# SADDLE-POINT APPROXIMATION

or

# STEEPEST-DESCENT APPROXIMATION

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Saddle-point approximation and steepest-descent approximation are alternative names for the same method. In strong-field work, most people seem to prefer "saddle-point", so we shall use that, abbreviated as SPA.

SPA refers to an approximate integration method that makes use of the properties of functions in the complex plane, even if the integral to be performed is real-valued. A few basic concepts in complex analysis are necessary:

- An *analytic function* is one that can be expressed as a convergent power series; or, equivalently, a function that is infinitely differentiable. Most functions encountered in physics are analytic functions. Examples: exponential, trigonometric, Bessel, Legendre,...and most other transcendental functions. Counter-examples are simple algebraic functions with a simple-pole singularity, like $1/(1-x)$ or an essential singularity like $\sqrt{x}$.

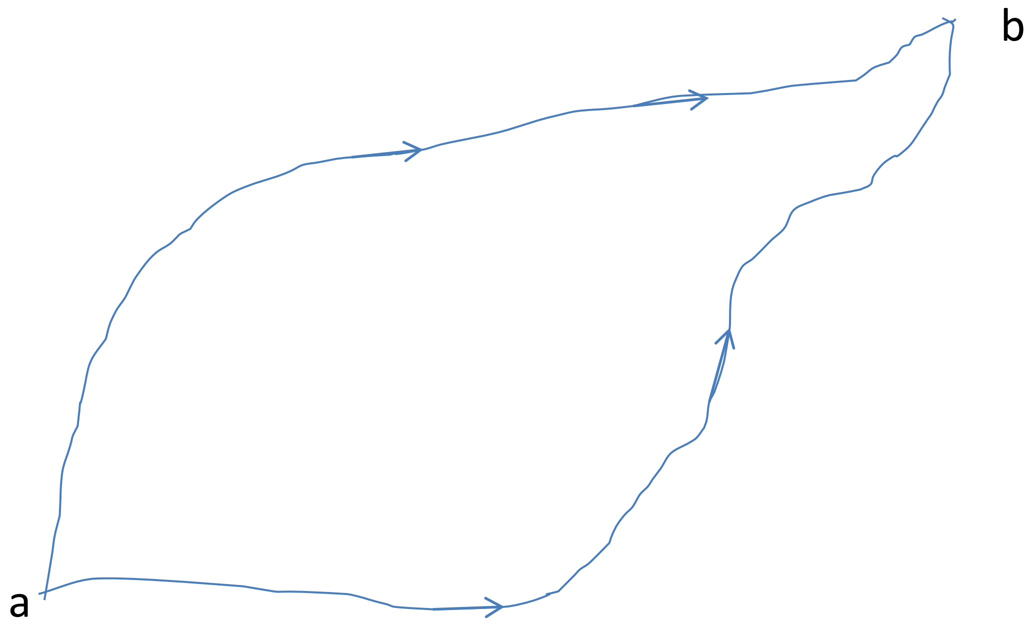
- A *holomorphic function* is a function that is analytic in the complex plane. This implies some non-obvious special properties. Specifically, a holomorphic function cannot have any extrema (minima or maxima) in the finite complex plane. Extrema can exist only at infinity.

- The *Cauchy theorem* states that any integral of a holomorphic function in a closed path gives zero. An alternative statement is that a path integral of a holomorphic function between any two points in a complex plane is independent of the path connecting those two points.

$$\oint f(z)\, dz = 0,$$

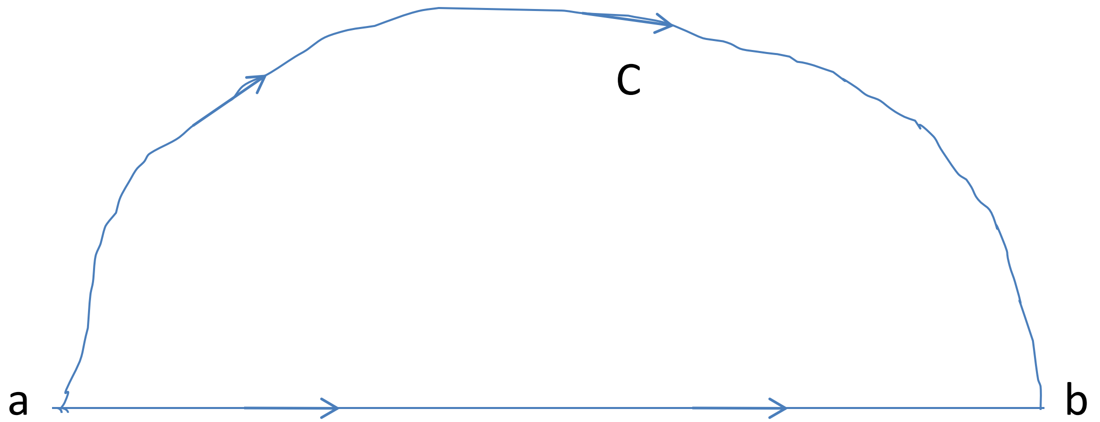
$$\int_{a,C}^{b} f(z)\, dz = \int_{a,C\prime}^{b} f(z)\, dz.$$

From the Cauchy Theorem:




Integral around the closed circuit = 0.

If a and b are both on the real axis, and the path of integration is also real, then the real-axis path can be deformed to any contour C without changing the value of the integral, if *f(z)* is analytic in the enclosed region.




$$\int_a^b dx f(x) = \int_C dz f(z)$$

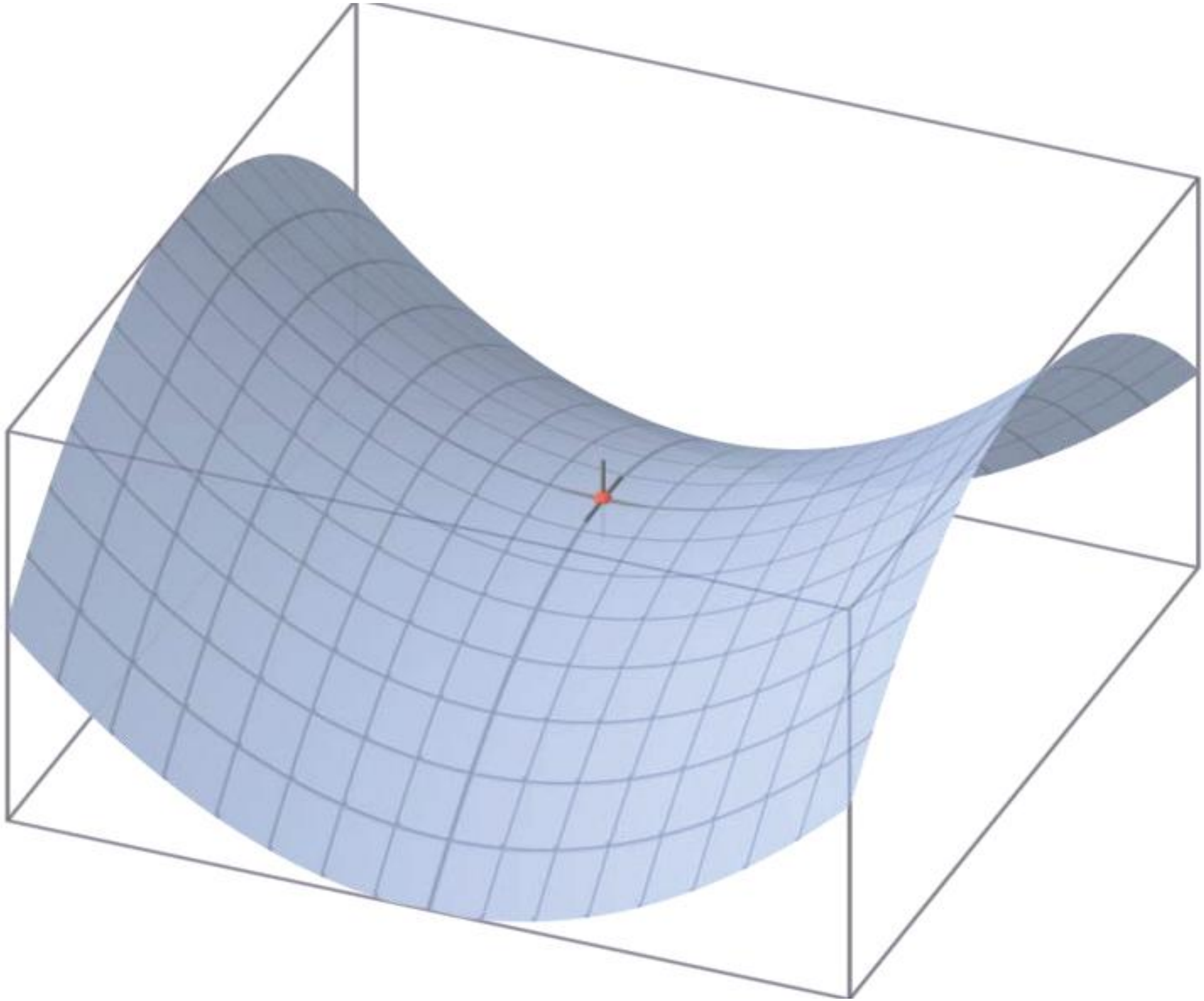
There might be a way to choose the contour C that is advantageous for evaluation of the integral.

## ABSENCE OF EXTREMA

A holomorphic function cannot have an extremum. As stated for a maximum, this means that there is no point in the complex plane that is higher than nearby points in the sense that there is no point from which a departure in every direction will lead to a smaller magnitude of the function. A completely analogous statement holds for the absence of a minimum.

## HOWEVER, *SADDLE POINTS* CAN EXIST.

A saddle point is such that there exist paths where a **maximum** of the function exists **along those paths**. There will exist a particular path along which the **descent from the maximum is steepest**. Perpendicular to such a path is another path where a **minimum** of the function exists **along that path**, and such that along that path the **ascent from the minimum is steepest**.



# PROBLEMS SUITED TO THE SADDLE-POINT METHOD

The saddle point method is suited to the evaluation of integrals that contain in the integrand an exponential function containing a large parameter. For instance:

$$\int_a^b dx \exp\left[\eta f\left(x\right)\right], \quad \eta \text{ real}, \quad \eta \gg 1.$$

Saddle points exist when $x$ is continued into the complex plane, and there are points where

$$\frac{d}{dz} f\left(z\right) = f'\left(z\right) = 0.$$

Let such a point be designated as $z_0$. A power series expansion of around the point $z_0$ gives

$$f\left(z\right) = f\left(z_0\right) + \frac{1}{2}\left(z - z_0\right)^2 f''\left(z_0\right) + ...$$

since the first-derivative term vanishes. The integral becomes

$$\int_a^b dx \exp\left[\eta f\left(x\right)\right] \approx \exp\left[\eta f\left(z_0\right)\right] \int_C dz \exp\left[\eta \frac{1}{2}\left(z - z_0\right)^2 f''\left(z_0\right)\right],$$

since the very large value of $\eta$ means that only the points nearest to the saddle point contribute,and the contour $C$ is a deformed contour starting with the original real path from the limits $a, b$ and deforming the path to pass through those saddle points that can be

One must now find that path through the saddle point where $(z-z_0)^2 f''(z_0)$ is real and

$$(z-z_0)^2 f''(z_0) < 0,$$

which signifies a maximum. The large value of $\eta$ means that the Gaussian fall-off from the maximum value will occur within a very short interval, so that one can simply extend the limits of integration to $\pm\infty$, which yields the well-known Gaussian integral

$$\int_{-\infty}^{\infty} dz \exp\left[\eta \frac{1}{2}(z-z_0)^2 f''(z_0)\right] = \frac{1}{\sqrt{-(\eta/2) f''(z_0)}} \int_{-\infty}^{\infty} dy \exp\left(-y^2\right),$$

where the substitution

$$y^2 = -\frac{\eta}{2}(z-z_0)^2 f''(z_0)$$

has been made, and one keeps in mind that the path of steepest descent is such that $y^2$ is real and positive. The value of the Gaussian integral is just

$$\int_{-\infty}^{\infty} dy \exp\left(-y^2\right) = \sqrt{\pi},$$

so the final result of the integration is

$$\int_a^b dx \exp[\eta f(x)] = \sum_{\text{saddle points}} \sqrt{\frac{2\pi}{-\eta f''(z_0)}} \exp[\eta f(z_0)].$$

## HOW TO FIND AN APPROPRIATE PATH

The distortion of the path from the original range of the integration to a final one, where the vicinity of the saddle points provide the only significant contribution, is specific to each particular problem.

There are reasonable rules for doing this selection. It is not mysterious.

The key is to examine the behavior of the integrand as the imaginary part of the variable goes to $\infty$.

An example will suffice to illustrate the general procedure.

# Nuclear beta decay induced by intense electromagnetic fields: Basic theory

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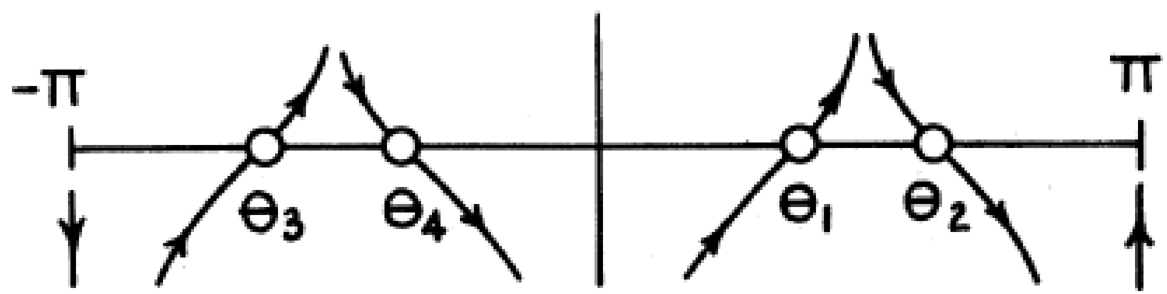


FIG. 1. Path of integration for evaluation of the asymptotic generalized Bessel function.

## PROCEDURE FOR THIS EXAMPLE

In this case the saddle points are on the real axis. That is not usual, but the procedure is independent of that.

Original range of the integration:

$$[-\pi, +\pi]$$

With $z = x + iy$,

$$\eta f(z) \to -\infty \quad \begin{cases} x = -\pi,\ y \to -\infty \\ x = -\pi/2,\ y \to +\infty \\ x = 0,\ y \to -\infty \\ x = +\pi/2,\ y \to +\infty \\ x = +\pi,\ y \to -\infty \end{cases}$$

The selection of path is:

- From the endpoint of the integration at $x = -\pi$, follow a path parallel to the imaginary axis to $y \to -\infty$
- Follow the path of steepest descent from $\theta_3$, starting at $y \to -\infty$ through the saddle point $\theta_3$
- Continue on the path of steepest descent through $\theta_3$ to $y \to +\infty$ and $x = -\pi/2$
- Follow the path of steepest descent from $\theta_4$, starting at $y \to +-\infty$ through the saddle point $\theta_4$
- Continue on the path of steepest descent through $\theta_4$ to $y \to -\infty$ and $x = 0$
- Follow the path of steepest descent from $\theta_1$, starting at $y \to -\infty$ through the saddle point $\theta_1$

- Continue on the path of steepest descent through $\theta_1$ to $y \to +\infty$ and $x = +\pi/2$
- Follow the path of steepest descent from $\theta_2$, starting at $y \to +-\infty$ through the saddle point $\theta_2$
- Continue on the path of steepest descent through $\theta_2$ to $y \to -\infty$ and $x = +\pi$
- From $y \to -\infty$, follow a path parallel to the imaginary axis to the endpoint of the integration at $x = +\pi$

Through periodicity of the integrand in this case, the paths to $y \to -\infty$ along $x = -\pi$ and $x = +\pi$ exactly cancel.

All other segments of the deformed path pass through saddle points, so that the vicinity of the saddle points is always dominant

SIGNIFICANCE OF THE SADDLE POINT METHOD FOR STRONG FIELD PHYSICS

In nonperturbative treatments of strong-field phenomena, almost all of the physics is contained in exponentials. Furthermore, the existence of a strong field usually results in the presence of a large multiplier in the exponential. These two phenomena lead to the nearly universal recourse to saddle-point methods in nonperturbative analytical approximations applied to strong-field problems.