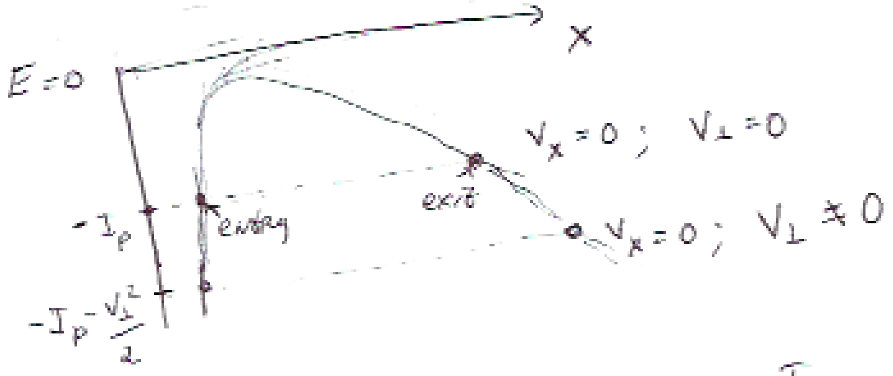
Lecture 5

Momentum distribution immediately after ionization



$$E = \frac{m}{2}\left(V_x^2 + V_\perp^2\right) + PE = -I_p$$

entry & exit from a tunnel @ $V_x = 0$ (classically forbidden region)

in a.u. : $PE = -I_p - \frac{1}{2}V_\perp^2$ (at entry & exit points)

$\Rightarrow$ height of the barrier

Last class: $P(\vec{v}, t=0) = \left|\langle \vec{v} | \Psi(t=0) \rangle\right|^2$

$$\sim \underbrace{e^{-\frac{2(2I_p)^{3/2}}{3F}}}_{\text{prefactor}} \times e^{-\frac{V_\perp^2}{2\sigma_\perp^2}} \times e^{-\frac{V_x^2}{2\sigma_x^2}}$$

where $\sigma_\perp = \sqrt{\frac{\omega_L}{2\gamma}}$; $\sigma_x = \sqrt{\frac{3\omega_L}{2\gamma^3}}$

Keldysh parameter : $\gamma = \frac{\sqrt{2I_p}}{F} \cdot \omega_L$

$\sigma_\perp \Rightarrow$ no frequency dependence

$\sigma_x \Rightarrow \frac{1}{\omega_L}$ frequency dependence

How to reconcile with the static tunneling picture with $V_x = 0$ @ the exit point?

⇒ all this can be understood from conservation of canonical angular momentum: $\vec{P}$

$$H = \frac{1}{2m}\underbrace{\left(\vec{P} - \frac{q}{c}A(t)\right)^2}_{\frac{1}{2}mv^2} + \cancel{q\phi(\vec{r})}$$ neglect coulomb potential

$$\dot{P}_x = \frac{\partial H}{\partial x} = \dot{P}_y = \frac{\partial H}{\partial y} = \dot{P}_z = \frac{\partial H}{\partial z} = 0$$ (no spatial dependence in the Hamiltonian!)

⇒ $P_x = \text{const} = C_1$; $P_y = C_2$; $P_z = C_3$

momentum distribution for linear polarization:

After ionization: $V_x = 0$, $V_\perp \neq 0$; $\frac{e\vec{A}}{c} = -V_0 \sin(\omega t)\,\hat{x}$

at exit point

$mV_x = P_x + V_0 \sin(\omega t)$

⇒ $0 = P_x + V_0 \sin(\omega t_e)$

$\boxed{V_\perp = P_\perp = \text{const}}$

$V_\perp$ unchanged for all t!

where t_e is the time of exit from the tunnel

$\boxed{P_x = -V_0 \sin(\omega t_e)}$ ⇒ can. ang. momentum determined by time of exit from the tunnel, t_e!

$\vec{E} = \underbrace{V_0\omega}_{F} \cos(\omega t)\,\hat{x}$ ⇒ peak @ $t=0$

⇒ ionization before the peak: $t_e < 0$ ⇒ $P_x > 0$

⇒ ionization after the " : $t_e > 0$ ⇒ $P_x < 0$

$\left\{\begin{array}{l} P_x = mV_x \text{ @ the detector! (since } mv_x = P_x + A(\underset{0}{t_{detector}})\text{)} \\ P_\perp = mV_\perp \text{ always (so @ the detector too)} \end{array}\right.$

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tunneling probability in the quasistatic limit:

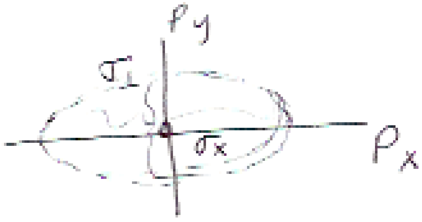
$$P \sim e^{\frac{-2\left[2\left(I_p+\frac{1}{2}V_\perp^2\right)\right]^{3/2}}{3F\cos(\omega t_e)}}$$

since $F\cos(\omega t_e)$ is the strength of $\vec{E}$ @ time t_e, where t_e is when the electron is "BORN" into the continuum (i.e. appears at exit point)

⇒ Peak of distribution @ $t_e = 0$, where $p_x = 0$, then exponential decrease for $p_x = mv_x \neq 0$ (given by: $e^{-V_x^2/2\sigma_x^2}$ where $\sigma_x = \sqrt{\frac{3\omega_L}{2\gamma^3}}$)

and @ $p_\perp = mV_\perp = 0$, then exponential decrease for $V_\perp \neq 0$ (given by $e^{-V_\perp^2/2\sigma_\perp^2}$ where $\sigma_\perp = \sqrt{\frac{\omega_L}{2\gamma}}$)

Linear pol. light:



$$\gamma = \frac{\sqrt{2I_p}}{F}\omega_L$$

$$P \sim e^{\frac{-2\left[2\left(I_p+\frac{1}{2}V_\perp^2\right)\right]^{3/2}}{3F\cos(\omega t_e)}} \approx \exp\left[\frac{-2(2I_p)^{3/2}}{3F\cos(\omega t_e)}\left(1+\frac{3V_\perp^2}{2I_p}\right)\right] =$$

$$= e^{\frac{-2(2I_p)^{3/2}}{3F\cos(\omega t_e)}} * \underbrace{e^{\frac{-2(2I_p)^{1/2}}{2F\cos(\omega t_e)}\cdot(V_\perp^2)}}_{= e^{-\frac{V_\perp^2}{2\sigma_\perp^2\cdot\cos(\omega t_e)}}}$$

⇒ longitudinal, V_x distribution:

$$P \propto e^{\frac{-2(2I_p)^{3/2}}{3F\cos(\omega t_e)}} \approx \exp\left[\frac{-2(2I_p)^{3/2}}{3F}\left(1+\frac{(\omega t_e)^2}{2}\right)\right]$$

where we used: $\frac{1}{\cos(\omega t)} \approx \frac{1}{\left(1-\frac{(\omega t)^2}{2!}\right)} \approx 1+\frac{(\omega t)^2}{2}$ for $\omega t \ll 1$

Now use 1) $V_x = 0$ @ exit point @ $t = t_e$

2) Angular can. momentum is <u>conserved</u> after ionization (neglecting Coulomb force)

⇒ Result: distribution @ the detector given by

$$\sigma_x = \sqrt{\frac{3\omega}{2\gamma^3}} \; !$$

<u>Derivation:</u>

$$P_x = \left(0 + \frac{F}{\omega} \sin(\omega t_e) \right) = V_x \text{ @ detector}$$

conserved canonical ang. momentum

exit time from tunnel

(the above follows from 1) and 2))

$$\Rightarrow V_x = \frac{F}{\omega} \sin(\omega t_e) \approx F \cdot t_e \quad \Rightarrow \omega t_e \approx \boxed{\frac{V_x}{F}\omega}$$

From page ③ ⇒

$$P \propto e^{-\frac{2(2I_p)^{3/2}}{3F\cos(\omega t)}} \approx \exp\left[-\frac{2(2I_p)^{3/2}}{3F}\left(1 + \frac{(\omega t_e)^2}{2}\right)\right]$$

$$\approx \underbrace{e^{-\frac{2(2I_p)^{3/2}}{3F}}}_{P_0} \cdot e^{-\frac{2(2I_p)^{3/2}}{3F}\left(\frac{V_x^2}{2F^2}\omega^2\right)} = P_0 e^{-\frac{\gamma^3}{3\omega}(V_x)^2}$$

$$= P_0 e^{-\frac{V_x^2}{2\sigma_x^2}} \; !$$

⇒ So we derived the same σ_x as from SFA using the <u>adiabatic</u> <u>tunneling formula</u>: $P \propto e^{-\frac{2(2I_p)^{3/2}}{3F\cos(\omega t_e)}}$ and assumptions ① & ② ! (it was much simpler!)

⇒ alternative derivation of the adiabatic tunneling formula?

yes: by solving for tunneling through the static triangular barrier: ① exact solution or ② WKB approximation

Momentum distribution (v_x, v_y, v_z) for elliptical polarization:

Elliptical light: $\vec{E} = F\cdot\left(\frac{1\,\hat{x}}{\sqrt{1+\varepsilon^2}}\cos(\omega t) \pm \frac{\varepsilon\,\hat{y}}{\sqrt{1+\varepsilon^2}}\sin(\omega t)\right]$

+ sign ⇒ counterclockwise (if looking along $-\hat{z}$)

− sign ⇒ clockwise

polarized in the $\hat{x}, \hat{y}$ plane (propagation along $\hat{z}$)

ε – ellipticity ⇒ $\varepsilon < 1$ (peak in ionization probability still occurs at $\omega t = 0, 180°$)

after ionization: $v_x = 0$ @ $t = t_e$ (exit time)

$$\Rightarrow mv_x = 0 = P_x + \frac{A_x(t_e)}{c}, \quad mv_y = P_y + \frac{A_y(t_e)}{c}$$

$$mv_z = P_z + \frac{A_z(t_e)}{c} = P_z \quad (\text{since } A_z = 0 \text{ for all } t)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \Rightarrow \frac{\vec{A}}{c} = \frac{F}{\omega}\left(\frac{1\,\hat{x}}{\sqrt{1+\varepsilon^2}}\sin(\omega t) \mp \frac{\varepsilon\,\hat{y}}{\sqrt{1+\varepsilon^2}}\cos(\omega t)\right)$$

$$\Rightarrow 0 = P_x + \frac{F}{\omega}\cdot\frac{\sin(\omega t_e)}{\sqrt{1+\varepsilon^2}} \Rightarrow mv_x = -\frac{F}{\omega}\frac{\sin(\omega t_e)}{\sqrt{1+\varepsilon^2}} \text{ @ detector}$$

peak of ionization @ $\omega t_e = 0, 180°$ the distribution @ detector is centered around $v_x = 0$ (same as for linearly polarized light)

$$\Rightarrow mv_y = P_y \mp \frac{F}{\omega}\cdot\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\cdot\cos(\omega t_e) \Rightarrow \text{distribution}$$

peaks @ $\omega t_e = 0, 180°$; $v_y = 0$ (remember $\sim e^{-\frac{v_\perp^2}{2\sigma_\perp^2}} = e^{-\frac{v_y^2+v_z^2}{2\sigma_\perp^2}}$

$$\Rightarrow P_y = \pm\frac{F}{\omega}\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \quad (\text{center of distribution along } y)$$

⇒ P_z centered at $P_z = 0$, since $P(v_z(t_e))$ is centered @ 0 + $\sim e^{-v_z^2/2\sigma_\perp^2}$

Distribution @ detector:

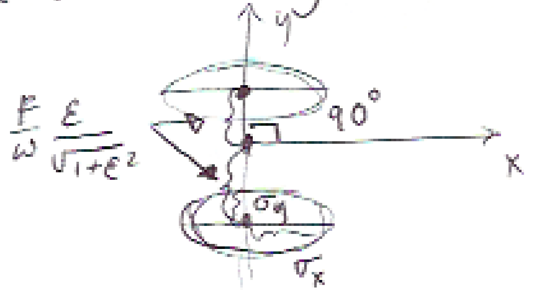
centered at $v_x = 0$; $v_z = 0$; $v_y = \pm \frac{F}{\omega} \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$

⇒ at ionization time: $t = t_e$, the distribution is centered around $v_x = v_z = v_y = 0$

⇒ the center along v_y shifts to $v_y = \pm \frac{F}{\omega} \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ at detector because $A_y \neq 0$ at $t_e = 0$, so the distribution is centered @ $p_x = 0 - \frac{A_x(t_e)}{c} = \pm \frac{F}{\omega}\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$

Coulomb effects?

⇒ lead to a shifting of an angle



direction of polarization along $\hat{x}$ @ $t = 0$ since $\vec{E} = F\left(\frac{\cos(\omega t)\hat{x}}{\sqrt{1+\varepsilon^2}} \pm \frac{\varepsilon \sin(\omega t)\hat{y}}{\sqrt{1+\varepsilon^2}}\right)$

clockwise

$$H = \underbrace{\frac{1}{2m}\left(p_x + \frac{F}{\omega}\sin(\omega t)\right)^2}_{\frac{1}{2}mv_x^2} + \underbrace{\frac{1}{2}m\left(p_y + \frac{F}{\omega}\cos(\omega t)\right)^2}_{\frac{1}{2}mv_y^2} + \underbrace{\frac{p_z^2}{2m}}_{\frac{1}{2}mv_z^2} - \underbrace{\frac{1}{r}}_{\text{correction due to Coulomb}}$$

⇒ spatial dependence, r, so $\frac{\partial H}{\partial r} \neq 0$ ⇒ no longer conservation of canonical angl. momentum

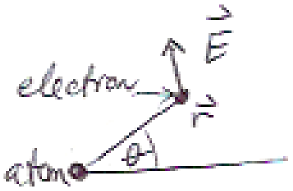
⇒ small correction $\Delta\theta$ to the $90°$ angle due to coulomb force.

Why? (since Coulomb force is central, why does it change the angle?)

Coulomb correction, $\Delta\theta$:

$$\vec{E} = F\,[\cos(\omega t)\,\hat{x} \pm \sin(\omega t)\,\hat{y}]$$ (circular polarization)

$$\vec{r} = r_0\,(\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y})$$ ⇒ location of electron in cylindrical coordinates



$$E_r = \frac{\vec{E}\cdot\vec{r}}{r_0} = F\,[\cos(\omega t)\cdot\cos\theta \pm \sin(\omega t)\sin\theta] = F\cos(\omega t \mp \theta)$$

Tangential force from laser field: $eE_\theta = -F\,(1-\cos^2(\omega t \mp \theta))^{1/2}$

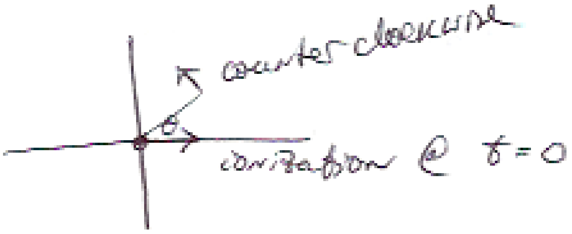
$$= -F\sin(\omega t \mp \theta)$$

$$\frac{F_{tan}}{m} = r\ddot{\theta} = -F\sin(\omega t \mp \theta)$$

$\Delta\theta = 90^\circ$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{F}{r}\sin(\omega t \mp \theta)} - F_c$$

r is smaller because of the attractive Coulomb force, so the rotation θ, is bigger!



$\ddot{\theta} \approx \pm\frac{F}{r}(\theta - \omega t)$ for $\theta - \omega t \ll 1$

Counter clockwise (direction of $\uparrow\theta$): $\Delta\theta > 0$

Clockwise (" " $\downarrow\theta$): $\Delta\theta < 0$