FILLING IN DETAILS

H_I with circular polarization

$$\mathbf{A}(t) = \frac{1}{2}A_0\left(\varepsilon e^{i\omega t} + \varepsilon^* e^{-i\omega t}\right); \quad \varepsilon^2 = \varepsilon^{*2} = 0; \quad \varepsilon \cdot \varepsilon^* = 1; \quad \varepsilon = \frac{1}{2^{1/2}}\left(\widehat{\mathbf{x}} \mp \widehat{\mathbf{y}}\right)$$

$$\mathbf{p} \cdot \varepsilon = |\mathbf{p} \cdot \varepsilon| e^{-i\varphi}; \quad \mathbf{A} \cdot \mathbf{p} = \frac{1}{2}A_0 |\mathbf{p} \cdot \varepsilon| \left(e^{i\omega t - i\varphi} + e^{-i\omega t + i\varphi}\right) = A_0 |\mathbf{p} \cdot \varepsilon| \cos(\omega t - \varphi)$$

$$\mathbf{A} \cdot \mathbf{A} = \frac{1}{4}A_0^2 (2\varepsilon \cdot \varepsilon^*) = \frac{1}{2}A_0^2; \quad \mathbf{p} \cdot \varepsilon = p_\perp; \quad A_0 = c\left(\frac{2z}{\omega}\right)^{1/2} = c\alpha_0$$

$$H_I = \frac{1}{c}\mathbf{A} \cdot \mathbf{p} + \frac{1}{2c^2}\mathbf{A} \cdot \mathbf{A} = \frac{1}{c}A_0 |\mathbf{p} \cdot \varepsilon| \cos(\omega t - \varphi) + \frac{1}{4c^2}A_0^2 = \zeta_c \omega \cos(\omega t - \varphi) + z\omega,$$

$$where\ \zeta_c \equiv \frac{1}{\omega c}A_0 |\mathbf{p} \cdot \varepsilon| = \frac{A_0}{\omega c}p_\perp = \frac{\alpha_0}{\omega}p_\perp; \quad z\omega = U_p = \frac{1}{2c^2}A_0^2$$

α_0 is the classical radius of motion of a free electron in a field of circular polarization.

Of particular importance: the $\boldsymbol{A}^2$ term with circular polarization leads only to U_p, whereas with linear polarization there is an additional double-frequency (2ω) term.

FILLING IN DETAILS

Transition rate from the squared amplitude

$$w = \lim_{T\to\infty} \left|M_{fi}^{SFA}\right|^2, \quad \left|M_{fi}^{SFA}\right|^2 \sim \left[\delta\left(\Delta E\right)\right]^2$$

In general:

$$f\left(x\right)\delta\left(x-a\right) = f\left(a\right)\delta\left(x-a\right) \quad \Longrightarrow \quad \delta\left(\Delta E\right)\delta\left(\Delta E\right) = \delta\left(0\right)\delta\left(\Delta E\right)$$

Introduce an integral representation for the delta function:

$$\delta\left(\Delta E\right) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T/2}^{T/2} dt\exp\left(-it\Delta E\right) = \lim_{T\to\infty}\frac{1}{-2\pi i\Delta E}\left[\exp\left(-iT\Delta E/2\right)-\exp\left(iT\Delta E/2\right)\right]$$

$$= \lim_{T\to\infty}\frac{1}{\pi}\frac{\sin\left(T\Delta E/2\right)}{\Delta E} = \frac{1}{\pi}\begin{cases} 0, & \Delta E \neq 0 \\ T/2, & \Delta E = 0 \end{cases}$$

$$\Longrightarrow \quad \lim_{T\to\infty}\frac{\delta\left(\Delta E\right)}{T} = \frac{1}{2\pi} \quad for\ \Delta E = 0$$

(As an alternative, the sine-function-limit representation of the delta function could be used directly.)

Hence, in the expression for *w*, one factor of the delta function remains, and the transition probability has become a transition rate.

PROPERTIES OF THE GENERALIZED BESSEL FUNCTION

The generalized Bessel function arises because of the simultaneous presence of single- and double-frequency terms coming from the simultaneous presence of $\boldsymbol{A}\cdot\boldsymbol{p}$ and $\boldsymbol{A}^2$. This is evident from the generating function and from the integral representation.

<u>Generating function:</u>

$$\exp\left[i\left(u\sin\theta+v\sin2\theta\right)\right]=\sum_{n=-\infty}^{\infty}e^{in\theta}J_n\left(u,v\right)$$

<u>Integral representation:</u>

$$J_n\left(u,v\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[i\left(u\sin\theta+v\sin2\theta-n\theta\right)\right]$$

<u>Series representation:</u>

$$J_n\left(u,v\right)=\sum_{k\to-\infty}^{\infty}J_{n-2k}\left(u\right)J_k\left(v\right)$$

<u>Limiting cases:</u>

$$J_n\left(u,0\right)=J_n\left(u\right);\quad J_n\left(0,v\right)=\begin{cases}J_{n/2}\left(v\right),\ n\ even\\ 0,\ n\ odd\end{cases}$$

<u>Parity properties:</u>

$$J_n\left(-u,v\right)=\left(-\right)^nJ_n\left(u,v\right);\quad J_n\left(u,-v\right)=\left(-\right)^nJ_{-n}\left(u,v\right)$$

plus many more properties that are generalizations of ordinary Bessel function properties. (See HRR PRA **22**, 1786 (1980) and Appendix J in Krainov, Reiss, Smirnov, "*Radiative Processes in Atomic Physics*", Wiley, 1997.

VISUALIZATION OF THE GENERALIZED BESSEL FUNCTION

HRR and VPK, J. Phys. A **36**, 5575 (2003)

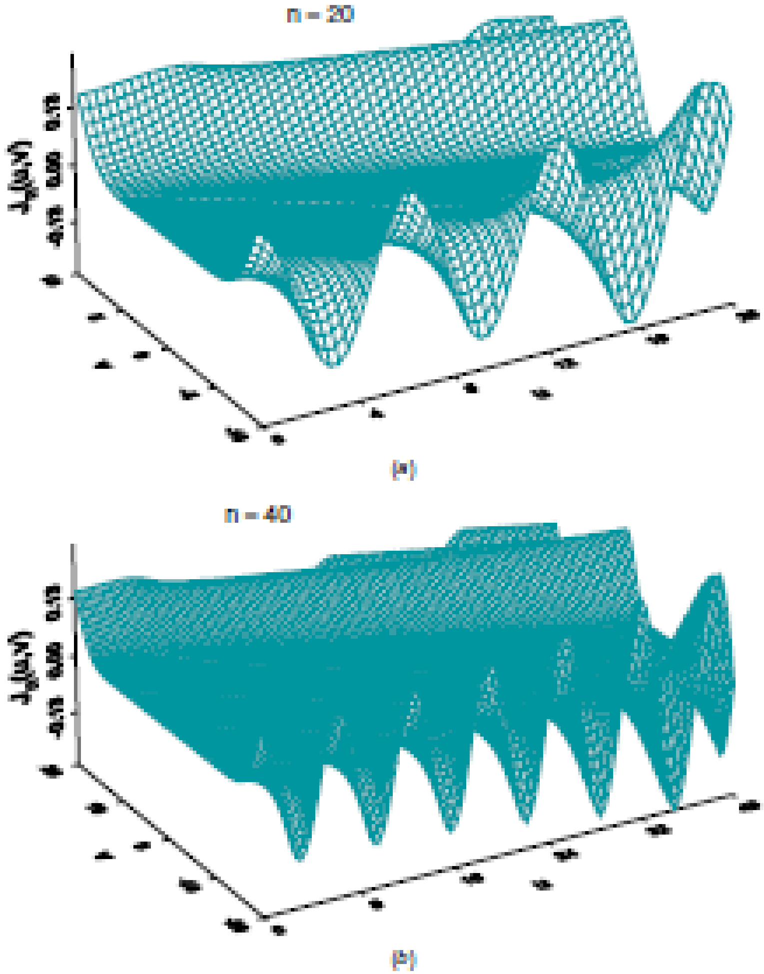

Figure 2. Behaviour of $J_n(u, v)$ as a function of u and v for a fixed order n. Figure 2(a) is for $n = 20$ and figure 2(b) is for $n = 40$. These figures illustrate the general properties that the amplitude of $J_n(u, v)$ is most significant in the region $v \lesssim -u/2$ corresponding to $z \lesssim z$; that $J_n(u, v)$ exhibits almost-periodic oscillations in this region with variable amplitude; and that there is a large domain with $|J_n(u, v)| \ll 1$ (the 'null domain') that increases in extent as n increases. Despite the increasing size of the null domain, the amplitude of the oscillations in $J_n(u, v)$ decreases only very slowly as n increases.