Lecture 6 28.10.13

Last time: QM analytical model of HHG (Lewenstein model)

Radiation spectra found by evaluating
$x(t) = \langle \Psi(t) | x | \Psi(t) \rangle$, where an approx. analytic solution for $|\Psi(t)\rangle$ is obtained from SFA

Results in:

$$x(t) = 2\,\mathrm{Re}\left\{ i \int_0^t dt' \int d^3p \cdot E\cos(t') \underbrace{\langle v(t')|X|0\rangle}_{d_x(t')} \cdot d_x^*(t) \cdot e^{-S(\vec{p},t,t')}\right.$$

using saddle-pt. method: $\nabla_{\vec{p}} S(\vec{p},t,t') = 0$ and integrating:

$$x(t) \approx i \int_0^\infty d\tau \left(\frac{\pi}{\epsilon + i\tau/2}\right)^{3/2} \overbrace{d_x^*\left(p_{st}(t,\tau) - A_x(t)\right)}^{d_x^*(t)} \times \underbrace{d_x\left(p_{st}(t,\tau) - A_x(t-\tau)\right)}_{d_x(t')} E\cos(t-\tau) \times e^{iS_{st}(p_{st}(t,\tau))},$$

where $\tau = t - t'$ ⇒ time spent in the continuum

where $p_{st}(t,\tau) = E\left(\dfrac{\cos t - \cos(t-\tau)}{\tau}\right)$

Note prefactor $\sim \left(\dfrac{\pi}{\epsilon + i\frac{\tau}{2}}\right)^{3/2}$ reflects $\dfrac{1}{\tau^{3/2}}$ diffusion of the wavepacket

⇒ cuts off contributions from trajectories with excursion times, τ, much larger than the cycle period.

⇒ Note: for a given τ, only the classical trajectory makes a significant contribution to the HHG signal

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$$S_{st}\left(p_{st}(t,\tau)\right) = \frac{1}{2}\int_{\underbrace{t-\tau}_{t'}}^{t} dt''\,(p_{st} - A(t''))^2 + I_p\tau$$

why τ and not $-t'$ like in SFA? (see below) page 3

$$= (I_p + U_p)\tau - 2U_p\left[1 - \frac{\cos(\tau)}{\tau}\right] - U_p\,C(\tau)\cdot\cos(2t-\tau) + I_p\tau$$

where $U_p = \frac{1}{4}\cdot\left(\frac{E}{\omega}\right)^2$ ⇒ ponderomotive energy

and $C(\tau) = \sin(\tau) - 4\frac{\sin^2(\tau/2)}{\tau}$

Saddle point Eqns.:

(1) $\nabla_p S(\vec{p},t,\tau) = x(t) - x(t-\tau) = 0$ ⇒ needed to perform the first integral over $\vec{p}$ (last class)

(2) $\frac{\partial S(\vec{p},t,\tau)}{\partial\tau} = \frac{1}{2}\left(\vec{p} - A(\underbrace{t-\tau}_{t'})\right)^2 + I_p = 0$ ⇒ was used to derive SFA tunneling rates $|a_v|^2 = |\langle v|\Psi\rangle|^2$ (a few classes ago)

t' complex for $I_p \neq 0$

(3) $\frac{\partial S(p,t,\tau)}{\partial t} = \frac{(p-A(t))^2}{2} - \frac{(p-A(t-\tau))^2}{2} = E = 2k+1$

⇒ 3 Eqns, 3 unknowns.

⇒ comes from conservation of energy, whereby a harmonic of energy $(2k+1)$th carries off the access energy of the electron.

⇒ From condition (2), if $I_p \approx 0$, then

$\frac{1}{2}\left(\vec{p} - A(t-\tau)\right)^2 = \frac{1}{2}v^2(t') = 0$ ⇒ corresponds with the semiclassical assumption of $v=0$ @ tunnel exit!

time of ionization

Because $I_p \neq 0$; t' is complex, with imaginary part corresponding to "under barrier motion".

Recall from SFA lecture:

$$a_v = \Psi(v,t) = -i\int_0^t dt'\, e^{-\frac{i}{2}\int_{t'}^{t}(p-A(t''))^2 dt''}\, d_x(t')\, e^{iI_p t'}$$

$$S(t,t') = \frac{1}{2}\int_{t'}^{t}(p-A(t''))^2\, dt'' - I_p t'$$

then $\frac{\partial S}{\partial t'} = 0 \Rightarrow \frac{1}{2}(p-A(t'))^2 + I_p = 0 \Rightarrow p_x - \frac{F}{\omega}\sin t' = i\sqrt{2I_p + v_\perp^2} \approx i\sqrt{2I_p}\left(1+\frac{v_\perp^2}{4I_p}\right)$

$\Rightarrow$ from this, the standard momenta distributions were derived: $|\langle p|\Psi(t)\rangle|^2 \propto e^{-2\,\mathrm{Im}\, S(t,t_{st})}$

$$|\langle p|\Psi(t)\rangle|^2 \propto e^{-\frac{2(2I_p)^{3/2}}{3F}}\, e^{-\frac{p_\perp^2}{2\sigma_\perp^2}} \times e^{-\frac{p_x^2}{2\sigma_x^2}}$$

$$\sigma_\perp = \sqrt{\frac{\omega_L}{2\gamma}} \;; \qquad \sigma_x = \sqrt{\frac{3\omega_L}{2\gamma^3}}$$

$t_{st} = \underbrace{t_R}_{\text{time electron appears @ tunnel exit}} + \underbrace{i\, t_{im}}_{\text{imaginary time}}$: for $\gamma \ll 1 \Rightarrow$

$t_{im} = \tau_{Keldysh} = \frac{\gamma}{\omega} = \frac{\sqrt{2I_p}}{F}$

$\Rightarrow$ How did $I_p\tau$ appear in $S_{st}(p_{st}(t,\tau))$ instead of $I_p t'$, like in $S(t,t')$?

From $x(t) = \langle\Psi(t)|x|\Psi(t)\rangle =$

$$\left(\langle 0|e^{-iI_p t} + \int d^3v' \underbrace{\langle\Psi|v'\rangle}_{\text{contributes: } e^{-I_p t'}}\langle v'|\right) x \left(\underbrace{e^{iI_p t}}\,|0\rangle + \int d^3v\, \underbrace{\langle v|\Psi(t)\rangle}_{\text{contributes } e^{iI_p t'}}\, |v\rangle\right)$$

where $t' = t - \tau$

(4)

$\Rightarrow$ hence, we get $I_p \cdot |t-t'| = I_p \tau$ instead of $-I_p t'$.

FROM (2) $-\frac{1}{2}(\vec{p} - A(t-\tau))^2 = I_p$

(3) becomes : $\frac{(p - A(t))^2}{2} + I_p = E_{kin}(t) + I_p = 2K+1$

$\Rightarrow$ conservation of energy $\Rightarrow$ the KE of electron upon return + ionization potential gives the energy of the emitted harmonic!

Cut-off harmonic: $(2K+1)_{max} = \max \mathrm{Re}\Big(\underbrace{\frac{(\vec{P}_{st}(\tau) - A(t(\tau)))^2}{2}}_{\text{maximum KE}}\Big) + I_p$

where, from (1), $p_{st}(t,\tau) = \frac{E(\cos t - \cos(t-\tau))}{\tau}$

With classical trajectories, maximum E_{kin} is given by $3.17 \times U_p$ ("the simpleman's model" or SM model)

$\Rightarrow$ However, SM assumes electron starts with $V=0$ & returns to the same point

$\Rightarrow$ In reality, tunnel exit: $x_0 \approx \frac{I_p}{E(t')}$ $\Rightarrow$ the electrons thus acquire additional KE as they move from x_0 to the origin

$\Rightarrow$ therefore, the exact Quantum cut-off law is actually $(2K+1)_{max} \approx 1.32\, I_p + 3.17\, U_p$, rather than $(2K+1)_{max} = I_p + 3.17\, U_p$

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Recap: Need to evaluate $x(t)$, which involves an integral over $\int_0^t dt' \int d^3\vec{p} \ldots$

Both integrals can be evaluated using saddle-pt. method: (1) $\nabla_{\vec{p}} S(\vec{p}, t, \tau) = 0$; (2) $\frac{\partial S(\vec{p}, t, \tau)}{\partial \tau} = 0$

$\Rightarrow$ SP method is asymptotically exact, provided U_p, I_p & K are large enough.

$\Rightarrow$ Harmonic intensity of $2K+1$ harmonic is given by the Fourier transform of $x(t)$ $\Rightarrow$ $X_{2K+1} = \frac{1}{2\pi} \int_{t_0}^{t_0+2\pi} dt \, x(t) \, e^{(2K+1)it}$

Conditions (1)-(3) can already be used to derive the QM. cut-off law, without explicitly computing $x(t)$.

$\Rightarrow$ Inserting $p_{st}(t, \tau) = \frac{E(\cos t - \cos(t-\tau))}{\tau}$ into (2)

or $\frac{1}{2}(p_{st} - E\sin(t-\tau))^2 + I_p = 0$, we get:

$\sin(t - \tau/2)\, a(\tau) - \cos(t - \tau/2)\, s(\tau) = i\sqrt{\frac{I_p}{2U_p}}$

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where $a(\tau) = \cos(\tau/2) - \dfrac{2\sin(\tau/2)}{\tau}$; $s(\tau) = \sin(\tau/2)$

Inserting into condition (3); results in:

$$\boxed{f(\tau) + i\sqrt{\frac{I_p}{2U_p}}\, g(\tau) = \frac{2K+1}{U_p}}$$ condition (3)

where $f(\tau) = 8\left(\dfrac{a^2(\tau)\, s^2(\tau)\left[s^2(\tau) + a^2(\tau) + \frac{I_p}{U_p}\right]}{\left[s^2(\tau) + a^2(\tau)\right]^2}\right)$

$$g(\tau) = 8\,\frac{a(\tau)\, s(\tau)\left[a^2(\tau) - s^2(\tau)\right]\sqrt{s^2(\tau) - a^2(\tau) + \frac{I_p}{2U_p}}}{\left(s(\tau)^2 + a^2(\tau)\right)^2}$$

Quantum cut-off given by:

$$(2K+1)_{max} = U_p \max Re \underbrace{\left(f(\tau) + i\sqrt{\frac{I_p}{2U_p}}\, g(\tau)\right)}_{\text{Note involves maximization wrt } \tau \text{ only}}$$

with a constraint: $Im\left(f(\tau) + i\sqrt{\dfrac{I_p}{2U_p}}\, g(\tau)\right) = 0$

$\Rightarrow$ Results in $(2K+1)_{max} = 3.17\, U_p + I_p\, F\left(\dfrac{I_p}{U_p}\right)$

For $I_p \ll U_p$; the cut-off is given by:

$$(2K+1)_{max} \approx 3.17\, U_p + 1.32\, I_p$$