Lecture 9

Recap from last lecture: Parabolic coords (tunneling)

$\Rightarrow$ extends the tunneling regime

HHG $\Rightarrow$ tunneling is the first step

low freq., strong field, Example: Ti-Sapphire

infrared {laser $\Rightarrow \lambda \sim 800$ nm

high $U_p$ {Intensity $\sim O(10^{15})\ \frac{W}{cm^2}$

$U_p = \frac{F^2}{4\omega^2} \Rightarrow$ KE of a free electron interacting with a laser field ("quiver energy")

HHG regime (also tunneling regime) $\Rightarrow \boxed{\hbar\omega \ll I_p \ll U_p}$

tunneling or multi-photon ionization

need to generate high freq harmonics, since

$E_{cut\text{-}off} \approx I_p + 3.\cancel{17}\, U_p$

$\hbar\omega_{cut\text{-}off}$

For Ti-Sapphire $\Rightarrow$ use

$\Rightarrow \omega_L(\text{in a.u.}) \approx \frac{45.56}{\lambda(\text{in nm})} = \frac{45.56}{800\text{ nm}} \approx .057\text{au} = \hbar\omega$

$I_p \approx .9$ au for Helium ; $U_p \approx \frac{F^2}{4\omega^2} = \frac{(.1)^2}{4(.057)} = 1.7$

($F = .1$ au corresponds to $\text{intensity}_{max} \approx 6.6 \times 10^{14}\ \frac{W}{cm^2}$ for LPL)

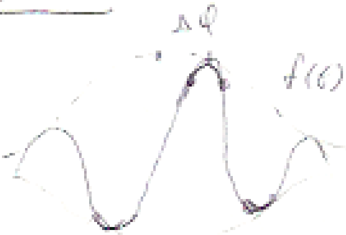
$\Rightarrow \hbar\omega \ll I_p < U_p$

also $\boxed{\hbar\omega_L \ll I_p \ll U_p}$ is a tunneling regime in terms of the Keldysh parameter: $\gamma < 1$

$\Rightarrow \gamma = \frac{(2I_p)^{1/2}}{F}\cdot\omega_L = \left(\frac{1}{2}\cdot\frac{I_p}{U_p}\right)^{1/2} < 1$ since $I_p < U_p$

also need $\hbar\omega_L \ll I_p$, so electron tunnels, rather than escapes by absorbing 1 photon

Laser pulse: $E(t) = F \cdot f(t) \cos(\omega t + \Delta\varphi)$




$f(t) \Rightarrow$ envelope   $\Delta\varphi \Rightarrow$ carrier envelope phase (CEP)

shortest $\Rightarrow$ few-cycle pulse ~5 fs width (ionization happens near the peaks)

$\Rightarrow$ Tunnel ionization probability: $P \sim e^{-\frac{2(2I_p)^{3/2}}{3F\cos(\omega t)}}$

$\omega t = \phi$ (probability of ionization depends on the phase of the field)

$F\cos(\omega t) \rightarrow F \cdot f(t)\cos(\omega t + \Delta\varphi) - |\vec{E}|$ laser field strength

max prob. when $|\vec{E}|$ is @ its maximum

$\frac{\partial |\vec{E}|}{\partial t} = 0$   For slowly varying envelope: $\omega f(t) \gg \frac{\partial f}{\partial t}$

$\Rightarrow$ peak ionization @ $\omega t + \Delta\varphi \approx 0$

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short (~2 optical cycle) laser pulse $\Rightarrow$ isolated attosecond pulse

HHG $\Rightarrow$ The semi-classical 3-step model

"simple man" model $\Rightarrow$ Corkum PRL '93

$E_{max} = I_p + 3.17\, U_p$ (cut-off freq.)

Quantum mechanical treatment - Lewenstein et al (incl. Corkum) '94 PRA (similar techniques used in Becker et al. PRA '90 but for "zero-range" potential)

$\Rightarrow$ extension of SFA (cf previous lectures $\Psi(r,t) \sim e^{iS_0}$) with additional constraints (to include only returning electrons)

$\Rightarrow$ saddle point method ~~$\Psi(r, t_0)$, ...~~

$\Psi_r \sim A_L e^{iS_0^L} + A_S e^{iS_0^S}$ (freq. of ultra-harmonic depends on interference btwn long & short trajectories)

cut-off energy: $E_{max} = \hbar\omega_{max} = 1.32 I_p + 3.17 U_p$

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"Simple man's" model, use $\vec{E} = F\cos(\omega t)\,\hat{x}$ (LPL)

step 1 ⇒ tunnel ionization: $P \sim e^{-\frac{2(2I_p)^{3/2}}{3F\cos\phi}}$

$\phi = \omega t$

step 2 ⇒ accel. by the laser field

step 3 ⇒ recollision & ~~xxxxx~~ emission of as pulse made up of phase-locked harmonics, where

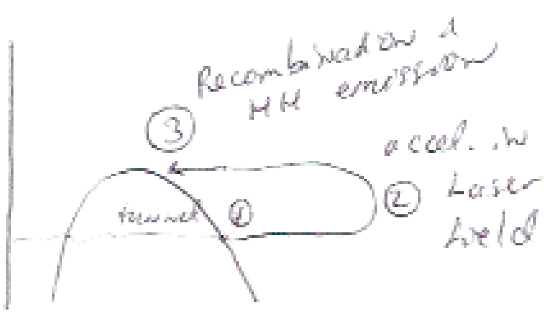
$\underbrace{\omega(\phi)}_{\text{freq. of HH depends on } \phi \text{ @ ionization}} = (2n+1)\,\omega_L$ (HH @ odd multiples of the laser frequency. Typical cut-off $\sim O(100)\times\omega_L$)

⇒ $\hbar\omega_{max} \approx 1.32 I_p + 3.17 U_p = (1.32)(0.9) + 3.17\cdot(1.7)$

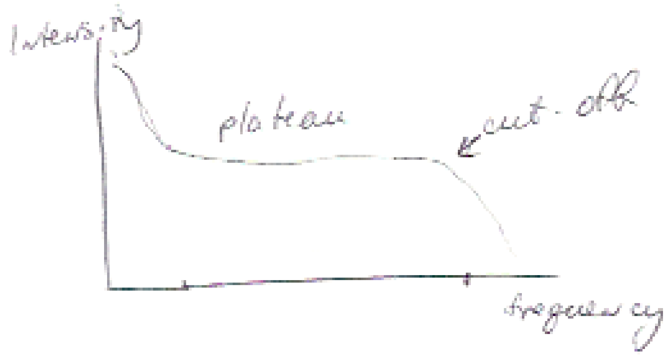
$\approx$ ~~xxx~~ $6.5 = (2n+1)\times .057$ ⇒ $2n+1 \approx 115$

⇒ cut-off @ 115th harmonic for Ti-sapphire $\lambda \sim 800$ nm, $F \approx .1$ au He experiment

3-step model




Harmonic spectrum




freq. of emitted photon determined by $\phi = \omega t_i$

ionization time ⇒ velocity @ recombination ⇒ $\hbar\omega = \frac{1}{2}mV(t_R)^2 + I_p$

Step 2 ⇒ classical model

free electron : $\ddot{x} = \underbrace{(e/m)}_{1\,au} \cdot F\cos(\omega t)$

general solution: $x(t) = -\frac{F}{\omega^2}\cos(\omega t) + \underbrace{V_0 t + x_0}$ determined by in. conditions ⇒ phase & velocity at ionization

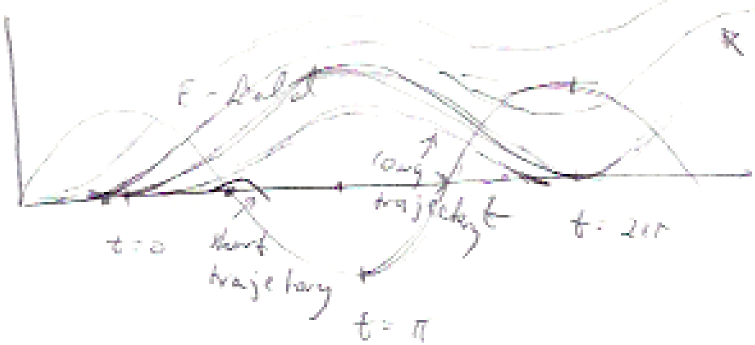
$V_x(t_i) = 0$ ⇒ electron appears into the continuum with zero velocity

$v_x(t) = \dot{x}(t) = \frac{F}{\omega}\sin(\omega t) + V_0$ ⇒ $V_0 = -\frac{F}{\omega}\sin(\underbrace{\omega t_i}_{\phi_i})$

Example: ionization @ the peak: $t_i = 0$ ; $\dot{x}(0) = 0$

⇒ $V_0 = 0$ ; $x_0 = \frac{F}{\omega^2}$

$x(t, \phi = 0) = -\frac{F}{\omega^2}\cos(\omega t) + \frac{F}{\omega^2}$




$\phi_i < 0$ ⇒ never returns to the atom!

⇒ electrons comes back to the atom @ $\omega t = 2\pi$ with zero energy ($\because V(t_i) = V(t_R) = 0$)

Case 1: ionization just before the peak ⇒ $t_i < 0$ $|\omega t_i| \ll 1$

$x(t, t_i < 0) \approx -\frac{F}{\omega^2}\cos(\omega t) + \underbrace{|F t_i|}_{V_0}\, t + \frac{F}{\omega^2} \neq 0$

for all $t > 0$ ⇒ never returns to the atom

Case 2: ionization after the peak : $t_i > 0$ $|\omega t_i| \ll 1$

$x(t, t_i > 0) \approx -\frac{F}{\omega^2}\cos(\omega t) - \underbrace{|F t_i|}_{V_0}\, t + \frac{F}{\omega^2} \sim 0$ for $\omega t < 2\pi$

Lecture 9



$v(t_R) \neq 0$

$\Rightarrow$ "long trajectory"

Case 3 there is another "short" trajectory for every "long trajectory" s.t. $v(t_R^{long}) = v(t_R^{short})$

where $t_R^{long} > t_R^{short}$

longest trajectories (highest prob of ionization, lower energy)

$\Rightarrow$ more spreading of the wave function since $t_R - t_i$ (travel time) is long, but high prob. of ionization

$\Leftrightarrow$

shortest trajectories (lowest prob of ionization, lower energy)

$\Rightarrow$ less spreading of the wave-function, but lower prob. of ionization

both contribute to the plateau ~

amplitude of wavefunction $\sim \frac{1}{\tau^{3/2}}$ where $\tau = t_R - t_i$

Energy @ $t = t_R$ $\Rightarrow$ $x(t_R) = 0 = -\frac{F}{\omega^2}\cos(\omega t_R) + v_0 t_R + x_0$

$$x(t_i) = 0 = -\frac{F}{\omega^2}\cos(\omega t_i) + v_0 t_i + x_0$$

$$\dot{x}(t_i) = 0 = \frac{F}{\omega}\sin(\omega t_i) + v_0 \Rightarrow v_0 = -\frac{F}{\omega}\sin(\omega t_i)$$

$$v(t_R) = \frac{F}{\omega}\left[\sin(\omega t_R) - \sin(\omega t_i)\right]$$

$$\Rightarrow \boxed{x(t_R) - x(t_i) = 0 = -\frac{F}{\omega^2}\left[\cos(\omega t_R) - \cos(\omega t_i)\right] - \frac{F}{\omega^2}\sin(\omega t_i)(t_R - t_i)}$$

Lecture 9 (6)

$t_r(t_i) \Rightarrow$ return time a function of ionization time

$$\frac{d\, v(t_R(t_i))\, \cancel{\text{ret}}}{dt_i} = 0 \Rightarrow \cancel{v_{max}} = v_{max}$$

maximum KE of the returning electron:

$$KE_{max} \approx 3.17 \cdot U_p$$

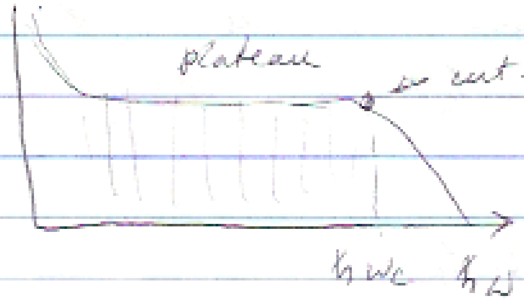
$$\frac{KE_{\cancel{max}}}{U_p} = \frac{\frac{1}{2}(F/\omega)^2\left(\sin(\omega t_R) - \cos(\omega t_i)\right)^2}{\frac{1}{4}(F/\omega)^2} =$$

$$f(t_i) = 2\left[\sin(\omega t_R) - \cos(\omega t_i)\right]^2 = 2\left[\sin(\omega t_R(t_i)) - \cos(\omega t_i)\right]^2$$

$$\Rightarrow \frac{df(t_i)}{dt_i} = 0 \Rightarrow 3.17 \text{ for } \omega t_i \approx 0.30 = 17^\circ$$

cut-off energy ~~of~~ $\Rightarrow$ $\boxed{\hbar\omega_c = I_p + 3.17 \cdot U_p}$

freq. of the highest harmonic!




plateau $\Rightarrow$ $(2n+1)\omega$ harmonics made up of interference of short + long trajectories $\Rightarrow$ short $\Rightarrow$ lower prob, but don't spread out as much as the long trajectories

$\Rightarrow$ spreading of the wavepacket $\sim \frac{1}{\tau^{3/2}}$, where $\tau$ is the return time: $\boxed{\tau = t_R - t_i}$

# Lecture 9

Plateau: ⇒ $\hbar\omega$ determined by phase, ~~but also~~ so $\phi$ @ ionization, so why the plateau?

⇒ high ionization, but longer $\tau$, more spreading of the wave-packet.

~~Intensity~~ Amplitude $\propto \underbrace{Amp(ion)}_{\Psi_0} \times \underbrace{\text{spreading}}_{\sim \frac{1}{\tau^{3/2}}} \approx$ const

works for short trajectories (Misha Ivanov)

(not exactly a plateau, either, downward sloping)

(Fig. 4.20 from the book)

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Quantum mechanical ⇒ "Lewenstein model" = ~~SFA~~ SFA

⇒ extension of SFA

$\langle v|\Psi\rangle \sim \int_{t_0'}^{t} e^{iS(t',t)} \underbrace{\langle\ \rangle}_{\text{prefactor}} dt' \sim e^{iS_0(t',t)}$

where $S_0(t',t)$ satisfies $\frac{\partial S}{\partial t'} = 0$

some difference with classical

cut-off ⇒ $\underbrace{3.17\,U_p + 1.32\,I_p}_{\text{Lewenstein}}$ vs $\underbrace{3.17\,U_p + I_p}_{\text{classical}}$

can explain discreet spectrum $(2n+1)\omega$ ⇒ in classical model ~~KE($\phi$) is a cos~~ KE of returning electron is a cont. function of $\phi$.

Have additional constraint since we're only interested in electrons that return to the atom:

1) $\underbrace{\frac{\partial S}{\partial t'} = 0}$ 2) $\frac{\partial S}{\partial p_x} = 0 \quad \frac{\partial S}{\partial p_\perp} = 0$

$t'$ ionization time , are canonical angular momenta

constraint that imposed multiply

3) $\frac{\partial S}{\partial t} = E(t) = (2n+1)\omega_L$ since $\frac{\partial S}{\partial t} = E(t)$

where $t$ is the recombination time

KE along x

$$S = \frac{1}{2}\int_{t'}^{t} (\underbrace{V_x - A(t)}_{P_x} + A(t'))^2 dt =$$

$$-I_p t' + \underbrace{\frac{V_\perp^2}{2}(t-t')}_{KE_\perp} = \frac{1}{2}\int_{t'}^{t} \overset{KE}{(\vec{P} + \vec{A}(t'))} dt - I_p t'$$

1) $$\frac{\partial S}{\partial t'} = \frac{1}{2}(\vec{p} + \vec{A}(t'))^2 + I_p = 0$$

ionization time ⇒ complex number imaginary part is the "tunneling time" + gives probability of ionization by making S complex $P_0 \propto e^{-2\mathrm{Im}(S)}$

before close to project onto $t=0$, we need to have $t = t_R$ (return time so we can calculate the phase $S(t', t^R)$

$$\Psi_{(2n+1)} \approx e^{iS_0(t'\underset{\text{long traj return time}}{\underbrace{t_L^R}})} + e^{iS_0(t' t_S^R)}$$ short traj return time

$\langle v | \Psi(t) \rangle$ can already get that from the classical model ⇒ same solution from Lewenstein

s.t. $\frac{1}{2}v^2 + I_p = (2n+1)\omega_L$ ⇒ project onto the v-basis (Volkov basis) that gives the odd-freq. harmonics,

emission spectra $\propto |\Psi_{(2n+1)}|^2$ ⇒ results from interference, diff phases of long + short trajectories $v(t_i, t_R) = \frac{E}{\omega}[\sin(\omega t_R) - \sin(\omega t_i)]$

2) $\frac{\partial S}{\partial p} = \int_{t_i}^{t_R} (P_x + A(t'))\, dt' = 0$ ⇒ $\underbrace{P_x^\circ(t_R - t_i)}_{-E/\omega \cdot \sin(\omega t_i)\ \text{for}\ v_x(t_i)=0} = \frac{E}{\omega^2}[\cos(\omega t_R) - \cos(\omega t_i)]$

$\Rightarrow$ justifies the $v_x(t_i) = 0$ assumption in the classical model

2) and $\frac{\partial S}{\partial P_\perp} = \frac{\partial S}{\partial v_\perp} = v_\perp (t_R - t_i) = 0 \Rightarrow v_\perp = 0$

(no contribution to HHG from $v_\perp \neq 0$

3) $\underbrace{\frac{\partial S}{\partial t_R}}_{\text{solving for the return of time}} = \frac{1}{2}(\vec{P} + \vec{A}(t_R))^2 = \frac{1}{2}(\underbrace{P_x + A(t_R)}_{\frac{E}{\omega}\sin(\omega t_R)})^2 = \underbrace{(2n+1)\omega - I_p}$

E @ return recombination so that harmonics @ freq (n+1) are emitten

since $\frac{\partial S}{\partial t_R} = E(t_R)$

solve for $t_i$, $t_R$ that gives discreet harmonics