KELDYSH – brief review from the previous lecture

L. V. Keldysh, JETP **47**, 1945 (1964) [Sov. Phys. JETP **20**, 1307 (1965)]

Keldysh is famous in solid-state physics. The introduction in his paper is written from the point of view of solid state physics. It contains almost every misconception that continues to plague strong-field (or high-field) laser physics to the present day.

List of problem concepts from pages 1 and 2 of the Keldysh paper:

- Atomic ionization is basically a tunneling process at low frequencies. *Stated without justification or apparent awareness that lasers do not produce longitudinal fields.*
- There is a transition between tunneling and multiphoton mechanisms that depends only on a specific frequency or an equivalent specific field intensity. *(To be explained further.)*
- There are "direct" transitions to the continuum, or indirect if there is an intervening resonance in the bound system. *This arises from a perturbative point of view.*
- The initial atomic state is Stark-shifted by the applied field. *This is again a perturbative notion, where the structure of quantum transition matrix elements are not inspected from a fundamental measurements point of view (as with S matrices).*
- All physical effects depend on electric field strength. *Laser effects are measured by energies.*

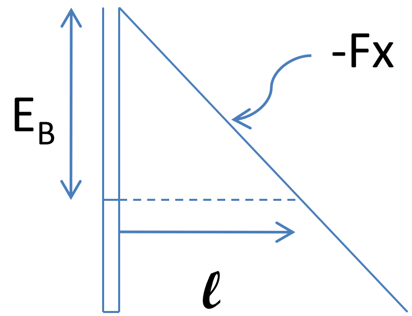
TUNNELING TIME, TUNNELING LENGTH, TUNNELING FREQUENCY

$\ell \approx E_B / F$

$v \approx (E_B)^{1/2}$

$\omega_t \approx F / (E_B)^{1/2}$

$\tau \approx 1 / \omega$



The frequency ω_t is taken to be the dividing line between tunneling ($\omega < \omega_t$) and multiphoton ($\omega > \omega_t$) processes.

There are statements like: "…the tunnel effect…is the most effective mechanism for the absorption of high-power radiation…in the region of frequencies… $\omega < E_B$ …"

Keldysh introduces a gauge-transformed Volkov solution as the wave function for the ionized electron with the statement: *"The difference between our procedure and the usual perturbation theory lies..only..in that we calculate the probability of transition not to a stationary final state, but to a state that takes exact account of the main effect of the electric field – the acceleration of the free electron."*

What is wrong with that statement?

The statement is completely QSE (quasistatic electric field) in nature. The Volkov solution for plane waves has nothing to do with net acceleration; it describes the oscillatory nature of an electron in a transverse PW field.

All of the descriptive statements by Keldysh describe a physical scenario that is unrelated to strong-field laser phenomena.

KELDYSH, continued

Keldysh now evaluates the transition amplitude. The form used is identical to the S-matrix form derived earlier:

$$M_{fi} \equiv (S-1)_{fi} = -i\int_{-\infty}^{\infty} dt(\Psi_f, H_I \Phi_i)$$

The Volkov solution in the LG will be used for the final state, ground-state hydrogen for the initial state, and the interaction Hamiltonian will be in the LG. (Keldysh doesn't bother with normalizing the wave functions until the end result is reached.)

$$\Psi_{VG}^{Volk} = \frac{1}{(2\pi)^{3/2}} \exp\left\{ i\mathbf{p}\cdot\mathbf{r} - \frac{i}{2}\int_{-\infty}^{t} d\tau \left[\mathbf{p} + \frac{1}{c}\mathbf{A}(\tau)\right]^2 \right\},$$

$$\Psi_{LG}^{Volk} = \frac{1}{(2\pi)^{3/2}} \exp\left\{ i\left[\mathbf{p} + \frac{1}{c}\mathbf{A}(t)\right]\cdot\mathbf{r} - \frac{i}{2}\int_{-\infty}^{t} d\tau \left[\mathbf{p} + \frac{1}{c}\mathbf{A}(\tau)\right]^2 \right\}$$

$$\Phi_{1s} = \frac{1}{\sqrt{\pi}} \exp\left(-r + iE_B t\right)$$

$$H_I = \mathbf{r}\cdot\mathbf{F}(t).$$

All dependence on $\boldsymbol{r}$ is in a simple form, which is exactly that of a Fourier transform. Keldysh calls it $V_0(\boldsymbol{p})$ (neglecting multiplicative constants):

$$V_0(\mathbf{p}) \equiv \int d^3r \exp(-i\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}\cdot\mathbf{F}\exp(-r)\,.$$

Keldysh now uses a clever mathematical physics trick. The integral to be evaluated is over d^3r, and the integrand contains projections of $\boldsymbol{r}$ onto both $\boldsymbol{p}$ and $\boldsymbol{F}$. The final result will then depend upon how $\boldsymbol{p}$ and $\boldsymbol{F}$ are oriented with respect to each other.

Keldysh gives the result in the form (neglecting multiplicative constants):

$$V_0(\mathbf{p}) \sim \mathbf{F}\cdot\nabla_{\mathbf{p}}\left(\frac{1}{(1+p^2)^3}\right),$$

where the result of the gradient operator with respect to $\boldsymbol{p}$ is in the direction of $\boldsymbol{p}$. (Details of evaluation of the integral are left to the reader.) Note that atomic units are used here, whereas Keldysh does not use a.u.

Looking back to the way in which $\boldsymbol{p}$ appears in the S matrix, it is always in the form

$$\boldsymbol{p} \to \boldsymbol{p} + \frac{1}{c}\boldsymbol{A}(\ t).$$

This means that $V_0\,(\boldsymbol{p})$ should be replaced by $V_0\,(\boldsymbol{p}+\boldsymbol{A}/c)$.

TRANSITION PROBABILITY, TRANSITION RATE

Keldysh forms a transition probability from the square of the transition amplitude at time $T \to \infty$, where T marks the time at which the experiment ends. He also starts the process at $t = 0$. These steps work, but they are not as rigorous as S-matrix techniques. In S-matrix procedures, there is always a reference to how measurements are actually made in the laboratory. Cutoffs in the experimental time and space parameters occur naturally by noting that measurements are made outside the space-time domain in which the system transition takes place.

It is also more usual in S-matrix work to calculate a transition rate (transition probability per unit time), rather than the transition probability itself.

As noted earlier, Keldysh employs some concepts that are alien to strong-field physics. For example, he refers to perturbation theory as if he were doing something of that nature (he isn't); he speaks of "direct transition to the continuum" as if he was omitting necessary intermediate steps (he isn't), and he speaks of the Volkov solution as representing the "…main effect of the electric field – the acceleration of the free electron." This statement appears to mean that he believes he is calculating the effects of a QSE field, which is a misrepresentation of what laser fields do.

MISUNDERSTANDING THE VOLKOV SOLUTION

Misunderstanding the Volkov solution is so widespread that it is difficult to make the point. Not only Keldysh, but almost all the Russians currently in the field make the same mistake. In fact, very few people are even aware that there IS a problem.

Consider the Schrödinger equation

$$i\partial_t \Psi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + V(r) + \mathbf{r}\cdot\mathbf{F}(t)\right]\Psi(\mathbf{r},t), \tag{1}$$

where $\mathbf{F}(t)$ is a periodic function of time. There is only one preferred direction in this problem, and that is the direction of the electric field $\mathbf{F}(t)$. There is only a scalar potential in this equation, and scalar potentials by themselves can describe only QSE fields. This is the Schrödinger equation for a QSE field; nothing more. It does **not** describe PW fields.

LONGITUDINAL AND TRANSVERSE FIELDS

In the $\boldsymbol{r} \cdot \boldsymbol{F}$ interaction Hamiltonian, a one-dimensional electric field is embedded in a 3-dimensional space. There can be only one direction as far as the field is concerned, so if there is any propagation, it has to be parallel (or anti-parallel) to the vector $\boldsymbol{F}$.

That is why the field is called a *longitudinal field* even when the field is static, so that there is no propagation direction. The static-field case is the <u>perfect</u> longitudinal field.

Transverse fields are those where the direction of propagation is perpendicular to the fields. The plane wave is the <u>perfect</u> transverse field.

Longitudinal and transverse fields are as opposite in character as fields can be.

A proper Volkov solution can be formulated only within the Dirac or Klein-Gordon equation. It describes the motion of a free electron in a plane-wave field. For this description, a vector potential $\mathbf{A}(\mathbf{r}, t)$ is employed that has the direction of the electric field, but depends upon the phase $\omega t - \mathbf{k} \cdot \mathbf{r}$. The direction of $\mathbf{k}$ is perpendicular to the electric field, and $\nabla \times \mathbf{A}$ gives the magnetic field that is perpendicular to both $\mathbf{F}$ and $\mathbf{k}$. A nonrelativistic Volkov solution is a strange object, since a PW field propagates with the velocity of light, and so one must consider that a charged particle coupled to this field may also be relativistic. An even stranger concept is a dipole-approximation Volkov solution, since a free particle has no bounds on its motion. Nevertheless, one can justify a nonrelativistic, dipole-approximation Volkov solution based on the limitations imposed when it is employed within a matrix element that is nonrelativistic and suited to the dipole approximation.

$$i\partial_t \Psi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + V(r) + \mathbf{r}\cdot\mathbf{F}(t)\right]\Psi(\mathbf{r},t)\,, \tag{1}$$

Equation (1) does not support a Volkov solution. When the dipole-approximation nonrelativistic Volkov solution is gauge-transformed to the LG, it is written in terms of a vector potential that does not exist in the LG. The schemes by which people persuade themselves that they have a Volkov solution in the LG (nonlocal representation in terms of the electric field; replacement of the vector potential amplitude by an electric field amplitude as a disguise) are schemes that delude them into thinking that they have an LG object. Thus, Keldysh (and many others) speak of the "Volkov solution" as describing the QSE acceleration of an electron in a QSE field, and so on. *This is a dangerous delusion. It leads to the near-universal and completely wrong conclusion that there is no difference between a tunneling problem and a Volkov-solution SFA problem.*

This was stated (without sufficient emphasis apparently) twenty years ago: HRR, Phys. Rev. A **42**, 1476 (1990); and many other times since.

CONTINUE WITH KELDYSH

Adopt his procedure of cutting off the interaction at time *T*, and then letting $T \to \infty$. However, we can replace his $t = 0$ with $t \to -\infty$. Also, the initial binding energy of the electron is E_B rather than I_0; what Keldysh calls $\boldsymbol{F}$ is what we call $\boldsymbol{F}_0$; and we use a.u.

$$w_0 = \frac{1}{(2\pi)^3} \lim_{T\to\infty} \mathrm{Re} \int d^3p \int_{-\infty}^{T} dt \cos\omega T \cos\omega t V_0 \left(\mathbf{p} + \frac{1}{c}\mathbf{A}(T)\right) V_0 \left(\mathbf{p} + \frac{1}{c}\mathbf{A}(t)\right)$$
$$\times \exp\left\{-i \int_t^T d\tau \left[E_B + \frac{1}{2}\left(\mathbf{p} + \frac{1}{c}\mathbf{A}(\tau)\right)^2\right]\right\}$$

The integral d^3p comes from an integration over phase space of all free-particle solutions; $(2\pi\hbar)^3$ is the volume of a cell in $\boldsymbol{r}$, $\boldsymbol{p}$ phase space.

It is generally useful to extract delta-function behavior from expressions of this type. To do this, take the t – dependent part, and expand in a Fourier series in t :

$$L(\mathbf{p},t) = V_0\left(\mathbf{p}+\frac{1}{c}\mathbf{A}_0\sin\omega t\right)\exp\left\{i\int^t d\tau\left[E_B+\frac{1}{2}\left(\mathbf{p}+\frac{1}{c}\mathbf{A}_0\sin\omega\tau\right)^2\right]\right\}$$

$$E_B+\frac{1}{2}\left(\mathbf{p}+\frac{1}{c}\mathbf{A}_0\sin\omega\tau\right) = E_B+\frac{p^2}{2}+\frac{\mathbf{p}\cdot\mathbf{A}_0}{2c}\sin\omega\tau+\frac{A_0^2}{2c^2}\frac{1}{2}\left(1-\cos 2\omega\tau\right)$$

$$\int^t d\tau\left[E_B+\frac{1}{2}\left(\mathbf{p}+\frac{1}{c}\mathbf{A}_0\sin\omega\tau\right)^2\right] = \left(E_B+\frac{p^2}{2}+\frac{A_0^2}{4c^2}\right)t-\frac{\mathbf{p}\cdot\mathbf{A}_0}{2c\omega}\cos\omega t-\frac{A_0^2}{8c^2\omega}\sin 2\omega t$$

$$L(\mathbf{p},t) = \sum_{n=-\infty}^{\infty}\exp\left[i\left(E_B+\frac{p^2}{2}+\frac{A_0^2}{4c^2}-n\omega\right)t\right]L_n(\mathbf{p}),$$

$$L_n(\mathbf{p}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}dx V_0\left(\mathbf{p}+\frac{1}{c}\mathbf{A}_0\sin x\right)\exp\left[\frac{i}{\omega}\left(n\omega x-\frac{\mathbf{p}\cdot\mathbf{A}_0}{2c\omega}\cos x-\frac{A_0^2}{8c^2\omega}\sin 2x\right)\right]$$

Note that *exp(-inωt)* is inserted into *L(**p**, t)*, but this is compensated for by the factor *exp [(i/ω) nωt]* in L_n *(**p**)*.

*L(**p**, t)* is in the form of a standard Fourier series in *t*.
L_n *(**p**)* is independent of *t*.

Another thing to observe is that

$$\int_t^T = \int_{-\infty}^T - \int_{-\infty}^t$$

This means that the expression for w_0 is exactly the product of

$$L(\mathbf{p}, t), \quad L^*(\mathbf{p}, T),$$

Furthermore, each $L(\boldsymbol{p},t)$ is multiplied by

$$\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

so that the end result will be of the form

$$\left|L_{n+1}(\mathbf{p}) + L_{n-1}(\mathbf{p})\right|^2$$

These terms can be combined by shifting the origin of the sum over n in opposite directions for the two terms so that there is a common value of n appearing in the delta functions.

The result is

$$w_0 = \frac{1}{(2\pi)^2} \int d^3p \, |L(\mathbf{p})|^2 \sum_{n=-\infty}^{+\infty} \delta\left(E_B + \frac{p^2}{2} + \frac{A_0^2}{4c^2} - n\omega\right)$$

Further manipulation involves introducing an integral representation for *L(**p**)* in terms of poles in a complex integration, and then doing a steepest descent calculation to get a final result.

The delta function indicates that n photons supply enough energy to provide the binding energy E_B, the kinetic energy $p^2/2$, and the ponderomotive energy $(A_0/2c)^2$.

This much is the same as VG results; not surprising because there was a gauge transformation to VG introduced. However, there are many differences.

VG, LG DIFFERENCES

Despite the fact that there is a gauge transformation involved, and gauge transformations preserve certain physically measurable quantities, the approximation of neglecting Coulomb corrections to the Volkov solution has a different meaning in the two gauges.

Physical interpretations are *very* different, starting with the fact that LG suggests tunneling and VG does not.