CONDITIONS IMPOSED BY KELDYSH

"The exponential [in the integrand] is rapidly oscillating … the integral can be calculated by the saddle-point method…"

This condition is: *Binding energy + Kinetic energy >>* $\hbar\omega$
This requires low frequencies. (Keldysh regards this as the key approximation, but it is the assumption of a scalar field that is most basic.)

"… contributions to the total probability of ionization are made only by small *p* satisfying the condition …" *Kinetic energy << Binding energy*
This is why tunneling momentum distributions and spectra are too narrow.

"… for very low frequencies and very strong fields, when $\gamma << 1$, the main contribution … is for large $n \sim 1/\gamma^3$ …"
Interesting, but yet another behavior unrelated to plane-waves.

ASSESSMENT TO THIS POINT

• Gauge choice is consequential because LG is limited to scalar fields.
• Theoretical methods are of two basic types: Analytical approximations and direct numerical solution of the equations of motion.
• Numerical solutions (TDSE) have difficulty with very low frequencies and with polarizations other than linear. Numerical treatment of the Dirac equation (i.e., relativistic equation of motion) is limited mostly to one dimension, with some two-dimensional results.
• Analytical approximations can be categorized as three-fold:

1. Tunneling theories.
2. Volkov-solution-based in the length gauge.
3. Volkov-solution-based in the velocity gauge. (The SFA)

The "SFA" terminology has become completely corrupted in meaning.
This has now become of basic importance.

GAUGES: WHY THEY HAVE BECOME OF SUCH BASIC IMPORTANCE

There is an almost universal attitude that all gauges give identical results, so any discussion of gauges is irrelevant, or wrong-headed, or an academic matter of no real significance. ***This is wrong!***

Use of the length gauge (LG) is the most serious problem because it is not a general gauge. It limits all discussion to scalar fields.

(There are other fundamental misunderstandings about gauges, based on the fact that potentials provide more information than do the electric and magnetic fields alone. This is a matter that is novel and controversial, but nevertheless of basic importance when fields are very strong.)

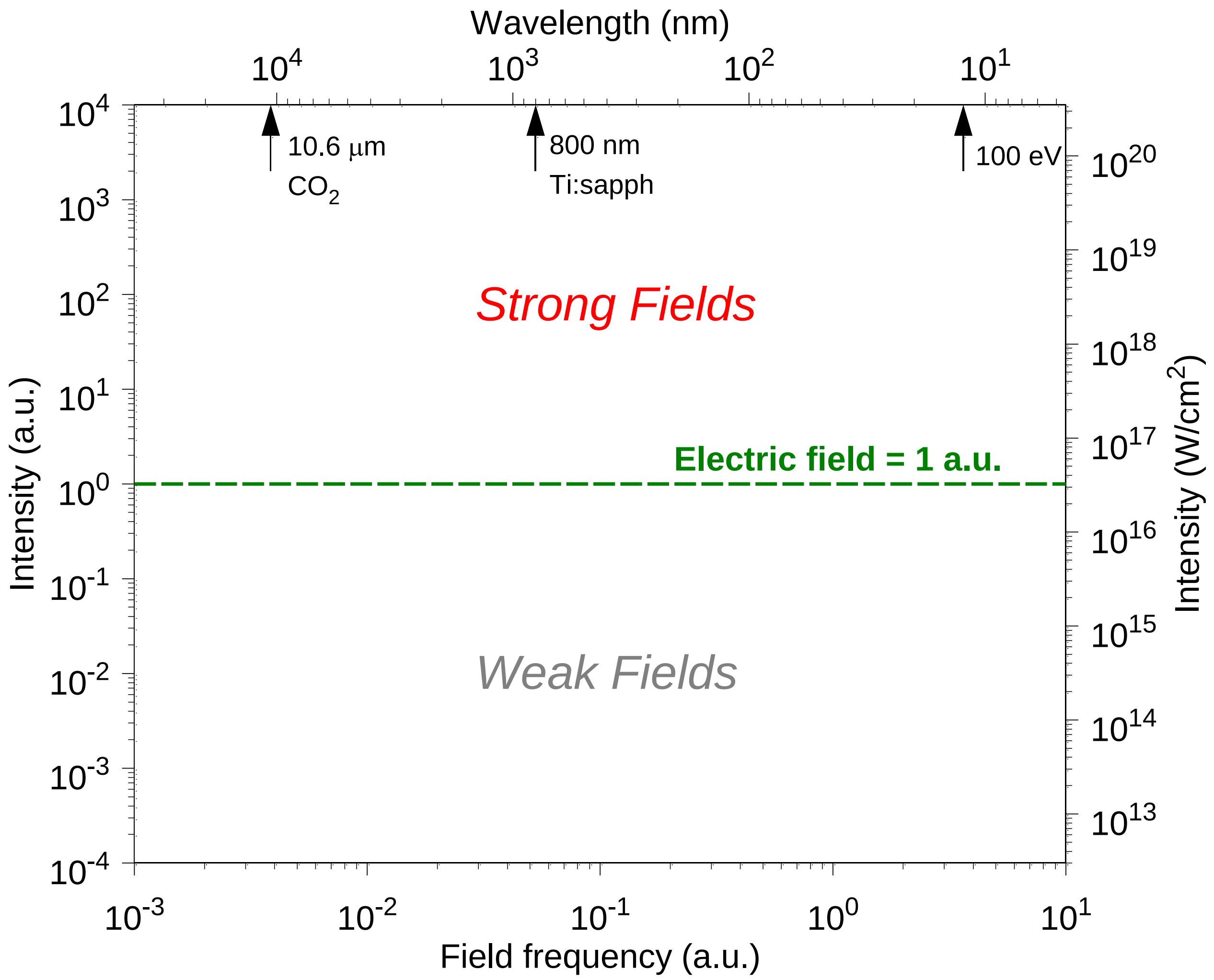
QUALITATIVE LENGTH-GAUGE AND COULOMB-GAUGE COMPARISONS

"Length gauge" limits discussion to static and quasistatic electric fields. An equivalent statement is that there is a limitation to scalar fields.

"Coulomb gauge" is universally applicable, but it is here confined to representation of plane-wave behavior.

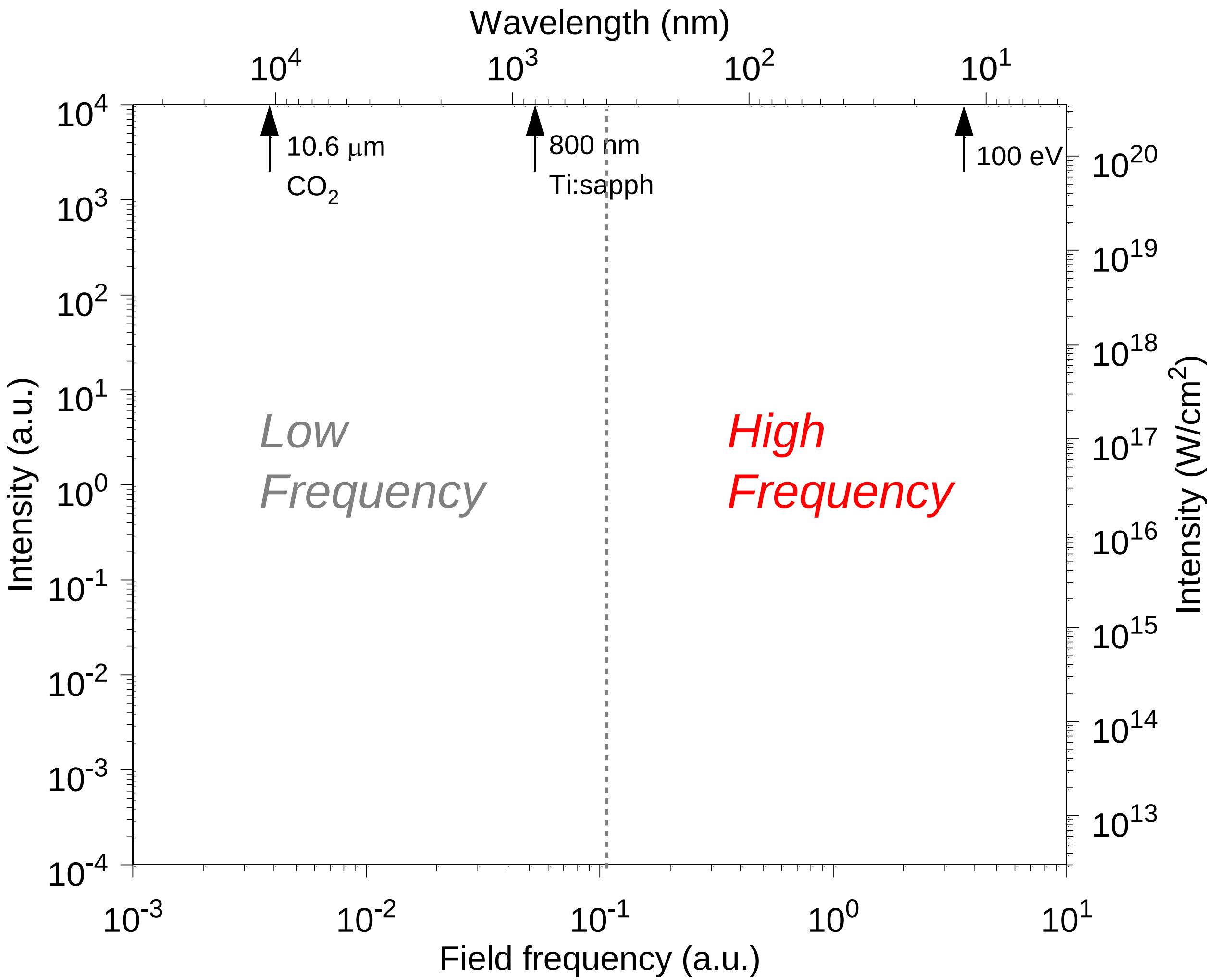
Qualitative descriptions are completely different in the two gauges, with the length-gauge description being very elementary, and nearly unrelated to plane-wave behavior as viewed in the Coulomb gauge.

LASER PHYSICS VIEWED FROM THE LENGTH GAUGE: FIELD INTENSITY

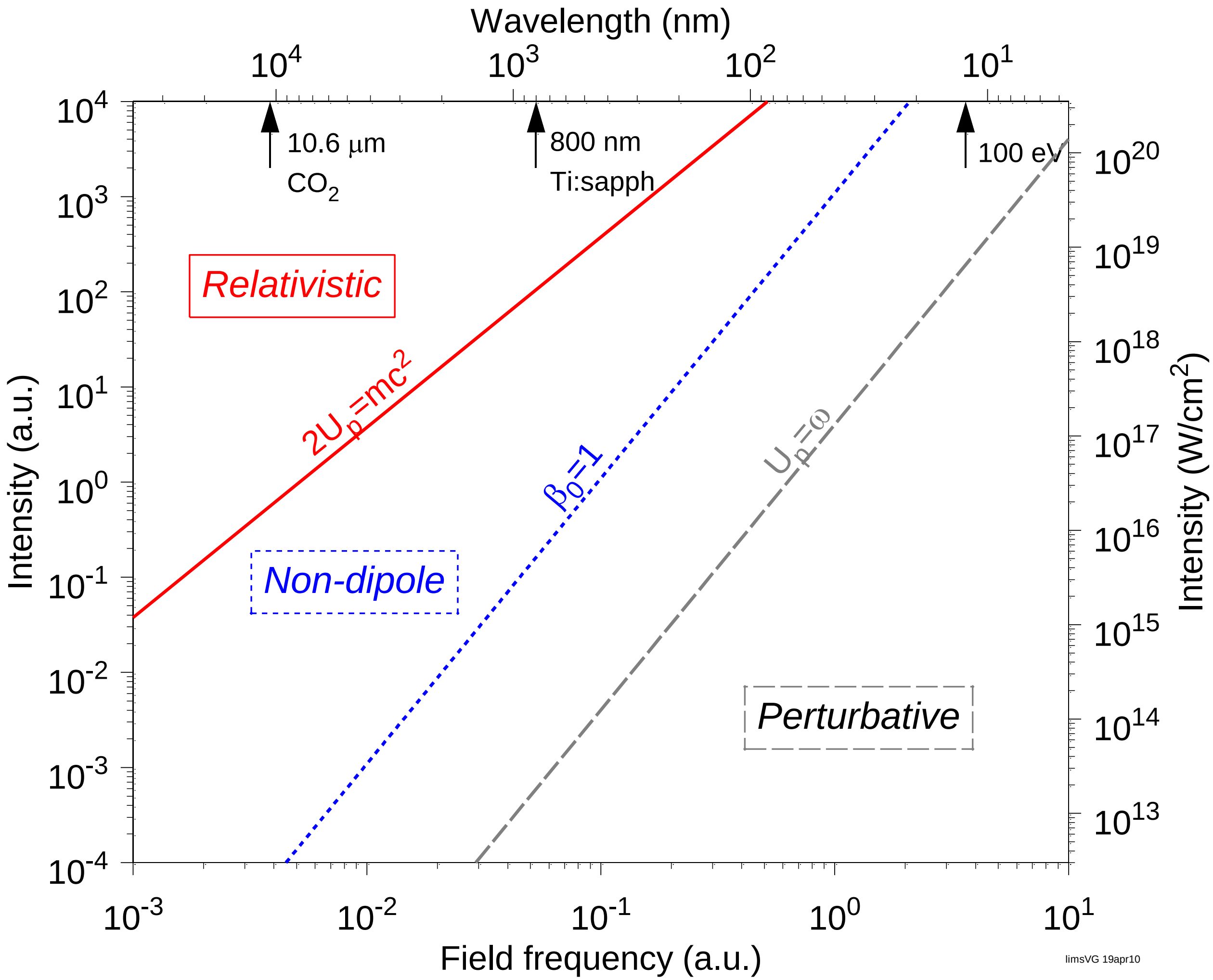


LASER PHYSICS VIEWED FROM THE LENGTH GAUGE: FIELD FREQUENCY

The boundary between high and low should really be taken to be E_B .



LASER PHYSICS VIEWED FROM THE COULOMB GAUGE

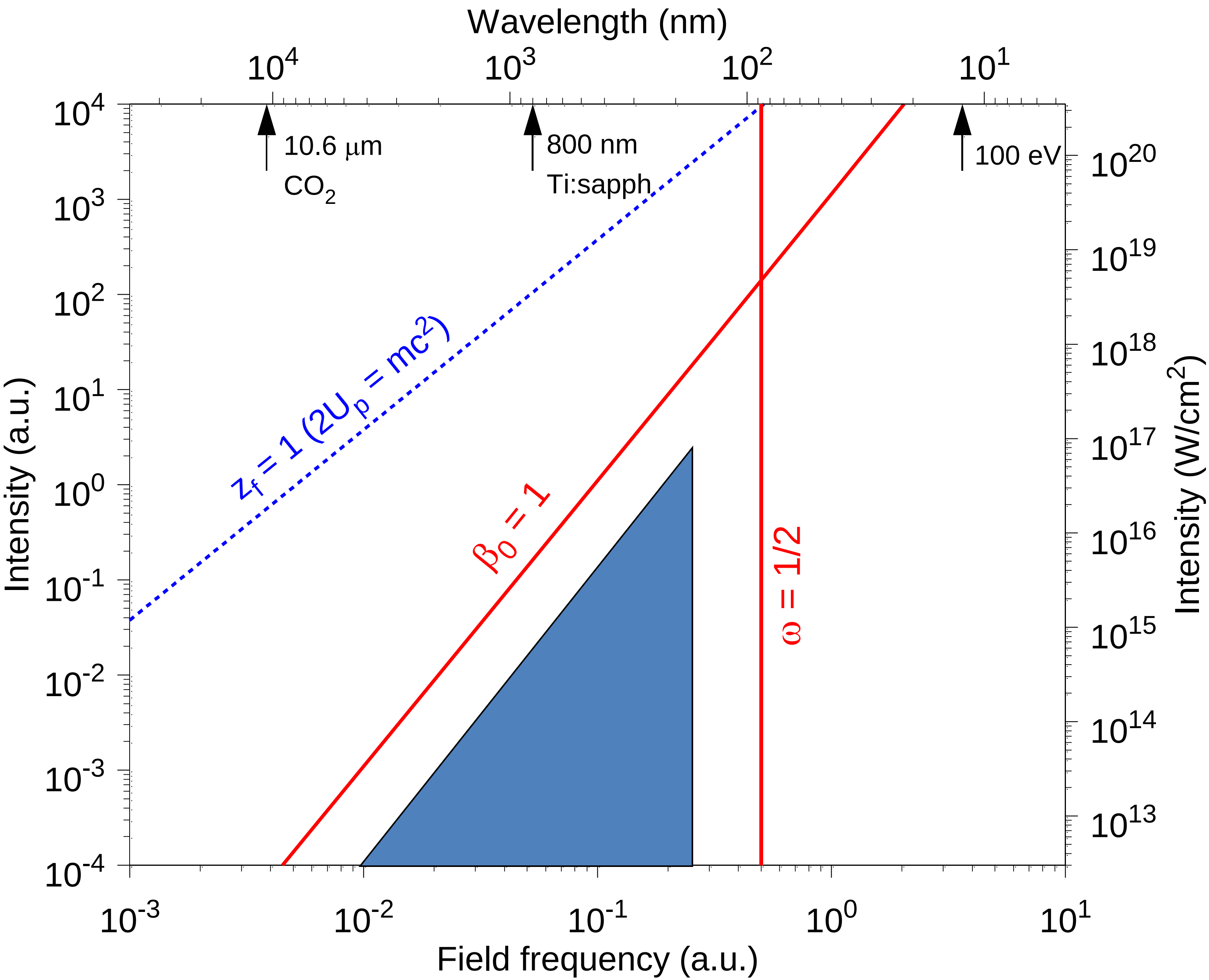


CONNECTION BY A GAUGE TRANSFORMATION

There is only a minor overlap region where nominal "gauge equivalence" exists, but even gauge equivalence pertains to specific quantities that are preserved in a gauge transformation.

Gauge equivalence applies only to the shaded area in the next slide.
HRR, PRL **101**, 043002 (2008).

Physical interpretations are strongly gauge-dependent.
This means that qualitative judgments are strongly gauge-dependent.


Wavelength (nm)
10^4
10^3
10^2
10^1
10.6 μm
CO2
800 nm
Ti:sapph
100 eV
zf = 1 (2Up = mc2)
β0 = 1
ω = 1/2
Intensity (a.u.)
Intensity (W/cm2)
10^20
10^19
10^18
10^17
10^16
10^15
10^14
10^13
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-3
10^-2
10^-1
10^0
10^1
Field frequency (a.u.)

NUMERICAL STRONG-FIELD THEORIES

TDSE (Numerical solution of the time-dependent Schrödinger equation).
Even here there are different perceptions based (usually) on academic background.
Theoretical chemists: <u>Harm Geert Muller, Andre Bandrauk</u>
Physicists: Ken Kulander, <u>Ken Taylor</u>, Dieter Bauer, Ken Shafer, + many others.

Low frequencies present a (*slowly moving*) barrier.

Another barrier is the perception by practitioners that everything that they do is <u>**exact**</u>, and so there is no reason to take note of analytical <u>**approximations**</u>.

Relativistic plasma formation.
Almost entirely numerical. Not treated further here.
Numerical solution of the Dirac equation.
Mostly from the Christoph Keitel group in Heidelberg; one- and two-dimensional.

TUNNELING ANALYTICAL APPROXIMATIONS

Tunneling approaches are dominated by the Russian effort:

L. V. Keldysh (1964); first and best known (starts with Volkov, but proceeds early to tunneling)

Nikishov and Ritus (1966); PPT before PPT; (very hard to read, even by Russian standards)

Vladimir Popov (PPT) (1966); basic tunneling (Popov is the principal exponent to the present)

Delone and Krainov (1982); (PPT-style theory with emphasis on practical application)

Others:

Kuchiev, Zaretsky, Goreslavsky, Manakov, Popruzhenko, …

VOLKOV-BASED ANALYTICAL APPROXIMATIONS

Relativistic

HRR (1962); noticed by Academician Ginzburg, of the Lebedev Institute, started the Russian effort

Nikishov and Ritus (1964); first Russian work

Brown and Kibble (1964); best-known in the West

Nonrelativistic

Faisal (1973); direct-time S matrix, high-frequency approximation

HRR (1980); basis of subsequent SFA work

Corkum (1993); semiclassical, assumes tunneling

Lewenstein (1994); application to HHG, "length-gauge SFA" – a misnomer

There is almost universal – and very damaging – confusion about the meaning of the SFA. This is current research that will be addressed later.

PPT

A. M. Perelomov, V. S. Popov, & M. V. Terent'ev, Sov. Phys. JETP **23**, 924 (1966).
A. M. Perelomov & V. S. Popov, Sov. Phys. JETP **24**, 207 (1967); **25**, 336 (1967).
V. S. Popov, Phys. Usp. **47**, 855 (2004).

Keldysh is given credit as the pioneer of tunneling methods, but PPT derive the method directly (rather than starting with a Volkov solution), and provide the basis for current tunneling work. ADK (Ammosov, Delone, Krainov), the most commonly used method, is based on PPT.

PPT start with a one-dimensional model. This is a good place for us to start.

Short-range (delta-function) potential, 1-D.

IONIZATION BY A CONSTANT ELECTRIC FIELD

(w = transition probability, F = electric field strength):

J. R. Oppenheimer, Phys. Rev. **31**, 66 (1928).

$$w_{stat} = \omega_0 \exp\left(-2F_0/3F\right)$$

$$\omega_0 = \kappa^2/2 = E_B$$

$$F_0 = \kappa^3 = (2E_B)^{3/2}$$

IONIZATION BY AN ALTERNATING ELECTRIC FIELD:

Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} - \kappa\delta\left(x\right) - f\left(t\right)x\right]\psi.$$

Initial condition, $t = t_0$ (zero-field stationary bound state):

$$\psi\left(x, t_0\right) = \sqrt{\kappa}\exp\left(-\kappa\left|x\right| + i\kappa^2 t/2\right).$$

To verify the solution without the oscillatory field:

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi\left(x, t_0\right) = -\frac{\kappa^2}{2}\psi\left(x, t_0\right); \quad x \neq 0$$
$$i\frac{\partial}{\partial t}\psi\left(x, t_0\right) = -\frac{\kappa^2}{2}\psi\left(x, t_0\right)$$

To examine what happens in the vicinity of $x = 0$, write $\left|x\right|$ as

$$\left|x\right| = \left[2\theta\left(x\right) - 1\right]x,$$

where $\theta\left(x\right)$ is the step function

$$\theta\left(x\right) = \begin{cases} 1, \; x > 0 \\ 1/2, \; x = 0 \\ 0, \; x < 0 \end{cases}.$$

Omitted in the $\partial^2/\partial x^2$ differentiation was the action on $\left[2\theta\left(x\right) - 1\right]$.

$$\frac{\partial}{\partial x}\left[2\theta\left(x\right) - 1\right] = 2\theta'\left(x\right) = 2\delta\left(x\right)$$

If this is followed for the two $\partial/\partial x$ operations, the additional term $-\kappa\delta\left(x\right)$ is the result.

IMAGINARY TIME AND OTHER REMARKS

Damping under the barrier can be reinterpreted as an imaginary time because of the appearance of the energy exponential.

This is not a real time; it serves as a shorthand way to express a transition rate. Unfortunately, the language used by PPT and others gives the impression that they are talking about an actual physical time under the barrier. This seems not to be the case.