TRANSFORMATION BETWEEN THE VELOCITY GAUGE AND THE LENGTH GAUGE

I. GAUGE TRANSFORMATION BASICS

In terms of a scalar potential ϕ and a vector potential $\mathbf{A}$, the electric and magnetic fields $\mathbf{F}$, $\mathbf{B}$ can be expressed as

$$\mathbf{F} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{1}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2}$$

The gauge transformation

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}, \tag{3}$$

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \tag{4}$$

leaves $\mathbf{F}$, $\mathbf{B}$ unchanged, as is easily verified by direct substitution. Note that

$$\nabla\times(\mathbf{A} + \nabla\chi) = \nabla \times \mathbf{A}, \quad \text{since} \tag{5}$$

$$\nabla \times \nabla = \mathbf{0}. \tag{6}$$

There is a condition (the Lorentz condition) on ϕ and $\mathbf{A}$ required to uncouple the wave equations for the potentials:

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \tag{7}$$

When the transformations (3) and (4) are substituted into Eq.(7), preservation of the Lorentz condition requires that the *generating function* χ for the gauge transformation must satisfy the homogeneous wave equation

$$\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi = 0. \tag{8}$$

II. VG $\longleftrightarrow$ LG TRANSFORMATION

Suppose we start in the dipole-approximation velocity gauge (VG), where

$$\phi^{VG} = 0, \tag{9}$$

$$\mathbf{A}^{VG} = \mathbf{A}^{VG}(t), \tag{10}$$

and introduce the gauge transformation induced by the generating function

$$\chi = -\mathbf{r} \cdot \mathbf{A}^{VG}. \tag{11}$$

Equation (3) gives

$$\phi^{LG} = -\frac{1}{c}\frac{\partial \chi}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{r} \cdot \mathbf{A}^{VG} = \mathbf{r} \cdot \frac{1}{c}\frac{\partial \mathbf{A}^{VG}}{\partial t} = -\mathbf{r} \cdot \mathbf{F}, \tag{12}$$

using Eq.(1). Equation (4) gives

$$\mathbf{A}^{LG} = \mathbf{A}^{VG} + \nabla\left(-\mathbf{r} \cdot \mathbf{A}^{VG}\right) = \mathbf{A}^{VG} - \mathbf{A}^{VG} = 0. \tag{13}$$

To summarize, Eqs.(12) and (13) give

$$\phi^{LG} = -\mathbf{r} \cdot \mathbf{F}, \tag{14}$$

$$\mathbf{A}^{LG} = 0, \tag{15}$$

in place of the original potentials (9) and (10).

To start in the LG with the potentials given in Eqs.(14) and (15), transformation to the VG is generated by the function

$$\chi' = \mathbf{r} \cdot \mathbf{A}^{VG}. \tag{16}$$

There is one peculiarity about the χ in Eq.(11) and the χ' in Eq.(16) that seems never to be mentioned. The homogeneous wave equation of Eq.(8) that must be satisfied to preserve the Lorentz condition of Eq.(8) is NOT satisfied. That is, although

$$\nabla^2 \chi = 0, \tag{17}$$

the other term

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi = -\frac{1}{c^2}\mathbf{r}\cdot\frac{\partial^2}{\partial t^2}\mathbf{A}^{VG} \neq 0. \tag{18}$$

Suppose

$$\mathbf{A}^{VG} = \mathbf{A}_0 \cos \omega t. \tag{19}$$

Then one has explicitly

$$-\frac{1}{c^2}\mathbf{r}\cdot\frac{\partial^2}{\partial t^2}\mathbf{A}^{VG} = \frac{\omega^2}{c^2}\mathbf{r}\cdot\mathbf{A}^{VG} = -\frac{\omega^2}{c^2}\chi. \tag{20}$$

This anomaly might be due to the fact that a plane wave does not require sources for its continued existence, whereas the LG “gauge-equivalent” potential represents a quasistatic electric field that cannot exist without sources. Once a plane wave has been generated, it can propagate in vacuum to infinity in time or space without any need for sources.

III. GAUGE TRANSFORMATION IN THE SCHRÖDINGER EQUATION

The Schrödinger equation is "form-invariant" under a gauge transformation if a solution Ψ in the original gauge is replaced by

$$\Psi' = \exp\left(iq\chi/c\right)\Psi, \quad or \tag{21}$$

$$\Psi = \exp\left(-iq\chi/c\right)\Psi' \tag{22}$$

in the transformed gauge. This can be checked by direct substitution. In the original gauge:

$$i\partial_t\Psi = \left\{\frac{1}{2}\left[\left(-i\nabla\right) - \frac{q}{c}\mathbf{A}\left(t\right)\right]^2 + q\phi\right\}\Psi. \tag{23}$$

Upon introduction of the transformed wave function, the exponential factor can be moved to the left-hand side, giving

$$i\partial_t\Psi = i\partial_t\left[\exp\left(-iq\chi/c\right)\Psi'\right] = \exp\left(-iq\chi/c\right)\left(i\partial_t + q\partial_t\chi/c\right)\Psi', \tag{24}$$

$$\left(-i\nabla - \frac{q}{c}\mathbf{A}\right)\Psi = \left(-i\nabla - \frac{q}{c}\mathbf{A}\right)\exp\left(-i\frac{q}{c}\chi\right)\Psi' = \exp\left(-i\frac{q}{c}\chi\right)\left(-i\nabla - \frac{q}{c}\mathbf{A} - \frac{q}{c}\left(\nabla\chi\right)\right)\Psi', \tag{25}$$

$$\left(-i\nabla - \frac{q}{c}\mathbf{A}\right)^2\Psi = \left(-i\nabla - \frac{q}{c}\mathbf{A}\right)\exp\left(-i\frac{q}{c}\chi\right)\left(-i\nabla - \frac{q}{c}\mathbf{A} - \frac{q}{c}\left(\nabla\chi\right)\right)\Psi' \tag{26}$$

$$= \exp\left(-i\frac{q}{c}\chi\right)\left(-i\nabla - \frac{q}{c}\mathbf{A} - \frac{q}{c}\left(\nabla\chi\right)\right)^2\Psi'. \tag{27}$$

In total, the transformation induced by Eq.(21) or (22) yields

$$\exp\left(-i\frac{q}{c}\chi\right)\left(i\partial_t + \frac{q}{c}\partial_t\chi\right)\Psi' = \exp\left(-i\frac{q}{c}\chi\right)\left\{\frac{1}{2}\left[\left(-i\nabla\right) - \frac{q}{c}\mathbf{A}\left(t\right) - \frac{q}{c}\left(\nabla\chi\right)\right]^2 + q\phi\right\}\Psi'. \tag{28}$$

After introduction of the gauge-transformed potentials given in (3) and (4), and neglecting the overall factor of $\exp\left(-iq\chi/c\right)$, the final result is

$$i\partial_t\Psi' = \left\{\frac{1}{2}\left[\left(-i\nabla\right) - \frac{q}{c}\mathbf{A}'\left(t\right)\right]^2 + q\phi'\right\}\Psi'. \tag{29}$$

The fact that Eq.(29) has exactly the same form as Eq.(23), but with the gauge-transformed potentials and wave function, is what is meant by *form invariance* of the Schrödinger equation upon a transformation of gauge.