Strong Field Ionization

Lecture 2: Feynman path Integrals

FPI gives deterministic paths!

Outline

⇒ Motivation a) FPI are used in semi-classical approx. of SFA

⇒ see Science 292, 902 (2001)

"Feynman's Path Integral Approach for Intense-Laser-Atom Interactions"

⇒ HHG using Lewenstein model! (PRA, '94, Ref [17])

FPI ⟷ Quantum orbits

b) a way to approach tunneling time

⇒ Problems that arise in extracting tunneling times

⇒ Fundamental to Q.M. ⇒ goes back to the 2-slit experiment that "lies @ the heart of Q.M."

Comparison / Correspondence to Noisy systems

(also "all possible paths" determined by noise)

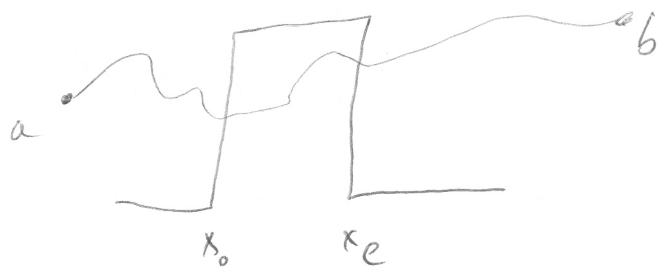
⇒ explains FPI

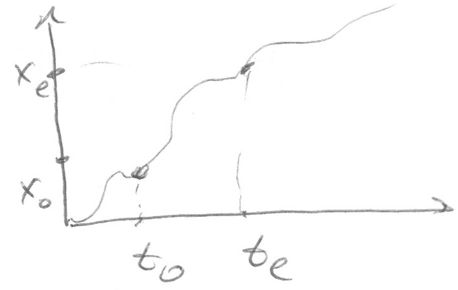
⇒ shows one important difference ⇒ interference effects.

Big Question:

How to extract tunneling time from the Schrodinger Equation?

Clear for classical trajectory





tunneling time: $\tau = t_e - t_0$

Deterministic trajectories → always have τ you can calculate

What if we could extract deterministic trajectories from Q.M.?

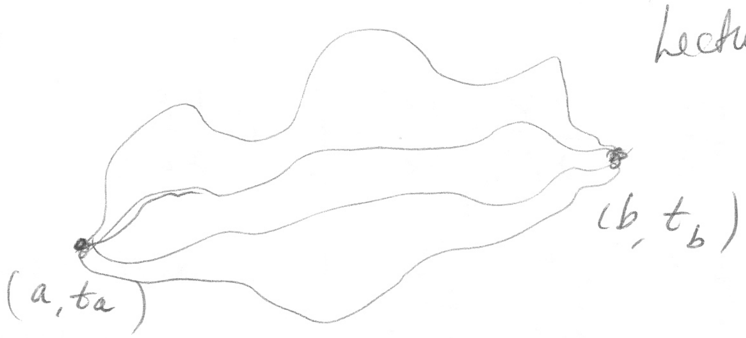
→ have that with semi-classical models ⇒ quantum orbits! Example: long & short paths in HHG!

→ What about exact expressions?

Answer: Feynman path integrals ⟺ Schrodinger wave mechanics (alternative formulation of Q.M.)

"sum of all possible paths"

→ histories approach → "all possible histories)



transition amplitude (propagator)

$$K(a, t_a \mid b, t_b) = \underbrace{\sum}_{\text{sum over all paths}} e^{iS[x_n(t)]/\hbar} \Rightarrow \underbrace{\int e^{i\frac{S[x(t)]}{\hbar}} d[x(t)]}_{x(t) \text{ stands for each path}}$$

$\Rightarrow$ each path $\overset{x(t)}{\vee}$ contributes a phase $\underbrace{S[x(t)]}$ to the integral

Background $\Rightarrow$ analogy to noisy systems
Norbert Wiener first developed the path integral approach $\Rightarrow$ for diffusion (Brownian motion)

Important point: Schrodinger Eqn. is a diffusion eqn. with an imaginary diffusion constant

Example
Heat Eqn.
$$\boxed{\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}} \quad \text{where } \alpha > 0$$

Schrodinger eqn:
$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \Rightarrow \underbrace{\boxed{\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2}}}_{\text{free particle}}$$

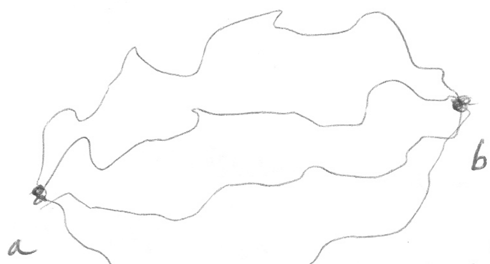
Same eqns. $\Rightarrow$ expect same solutions!

$t' = it \qquad \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial t'} \cdot \frac{\partial t'}{\partial t} = i \frac{\partial \Psi}{\partial t'}$

$+\hbar \frac{\partial \Psi}{\partial t'} = \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} \Rightarrow$ diffusion eqn. in imaginary time!

$\Rightarrow$ same solution as a heat eqn.; only remember to substitute $t' = it$ after you solve for $\Psi(x, t')$

Heat equation $\Rightarrow$ Noisy system (individual trajectories driven by noise)



in principle, any trajectory is possible be of the noise: $f(t)$

$P(f(t)) = e^{-\int \frac{f^2(t)\,dt}{2\sigma^2}}$ } probability of any specific realization of $f(t)$

Remember $P(\Delta x) = e^{-\frac{\Delta x^2}{2\sigma^2}}$ (Gaussian probability)

$P(a \to b) = P(a, t_a, b, t_b) = \int e^{-\int_{t_a}^{t_b} f^2(t)\,dt/2\sigma^2} \, d[f(t)]$

In Q.M $\qquad K(a, t_a | b, t_b) = \int e^{i\int_{t_a}^{t_b} L(t)\,dt/\hbar} \, d[x(t)]$

$\Rightarrow$ propagator

$S = \int_{t_a}^{t_b} L(t)\,dt$ (action in C.M)

($L(t)$ is the Lagrangian)

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minimizing action $\Rightarrow$ classical trajectory

K is the propagator $\Rightarrow$ probability transfer amplitude from point a to b

can take any $\psi(x', t')$ & propagate it in time to $\psi(x,t)$

x can be anything

$$\left\{ \psi(x,t) = \int K(x,t \mid x' t') \, \psi(x', t') \, dx' \, dt' \right.$$

$\delta(x - x_a, t_a)$ @ $t = t_a$

Difference with noisy system: $P(a \, t_a \mid b \, t_b)$

$\Rightarrow$ all $e^{-\int \frac{f(t)^2 \, dt}{2\sigma^2}}$ are real, > 0 paths add)

but $e^{iS[x(t)]/\hbar}$ are complex $\Rightarrow$ can take + & - values $\Rightarrow$ different paths interfere with each other!

Saddle point methods = $\hbar \ll$ & small

$$\delta S^{cl}[x(t)] = 0$$

(fast oscillations cancel out)

classical limit $\hbar \ll 1$

minimization of action $\Rightarrow$ classical Lagrangian mechanics

Euler-Lagrange eqns:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \quad \text{where } L = T - V$$

Quantum orbits

(3) Diffusion eqn. (heat eqn): $\boxed{\dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2}}$ where $\alpha > 0$

Schrodinger eqn: $\underbrace{i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \psi}{\partial x^2}}_{\text{free particle}} \Rightarrow \boxed{\dfrac{\partial \psi}{\partial t} = \dfrac{i\hbar}{2m} \dfrac{\partial^2 \psi}{\partial x^2}}$

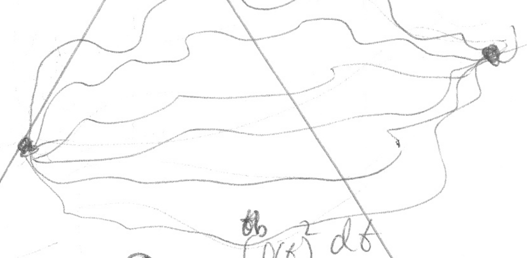
to normalize

$$\frac{P(a \to b)}{P(a \to \text{anything})} = \frac{\int e^{-\int_{t_a}^{t_b} \frac{f(t)^2 dt}{2\sigma^2}} d[f(t)_{a\to b}]}{\int e^{-\int_{t_a}^{t_b} \frac{f^2(t)\, dt}{2\sigma^2}} d[f(t)_{a\to \text{anywhere}}]}$$

$e^{-x^2/2\sigma^2}$

same as the heat eqn. except for the imaginary i (what allows different paths to interfere by introducing a phase.

Example: ~~Escape noise-induced escape from a 1-D well~~



$P(a, t_a, b, t_b)$

in noisy system $\cancel{P(a \to b)} = \int \underbrace{e^{-\int_{t_a}^{t_b} \frac{f(t)^2 dt}{2\sigma^2}}}_{\text{probability of each path}} d[f(t)]$

$f(t) \Rightarrow$ all realizations of noise that take the ~~electron~~ particle from point a to pt. b

in Q.M system $K(a, t_a, b, t_b)$

$$= \int e^{i\tilde{S}[x(t)]/\hbar} d[x(t)] \sim e^{iS_{cl}} \text{ for } S \gg \hbar$$

$S[x(t)]$ is the classical action $S = \int_{t_a}^{t_b} L(x, t) \, dt$

minimizing $\int_{t_a}^{t_b} f^2(t) dt \Rightarrow$ most probable trajectory from a to b

Lecture 2

Note: in tunneling, we also use the saddle point method

SFA amplitude (to exponential accuracy) (Misha Ivanov Anatomy of SFA)

$$\Psi(v,t) \sim i\int_0^t dt' e^{iS(t,t')} \sim e^{iS_0(t,t_0^*)}$$ ← complex time

where S_0 satisfies $\delta S = 0$

$$e^{+iS_0(t,t_0^*)} \simeq e^{\underbrace{-\frac{(2I_p)^{3/2}}{3F}}_{\text{real!}}}$$ } Keldysh exponent

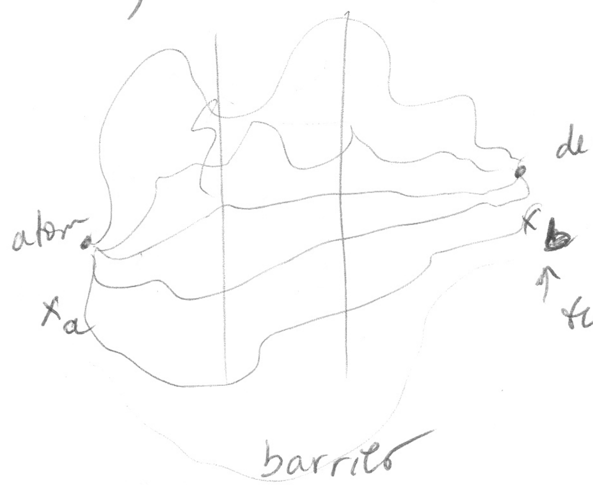
real! → doesn't correspond to any Feynman path → since any actual path has real S!

→ makes sense since the barrier is a classically forbidden region → no classical path!

Nevertheless could imagine a situation where there is a real path S through the tunnel from which we could extract tunneling time! (~~that~~ not the case!)

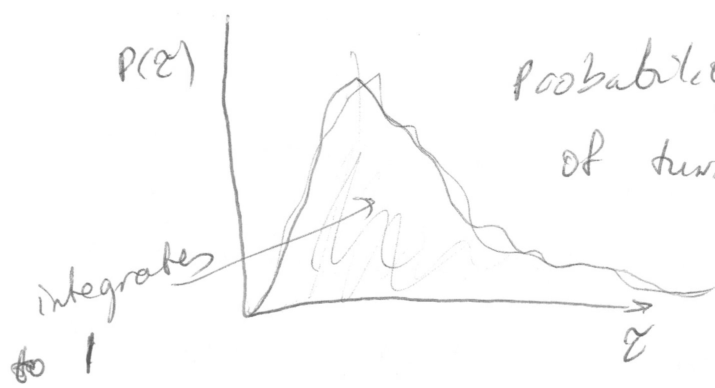
→ "tunneling time is imaginary"

Back to FPI $\Rightarrow$ why not try to extract probability distribution on tunneling time from Feynman paths?



Could do it in a noisy analogy (noise pushing particle from $a \to b$ (also through the forbidden region)

Calculate $P(\tau) = \dfrac{P(a, t_a \mid b, t_b, \tau)}{\underbrace{P(a, t_a \mid b, t_b)}_{\text{normalization}}}$



Probability distribution of tunneling times

why not do: $K(a, t_a, b, t_b \mid \tau) = \dfrac{\overbrace{K(a, t_a \mid b, t_b, \tau)}^{\text{can in fact calculate it for simple cases!}}}{K(a, t_a \mid b, t_b)}$

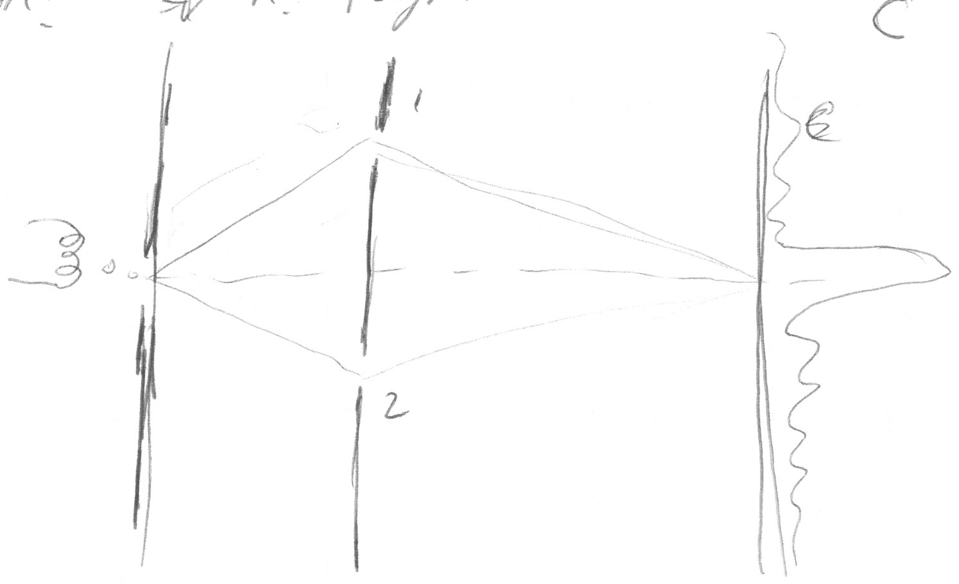
K is complex! $\Rightarrow$ doesn't behave like a probability distribution!

Note:

So from the math perspective, all the "mystery of Q.M." is just the imaginary # in the Schrodinger eqn! (makes it different from diffusion eqn ⇒ leads to interference effects).

Different Feynman paths interfere!

2-slit experiment that lies @ the heart of Q.M. ⇒ R. Feynman



hole 2 covered hole 1 covered

Probability amplitudes are represented by complex #'s

$P_1 = |\phi_1|^2 \qquad P_2 = |\phi_2|^2 \qquad \phi = \phi_1 + \phi_2 \Rightarrow |\phi|^2 \neq P_1 + P_2$

Can ~~bring~~ think of the 2 paths ~~are~~ as 2 Feynman paths (~~the two that minimize~~ the action)

So extracting tunneling time is like trying to determine which slit the particle went through

⇒ ~~cannot assign~~ can calculate

$K(a, b, \tau)$ → analogous to Φ_1 ; Φ_2

but $|K(a, b|\tau)|^2$ doesn't give us the desired probability distribution

⇒ Nevertheless, we can try to use $K(a, b|\tau)$ to extract something like the tunneling time

⇒ different definitions of tunneling time come out of it

4 well known ones

Buttiker-Landauer, $\tau_{BL} = -\hbar \frac{\partial \ln(T)}{\partial V}$

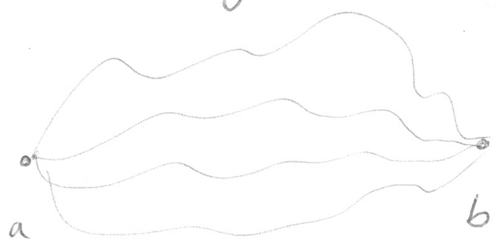
Larmor ~~clock~~ time ; $t_{Lm} = -\hbar \frac{\partial \theta}{\partial V}$

Bohm-Wigner (phase time) $\tau_{BW} = \hbar\, \partial\theta / \partial E$

Pollack-Miller time ; $\tau_{pm} = \hbar \frac{\partial \ell(T)}{\partial E}$

Strong Field Ionization, Lecture 3

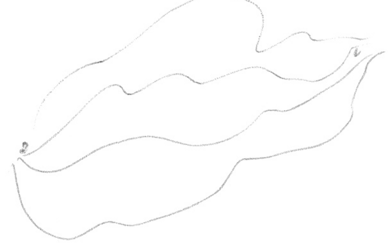
Compa btwn Feynman paths & noisy systems



$$P(f(t)) = e^{-\frac{\int_{t_a}^{t_b} f^2(t)\,dt}{2\sigma^2}}$$

probability of any path $f(t)$

Transition amplitude (Feynman)



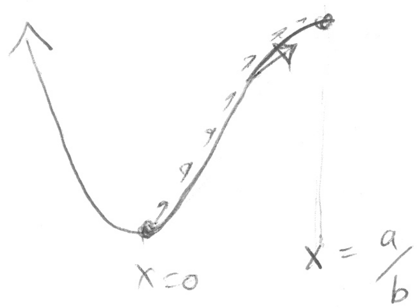
$$P[L(t)] = e^{i\int_{t_a}^{t_b} \frac{L\,dt}{\hbar}}$$

$$f^2(t) \Leftrightarrow L$$

Illustrative example: $\dot{x}(t) = \underbrace{K(x,)}_{\text{deterministic drift}} + \underbrace{f(t)}_{\text{white noise}}$

$\boxed{f(t) = D\,\delta(t-t')}$

$$K(x,t) = -ax + bx^2$$



escape from overdamped potential

"likeliest escape path" ⇒ from $x=0$ to $x=\frac{a}{b}$

⇒ in the limit $\sigma \to 0$ ⇒ likeliest escape path (Wentzel)

2

minimize $P(f(t)) = e^{-\int_{t_a}^{t_b} f^2(t)\, dt / 2\sigma^2}$ ← in the limit $\sigma \to 0$, every other path is essentially zero probability

$f(t) = (\dot{x}(t) - K) = (\dot{x}(t) + ax - bx^2)$

$L(x, t) = f^2(t) = (\dot{x} - K(x))^2$ ← Lagrangian = noise

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$\Rightarrow \quad \dot{x}_{esc} = -K(x) = x(a - bx)$

$$x_{esc}(t) = \frac{a}{b}\left[\frac{e^{a^2 t}}{1 + e^{a^2 t}}\right]$$

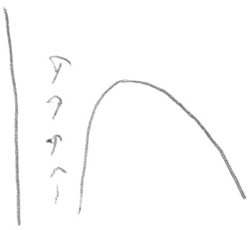
$$f_{opt} = \frac{2a^2}{b}\left[\frac{e^{a^2 t}}{(1 + e^{a^2 t})^2}\right]$$

$\Rightarrow$ Quantum case: also only classical path dominates (the one determined by eqns $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$) when $\hbar \to 0$

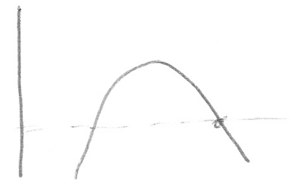
Both are paths taken when $\sigma \to 0$ or $\hbar \to 0$

SFI Lecture 3

multi-photon



Tunneling regime



appears in the continuum with (lowest) $\vec{v} = 0$

Above threshold ionization (if more electrons than needed are absorbed)

ATI : $v^2 > 0$ (by definition)

$\gamma > 1$ & $\gamma < 1$

where $\gamma = \frac{\sqrt{2 I_p}}{F} \cdot \omega$ $U_p = \frac{1}{4}\left(\frac{F}{\omega}\right)^2$ (oscillation energy of the electron in an E&M wave

$= \sqrt{\frac{I_p}{2 U_p}}$

$\Rightarrow$ ATI at lower intensities, shorter wavelength (adiabatic assumption fails)

(no "nice" initial condition, like in tunneling

Tunneling Probability ($\gamma < 1$ regime)

$\vec{F} = F_0 (\underbrace{\cos(\omega t)\,\hat{x}}_{\text{major axis}} + \underbrace{\epsilon \sin(\omega t)\,\hat{y}}_{\text{minor axis}})$

Tunneling Probability

derived from SFA $\Big\{$ $P(v_y, v_z | F(t_0)) = \underbrace{C}_{\text{obtained from PPT (Coulomb correction)}} e^{-\frac{2 (2 I_p)^{3/2}}{3 F(t_0)}} \, e^{-\frac{v_y^2}{2\sigma_\perp}} \cdot e^{-\frac{v_z^2}{2\sigma_\perp}}$

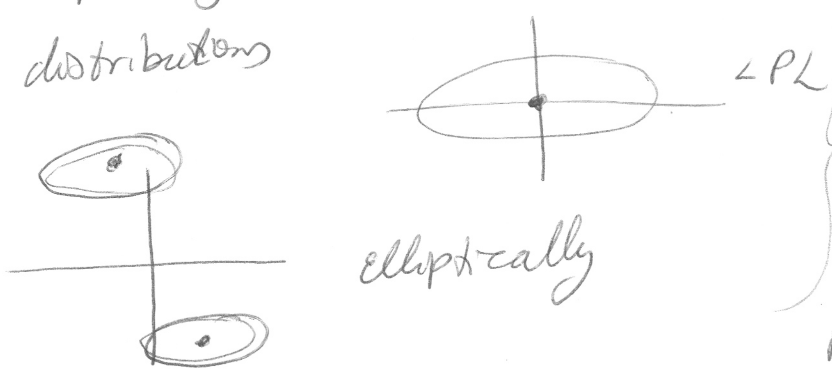
$v_\perp$ is given by ADK probability distribution

$P(v_y, v_z)$ is clearly maximized @ $v_z = v_y = 0$ &
$F(t_0) = F_0$ (so that $t_0 = 0$ ⇒ peak of the E-field!

So initial condition for tunneling (most probable trajectory:

⇒ electron appears with $\vec{v} = 0$ in the continuum @ the peak of the field ($t=0$)

⇒ in principle know the exact trajectory corresponding to the peak in your electron momenta distributions



in both LPL & elliptical polarization, the ~~center~~ corresponds to ionization @ the peak with $\vec{v} = 0$

Not the case in ATI!

1) by definition $\vec{v} \neq 0$ in the continuum
2) Not the same as tunneling, so where is the exit point? (Normally still taken to be the tunnel exit point)
3) When is ionization most ⇒ likely to happen? (may not be @ the peak of the field)

Both ATI & tunneling can be derived within SFA
(the tunneling formula comes from SFA & imposing $\gamma << 1$)

Within SFA; for $\gamma >> 1$

ionization probability: $\Gamma(t) = X(2\gamma)^{-2N} \propto F^{2N}$

$N = \frac{I_p}{\omega} \Rightarrow$ # of photons required to reach the continuum threshold

$\Rightarrow$ So still most probable to be ionized @ the peak ($F = F_0$), with just enough photons to reach the continuum ($\vec{v} \approx 0$)

What about ATI where $\gamma \sim 1$ (not $\gamma >> 1$)?

Illustrative example (again ionization probability given by SFA)
Goreslavski et. al. PRL 93, 233002 (2004)
"Coulomb asymmetry in above-threshold ionization"

1) Uses ATI ionization probability derived from SFA (modified Keldysh exponent)
   * still in the "tunneling regime", but does not use a quasistatic assumption (non-adiabatic)

2) Coulomb correction after ionization

3) Good agreement with data (at least for certain η)

Experimental regime.

Xe : $I_p = 12.13$ Intensity $\approx 0.9 \times 10^{14}$ W/cm^2

$\lambda \approx 800$ nm $\epsilon = 0.56$ & 0.36

Eckle Science paper $\Rightarrow$ Intensity $2.3 \times 10^{14} - 3.5 \times 10^{14}$

$\gamma = \frac{\sqrt{2 I_p}}{F} \cdot \omega$ $F \propto \sqrt{\text{Intensity}}$

He $I_p \approx 24$; $\gamma_{\text{Eckle}} \approx 1.45 - 1.17$

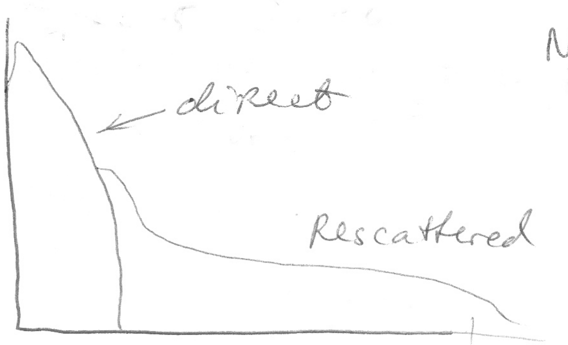
(Eckle Science paper) Science Vol. 322 (2008)

$\gamma_{\text{Goreslavsy}} \approx \gamma_{\text{Eckle}} \cdot \frac{1}{\sqrt{2}} \times 2 \approx 1.65$

$\Rightarrow$ not much higher then the "tunneling regime" paper in Science

$\Rightarrow$ Concentrate on "low energy" direct electrons (no rescattering) $\Rightarrow$ $E < 25$ eV

Nature Phys. Vol. 5 May 2009 (Blaga et. al.)



cut-off $\sim 10 U_p$

cut-off $\sim 2 U_p$ (corresponds to drift in CPL)

1 a.u. $\approx$ 27 eV

1 $U_p \approx$ 1 au for $F \approx 0.1$, $\omega \approx 0.05$

drift momentum (momentum measured @ the detector): $\frac{p^2}{2} = (N+S)\omega - I_p - \underbrace{\underbrace{\frac{F^2(1+\xi^2)}{4\omega^2}}_{\text{ponderomotive energy: } U_p}}_{\text{ionization threshold in the field}}$

N - minimum # absorbed to reach a continuum

$S = 0, 1, 2 ..$ is the # of the ATI peak

Can be calculated within SFA (= in the tunneling but not adiabatic regime)

Probability of ionization: $P(\vec{V}_0, F(t_0))$

$\vec{V}_0 \perp$ to the direction of ionization $\Rightarrow \vec{F}(t_0) \cdot \vec{V}_0 \Rightarrow \vec{V}_0 = \vec{V}_\perp$

$\vec{V}_0$: initial velocity; t_0: time of ionization

Prefactor: $C = \frac{2(2I_p)^3 \omega \sqrt{1 + V_\perp \dot{F}(t_0)/F^2(t_0)}}{\pi^2 \sqrt{1 + V_\perp^2/2I} \; F^2}$

Coulomb correction for sub-barrier motion

actual PPT depends on l, m — approximation of PPT? (valid for multi-photon as well as tunnel ionization)

$$P(\vec{V}_0, \vec{F}(t_0)) = C \cdot \exp\left[- \frac{2(2I_p)^{3/2}}{3\sqrt{F^2(t_0) + V_\perp \dot{F}(t_0)}} \underbrace{\left(1 + \frac{V_\perp^2}{2I_p}\right)^{3/2}}_{\text{from expanding, get ADK formula for } V_\perp \text{ distribution } e^{-\frac{V_\perp^2}{2\sigma_\perp^2}}} \right] \quad (1)$$

Difference from the tunneling formula:

term: $V_\perp \dot{F}(t_0)$

⑧

if $\dot{F}(t_0) \approx 0$ (adiabatic regime)

then: $P(v_0, F(t_0)) = C(\dot{F}=0) \times \exp\left[\frac{-2(2I_p)^{3/2}}{3\, F(t_0)} \left(1 + \frac{v_\perp^2}{2I_p}\right)^{3/2} \right]$

$\Rightarrow$ Quasi-static limit!

$1 + \frac{v_0^2}{2I_p} \Rightarrow 2I_p + v_0^2 \Rightarrow$ just means that transverse velocities affectively raises the ionization potential (well-known)

$\dot{F}(t_0) \neq 0 \Rightarrow$ might mean maximum of ionization probability is not @ the peak and not $v_\perp = 0$!

After ionization:

@ the exit point: $\vec{r}_e = \frac{I_p}{F(t_0)} \hat{n}$ where $\hat{n} = \frac{\vec{F}(t_0)}{F(t_0)}$

(triangular approximation)

$v_\parallel = 0$ (direction of the field)

$v_\perp$ given by $P(v_\perp, F(t_0))$ (see Eqn. (1) on previous page)

Coulomb correction:

within SFA $\vec{P}_{SFA} = \frac{-\vec{A}(t_0)}{c} + \vec{v}_\perp(t_0)$

momentum observed @ the detector

momentum with a Coulomb correction:

$\vec{P} = \vec{P}_{SFA} + \vec{P}_c$

where $\vec{P}_c = -\int_{t_0}^{\infty} dt \, \frac{\vec{r}_L}{r_L^3}$

$\vec{r}_L \Rightarrow$ trajectory of the electron after ionization in a FROZEN laser field

$\Rightarrow$ assumes most of the Coulomb correction happens @ the exit pt.

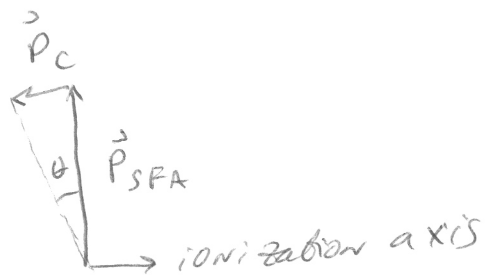
$\vec{r}_L = \vec{r}_e + \vec{v}_\perp(t_0)\, t + \frac{1}{2} F(t_0) t^2$

This is just a trajectory in a constant E-field (compare to $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$)

won't work rescattered electrons!

Substituting for $\vec{r}_L$ into the integral, we get:

$\vec{p} = \vec{P}_{SFA} - \frac{\pi \mathcal{E}_0}{4 r_0^2} \hat{n} \Rightarrow \hat{n} \perp \vec{P}_{SFA}$ since $\vec{P}_{SFA} \parallel \vec{A}(t_0)$



total angle $= 90° + \theta$

simulations fairly accurate for $s = 5-7$ (and greater?)

exponent minimized when $v_\perp \dot{F}(t_0) > 0 \Rightarrow$ velocity @ the exit pt. in the same direction as the correction drift (since $\vec{P}_{SFA} = -\frac{\vec{A}(t_0)}{c} + \vec{v}_p(t)$ drift)

$\dot{\vec{F}}(t_0) = -\frac{\partial^2 \vec{A}(t_0)}{\partial t^2}$

(corresponds to more s absorbed!)

10

$$\vec{L} = \hbar n \omega = \vec{p} \times \vec{r}$$

final
ang. momentum

\# of
photons absorbed