Lecture 4

SFA: $\langle v|\Psi(t)\rangle \sim e^{-iS_v(t,t_0')}$

where t_0' is found by solving: $\frac{\partial S_v}{\partial t'} = 0$

where $S_v(t,t') = \frac{1}{2}\int_{t'}^{t} dt'' \left[V_x - V_0 \sin(\omega_L t) + V_0 \sin(\omega_L t'')\right]^2$

$$- I_p t' + \frac{V_\perp^2}{2}(t - t')$$

from $\frac{\partial S_v}{\partial t'} = 0 \Rightarrow \left(\frac{V_x}{V_0} - \sin(\omega_L t) + \sin(\omega_L t')\right)^2 + (2I_p + V_\perp^2) \times \left(\frac{\omega_L}{F}\right)^2 = 0$

for $V_\perp = V_x = 0 \Rightarrow \boxed{\underbrace{|\Psi(v=0,t)|^2}_{\text{probability}} \sim e^{-2\frac{(2I_p)^{3/2}}{3F}}}$

Transverse and longitudinal velocity distributions: $V_\perp$; V_x :

Transverse: $V_\perp$ distribution $\Rightarrow V_\perp^2 = V_y^2 + V_z^2$ for

$\vec{A} = v_0 \sin(\omega t)\,\hat{x}$

$\Rightarrow$ just substitute $(2I_p + V_\perp^2)$ for $2I_p$ into:

$|\Psi(v=0,t)|^2 = P_0 \sim \exp\left[\frac{-2(2I_p)^{3/2}}{3F}\right]$

(remember $V_\perp \neq 0$ is like increasing ionization potential, I_p

$$(2I_p + V_\perp^2)^{3/2} = (2I_p)^{3/2}\left(1 + \frac{V_\perp^2}{2I_p}\right)^{3/2} \qquad \text{for } \frac{V_\perp^2}{2I_p} \ll 1$$

$$\approx (2I_p)^{3/2}\left(1 + \left(\frac{3}{2}\right)\frac{V_\perp^2}{2I_p}\right)$$

$$P(V_\perp) \approx \exp\left[\frac{-2[2I_p]^{3/2}\left(1 + \left(\frac{3}{2}\right)\frac{V_\perp^2}{2I_p}\right)}{3F}\right] =$$

$$P(V_\perp) \approx P_0 \cdot \exp\left[-\frac{(2I_p)^{3/2}}{F} \cdot \frac{V_\perp^2}{2I_p}\right] = \underbrace{P_0 \cdot \exp\left[\frac{-V_\perp^2}{2\sigma_\perp^2}\right]}_{\text{Gaussian distribution}}$$

where $\sigma_\perp^2 = \frac{1}{2} \cdot \frac{F}{(2I_p)^{1/2}} = \frac{1}{2} \cdot \frac{\omega_L}{\gamma}$

since $\gamma = \frac{(2I_p)^{1/2}}{F} \cdot \omega_L$

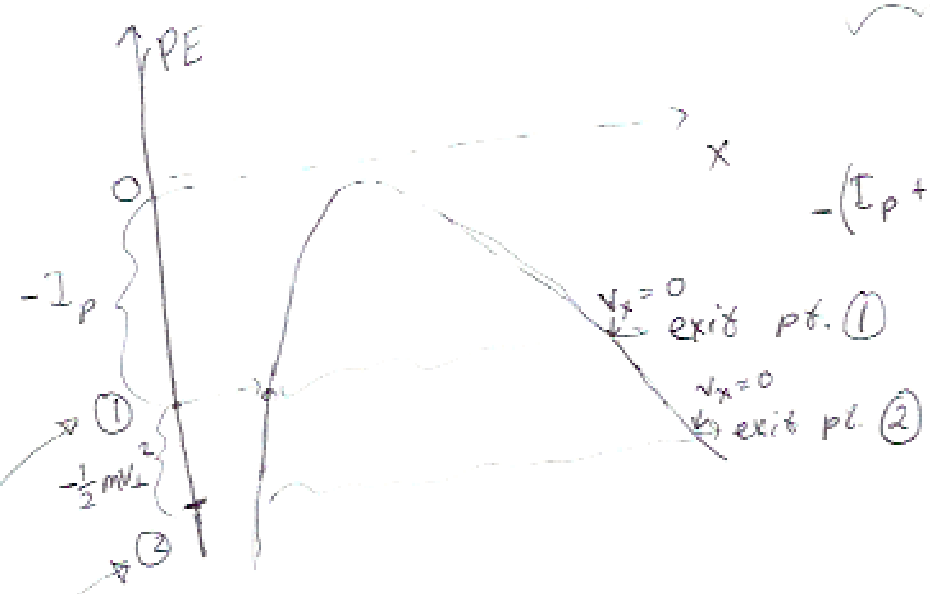
$$\Rightarrow \boxed{\sigma_\perp = \sqrt{\frac{\omega_L}{2\gamma}}}$$

$$P(v_x = 0, V_\perp) \sim e^{\frac{-2(2I_p)^{3/2}}{3F}} \cdot e^{-\frac{V_\perp^2}{2\sigma_\perp^2}}$$ (ionization @ the peak of the $\vec{E}$ field)

Why does $v_\perp \neq 0$ decrease ionization probability?

③

bound energy: $E_{electron} = -I_p = \underbrace{\frac{1}{2}m(V_x^2 + V_\perp^2)}_{KE} + V(x)$



$-(I_p + \frac{1}{2}mV_\perp^2) = \frac{1}{2}mV_x^2 + V(x)$

Effective height of the barrier: $I_p + \frac{1}{2}mV_\perp^2$

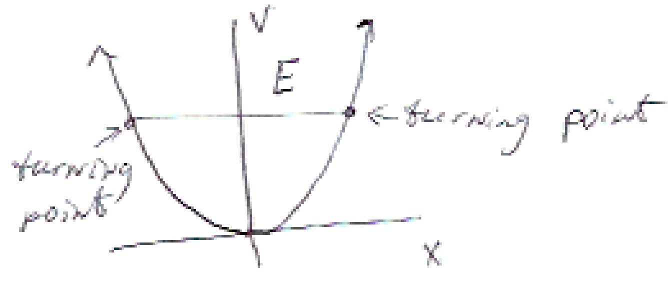
point ① $V_\perp^2 = 0$

point ② $V_\perp^2 \neq 0$

Entry point into a tunnel for $V_\perp^2 = 0 \Rightarrow$ barrier height $= I_p$

(remember barrier is a "classically inacessible region" $\Rightarrow$ a turning point where $E = PE$)

Example : SHO



Entry into a barrier at $V_\perp \neq 0$

barrier height $= (+I_p + \frac{1}{2}mV_\perp^2)$

for $V_\perp \neq 0 \Rightarrow$ lower transition amplitudes

$\Rightarrow$ farther exit points

(the electron has to tunnel deeper under a barrier)

Longitudinal $P(V_x)$ (parallel to the $\vec{E}$ field)

$$\left[\left\{\frac{V_x}{V_0} - \sin(\omega_L t)\right\} + \sin(\omega_L t')\right] = \pm i\gamma'$$

, where t' is now complex: $t' = t_R + i\Delta_i$

$$\sin(\omega_L t') = \frac{1}{2i}\left(e^{i(\omega_L t')} - e^{-i\omega_L t'}\right) =$$

$$\underbrace{\frac{1}{2}\sin(\omega_L t_R)\left(e^{-\omega_L \Delta_i} + e^{+\omega_L \Delta_i}\right)}_{\text{real part}} - \frac{i}{2}\cos(\omega_L t_R)\underbrace{\left(e^{-\omega_L \Delta_i} - e^{+\omega_L \Delta_i}\right)}_{\text{imaginary part}}$$

$$\boxed{\sin(\omega_L t') \approx \sin(\omega_L t_R) + i\cos(\omega_L t_R)(\omega_L \Delta_i)}$$

1 Real: $\boxed{\left\{\frac{V_x}{V_0} - \sin(\omega_L t)\right\} = -\sin(\omega_L t_R)}$

(Note if $t=0$, t_R is before the peak & $\frac{V_x}{V_0} \approx -\omega_L t_R$)

$\Rightarrow$ $V_x > 0$ for ionization before the peak (remember V_x @ $t=0$ is the same as V_x @ large t due to $[V_x - A_x(t)] = [V_x(t\to\infty)]$ $\Rightarrow$ conserved canonical angular momentum

(both @ $t=0$ & $t\to$ large, $A_x = 0$)

$\Rightarrow$ For linearly polarized light: $P(v, t=0) = P(v, t\to\text{large})$

$\Rightarrow$ same distribution seen @ $t=0$ as @ the detector!

2 Imaginary part: $\pm\frac{\gamma'}{\omega_L} = \cos(\omega_L t_R)\cdot\Delta_i$

From 1, @ $t=0$ $\Rightarrow \sin(\omega_L t_R) = -\frac{V_x}{V_0}$

$$\Rightarrow \cos(\omega_L t_R) = \left[1 - \left(\frac{V_x}{V_0}\right)^2\right]^{1/2}$$

$$\boxed{\Delta_i = \pm \frac{\sqrt{2 I_p}}{F} \cdot \frac{1}{\cos(\omega_L t_R)}}$$

$$S_v(0, t') = \frac{1}{2}\int_{t'}^{0} dt'' \left[v_x - \cancel{v_0 \sin(\omega_L t)} + v_0 \sin(\omega_L t'')\right]^2$$

$t=0$

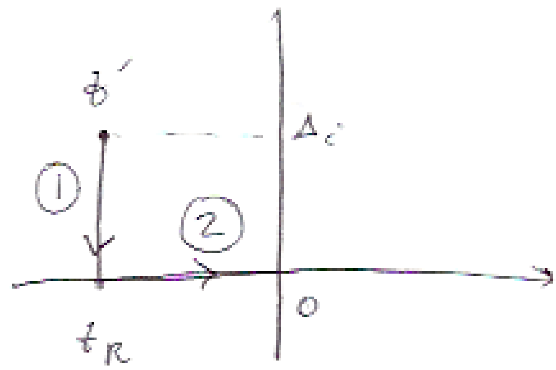
$$\cancel{- I_p t' + \frac{v_x^2}{2}(0 - t')}$$

on

(interested in v_x terms only)

where $t' = t_R + i\Delta_i$

$$\int_{t'}^{t} \Rightarrow \underbrace{\int_{t_R + i\Delta_i}^{t_R}}_{1} + \underbrace{\int_{t_R}^{0}}_{2}$$



Only interested in 1, since 2 only contributes to the phase (comes out all real)

$$\Rightarrow \sin(\omega_L t') \approx \sin(\omega_L t_R) + i\cos(\omega_L t_R)(\omega_L \Delta_i)$$

$$\Rightarrow \frac{1}{2}\int_{t_R + i\Delta_i}^{t_R} (\cancel{C_1}^{\,0} + C_2 \Delta_i)^2 \, dt''$$

where $C_1 = v_x + v_0 \sin(\omega_L t_R) = 0$

$C_2 = v_0 \cos(\omega_L t_R) \cdot \omega_L$

$$\underbrace{\frac{1}{2}\int_{t_R+i\Delta_i}^{t_R} dt\,\left[F^2\cos^2(\omega_L t_R)\cdot\Delta_i\right]^2}_{\text{Integral}} - iI_p\,\frac{\sqrt{2I_p}}{F}\cdot\frac{1}{\cos(\omega_L t_R)} = \mathrm{Im}(S)$$

Integral $$= \frac{1}{2}F^2\cos^2(\omega_L t_R)\cdot\frac{(\Delta_i)^3}{3}\Bigg|_{i\Delta_i}^{0} = \frac{i}{2}\,\frac{(2I_p)^{3/2}}{3F\cos(\omega_L t_R)}$$

(From page 5: $\Delta_i = \dfrac{\sqrt{2I_p}}{F}\cdot\dfrac{1}{\cos(\omega_L t_R)}$)

$$\mathrm{Im}(S) = \frac{i}{2}\,\frac{(2I_p)^{3/2}}{3F\cos(\omega_L t_R)} - \frac{i}{2}\,\frac{(2I_p)^{3/2}}{F\cos(\omega_L t_R)} = \frac{-i\,(2I_p)^{3/2}}{3F\cos(\omega_L t_R)}$$

bottom of page 4 $\Rightarrow$ $\dfrac{1}{\cos(\omega_L t_R)} = \dfrac{1}{\left(1-\left(\frac{v_x}{v_0}\right)^2\right)^{1/2}} \approx 1+\dfrac{1}{2}\left(\dfrac{v_x}{v_0}\right)^2$ (for $(v_x/v_0)^2 \ll 1$)

$$\Rightarrow \boxed{\mathrm{Im}(S) \approx \frac{-i\,(2I_p)^{3/2}}{3F}\times\left(1+\frac{1}{2}\left(\frac{v_x}{v_0}\right)^2\right)}$$

$$P = |\Psi|^2 \propto e^{-2i\,\mathrm{Im}(S)}$$

v_x-dependent part of $-2i\,\mathrm{Im}(S)$ $= -\dfrac{(2I_p)^{3/2}}{3F}\times\left(\dfrac{v_x}{v_0}\right)^2$

note: v_x is scaled by v_0!

$$\Rightarrow \boxed{= -\frac{(2I_p)^{3/2}}{3F}\times\left(\frac{v_x}{\underbrace{F/\omega_L}_{v_0}}\right)^2 = -\frac{\gamma^3}{3}\cdot\frac{1}{\omega_L}\cdot v_x^2 = \frac{-v_x^2}{2\sigma_x^2}}$$

Note: $\gamma = \dfrac{(2I_p)^{1/2}}{F}\,\omega_L$

$\Rightarrow \boxed{\sigma_x = \sqrt{\dfrac{3\omega_L}{2\gamma^3}}}$ from page 2 $\Rightarrow \boxed{\sigma_\perp = \sqrt{\dfrac{\omega_L}{2\gamma}}}$

probability distribution $\Rightarrow$ $$P(\vec{v}, t=0) = e^{-\frac{2(2I_p)^{3/2}}{3F}} \times e^{-\frac{v_\perp^2}{2\sigma_\perp^2}} \times e^{-\frac{v_x^2}{2\sigma_x^2}}$$