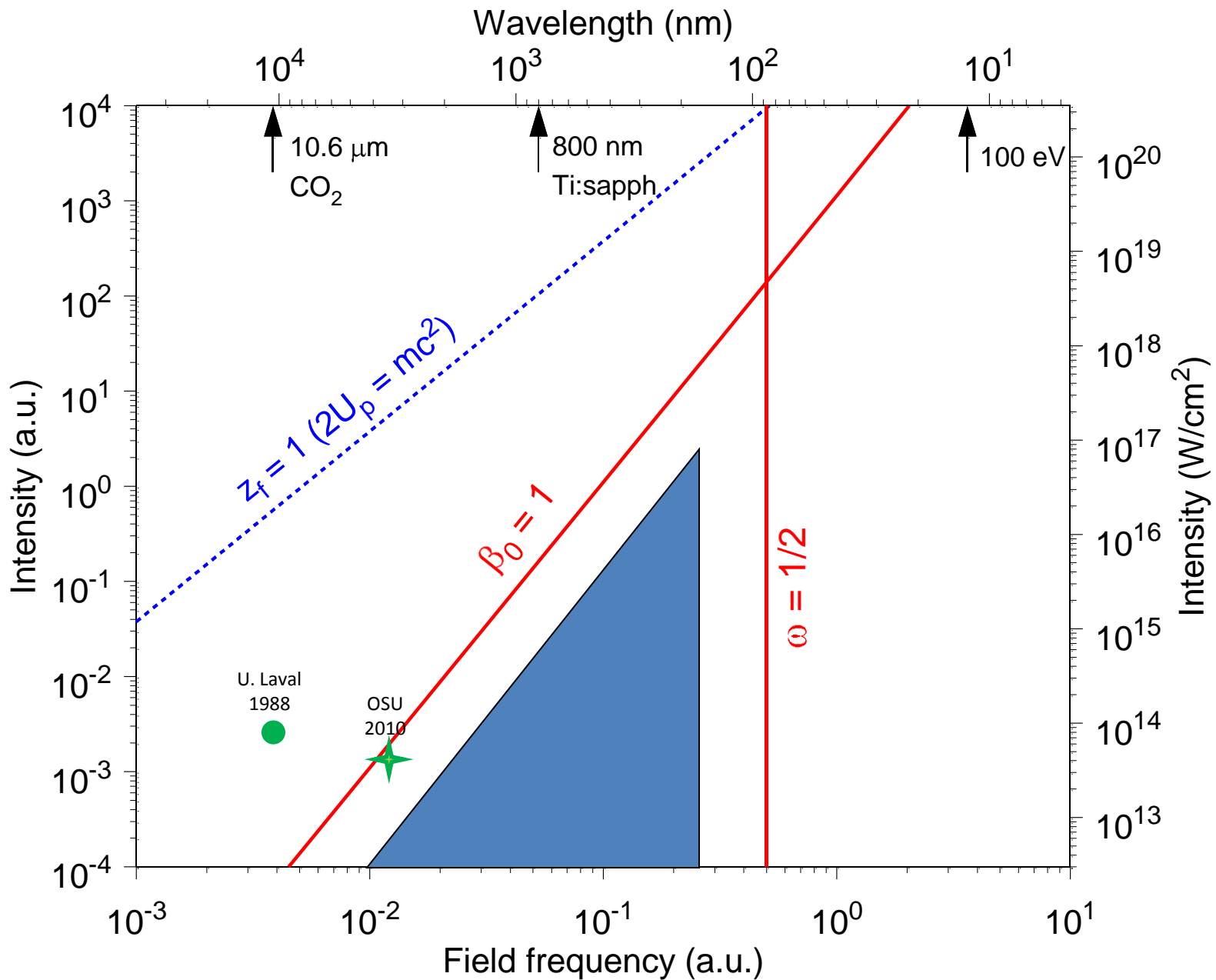


# RELATIVISTIC AND NON-DIPOLE PHOTOIONIZATION

## NECESSITY FOR GOING BEYOND THE DIPOLE APPROXIMATION

- It has already been shown that nonrelativistic dipole-approximation considerations fail as intensity increases and also as frequency decreases.
- The high-intensity, high-frequency case has the characteristic that multiple ionization will dominate, and single-active-electron (SAE) theories are not useful until one reaches inner shells.
- It is the low-frequency non-dipole regime that is now starting to be investigated (although for the wrong reasons in some cases, as in the “tunneling-limit papers”).
- One way to treat non-dipole, but still nonrelativistic problems is simply to introduce fully relativistic theories. A fully relativistic theory will be inclusive of all non-dipole as well as dipole-approximation effects.
- In the figure that follows, it is seen that both the 1988 experiments at  $10.6 \mu\text{m}$  and the 2009-2010 experiments at  $3.6 \mu\text{m}$  are beyond the reach of tunneling theories. (VG theory works to some extent.)

TUNNELING THEORIES FOR LASER-INDUCED PROCESSES ARE LIMITED TO THE SHADED AREA



## BASICS OF RELATIVISTIC CONVENTIONS

**Fundamental postulate of special relativity: The velocity of light is the same in all Lorentz frames of reference.** A “Lorentz frame of reference” is also called an “inertial frame of reference”; simply meaning a frame of reference in which the dynamics that will be developed are correct. For our purposes, it is adequate to refer to any terrestrial laboratory as an inertial frame of reference.

A **Lorentz transformation** is any transformation to another frame of reference moving at constant velocity with respect to the initial frame. All such **Lorentz frames** obey the same relativistic dynamics.

**The velocity of light is the same in all Lorentz frames:**

$$\left| \frac{d}{dt} \mathbf{r} \right| = c, \quad \text{or} \quad c^2 (dt)^2 - |d\mathbf{r}|^2 = 0,$$

where  $\mathbf{r}$  is a point on a “light front” – the locus of points at the front of a beam of light emanating from a light source; that is, all points corresponding to emission at a common time.

It is useful to introduce a 4-dimensional space

$$dx^\mu : (cdt, d\mathbf{r}), \quad \text{or} \quad x^\mu : (ct, \mathbf{r})$$

that satisfies the fundamental relativity postulate for a point on a light front. This amounts to defining a **metric** for the space; that is, a consistent system of measuring the distance between any 2 points. For a “light-like” interval:

$$c^2(dt)^2 - |d\mathbf{r}|^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = 0, \quad \text{or}$$
$$(ds)^2 \equiv (dx^0)^2 - (d\mathbf{r})^2 = \text{constant} = \text{Lorentz scalar}.$$

where this result is the same in any Lorentz frame. The quantity  $dx^\mu$  is a **Lorentz vector** in this space.

It is possible to have also “time-like” intervals and “space-like” intervals where  $(ds)^2 > 0$  and  $(ds)^2 < 0$ , respectively.

Of special note: the squared terminology  $(ds)^2$  no longer requires a positive result.

## METRIC TENSOR

In any space, there exists a squared differential length

$$(ds)^2 = g_{ij} dx^i dx^j, \quad g_{ij} \equiv \textit{metric tensor}$$

where the **summation convention** is used: **if any index is repeated in a product, that signifies a summation over all values of that index**. In this case, for example

$$g_{ij} dx^i dx^j \equiv \sum_i \sum_j g_{ij} dx^i dx^j.$$

Familiar examples of the matrix representation of the metric tensor:

Rectangular coordinates:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical coordinates:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \sin \theta \end{pmatrix}; \quad dx^1 = dr, \quad dx^2 = r d\theta, \quad dx^3 = r \sin \theta d\varphi$$

For special relativity:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The most common (*but **not** universal*) convention is to use  $i = 1, 2, 3$ ;  $\mu = 0, 1, 2, 3$

When imaginary time is used:  $i = 1, 2, 3, 4$ , where  $dx_4 = icdt$

(The imaginary time convention was once dominant; it is now almost dead.)

A scalar product is, in general, formed with the metric tensor. For any 4-vectors  $A^\mu$ ,  $B^\mu$

$$A^\mu : (A^0, \mathbf{A}); \quad B^\mu : (B^0, \mathbf{B})$$

$$(\mathbf{A})^2 = g_{\mu\nu} A^\mu A^\nu = (A^0)^2 - (\mathbf{A})^2$$

$$\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$$

## DUAL SPACE

It is convenient to define a *dual space*:

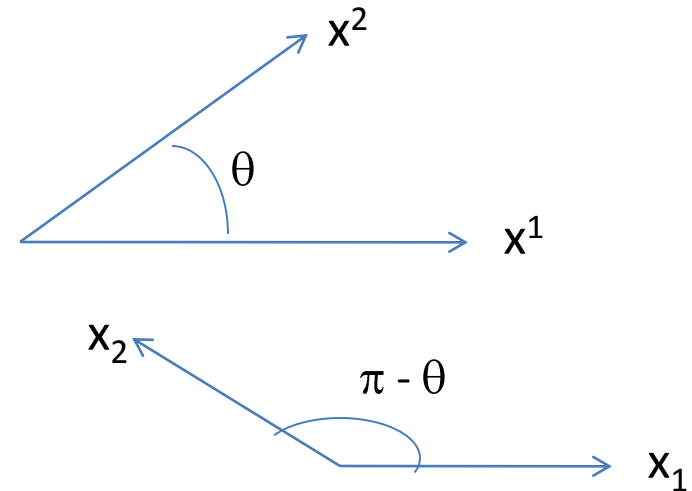
$$dx_{\mu} = g_{\mu\nu} dx^{\nu}, \quad \Rightarrow$$
$$(ds)^2 = dx_{\mu} dx^{\mu}.$$

The convention in special relativity is

$$dx^0 = dct, \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz, \quad \text{so that}$$
$$dx_0 = dct, \quad dx_1 = -dx, \quad dx_2 = -dy, \quad dx_3 = -dz$$

It is also conventional to call  $dx^{\mu}$  the **contravariant vector**,  
and  $dx_{\mu}$  the **covariant vector**.

An exercise for anyone with time to spare: If one has a 2-dimensional non-orthogonal coordinate system  $x^1, x^2$



There is a dual system  $x_1, x_2$  such that,  
for any increment of length  $ds$ , it is true  
that  $(ds)^2 = dx_i dx^i$ .



## THE RULES TO THIS POINT

*contravariant vector:  $A^\mu$*

*covariant vector:  $A_\mu$*

*contra- to co- conversion:  $A_\mu = g_{\mu\nu} A^\nu$*

*co- to contra- conversion:  $A^\mu = g^{\mu\nu} A_\nu$*

*scalar product:  $A \cdot B = A_\mu B^\mu = A^\mu B_\mu$*

*meaningless expressions: sums over 2 co- or contravariant indices:  ~~$A^{\mu\mu}$~~  or  ~~$A_\mu B_\mu$~~*

*more than 2 appearances of the same index:  ~~$A^\mu B_\mu C^\mu$~~*

Derivatives: Let  $a_\mu$  be a constant vector, and form the product  $a_\mu x^\mu$ . Then

$$\frac{\partial}{\partial x^\mu} a_\nu x^\nu = a_\mu, \quad \text{since } \frac{\partial}{\partial x^\mu} x^\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases} = \delta_\mu^\nu$$

Hence  $\frac{\partial}{\partial x^\mu}$  is covariant, so one can write  $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$  and  $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$

For example:

$$A^\mu : (A^0, \mathbf{A}), \quad B_\mu : (B^0, -\mathbf{B}) \quad \Rightarrow \quad A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} : \left( \frac{\partial}{\partial ct}, -\nabla \right) \quad \Rightarrow \quad \partial^\mu A_\mu = \frac{\partial A^0}{\partial ct} + \nabla \cdot \mathbf{A}$$

## COVARIANCE

A covariant expression is one in which every term has (the same) explicit tensor rank: *scalar* (a quantity that is identical in all Lorentz frames); *vector* (contravariant or covariant); *second-rank tensor* (2 contravariant indices, or 2 covariant indices, or mixed indices with contravariant first and covariant second, or vice versa).

In general,  $A_{\mu}^{\nu} \neq A^{\nu}_{\mu}$

The definition of a Lorentz vector is any quantity that has the same Lorentz transformation properties as  $x^{\mu} : (ct, \mathbf{r})$ ; where  $x'^{\mu} = L^{\mu}_{\nu} x^{\nu}$

**Proper Time**  $\tau$  is time measured in a coordinate system in which a particle is at rest; that is  $dt = d\tau$  when  $d\mathbf{r}=0$ .

The time dilation factor  $\gamma = 1/(1-v^2)^{1/2}$  (not derived here) means that  $dt = \gamma d\tau$  is a longer time interval in any system moving at velocity  $v$  with respect to the rest system.

**Covariant Velocity** is  $u^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma \frac{dx^{\mu}}{dt}$

**Covariant momentum** is  $p^{\mu} = mu^{\mu} \left[ = m\gamma \frac{dx^{\mu}}{dt} = (m\gamma) \frac{dx^{\mu}}{dt} \right]$

***There is no need for the concepts of “rest mass” and “variable mass”, which come from associating the relativistic  $\gamma$  factor with the mass, where it does not belong. The mass  $m$  is a Lorentz scalar, and the variable mass concept is needless and confusing.***

ENERGY – MOMENTUM 4-VECTOR

From relativistic classical mechanics:

$$p^\mu : \left( \frac{E}{c}, \mathbf{p} \right)$$

$$p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = \text{relativistic invariant} = m^2 c^2, \quad \text{or}$$

$$p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2 \quad \text{in "natural units" } \hbar=c=1.$$

This last expression is called the “mass shell” for the electron:  $E^2 - \mathbf{p}^2 = m^2$

In analogy to the nonrelativistic energy and momentum quantum operators:

$$\hat{p}^\mu \rightarrow i\hbar\partial^\mu : (i\hbar\partial^0, -i\hbar\nabla) \quad \text{or}$$

$$\hat{p}^\mu \rightarrow i\partial^\mu : (i\partial^0, -i\nabla) \quad \text{in natural units}$$

## RELATIVISTIC EQUATIONS OF MOTION: KLEIN-GORDON

The Schrödinger equation has a first-order derivative in time and second-order derivatives in spatial coordinates. Relativistic equations of motion have to have the same orders in time and space.

1. The Dirac equation is first-order in time and space: it describes fermions.
2. The Klein-Gordon equation is second order in time and space: it describes bosons.

Klein-Gordon equation (1926):

$$\left[ \left( i\hbar\partial^\mu - \frac{q}{c} A^\mu \right)^2 - m^2 c^2 \right] \Phi = 0, \text{ or, in "natural units" } (\hbar=1, c=1):$$
$$\left[ (i\partial^\mu - qA^\mu)^2 - m^2 \right] \Phi = 0$$

For free particles ( $A^\mu = 0$ ), there are solutions for  $E \geq m$  and for  $E \leq -m$ .

For bosons ( $spin = 0$ ), there is nothing to prevent spontaneous decay into negative energy states. This put the KG equation in bad repute until **Pauli and Weisskopf (1934)** showed that negative energy states of a (zero-spin) particle can be interpreted as positive energy states of the antiparticle.

Except as a convenience, ***there are no negative energy states; "hole theory" is unnecessary.***

## NEGATIVE ENERGY STATES AND SECOND QUANTIZATION

The last comment, that hole theory is unnecessary, is usually directed only to the negative energy problem for bosons, where there is no Pauli Principle to prevent positive energy bosons from making spontaneous transitions into the negative energy continuum. It applies equally well to fermions, and this was the original motivation for the Pauli & Weisskopf paper.

W. Pauli & V.F. Weisskopf, *Helv. Phys. Acta* **7**, 709 (1934).

One still finds journal articles and textbooks that continue to state that all negative-energy electron states have to be filled to prevent spontaneous decay. From this they deduce that every problem is a many-body problem, and they then make the further inference that a second-quantized theory is a requirement.

(Second-quantization  $\rightarrow$  the electromagnetic field is quantized, such that there exists a number operator for photons.)

This is specifically unnecessary when the fields are weak (perturbation theory) and when the fields are very strong (as in high-field laser physics).

Z. Fried & J.H. Eberly, *Phys. Rev.* **136**, B871 (1964).

J.H. Eberly & HRR, *Phys. Rev.* **145**, 1035 (1966).

## SIMPLE DERIVATION OF THE DIRAC EQUATION

The KG equation could be derived simply from the mass-shell expression

$$p^\mu p_\mu = m^2 \quad \text{or} \quad (p^\mu - qA^\mu)(p_\mu - qA_\mu) = m^2 \quad \text{with} \quad p^\mu \rightarrow i\partial^\mu.$$

Can one find a relativistic quantum equation that is first order in  $p^\mu$ ?

Consider first the free particle equation ( $A^\mu = 0$ ), since it is easy to extend to the general case.

**Problem: can we extract the square root of  $p^\mu p_\mu = m^2$ ?**

This can't be done directly, but try introducing a set of square matrices to accomplish this, and then find the necessary properties of the matrices.

$$\pm m = \sqrt{p^\mu p_\mu} = \gamma^\mu p_\mu$$

The  $\mu$  index is the Lorentz-space index:  $\mu = 0, 1, 2, 3$ . Each of the  $\gamma^\mu$  square matrices also has matrix indices that are normally omitted to avoid clutter.

$$\sqrt{p^\mu p_\mu} \delta_{\alpha\beta} = \gamma_{\alpha\beta}^\mu p_\mu.$$

To find the properties of the  $\gamma^\mu$ , square both sides and then symmetrize:

$$p^\mu p_\mu = \gamma^\mu p_\mu \gamma^\nu p_\nu = \frac{1}{2}(\gamma^\mu p_\mu \gamma^\nu p_\nu + \gamma^\nu p_\nu \gamma^\mu p_\mu)$$

It cannot be assumed that  $\gamma^\mu$  and  $\gamma^\nu$  commute with each other, but this does not apply to the momentum operators.

$$p^\mu p_\mu = \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) p_\mu p_\nu \Rightarrow$$
$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

The  $\gamma^\mu$  are the **Dirac matrices**: The superscript  $\mu$  is a Lorentz index, so there are 4 Dirac matrices:  $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ .

Each of these Dirac matrices has the indices of a *Dirac matrix space*:  $\gamma_{\alpha\beta}^\mu$

The dimensionality of this space can be any even number  $\geq 4$ .

These  $\alpha, \beta$  indices are usually suppressed, and it is also generally possible to avoid using an explicit representation of these matrices. When necessary to use a representation, there is a standard generalization of the 2 X 2 Pauli spin matrices to a set of 4 X 4 matrices.

The condition satisfied by the Dirac matrices is an *anticommutator*, defined in analogy to the familiar *commutator* of nonrelativistic quantum mechanics.

$$\text{Commutator: } [A, B] \equiv AB - BA$$

$$\text{Anticommutator: } \{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu$$

Recap: Found properties of  $\gamma$  matrices such that

$$\gamma^\mu p_\mu = \pm m, \quad \text{where } \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

This implies the free-particle Dirac equations

$$(\gamma^\mu p_\mu - m)\Psi = 0 \quad \text{or} \quad (\gamma^\mu p_\mu + m)\Psi = 0$$

Both equations have the same physical content, but it is customary to employ the first. With the operator form for the momentum, the free-particle Dirac equation is

$$(\gamma^\mu i\partial_\mu - m)\Psi = 0 \quad \text{or} \quad (\gamma^\mu_{\alpha\beta} i\partial_\mu - m\delta_{\alpha\beta})\Psi_\beta = 0,$$

Lorentz-space indices  $\mu=0,1,2,3$ ; Dirac-space indices  $\alpha,\beta=1,2,3,4$

Including interaction with a 4-vector potential  $A^\mu$  :

$$[\gamma^\mu (i\partial_\mu - qA_\mu) - m]\Psi = 0.$$



## DIRAC – PAULI REPRESENTATION OF THE DIRAC MATRICES

Explicit representations can often be avoided completely, but the standard Dirac-Pauli representation consists of 4 X 4 matrices composed of 2 X 2 Pauli matrices and the 2 X 2 unit matrix:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma^0 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad \text{or } \boldsymbol{\gamma} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix}$$

A combination associated with electron spin is:

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

A combination that arises in the weak interactions (like beta decay) is:

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

## SECOND-ORDER DIRAC EQUATION

There were two forms of the Dirac equation following from  $\pm m$ . If these two first-order operator expressions are multiplied, does the Klein-Gordon equation emerge?

$$[\gamma^\mu (p_\mu - qA_\mu) + m] \cdot [\gamma^\nu (p_\nu - qA_\nu) - m] = (p^\mu - qA^\mu)(p_\mu - qA_\mu) - m^2 + \frac{q}{2} \frac{i}{2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}$$

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu = \text{electromagnetic field tensor}$$

Introduce the expression:

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

The second-order Dirac equation and the Klein-Gordon equation are:

$$[(p^\mu - qA^\mu)(p_\mu - qA_\mu) - m^2 + \frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu}] \Psi = 0$$

$$[(p^\mu - qA^\mu)(p_\mu - qA_\mu) - m^2] \Phi = 0$$

The extra term in the Dirac equation can be shown to be connected with the spin of the electron. Also, while  $\Phi$  is a scalar function,  $\Psi$  is a 4-component spinor, or column matrix. (There is a redundancy in the components; only 2 are independent, whereas all 4 components are independent in the first-order Dirac equation.)

## A NOTE ON THE ELECTROMAGNETIC FIELD TENSOR

The electromagnetic field tensor is the covariant expression of how the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  are formed from the scalar and vector potentials  $\phi$  and  $\mathbf{A}$ .

(To avoid any confusion between the electric field and the electromagnetic field tensor, the electric field is here expressed as  $\mathbf{E}$  rather than the  $\mathbf{F}$  used elsewhere.)

$$A^\mu : (\phi, \mathbf{A})$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu; \quad F_{\mu\nu} = g_{\mu\lambda} F^{\lambda\rho} g_{\rho\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The electromagnetic Lorentz invariant of greatest interest in these lectures is formed from the scalar product of  $F^{\mu\nu}$  with itself, summing over both Lorentz indices:

$$F^{\mu\nu} F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$$

## LIMITING CASES OF THE SECOND-ORDER DIRAC EQUATION

If the 4-spinor  $\Psi$  is reduced to the 2-spinor  $\Psi_{(2)}$ , the second-order Dirac equation is

$$[(p^\mu - qA^\mu)(p_\mu - qA_\mu) - m^2 + \boldsymbol{\sigma} \cdot (\mathbf{B} + iq\mathbf{F})]\Psi_{(2)} = 0$$

The nonrelativistic limit of the second-order Dirac equation is

$$i\partial_t \Psi_{(2)} = \left\{ \frac{1}{2m} [(-i\nabla) - q\mathbf{A}]^2 + q\phi + \frac{q}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \right\} \Psi_{(2)}$$

$$A^\mu : (\phi, \mathbf{A})$$

This is exactly the Pauli equation of nonrelativistic quantum mechanics with spin included. The last term is sometimes called the Stern-Gerlach term.

## A NOTE ABOUT SPINORS

Spinors were introduced into physics by Paul Dirac for the description of spin-1/2 particles. (They were developed independently by mathematicians.)

A spin-1/2 particle (such as an electron) has two independent states: spin-up and spin-down. In “geometric space”, these two states are 180° apart.

However, in spinor space, these two states are orthogonal (independent) states, so they are 90° apart.

Hence a rotation of 720° in geometric space is necessary for a full rotation of 360° in spinor space.

$$\text{spin up: } u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{spin down: } d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u = \uparrow \quad d = \downarrow \quad \text{anti-parallel (180}^\circ\text{)}$$

$$u^\dagger d = (1 \quad 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad d^\dagger u = (0 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \text{orthogonal (90}^\circ\text{)}$$

Relativistic spinors describe both particles and antiparticles (positive and negative energy states), and require 4 components rather than 2.

## ELECTRON MASS SHELL

For a free electron:

$$p^\mu p_\mu = \text{constant} = m^2 \quad (c = 1)$$

$$E^2 - \mathbf{p}^2 = m^2 \quad \text{or} \quad \left(\frac{E}{m}\right)^2 - \left(\frac{\mathbf{p}}{m}\right)^2 = 1$$

In a 2-dimensional space  $E/m$ ,  $p/m$ , where one axis is taken to be in the direction of  $\mathbf{p}$ , this is the equation of a hyperbola. This is the “*mass shell*” for the electron, envisioned to be in 4 dimensions.

For a free electron in a monochromatic plane-wave field, the mass shell is altered to

$$(p^\mu - nk^\mu)(p_\mu - nk_\mu) = m^2(1 + z_f); \quad z_f = \frac{2U_p}{m}; \quad k^\mu : (\omega, \mathbf{k})$$

where  $n$  is an integer,  $k^\mu$  is the propagation 4-vector for the plane-wave field, and  $m^2 z_f$  is the “*strong-field mass-shift*”.

strong-field mass shift: HRR, J. Math. Phys. **3**, 387 (1962);

strong-field mass shell: HRR & J.H. Eberly, Phys. Rev. **151**, 1058 (1966).

## THE VOLKOV ELECTRON

There are exact solutions of both the Klein-Gordon equation and the Dirac equation for an electron in a plane wave field. These solutions are often both called Volkov solutions because Volkov was the first to solve the Dirac problem. However, Gordon solved the KG problem earlier, and the Volkov solution now widely employed in the nonrelativistic, dipole-approximation form is actually more closely related to Gordon.

W. Gordon, Z. Phys. **40**, 117 (1926).

D. M. Volkov, Z. Phys. **94**, 250 (1935).

Both solutions share the mass-shell properties just exhibited. That is, both the Gordon and Volkov solutions describe an electron with  $n$  photons acting as if they were an inherent part of the electron, and with an apparent additional mass. Both effects can be large for strong fields.

The scalar (spin-zero) Gordon solution is

$$\Psi^{scalar} = C \exp \left[ -ip \cdot x - i \int_{-\infty}^{k \cdot x} d(k \cdot x) \left( \frac{qA \cdot p}{p \cdot k} - \frac{q^2 A^2}{2p \cdot k} \right) \right]$$

$$p \cdot x = Et - \mathbf{p} \cdot \mathbf{r}, \quad k \cdot x = \omega t - \mathbf{k} \cdot \mathbf{r}, \quad A \cdot p = -\mathbf{A} \cdot \mathbf{p}, \quad p \cdot k = E\omega - \mathbf{p} \cdot \mathbf{k}$$

The spinor (spin-1/2) Volkov solution is

$$\Psi^{spinor} = \left( 1 + \frac{qk_{\mu} A_{\nu} \gamma^{\mu} \gamma^{\nu}}{2p \cdot k} \right) u \Psi^{scalar}, \quad \text{where } u \text{ is a spinor that satisfies } (p_{\mu} \gamma^{\mu} - m)u = 0.$$

The second term in the parentheses mixes spin states and particle and antiparticle states in an intensity-dependent way.

The relativistic SFA is based on the relativistic S-matrix transition amplitude

$$(S - 1)_{fi} = -i \int d^4x \bar{\Psi}_f^{spinor} qA_{\mu} \gamma^{\mu} \Phi_i, \quad \bar{\Psi} \equiv \Psi^{\dagger} \gamma^0$$

HRR, J. Opt. Soc. Am B **7**, 574 (1990).

HRR, Prog. Quant. Electr. **16**, 1 (1992).

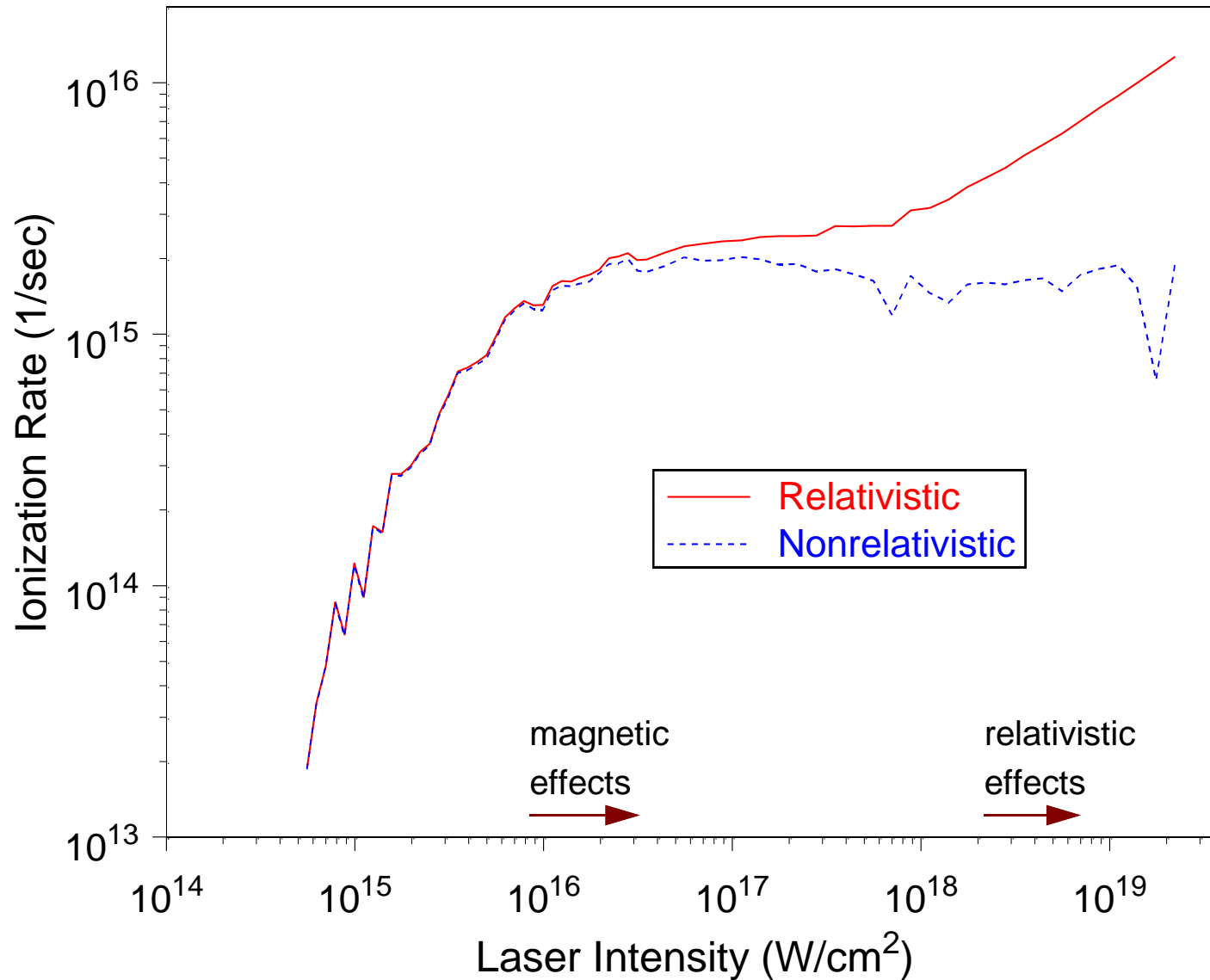
To maintain a completely relativistic theory, the work to date has all involved the known solutions of the Dirac equation for the hydrogen atom as the initial field-free states  $\Phi_i$ .

Examples of specific relativistic effects are shown in the following slides.



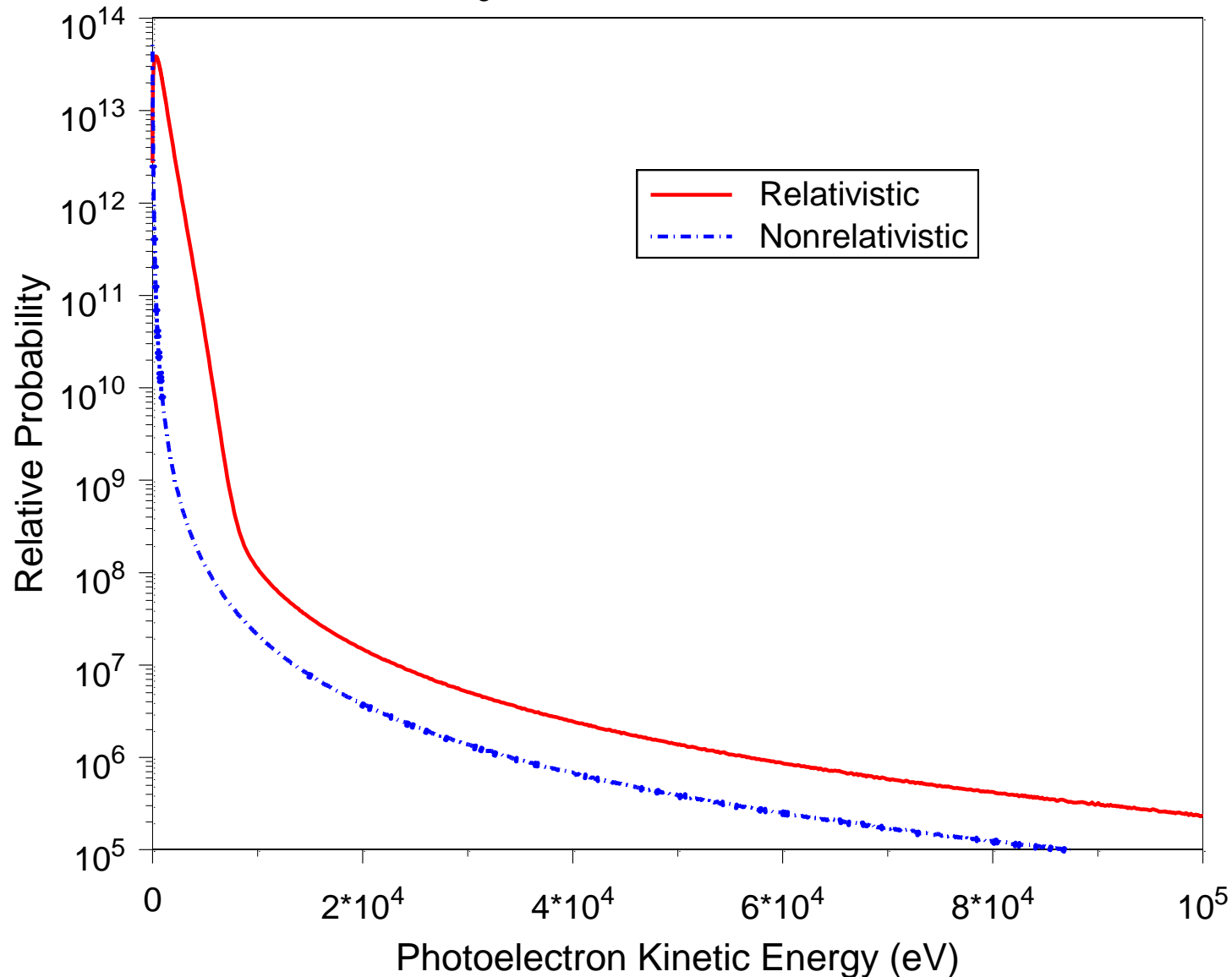
# MONOCHROMATIC TRANSITION RATES

Linear Polarization,  $\omega = 0.125$  a.u. = 3.40 eV



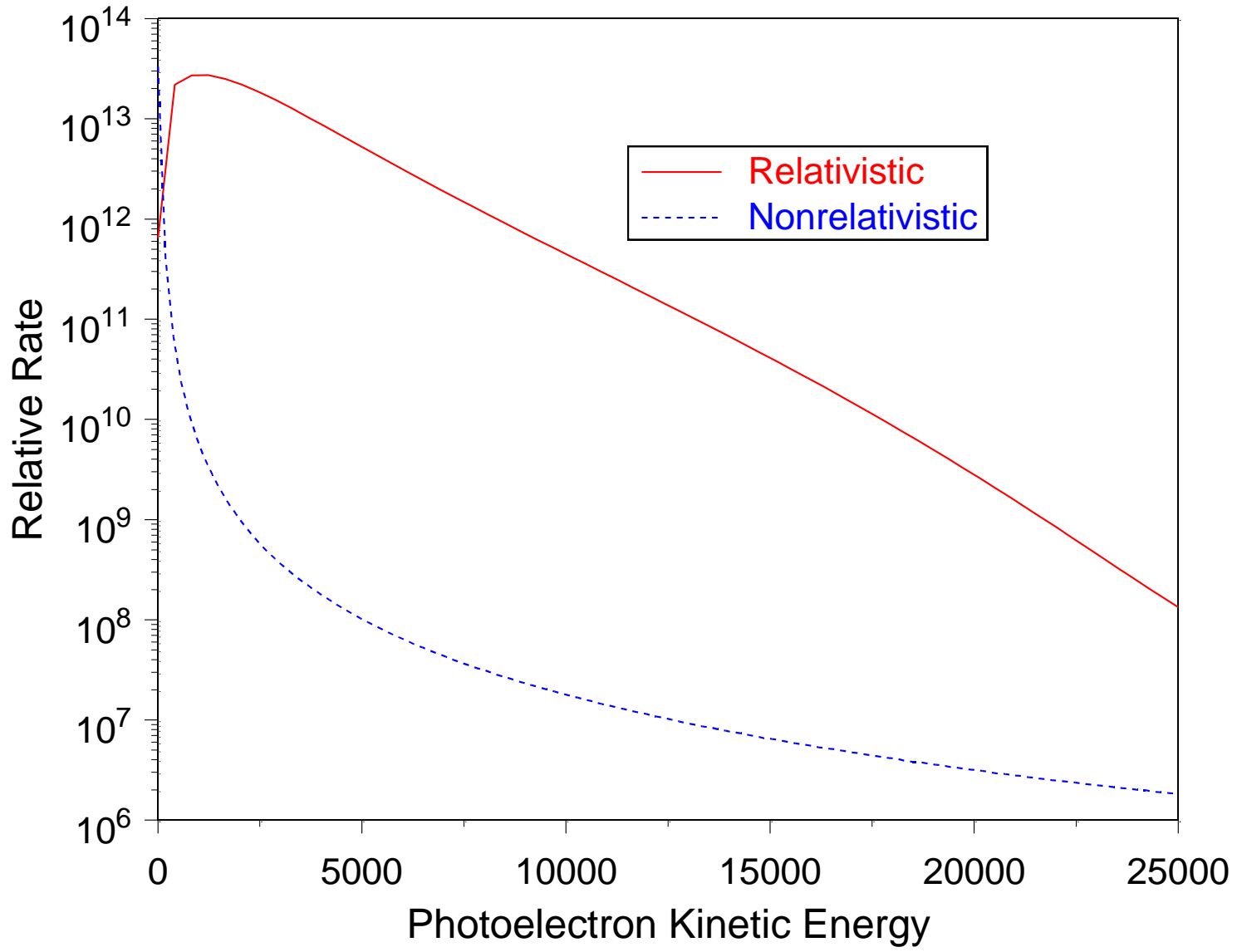
# Ionization of Ground-State Hydrogen, Linear Polarization

Laser frequency= $1/8$  a.u., Intensity= $10^{2.5}$  a.u. =  $1.1 \cdot 10^{19}$  W/cm<sup>2</sup>



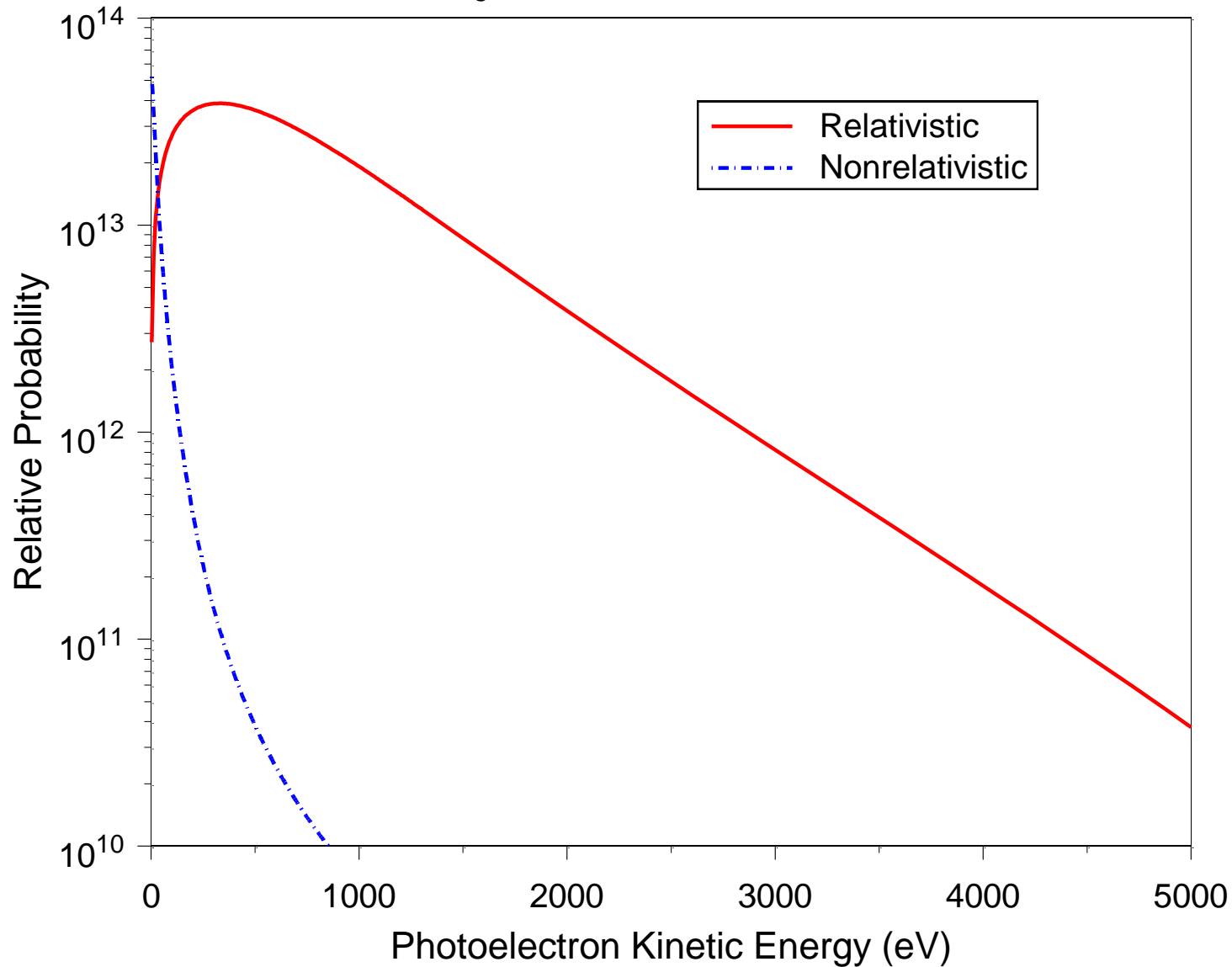
# PHOTOELECTRON SPECTRUM, GROUND STATE HYDROGEN

Linear Polarization,  $\omega=0.125$  a.u.=3.40 eV,  $I=2.2\times 10^{19}$  W/cm<sup>2</sup>



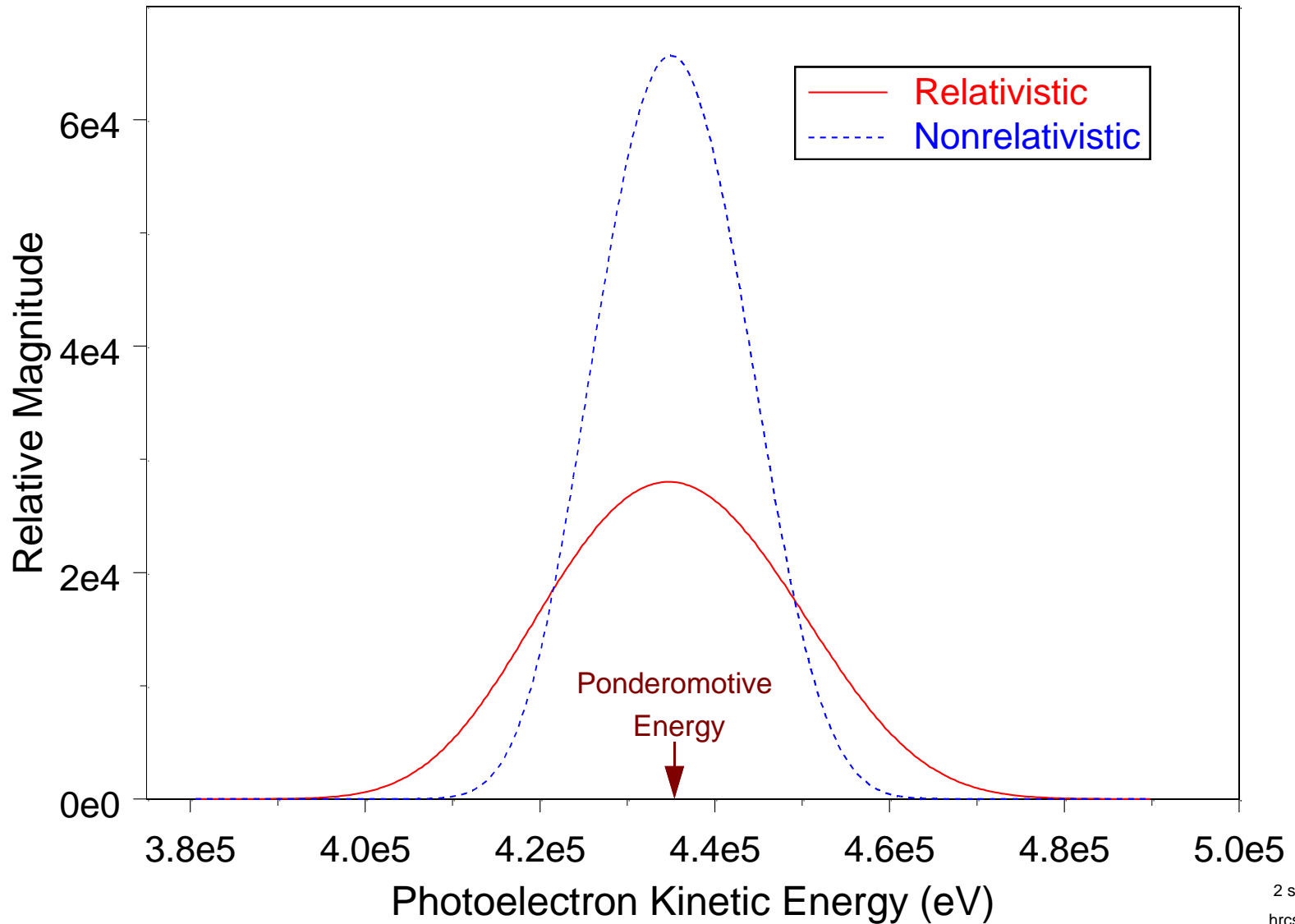
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Laser frequency= $1/8$  a.u., Intensity= $10^{2.5}$  a.u. =  $1.1 \cdot 10^{19}$  W/cm<sup>2</sup>



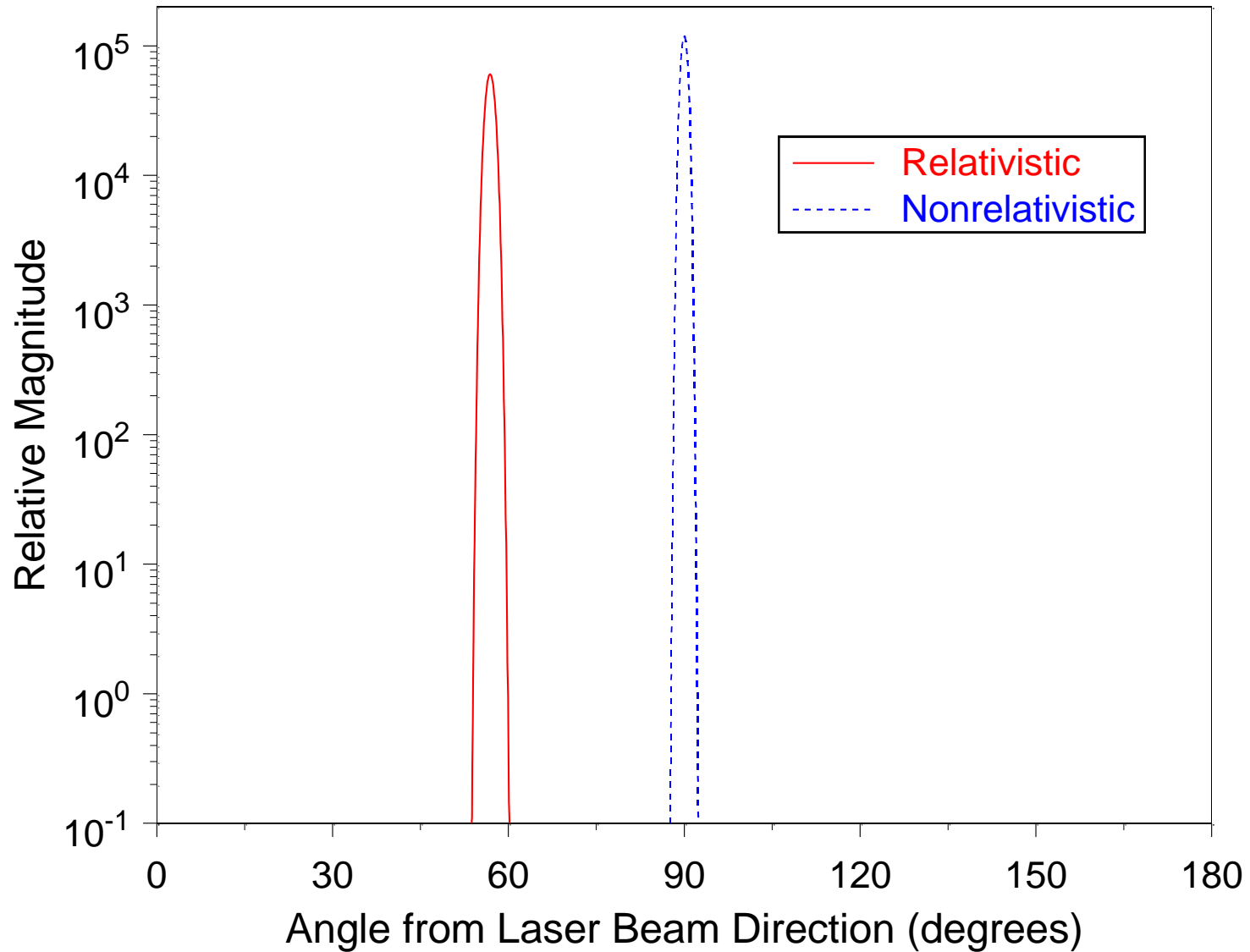
# PHOTOELECTRON SPECTRUM

Circular Polarization,  $\omega = .125$  a.u. = 3.40 eV,  $I = 3.5 \times 10^{19}$  W/cm<sup>2</sup>



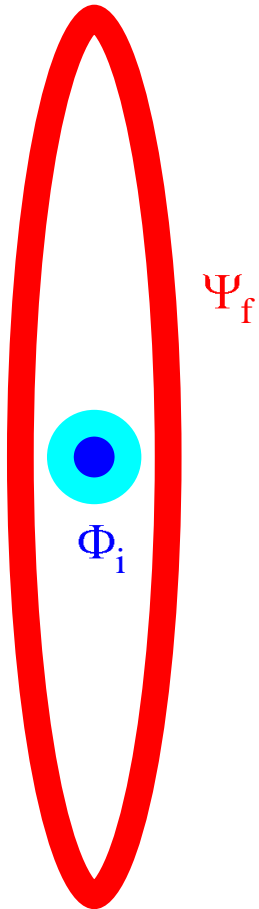
# PHOTOELECTRON ANGULAR DISTRIBUTION

Circular Polarization,  $\omega=0.125$  a.u.=3.40 eV,  $I=3.5\times 10^{19}$  W/cm<sup>2</sup>

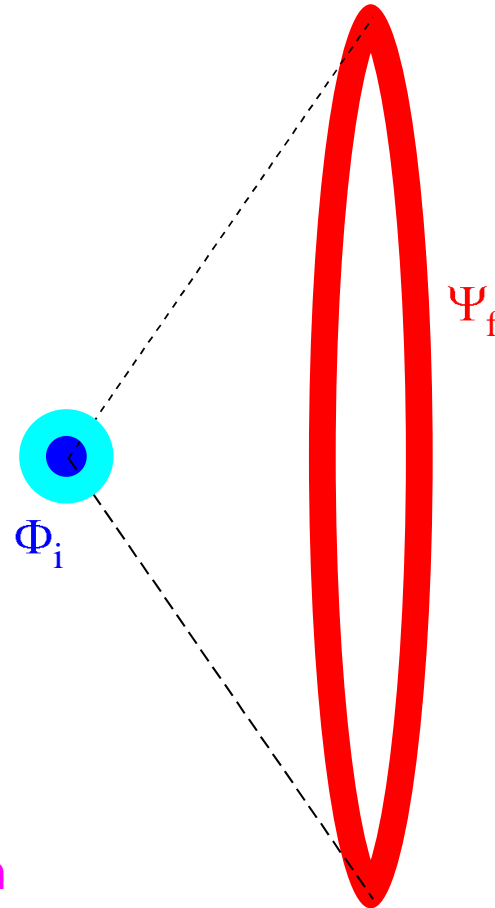


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(a) Nonrelativistic



(b) Relativistic



Laser Beam



From: HRR, Optics Express **8**, 99 (2001)

## FINAL REMARK

The best environment for the exploration of relativistic effects is at low frequencies where single-active-electron (SAE) conditions exist even in the relativistic domain, and where saturation effects are largely absent.



### **Photon-Multiphoton Pair Production**

*Basic Theory:* HRR, J. Math. Phys. **3**, 59 (1962).

*Detailed proposal for doing the experiment at SLAC, and noting that such an experiment would be the first observation of creating mass from a zero-mass state:* HRR, Phys. Rev. Lett. **26**, 1072 (1971).

*First experimental observation at SLAC:* D. Burke, et al., Phys. Rev. Lett. **79**, 1626 (1997).

### **Strong-Field Added Mass**

*Basic Theory:* HRR, J. Math. Phys. **3**, 387 (1962).

*Laboratory observation:* C. Bamber, et al., Phys. Rev. D **60**, 092004 (1999).

### **Mistakes in the Work of Burke et al. and of Bamber et al.,**

#### **+ Early History of Strong-Field Physics**

HRR, Eur. Phys. J. D **55**, 365 (2009).