NONRELATIVISTIC STRONG-FIELD APPROXIMATION (SFA)

Note: "SFA" will automatically be taken to mean Coulomb gauge (relativistic or non-dipole) or VG (nonrelativistic, dipole-approximation). If LG is intended (rarely), it will be explicitly identified.

Start with the time-reversed S matrix (temporarily introduce the "hat" to distinguish operators from eigenvalues):

$$M_{fi} = (S-1)_{fi} = -i \int_{-\infty}^{\infty} dt \left(\Psi_f(\mathbf{r},t), \widehat{H}_I(t)\, \Phi_i(\mathbf{r},t) \right)$$

Approximate the fully interacting final state by a Volkov solution:

$$\Psi_f \to \Psi_f^{Volk}(\mathbf{r},t) = \frac{1}{V^{1/2}} \exp\left\{ i\left[\mathbf{p}\cdot\mathbf{r} - \frac{1}{2}p^2 t - \int_{-\infty}^{t} d\tau H_I(p,\tau) \right] \right\}$$

$$H_I(p,t) = \mathbf{A}(t)\cdot\mathbf{p} + \frac{1}{2}A^2(t)$$

Within the matrix element, the interaction Hamiltonian operator

$$\widehat{H}_I(t) = \mathbf{A}\cdot\left(-i\widehat{\nabla}\right) + \frac{1}{2}A^2$$

has the Volkov solution as an eigenvector:

$$\widehat{H}_I(t)\,\Psi_f^{Volk} = H_I(\mathbf{p},t)\,\Psi_f^{Volk}$$

For the sake of simplicity, one detail of S-matrix theory has been overlooked. Because of the time-reversed nature of the S matrix, the "in-state" that was used for the Volkov solution should actually be replaced by an "out-state".
(The final answer is actually unchanged if we ignore this distinction.)
Rather than using a Volkov solution that proceeds from $t \rightarrow -\infty$ to the "current" or "laboratory time" t , it is necessary to trace the Volkov solution backwards from $t \rightarrow +\infty$ to the laboratory time t. This amounts to the replacement of

$$\Psi_f^{(+)Volk}(\vec{r},t) = \frac{1}{V^{1/2}} \exp\left\{i\left[\vec{p}\cdot\vec{r} - \frac{1}{2}p^2 t - \int_{-\infty}^{t} d\tau H_I(p,\tau)\right]\right\}$$

by

$$\Psi_f^{(-)Volk}(\vec{r},t) = \frac{1}{V^{1/2}} \exp\left\{i\left[\vec{p}\cdot\vec{r} - \frac{1}{2}p^2 t + \int_{t}^{\infty} d\tau H_I(p,\tau)\right]\right\}$$

The transition amplitude is in a compact form:

$$M_{fi} = -i\int_{-\infty}^{+\infty} dt H_I(\vec{p},t)\left(\Psi_f^{(-)}(\vec{r},t),\Phi_i(\vec{r},t)\right).$$

When the initial atomic state is written in stationary-state form, it becomes possible to separate space and time parts.

$$\Phi_i\left(\vec{r},t\right)=\phi_i\left(\vec{r}\right)\exp\left(-iE_it\right)$$

All spatial dependence comes from $\phi_i\left(\vec{r}\right)$ and from *exp(i**p·r**)* in the Volkov function.

This gives the bound-state wave function in momentum space:

$$\tilde{\phi}_i\left(p\right)=\left(\exp\left(i\vec{p}\cdot\vec{r}\right),\phi_i\left(\vec{r}\right)\right)$$

Remaining factors constitute the time-dependent part.

$$M_{fi}^{SFA}=-\frac{i}{V^{1/2}}\,\tilde{\phi}_i\left(p\right)\int_{-\infty}^{+\infty}dt\exp\left[i\left(\frac{p^2}{2}-E_i\right)t\right]H_I\left(p,t\right)\exp\left[i\int_t^{\infty}d\tau H_I\left(p,\tau\right)\right],$$

This will turn out to be a very important feature of the SFA.
The momentum-space wave function of the initial atomic state plays a vital role in the final result.
This property gives basic information about photoelectron momentum distributions that is not to be found in the LG.

There is an efficient way to do this that maintains a normalized amplitude:

$$\vec{A}(t) = \frac{1}{2} A_0 \left(\vec{\varepsilon} e^{i\omega t} + \vec{\varepsilon}^* e^{-i\omega t} \right); \quad \vec{\varepsilon}^2 = \vec{\varepsilon}^{*2} = 0; \quad \vec{\varepsilon} \cdot \vec{\varepsilon}^* = 1.$$

If the laser field propagates in the direction of the z axis, then

$$\vec{\varepsilon} = \frac{1}{2^{1/2}} \left(\hat{x} \mp i\hat{y} \right),$$

where $\hat{x}, \hat{y}$ are unit vectors in the directions of the x and y axes.
The interaction term is

$$H_I(p,t) = -\zeta_c \omega \cos(\omega t - \varphi) + z\omega$$

where the phase angle φ is defined such that ζ is real and

$$\zeta_c = \frac{A_0 \vec{p} \cdot \vec{\varepsilon} \exp(i\varphi)}{c\omega},$$

$$z = \frac{A_0^2}{4\omega c^2} = \frac{U_p}{\omega}$$

The Volkov exponential contains a trigonometric function.
This is a sure indicator of Bessel function behavior.
The generating function for Bessel functions can be introduced in the form

$$\exp\left[-i\zeta_c \sin(\omega t - \varphi)\right] = \sum_{n=-\infty}^{\infty} J_n(\zeta_c) \exp\left[-in(\omega t - \varphi)\right]$$

The time-dependent factor in the S matrix now has the form

$$\int dt... = \sum_n J_n(\zeta_c) \int dt \exp\left[i\left(p^2/2 - E_i + z\omega\right)\right]$$

$$\times \left[ze^{-in(\omega t - \varphi)} - (\zeta_c/2) e^{-i(n-1)(\omega t - \varphi)} - (\zeta_c/2) e^{-i(n+1)(\omega t - \varphi)} \right]$$

A common exponential factor can be achieved if the origin of the sum over the infinite range of the *n* index is shifted by ±1. The result is

$$\int dt... = \sum \int dt \exp\left[i\left(p^2/2 - E_i + z\omega\right) - in(\omega t - \varphi)\right] \times \left[zJ_n - (\zeta_c/2)(J_{n+1} + J_{n-1}) \right]$$

A Bessel function recursion relation can now be used to give a common Bessel function factor:

$$J_{n+1}(\zeta_c) + J_{n-1}(\zeta_c) = \frac{2n}{\zeta_c} J_n(\zeta_c)$$

The time integral is now

$$\int dt \ldots = \sum J_n(\zeta_c) \exp(in\varphi)(n-z) 2\pi\delta\left(p^2/2 - E_i - n\omega + z\omega\right).$$

The energy E_i is for an initial bound state, so it is negative. It can be replaced by a positive quantity, sometimes called ionization potential and represented either by *IP* or by I_p. The preferred notation here is the binding energy E_B.

$$E_i \rightarrow -E_B$$

The delta function from the time integration can be written as the conservation condition

$$\frac{p^2}{2} = n\omega - E_B - U_p$$

In words, the conservation condition is:

*kinetic energy of the photoelectron = energy supplied by n photons
minus energy required to overcome the binding energy
minus energy required for interaction energy of the free electron with the field*

A basically important point, almost never mentioned in laser physics, is that the need to supply ponderomotive energy to maintain a physical free electron immersed in a strong plane-wave field, cannot be found in any finite order of perturbation theory.

See:
J. H. Eberly and HRR, Phys. Rev. **145**, 1035 (1966).
HRR and J. H. Eberly, Phys. Rev. **151**, 1058 (1966).
The first paper deduced the result from the finite exact sum of an infinite series of divergent Feynman diagrams.
The second paper (actually the first chronologically) showed the result by direct analytical means as a finite mass renormalization. This, in turn, was the outcome of the mass-shift effect explored in two HRR papers of 1962.

Substitution of the result of the time-dependent factor into the full S-matrix expression gives

$$M_{fi}^{SFA} = \frac{2\pi i}{V^{1/2}} \tilde{\phi}_i(p)\omega\sum(n-z)J_n(\zeta_c)\exp(in\varphi)\delta\left(\frac{1}{2}p^2 + E_B + U_p - n\omega\right)$$

$$\omega(n-z) = n\omega - U_p = \frac{1}{2}p^2 + E_B$$

Transition probability is $|M_{fi}^{SFA}|^2$, transition rate is

$$w = \lim_{t\to\infty}\frac{1}{t}\left|M_{fi}^{SFA}\right|^2.$$

The total transition rate requires an integration over the phase space of all possible final states, divided by $(2\pi\hbar)^3$, the volume of a unit cell in the phase space.

$$W = \int\frac{wVd^3p}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int wVp^2dpd\Omega,$$

where Ω is the solid angle into which the photoelectron is emitted.

$$\frac{dW}{d\Omega} = \frac{1}{(2\pi)^2} \sum_n \int p^2 dp \left(\frac{p^2}{2} + E_B \right)^2 \left| \tilde{\phi}_i (p) \right|^2 J_n^2 \left(\alpha_0^{cir} p \sin\theta \right) \delta \left(\frac{p^2}{2} + E_B + U_p - n\omega \right),$$

$$\alpha_0^{cir} = \left(\frac{2z}{\omega} \right)^2 = \frac{\left(2U_p \right)^{1/2}}{\omega} = \frac{1}{\omega^2} \left(\frac{I}{2} \right)^{1/2}$$

α_0^{cir} is the classical radius of the circular orbit of an electron in a circularly polarized field, as expressed in several ways. The angle θ is the angle in which the photoelectron is emitted with respect to the propagation vector of the laser field.

There is a lower limit on the sum over n that follows from the necessity that the final kinetic energy has to be a positive number.

$$\frac{p^2}{2} = n\omega - E_B - U_p \geq 0 \quad \Rightarrow \quad n \geq n_0 \equiv \left[\frac{E_B + U_p}{\omega} \right],$$

where the square bracket means the "smallest integer containing" the quantity in the bracket.

The delta function can be used to perform the integration over *p*. This can be done using the relation

$$\delta\left(\frac{p^2}{2}+E_B+U_p-n\omega\right)=\frac{1}{\sqrt{2}\sqrt{n\omega-E_B-U_p}}\delta\left(p-\sqrt{2}\sqrt{n\omega-E_B-U_p}\right)$$

The final result for the total ionization (or photodetachment) rate by a circularly polarized laser field is

$$\frac{dW}{d\Omega}=\frac{1}{(2\pi)^2}\sum_{n=n_0}^{\infty}p\left(\frac{p^2}{2}+E_B\right)^2\left|\tilde{\phi}_i(p)\right|^2 J_n^2\left(\alpha_0^{cir}p\sin\theta\right).$$

$$p=\sqrt{2\left(n\omega-E_B-U_p\right)}$$

This is an analytically simple result that gives remarkably good agreement with experimental measurements.

The analytical nature of the approximation is elementary:

$$M_{fi}^{exact} \sim \left(\Psi_f^{exact}, H_I \Phi_i \right) \quad \rightarrow \quad M_{fi}^{SFA} \sim \left(\Psi_f^{Volkov}, H_I \Phi_i \right)$$

The statement amounts to saying that the effects of the laser field on the detached electron dominate the residual effects of the binding potential.

In quantum mechanics, everything is judged in terms of energies and not in terms of field strengths. $H = T + V$ and all that kind of thing. That is, the Schrödinger equation is an energy equation. SFA validity is assured if

$$U_p \gg E_B$$

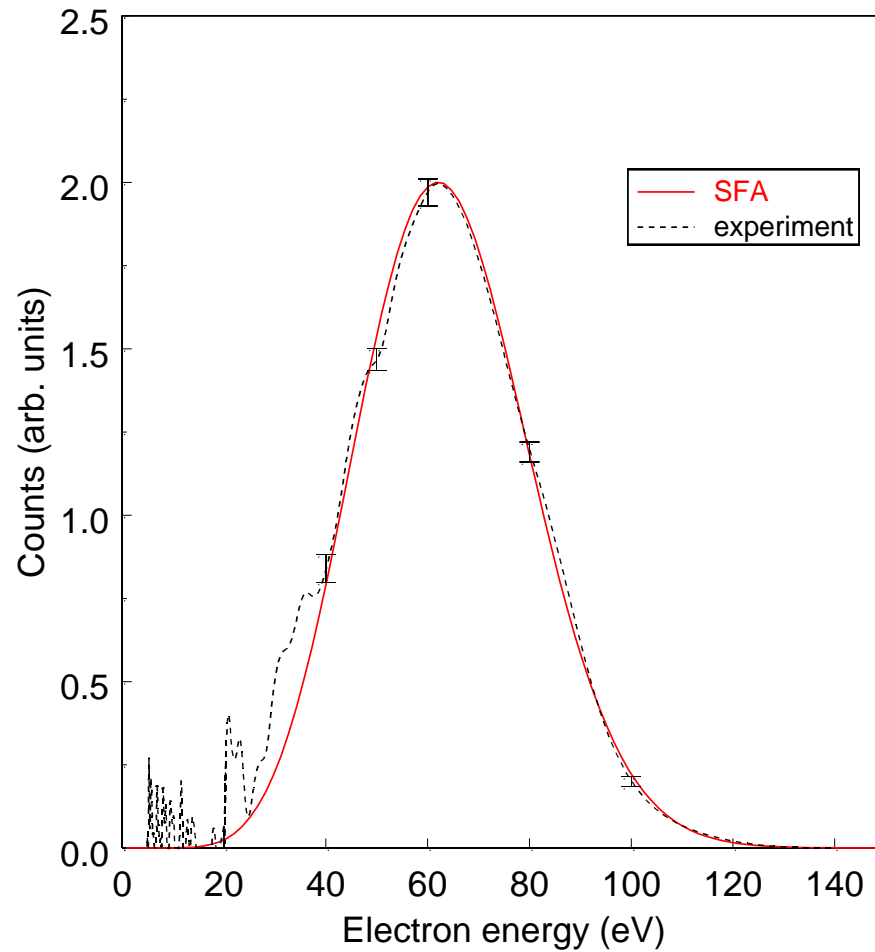In practical application, it seems to be sufficient to require only
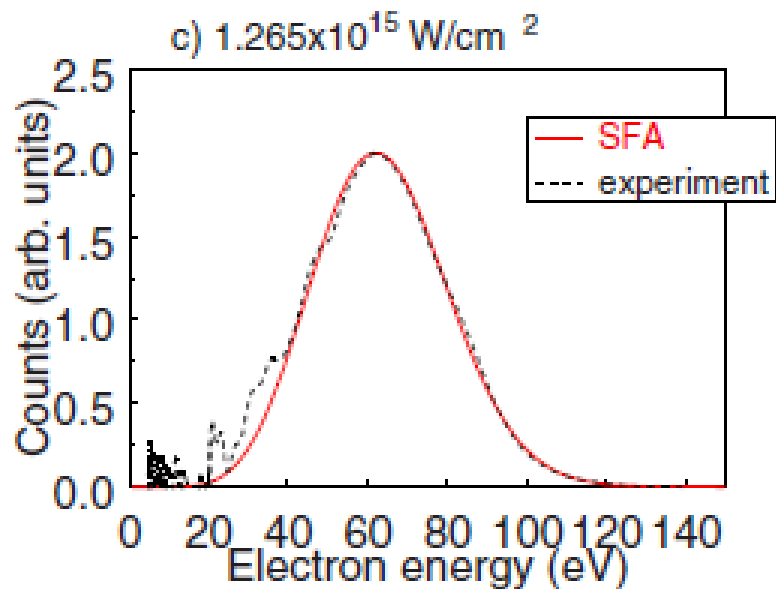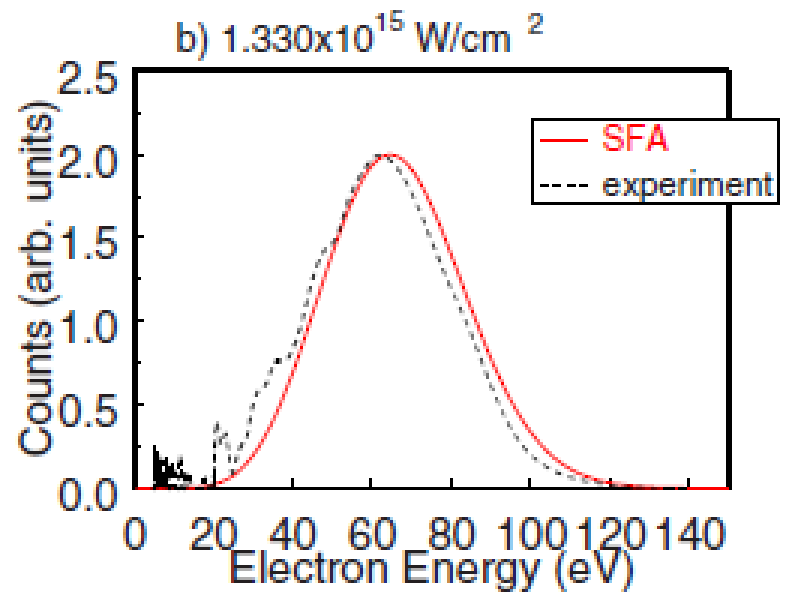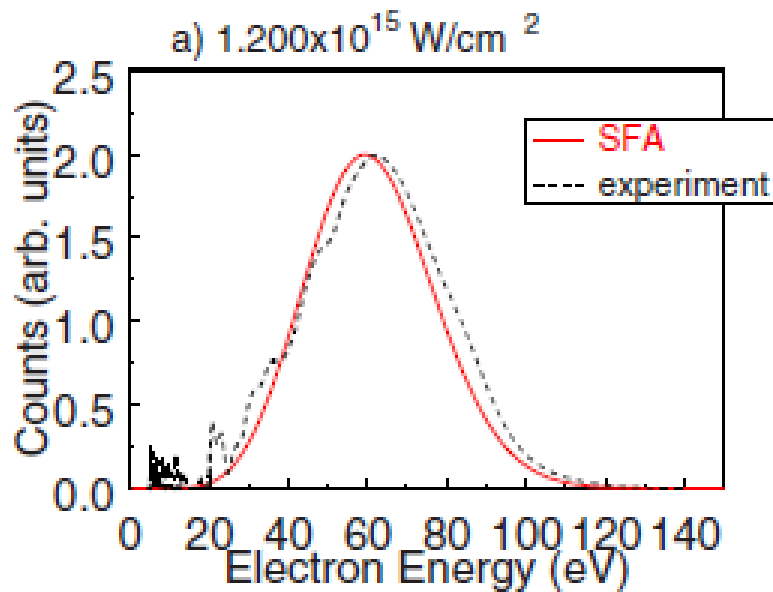
$$U_p \approx E_B$$

in order to obtain accurate spectra.

See U. Mohideen *et al.,* Phys. Rev. Lett. **71**, 509 (1993).
HRR, Phys. Rev. A **54**, R1765 (1996).

Helium, $E_B = 0.904$ a.u., $U_p = 2.88$ a.u., 815 nm, 1.265e15 w/cm²

From: HRR, Phys. Rev. A **76**, 033404 (2007).

a) 1.200×10^{15} W/cm^2

b) 1.330×10^{15} W/cm^2

c) 1.265×10^{15} W/cm^2

What the preceding slide shows is that the SFA for circular polarization is sufficiently accurate that it could be used to measure the actual peak intensity of the laser to within better accuracy than other means.

Part (c) is the best fit.
Part (a) assumes 5% less peak intensity and part (b) assumes 5% more peak intensity than the best-fit intensity. The differences are clear.

All calculations are done with a Gaussian time profile of intensity, and with the spatial intensity profile of a Gaussian laser focus. (This is generally called "focal averaging".)

Depletion effects are fully accounted for.

In the Mohideen helium experiments, the peak U_p is 78.5 eV, the peak in the spectrum is at 61.9 eV, or at about 80% of U_p .

In a single-intensity calculation, the high-intensity case would have the peak at U_p . The reason for the difference is that at the peak intensity for which the actual experiment was done, less-than-maximum intensities in the focal distribution account for a significant part of the total.

The SFA accounts accurately for the focal averaging effect. Mohideen *et al.* made their own attempt to fit the data using a single intensity, and were unable to find a satisfactory fit.

An important point to make here is that the peak of the spectrum occurs at an energy $>> E_B$. This difference is sufficient to justify the SFA without regard to other criteria. Consider the next case from one of the earliest ATI experiments.
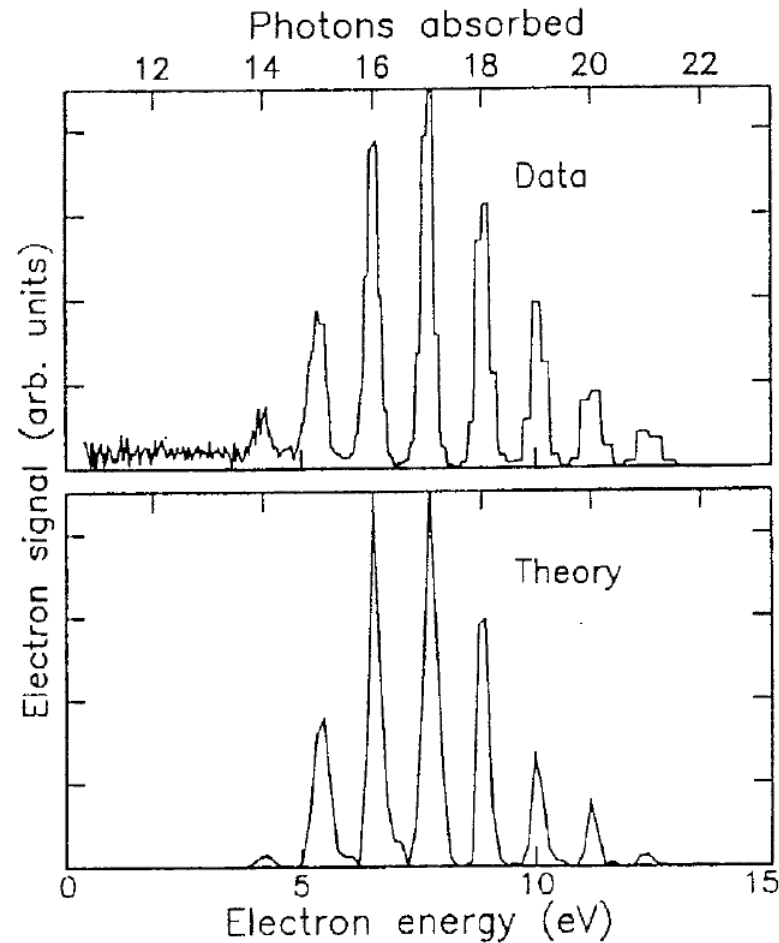
Figure 10. Top: ATI spectrum for circularly polarized 1064 nm light in xenon (from Bucksbaum 1986). Bottom: A computer simulation of the experiment, using ionization rates from the Keldysh theory of H. Reiss (1980), integrated over the measured temporal and spatial profile of the laser pulse.

Wavelength: λ = 1064 nm; Peak intensity = 2.3e13 W/cm² ; Xenon: E_B = 12.13 eV

This amounts to U_p = 2.43 eV $\Rightarrow$ $U_p << E_B$
That is, the SFA should NOT be applicable.

Also, $\gamma_{Keldysh}$ = 1.58. According to tunneling terminology, this is in the "multiphoton domain", so how is it that a "Strong-Field Approximation" works so well?

The peak in the spectrum occurs at 20 eV, and this is large as compared to the binding energy of 12.13 eV.

Hence the SFA validity criterion of $U_p >> E_B$ is not a rigorous limit.
An energetic spectrum serves as well.

However, this is not so effective with linear polarization.

The vector potential for linear polarization can be taken to be

$$\vec{A} = A_0 \vec{\varepsilon} \cos \omega t, \quad \vec{\varepsilon}^2 = 1$$

An important difference between circular and linear polarization comes from A^2.

$$\vec{A}^2(t) = A_0^2 \cos^2 \omega t = \frac{1}{2} A_0^2 (1 + \cos 2\omega t)$$

There now exists a double-frequency term $cos2\omega t$ that did not exist for circular polarization. This is the dipole-approximation residuum of the figure-8 motion of a free electron in a plane-wave field. For each wave period, there are two lobes generated in the figure-8, and these come from the non-dipole generalization of A^2.

Proceeding as in the circular polarization case, the transition amplitude is

$$M_{fi} = \frac{i}{V^{1/2}} \tilde{\phi}_i(p) \left(\frac{p^2}{2} + E_B \right) \int_{-\infty}^{+\infty} dt \exp \left[i \left(\frac{p^2}{2} + E_B + U_p \right) t - i\zeta_l \sin \omega t + i \frac{U_p}{2\omega} \sin 2\omega t \right]$$

$$\zeta_l = A_0 \vec{p} \cdot \vec{\varepsilon} / \omega$$

Now the Volkov exponential contains trigonometric functions of ωt and $2\omega t$. This leads to the generalized Bessel function, with the generating function

$$\exp\left[i\left(-\zeta_l\sin\omega t + \frac{z}{2}\sin 2\omega t\right)\right] = \sum_{n=-\infty}^{+\infty} J_n\left(\zeta_l, -\frac{1}{2}z\right)\exp\left(-in\omega t\right)$$

$$z = U_p / \omega$$

The generalized Bessel function $J_n\,(u,v)$ first arose in the 1962 papers of HRR; it was then used also by Nikishov and Ritus in 1964. Functional properties were systematized in the paper HRR, Phys. Rev. A **22**, 1786 (1980). See Appendices B-D. See also Appendix J in Krainov, Reiss, Smirnov, "Radiative Processes in Atomic Physics" (Wiley, NY, 1997); published online 2005.

Dattoli and Torre in Frascati have since studied it in great detail, have found further generalizations, and have found precursors in the mathematical literature of the 19th century.

These functions are difficult, but always occur with Volkov solutions for linear polarization.

Following the same procedures as with circular polarization, the total transition rate (that is, after integration over final phase space) is:

$$\frac{dW}{d\Omega} = \frac{1}{(2\pi)^2} \sum_{n=n_0}^{\infty} p\left(\frac{p^2}{2} + E_B\right)^2 \left|\tilde{\phi}_i(p)\right|^2 J_n^2\left(\alpha_0^{lin} p\cos\theta, -\frac{1}{2}z\right)$$

$$p = \sqrt{2\left(n\omega - E_B - U_p\right)}, \quad \alpha_0^{lin} = 2\left(\frac{z}{\omega}\right)^2 = \frac{2}{\omega}\left(U_p\right)^{1/2} = \frac{1}{\omega^2}\left(I\right)^{1/2}$$

The quantity α_0^{lin} is the classical amplitude of oscillation of a free electron in a linearly polarized plane-wave field.
The analogy of this result to the circular polarization case is very close.

Will give an example of application to spectra; then develop the method for calculating momentum distributions; and then give some details about how depletion effects are taken into consideration.
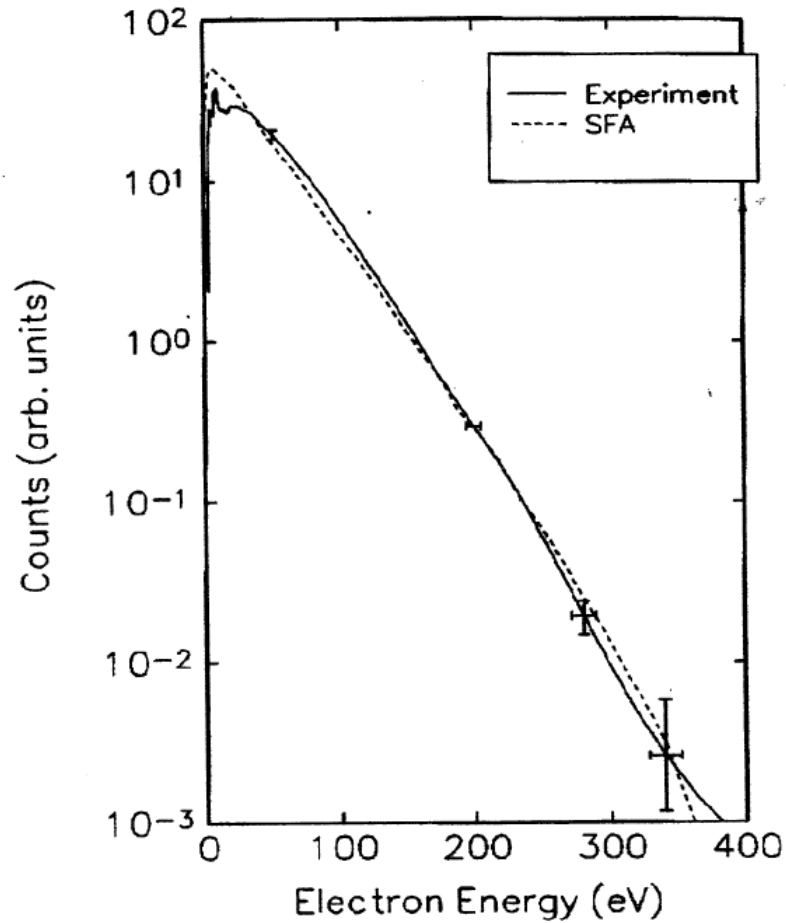
FIG. 1. Linear polarization spectrum for He^+ compared with Ref. [1]. Error bars are as given in [1]. The 7×10^{15} W/cm^2 cited in [1] is reduced to 3.6×10^{15} W/cm^2 in the theory.