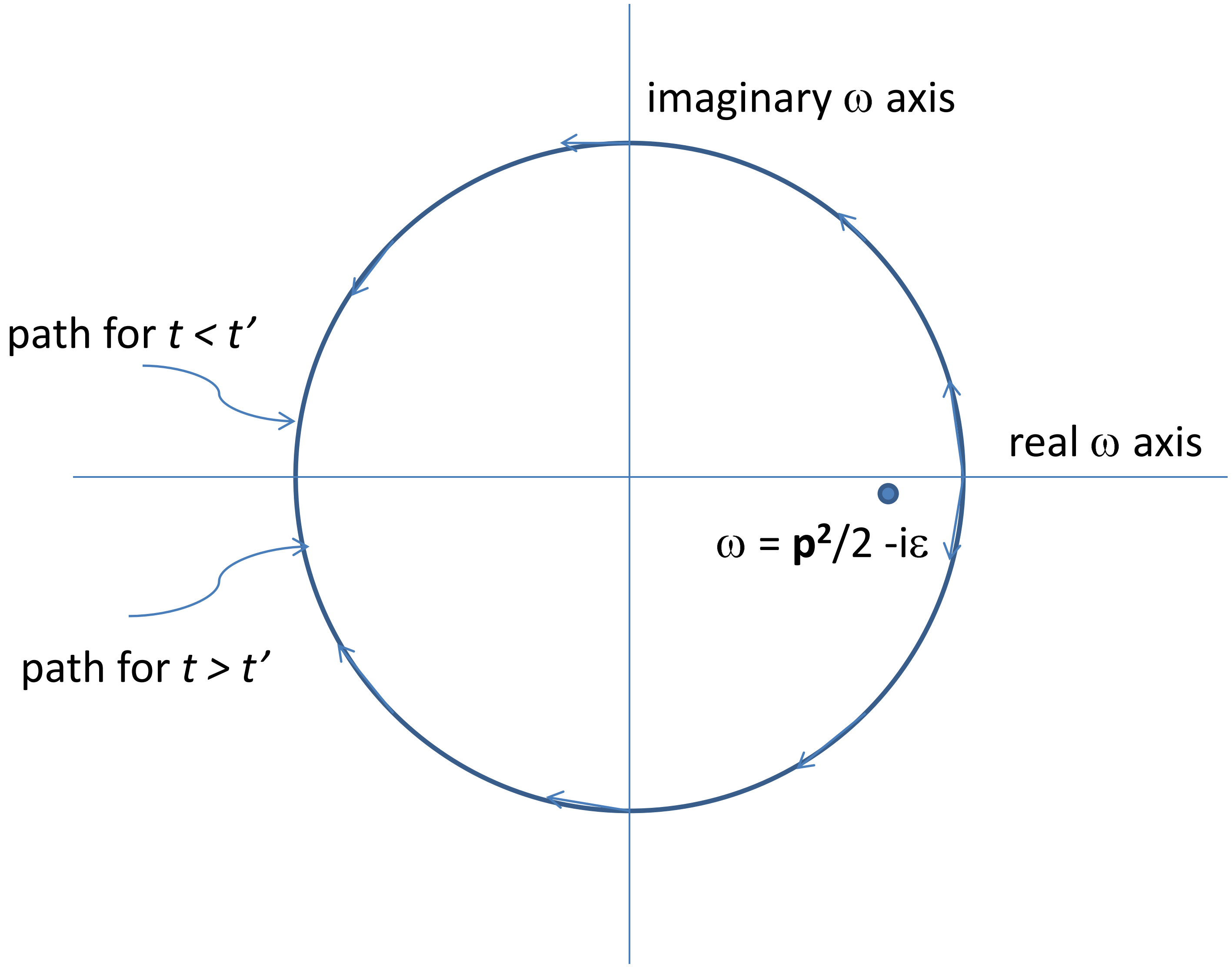
The integration over ω therefore has a singularity at $\omega = \boldsymbol{p}^2/2$.

Different physical solutions correspond to different ways of avoiding the singularity.

For a causal solution:

$$\omega - \frac{\boldsymbol{p}^2}{2} \rightarrow \lim_{\varepsilon\rightarrow 0^+}\left(\omega - \frac{\boldsymbol{p}^2}{2} + i\varepsilon\right)$$

CONTOUR FOR THE PATH OF INTEGRATION IN THE COMPLEX ω PLANE



That contour is selected such that

$$\exp[-i\omega(t-t')] \to 0 \quad as \quad |\mathrm{Im}(\omega)| \to \infty$$

This gives the result

$$\oint d\omega = \int_{-\infty}^{\infty} d\omega + \oint_{lower\,half-plane} d\omega$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} d\omega = 2\pi i Res\left(\omega = \frac{p^2}{2}\right)$$

The Green's function is then

$$G(\mathbf{r}-\mathbf{r}';t-t') = -\frac{i\theta(t-t')}{(2\pi)^3}\int d^3p \exp[-i(p^2/2)(t-t') + i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')],$$

where $\theta(t-t')$ is called the *step function* or the *theta function*.

STEP FUNCTIONS; THE THETA FUNCTION

The theta function is defined to be:

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases}$$

with the useful property that

$$\delta(x) = \frac{d\theta}{dx}$$

The specification of the *x = 0* option is sometimes omitted. It is better not to omit it, since the delta-function property of the derivative of the theta function requires it in order to apply to the special case of an integration by parts with one zero limit:

$$\int_a^b dx\, f(x)\delta(x) = f(0) \; if \;\; a < 0 < b; \quad \int_0^b dx\, f(x)\delta(x) = \frac{1}{2} f(0) \; if \;\; 0 < b$$

$$\int_0^b dx\, f(x)\delta(x) = \int_0^b dx\, f(x)\frac{d\theta}{dx} = f(x)\theta(x)\Big|_0^b - \int_0^b dx\frac{df}{dx}\theta(x)$$

$$= f(b) - \frac{1}{2} f(0) - \int_0^b dx\frac{df}{dx} = f(b) - \frac{1}{2} f(0) - f(b) + f(0) = \frac{1}{2} f(0)$$

STEP FUNCTIONS; THE SIGN FUNCTION

Another step function often encountered is the *epsilon function* or *sign function*, usually written as *sgn(x).*

$$\varepsilon(x) = \operatorname{sgn}(x) = 2\theta(x) - 1$$

$$= \begin{cases} +1, \ x > 0 \\ 0, \ x = 0 \\ -1, \ x < 0 \end{cases}$$

FINAL RESULT FOR THE FREE-PARTICLE GREEN'S FUNCTION

$$G(\boldsymbol{r}-\boldsymbol{r}';t-t') = -\frac{i\theta(t-t')}{(2\pi)^3}\int d^3p\,\exp[-i(p^2/2)(t-t')+i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')]$$

The three-fold integration d^3p can be done in closed form, with the result:

$$G(\boldsymbol{r}-\boldsymbol{r}';t-t') = -i\theta(t-t')\left(\frac{1}{2\pi i(t-t')}\right)^{3/2}\exp\left[\frac{i(\boldsymbol{r}-\boldsymbol{r}')^2}{2(t-t')}\right] \quad or$$

$$G(\boldsymbol{r};t) = -i\theta(t)\left(\frac{1}{2\pi i t}\right)^{3/2}\exp\left(\frac{i\boldsymbol{r}^2}{2t}\right)$$

COMPLETE GREEN'S FUNCTION

The complete Green's function is useful for formal purposes.

$$[i\partial_t - H_0(\mathbf{r},t) - H_I(\mathbf{r},t)]\mathcal{G}(\mathbf{r},\mathbf{r}';t,t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

A simpler combined form is

$$[i\partial_t - H_0(x) - H_I(x)]\mathcal{G}(x;x') = \delta^4(x-x'),$$

with a solution that can be written in terms of the ordinary Green's function as

$$\mathcal{G}(x,x') = G(x,x') + \int d^4x'' G(x,x'')H_I(x'')\mathcal{G}(x'',x).$$

Verify by substitution:

$$[i\partial_t - H_0(x)]\mathcal{G}(x,x') = [i\partial_t - H_0(x)]G(x,x') + \int d^4x'' \ (i\partial_t - H_0(x))G(x,x'') \ H_I(x'')\mathcal{G}(x'',x')$$

$$= \delta^4(x-x') + \int d^4x''[\delta^4(x-x'')]H_I(x'')\mathcal{G}(x'',x')$$

$$= \delta^4(x-x') + H_I(x)\mathcal{G}(x,x')$$

By re-ordering this expression, the second equation above is reproduced. *q.e.d.*

APPLICATION: DERIVATION OF PERTURBATION THEORY

Perturbation theory can be obtained efficiently, starting from the expression:

$$\mathcal{G}(x,x') = G(x,x') + \int d^4x'' G(x,x'') H_I(x'') \mathcal{G}(x'',x)$$

As a first approximation, insert the first term of $\mathcal{G}$ into the integral equation:

$$\mathcal{G}^{(1)}(x,x') = G(x,x') + \int d^4x'' G(x,x'') H_I(x'') G(x'',x')$$

$$+ \int d^4x''' \int d^4x'' G(x,x'') H_I(x'') G(x'',x') \mathcal{G}(x''',x'') \mathcal{G}(x''',x')$$

and so on … In condensed notation:

$$\mathcal{G} = G + \int G H_I G + \int\int G H_I G H_I G + \dots$$

This is an expansion in powers of H_I. For wave functions:

$$(i\partial_t - H_0 - H_I)\Psi = 0; \qquad (i\partial_t - H_0)\Phi = 0.$$

The integral equation solution is:

$$\Psi(x) = \Phi(x) + \int d^4x' \mathcal{G}(x,x') H_I(x') \Phi(x')$$

Inserting the expansion for $\mathcal{G}$ gives the perturbation expansion for Ψ:

$$\Psi = \Phi + \int G H_I \Phi + \int\int G H_I G H_I \Phi + \int\int\int G H_I G H_I G H_I \Phi + \dots$$

PPT TUNNELING

The first PPT paper is entitled: "Ionization of Atoms in an Alternating Electric Field". That is exactly correct; that is what they calculate. The problem is that they lose sight of that restriction when they apply their theory to laser fields.

There are 2 fundamental Lorentz-invariant quantities that define an electromagnetic field. The significant one here is $\boldsymbol{F}^2 - \boldsymbol{B}^2$.

$$QSE: \quad \boldsymbol{F}^2 - \boldsymbol{B}^2 = \boldsymbol{F}^2$$

$$PW: \quad \boldsymbol{F}^2 - \boldsymbol{B}^2 = 0$$

Particularly in the case of strong fields, this difference is essential.
However, there is a gauge transformation that connects the 2 types of fields ***when the dipole approximation is valid.***
M. Göppert-Mayer, Ann. Phys. (Leipzig) **9**, 273 (1931).
The principle of gauge invariance guarantees that direct physical measurables will have the same values in the two gauges.

This gauge transformation has caused much confusion, since there is a tendency to forget the limitation to the dipole approximation, as well as more subtle matters.

PPT VALIDITY CONDITIONS

PPT1 states 3 validity conditions:

- Low frequencies and sufficiently weak fields that tunneling times are >> atomic times.

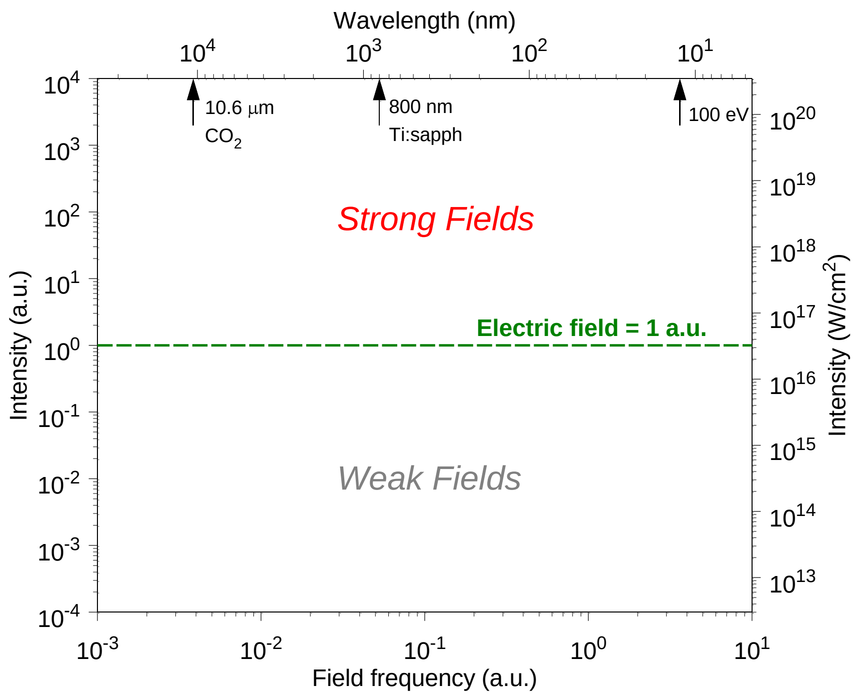
$$\textit{Terminology}: \omega_0 \equiv E_B; \quad \omega_H = \frac{1}{2} a.u.; \quad F_H = \frac{1}{2} a.u.; \quad F_0 \equiv \left(\frac{\omega_0}{\omega_H} \right)^{3/2}$$

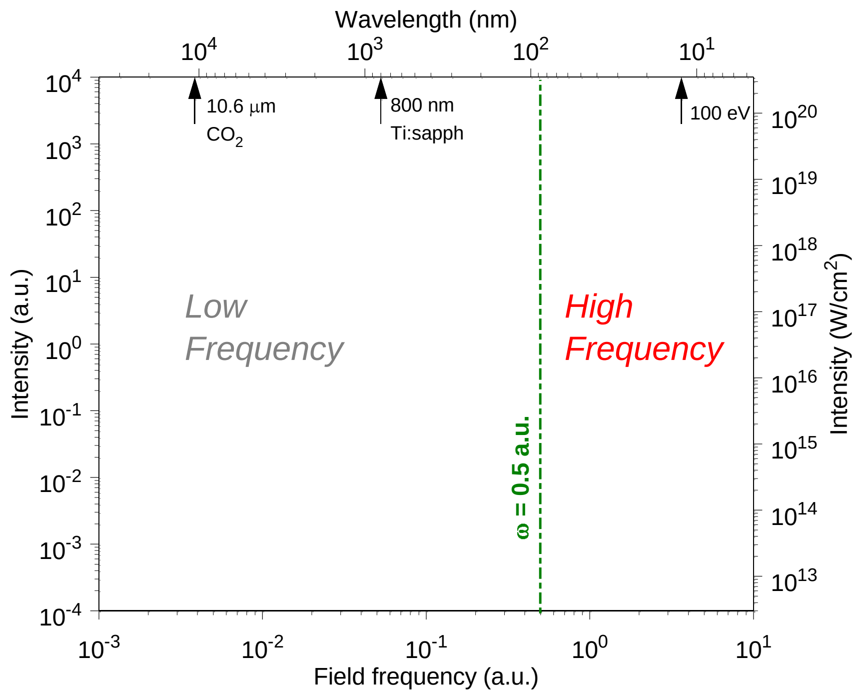
$$\textit{Conditions}: \omega << \omega_0, F << F_0$$

- Long wavelengths: $\lambda >> 1$. This is essentially a repetition of the low-frequency condition. It is imposed for the express purpose of being able to treat the magnitude of the electric field as a constant.
- Speed of the electron is nonrelativistic: $(v/c)^2 << 1$. PPT explain this as the condition for the neglect of the magnetic field. That is, they do not distinguish between relativistic conditions and magnetic field effects.

Although PPT specify that they are neglecting the magnetic field, they do not recognize the correct condition for this constraint. Furthermore, even had they used the correct condition, they violate this limitation in later applications.

PPT place limits on the validity of their tunneling theory as shown in the graphs. There is a limitation to weak fields (as defined in the left-hand graph) and low frequencies (as defined in the right-hand graph).





TUNNELING TRANSITION RATE

PPT start with the known expression for tunneling ionization by a static field.
J.R. Oppenheimer, Phys. Rev. **31**, 66 (1928).

$$w = 8\omega_H \frac{F_H}{F} \exp\left(-\frac{2}{3}\frac{F_H}{F}\right) = \frac{4}{F}\exp\left(-\frac{2}{3F}\right) (in\, a.u.)$$

This type of negative exponential, with the electric field in the denominator of the argument, characterizes all tunneling results.

PPT then examine the *"adiabatic case"*, later known as the *"tunneling domain"*: γ<<1.

$$w(F,\omega) = \left(\frac{3F}{\pi F_0}\right)^{1/2} w_{static}(F),$$

$$w_{static}(F) \sim \exp\left(-\frac{2}{3F_0\left|\cos\omega t\right|}\right).$$

cosωt will have zeroes, but the exponential goes to zero very rapidly as the argument $\rightarrow \infty$, so ionization will occur predominantly when $\omega t \approx 0, \pi, 2\pi, 3\pi,$ ... That is, ionization occurs predominantly when the field reaches its maximum.

Circular polarization: The amplitude of ***F*** is constant and ***F*** rotates uniformly with time, so the result is always just the static result.

One-dimensional zero-range potential: Zero range means that the potential is a delta function. This avoids the problems of a long-range Coulomb force.

$$V(x) = -\kappa\delta(x); \quad E_B = \kappa^2 / 2$$

A delta function potential has only one bound state.

Static case:

$$w_{static}(F) = 2\omega_0 \exp\left(-\frac{2}{3}\frac{F_0}{F} \right); \quad \omega_0 = E_B, F_0 = \left(\frac{\omega_0}{\omega_H} \right)^{3/2} F_H = (2\omega_0)^{3/2} \textit{ in a.u.}$$

Adiabatic case (tunneling domain):

$$\omega << \omega_t; \quad \omega_t = F / \kappa$$

$$w_{ad} = 2\omega_0 \left(\frac{3F}{\pi F_0} \right)^{1/2} \exp\left(-\frac{2}{3}\frac{F_0}{F} \right)$$

PPT are unaware that $\omega << 1$ means that the magnetic field is important.