Ultrafast Laser Physics

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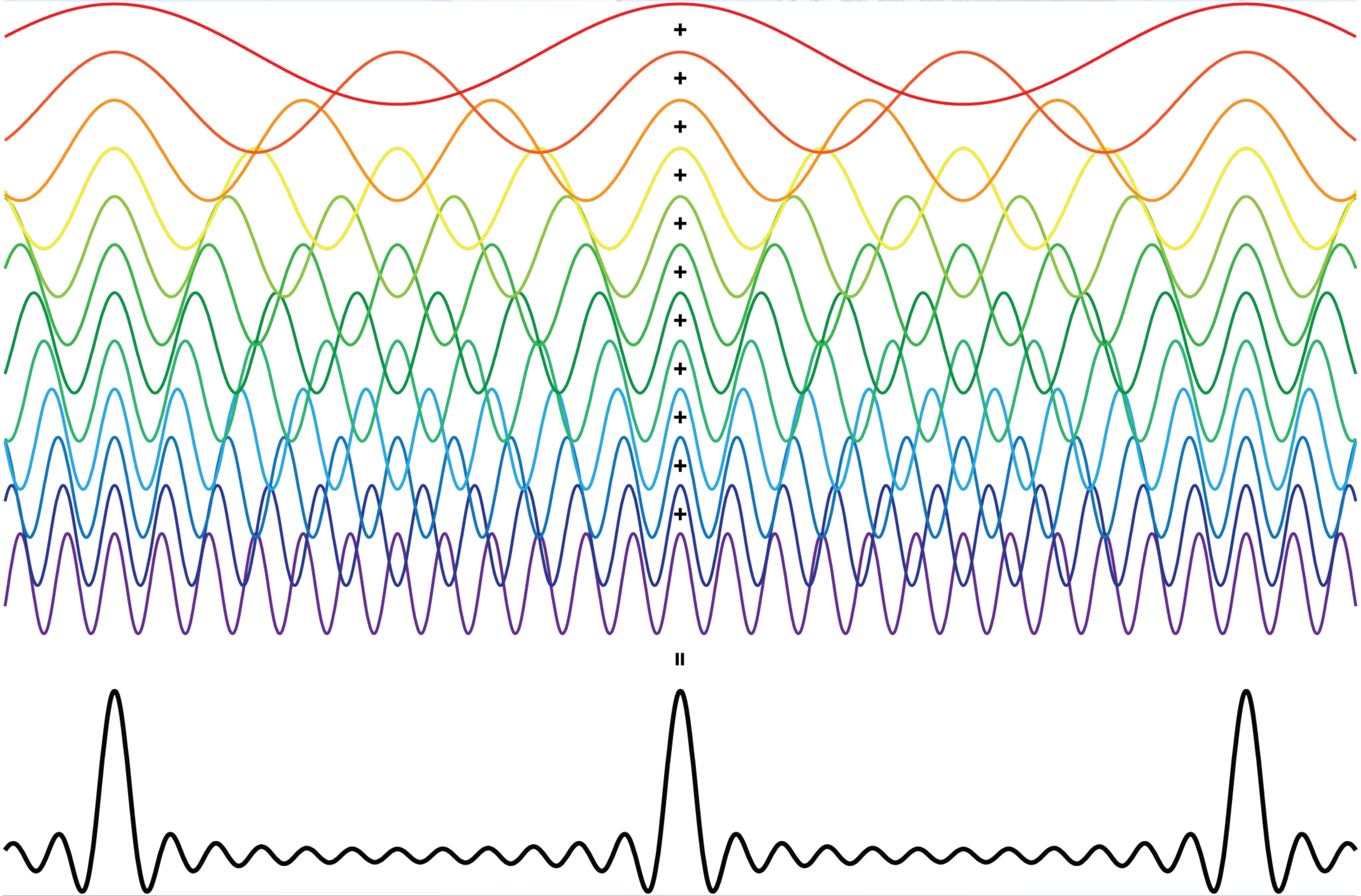
Linear pulse propagation

Superposition of many monochromatic waves



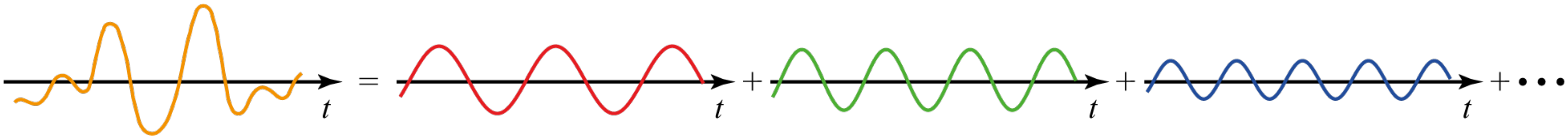
Fourier transformation

Plane wave, monochromatic $E(z,t) = E_0 e^{i(\omega t - kz)}$

Inverse Fourier Transformation (for every position in space)

$$E(z,t) = F^{-1}\left\{\tilde{E}(z,\omega)\right\} = \frac{1}{2\pi}\int \tilde{E}(z,\omega)e^{i\omega t}d\omega$$

$$E(t) \quad = \quad \tilde{E}(\omega_1) \quad + \quad \tilde{E}(\omega_2) \quad + \quad \tilde{E}(\omega_3) \quad + \ldots$$

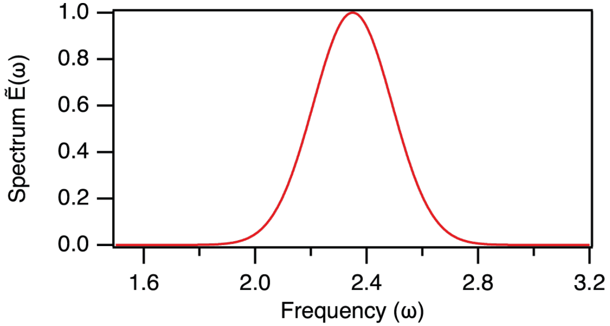


$$E(t) = \frac{1}{2\pi}\int \tilde{E}(\omega)e^{i\omega t}d\omega$$

Fourier Transformation

$$\tilde{E}(z,\omega) \equiv F\{E(z,t)\} = \int E(z,t)e^{-i\omega t}dt$$

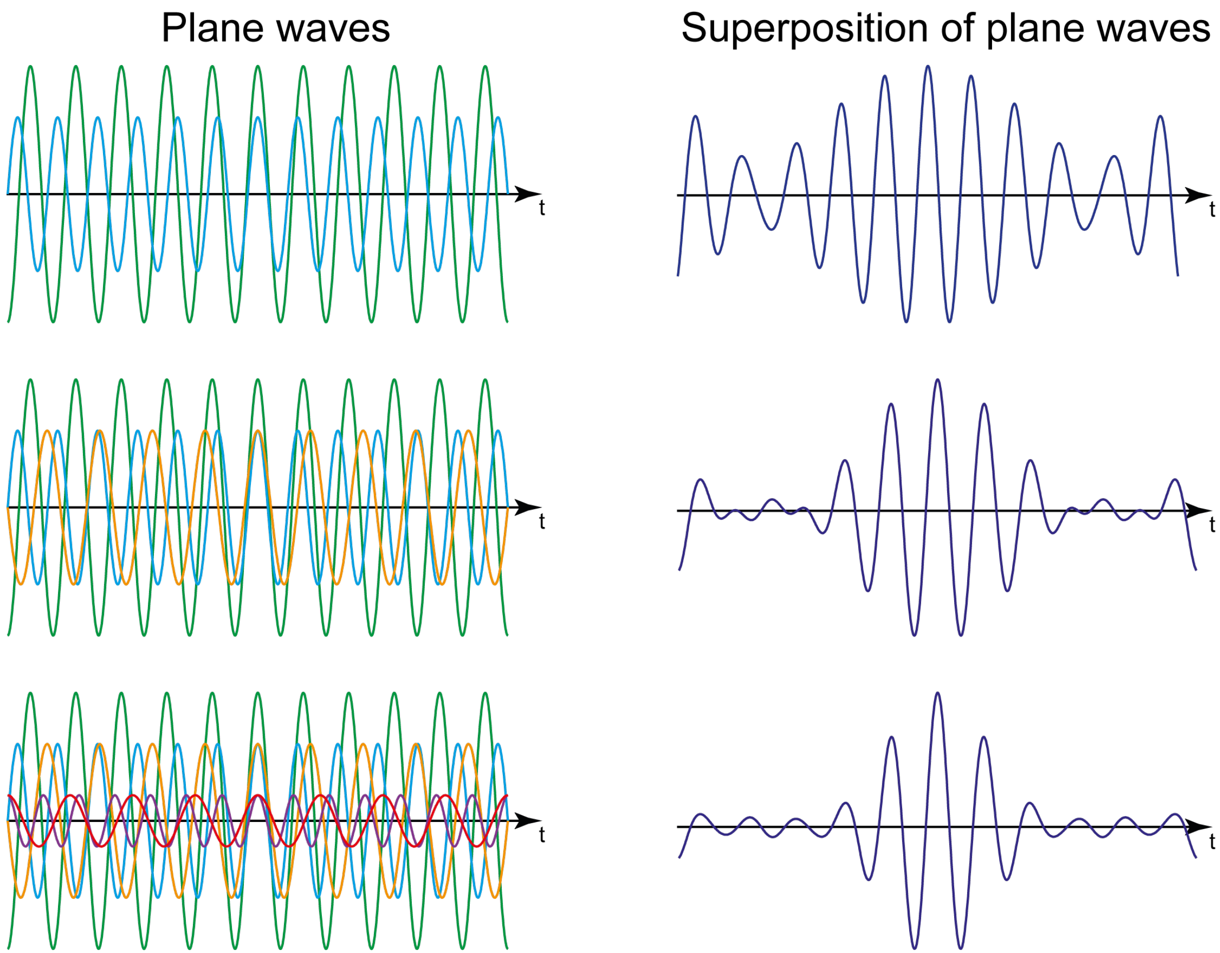
Superposition of monochromatic waves



Light pulse

Wave packet

$$E(t) = \frac{1}{2\pi} \int \tilde{E}(\omega) e^{i\omega t} d\omega$$



2 waves

3 waves

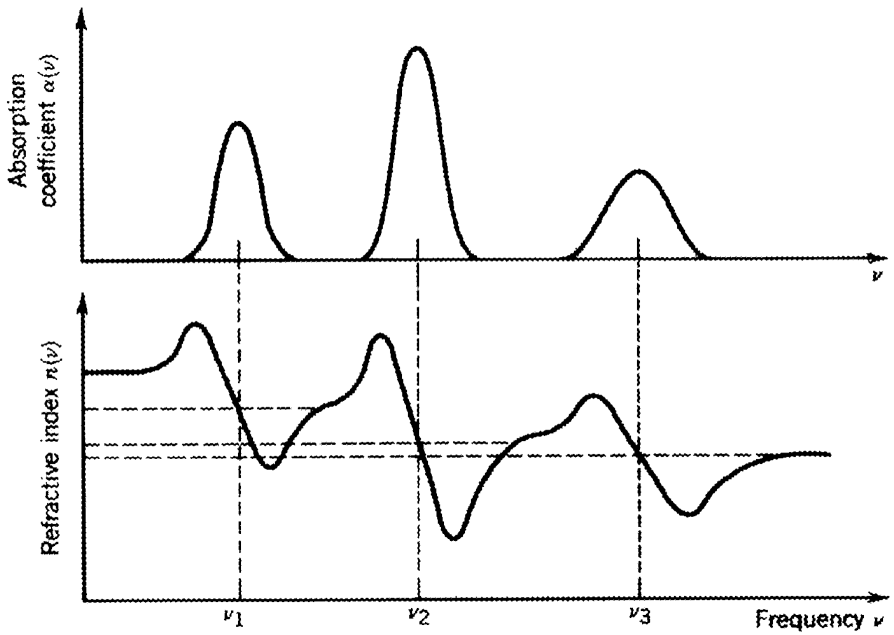
5 waves

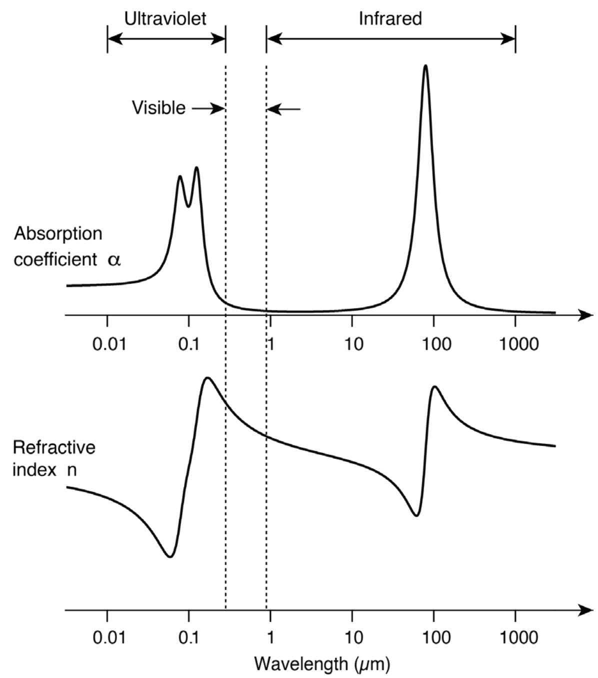
Monochromatic plane wave

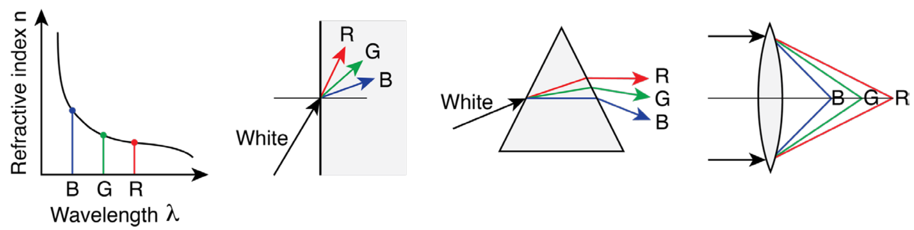
$$E(z,t) = E_0 e^{i(\omega t - k_n z)}$$

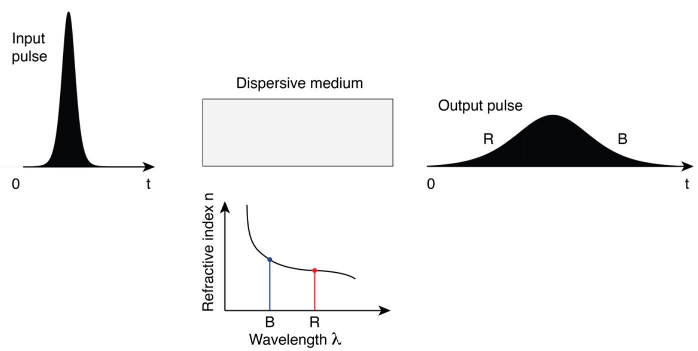
	Vacuum	Dispersive material
Frequency	ν	ν
Period	$T = 1/\nu$	$T = 1/\nu$
Phase velocity	$v_p = c$	$v_p = c_n = c/n$
Wave number	$k = \frac{\omega}{c}$	$k_n = \frac{\omega}{v_p} = \frac{\omega}{c} n = kn$
	$k = \frac{2\pi}{\lambda}$	$k_n = \frac{2\pi}{\lambda_n} = kn$
Wavelength	λ	$\lambda_n = \frac{\lambda}{n}$

Optical dispersion









Helmholtz equation

$$E(z,t) = E_0 e^{i(\omega t - kz)}$$

Fourier transformation

$$\tilde{E}(z,\omega) \equiv F\{E(z,t)\} = \int E(z,t) e^{-i\omega t} dt$$

$$E(z,t) = F^{-1}\left\{\tilde{E}(z,\omega)\right\} = \frac{1}{2\pi}\int \tilde{E}(z,\omega) e^{i\omega t} d\omega$$

$$\omega \Leftrightarrow -i\frac{\partial}{\partial t}$$

$$\omega^2 \Leftrightarrow -\frac{\partial^2}{\partial t^2} = \left(-i\frac{\partial}{\partial t}\right)^2$$

$$\omega^3 \Leftrightarrow i\frac{\partial^3}{\partial t^3} = \left(-i\frac{\partial}{\partial t}\right)^3$$

$$\frac{\partial^2}{\partial z^2}E(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t) = \mu_0 \frac{\partial^2}{\partial t^2}P(z,t)$$

$$\frac{\partial^2}{\partial t^2} \Leftrightarrow -\omega^2$$

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + \frac{\omega^2}{c^2}\tilde{E}(z,\omega) = -\mu_0 \omega^2 \tilde{P}(z,\omega)$$

$$\tilde{P}(z,\omega) = \chi(\omega)\varepsilon_0 \tilde{E}(z,\omega) = [\varepsilon(\omega) - 1]\,\varepsilon_0 \tilde{E}(z,\omega)$$

$$k_n(\omega) = \frac{\omega}{c} n(\omega)$$

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + [k_n(\omega)]^2\, \tilde{E}(z,\omega) = 0$$

$$\tilde{E}(z,\omega) = \tilde{E}_0^+(\omega) e^{-ik_n(\omega)z} + \tilde{E}_0^-(\omega) e^{ik_n(\omega)z}$$

Analogy to the Schrödinger equation

Helmholtz equation:

Wave equation in the spectral domain

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + [k_n(\omega)]^2 \, \tilde{E}(z,\omega) = 0$$

Time independent Schrödinger equation:

free particle

$$\frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0$$

particle in a potential field

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2} [E - E_p(x)] \, \psi = 0$$

Dispersion for quantum mechanical particles

Photon in vacuum: $E = cp \quad \Leftrightarrow \quad \omega(k) = ck$

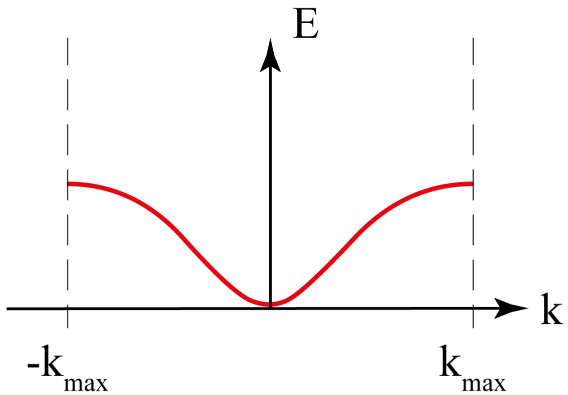
Photon in medium: $E = c_n p_n \quad \Leftrightarrow \quad \omega(k) = c_n k_n$

Free electron: $E = \dfrac{p^2}{2m} \quad \Leftrightarrow \quad \omega(k) = \dfrac{\hbar}{2m}k^2$

$$v_p = \frac{\omega}{k} = \frac{\hbar k}{2m}$$

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = v_{\text{class}}$$

Electron in conduction band: $E(k) \approx \tilde{E}_{at} - 2E_s \cos ka \quad \Leftrightarrow \quad \omega(k) \approx \tilde{\omega}_{at} - 2\omega_s \cos ka$



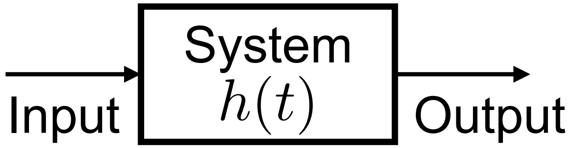
$$\pm k_{\max} = \pm\pi/a$$

$$v_g\left(\pm k_{\max}\right) = 0$$

Phonon in 1-dimensional lattice: $\omega(k) = 2\sqrt{\dfrac{\beta}{M}}\sin\left(\dfrac{1}{2}|\boldsymbol{k}|a\right)$

Linear system theory

$E_{in}(t)$



$E_{out}(t)$

impulse response $h(t)$

$$E_{out}(t) = \int h(t')E_{in}(t-t')\,dt' = h(t) * E_{in}(t)$$

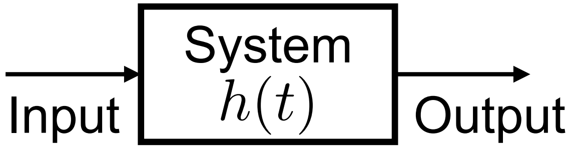
$$E_{in}(t) = \delta(t) \longrightarrow E_{out}(t) = \int h(t')\delta(t-t')\,dt' = h(t)$$

Examples of linear systems:

- pulse propagation in dispersive media
- photo detector (impulse response links photo current with light power or intensity)
- active light modulator in the linear regime
- image propagation through a lens systems
- stochastic processes such as amplitude and phase noise (linearized as a perturbation on a much stronger signal)

Linear system theory

$E_{in}(t)$



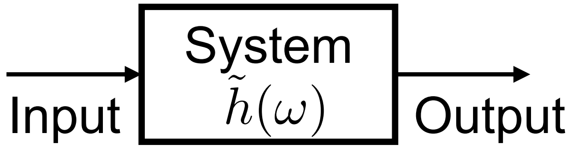
$E_{out}(t)$

impulse response $h(t)$

$$E_{out}(t) = \int h(t') E_{in}(t - t')\, dt' = h(t) * E_{in}(t)$$

$$E_{in}(t) = \delta(t) \longrightarrow \quad E_{out}(t) = \int h(t') \delta(t - t')\, dt' = h(t)$$

$\tilde{E}_{in}(\omega)$

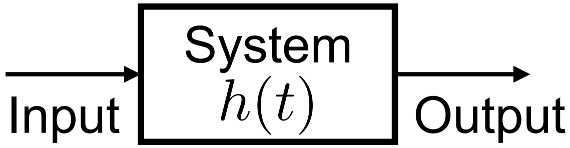


$$\tilde{E}_{out}(\omega) = \tilde{h}(\omega) \tilde{E}_{in}(\omega)$$

transfer function $\tilde{h}(\omega)$

Linear system theory

$E_{in}(t)$



$E_{out}(t)$

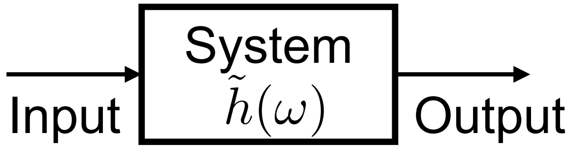
impulse response $h(t)$

$$E_{out}(t) = \int h(t')E_{in}(t-t')\,dt' = h(t) * E_{in}(t)$$

$E_{in}(t) = \delta(t)$ →

$$E_{out}(t) = \int h(t')\delta(t-t')\,dt' = h(t)$$

$\tilde{E}_{in}(\omega)$

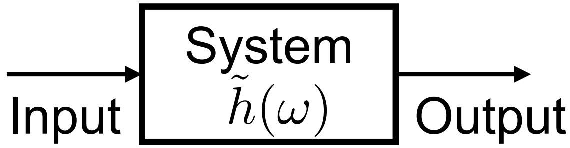


$$\tilde{E}_{out}(\omega) = \tilde{h}(\omega)\tilde{E}_{in}(\omega)$$

transfer function $\tilde{h}(\omega)$

Spectral power density:

$$P_{in}(\omega) = \left|\tilde{E}_{in}(\omega)\right|^2$$



$$P_{out}(\omega) = \left|\tilde{E}_{out}(\omega)\right|^2$$

$$S(\omega) \equiv \left|\tilde{h}(\omega)\right|^2$$

$$P_{out}(\omega) = S(\omega)P_{in}(\omega)$$

Linear system – example: dispersion

What is the impulse response or transfer function?

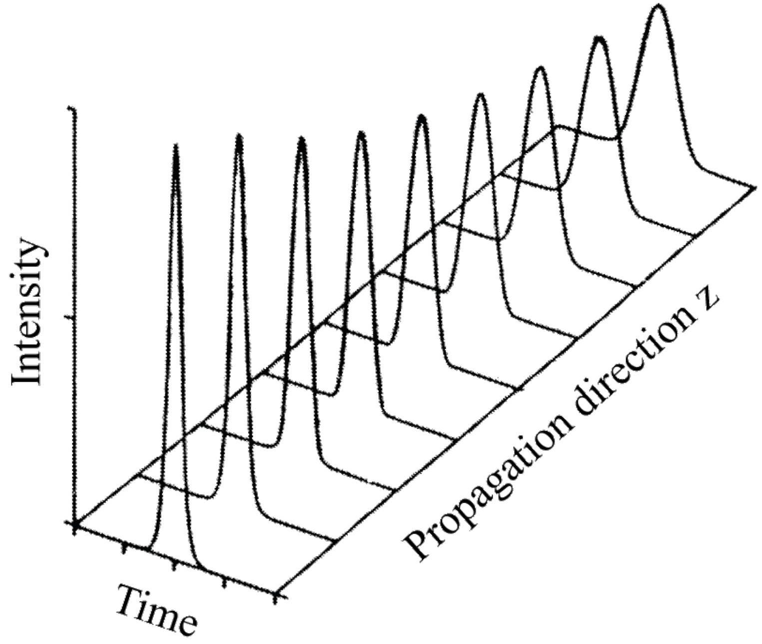
What is the pulse form during propagation?

How is the spectrum of the pulse changing ?

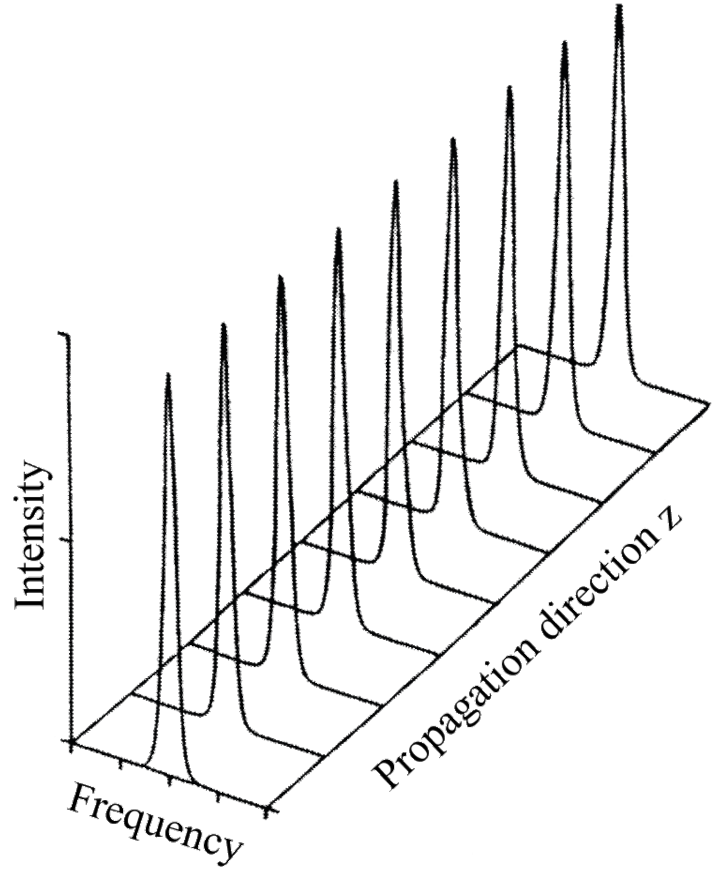
Solution:

Ultrafast Lasers book, Chapter 2, page 34

Dispersive pulse broadening



Time domain

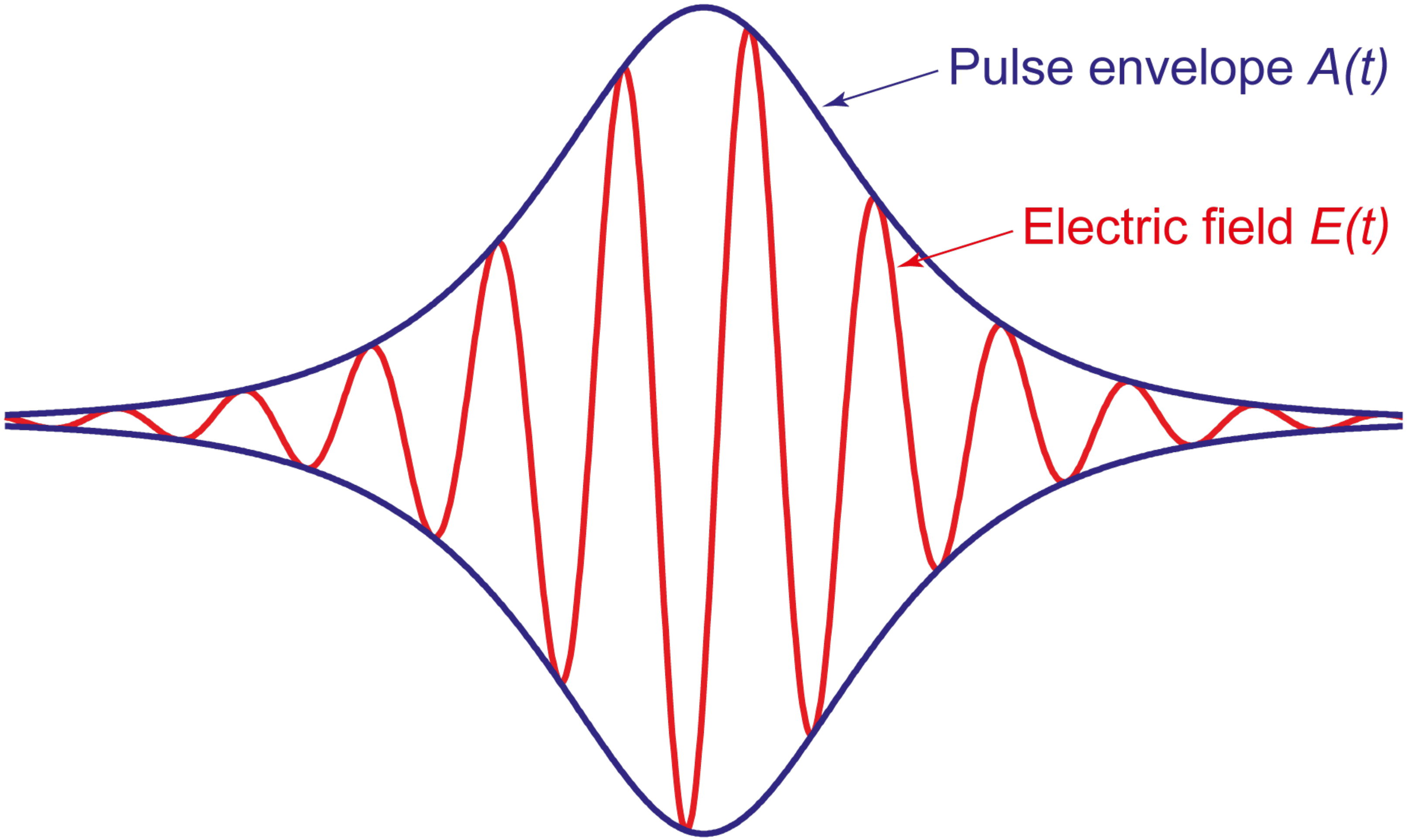


Spectral domain

Linear pulse broadening:

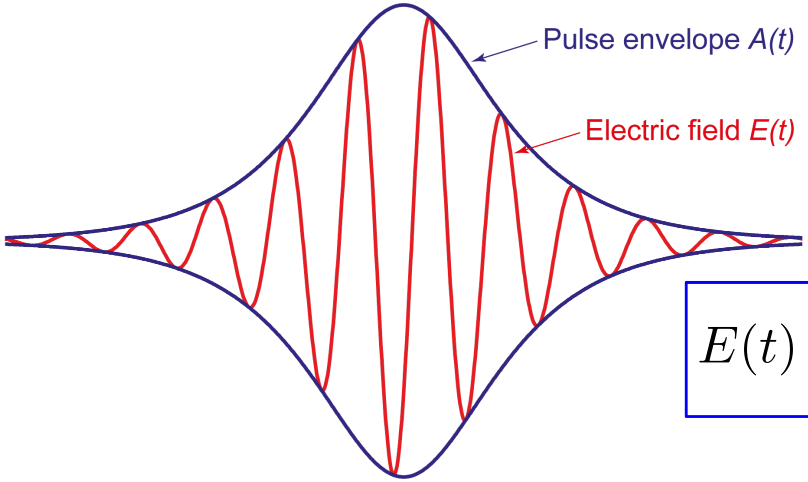
Transform-limited pulse is broadening in the time domain but its spectrum remains unchanged.

Light pulse



$$E(t) = A(t)e^{i\omega_0 t}$$

Laser pulse

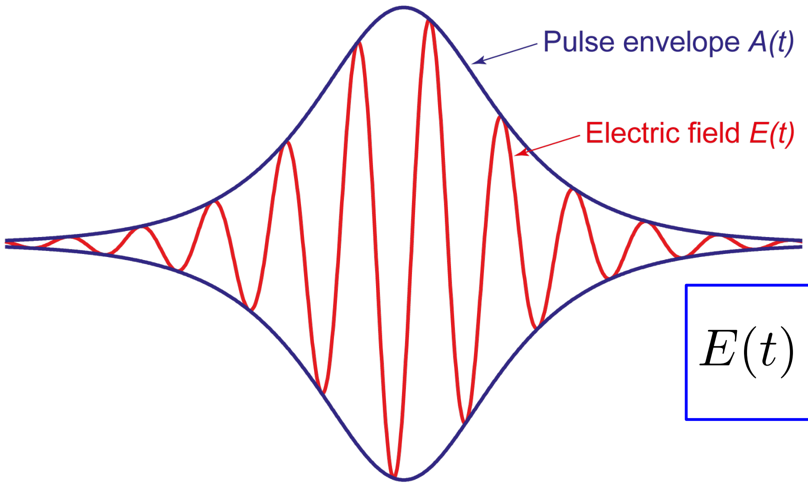


$$E(t) = \frac{1}{2\pi} \int \tilde{E}(\omega) e^{i\omega t} d\omega$$

$$E(t) = A(t) e^{i\omega_0 t} \quad \text{where} \quad A(t) = \frac{1}{2\pi} \int \tilde{A}(\Delta\omega) e^{i\Delta\omega t} d\Delta\omega$$

$$\Delta\omega \equiv \omega - \omega_0$$

Laser pulse

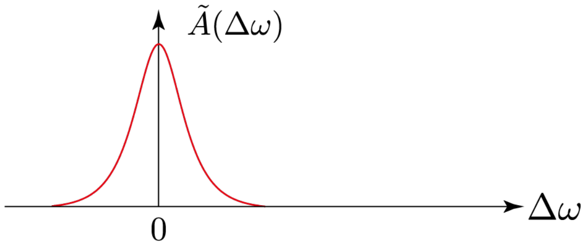


$$E(t) = \frac{1}{2\pi}\int \tilde{E}(\omega)e^{i\omega t}d\omega$$

$$E(t) = A(t)e^{i\omega_0 t} \quad \text{where} \quad A(t) = \frac{1}{2\pi}\int \tilde{A}(\Delta\omega)e^{i\Delta\omega t}d\Delta\omega$$

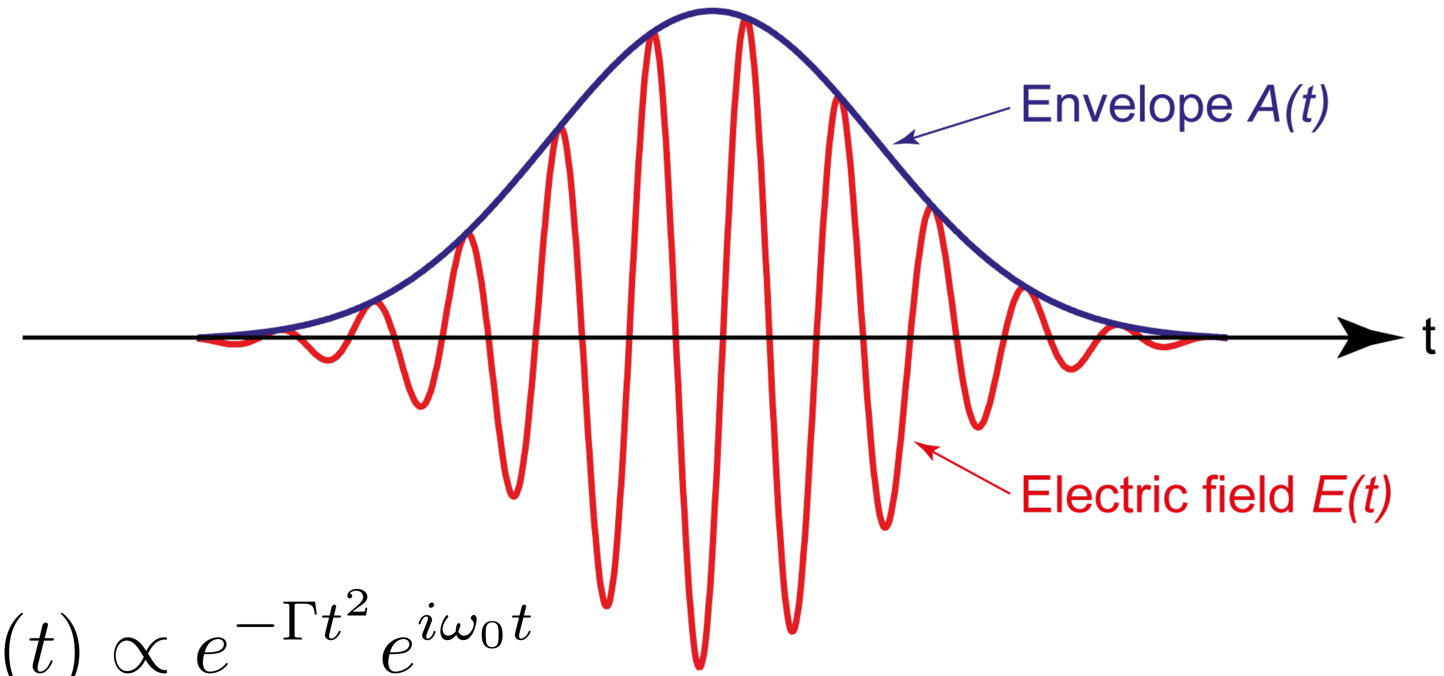
$$\Delta\omega \equiv \omega - \omega_0$$

$$E(t) = \frac{1}{2\pi}\int \tilde{E}(\omega_0 + \Delta\omega)e^{i(\omega_0+\Delta\omega)t}d\Delta\omega = \frac{1}{2\pi}e^{i\omega_0 t}\int \tilde{A}(\Delta\omega)e^{i\Delta\omega t}d\Delta\omega$$



$$\tilde{A}(\Delta\omega) = \tilde{E}(\omega_0 + \Delta\omega)$$

Example: Gaussian pulse



$$E(t) \propto e^{-\Gamma t^2} e^{i\omega_0 t}$$

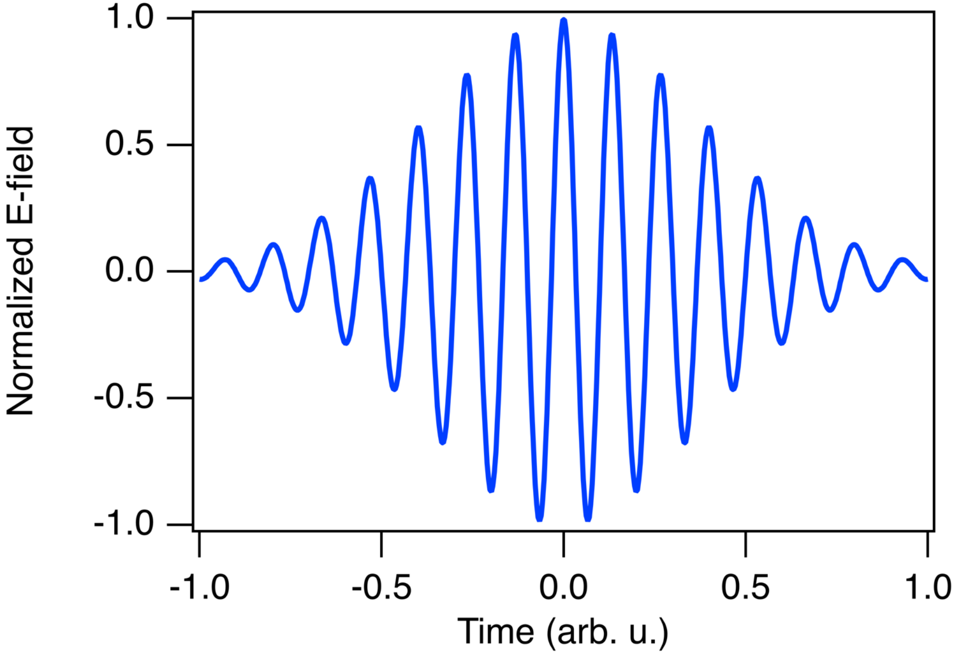
$$A(t) \propto e^{-\Gamma t^2}, \quad \Gamma \equiv \Gamma_1 - i\Gamma_2$$

$$\tau_p = \sqrt{\frac{2\ln 2}{\Gamma_1}}$$

$$\omega(t) \equiv \frac{d\phi_{tot}(t)}{dt} = \omega_0 + 2\Gamma_2 t$$

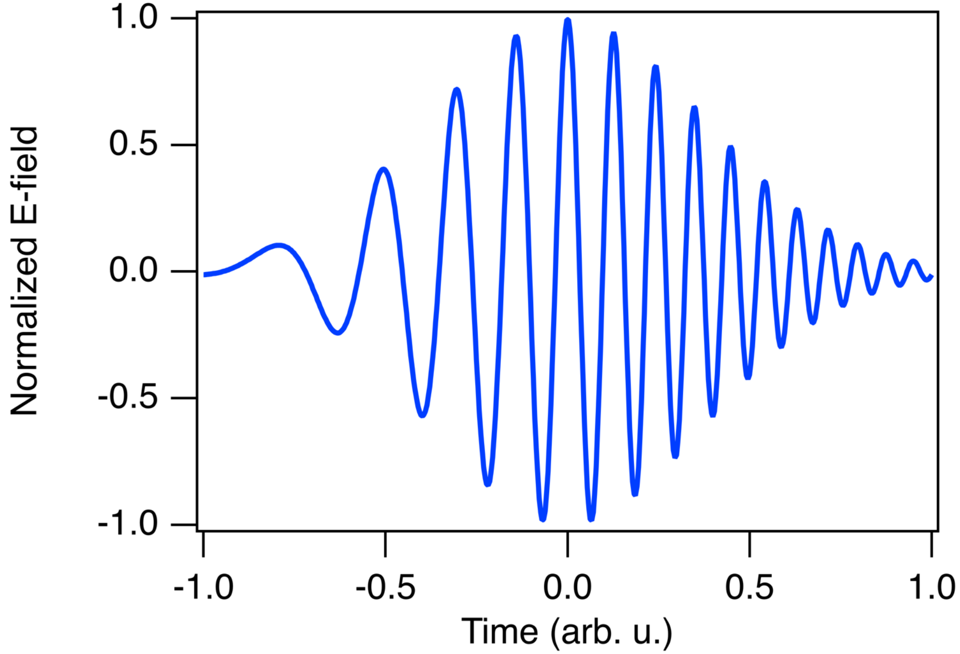
Chirped Gaussian pulse

$$E(t) = A(t) e^{i\omega_0 t} \propto e^{-\Gamma t^2} e^{i\omega_0 t}$$



$$\Gamma \equiv \Gamma_1 - i\Gamma_2$$

$$\Gamma_2 = 0$$



$$\Gamma \equiv \Gamma_1 - i\Gamma_2$$

$$\Gamma_2 \neq 0$$

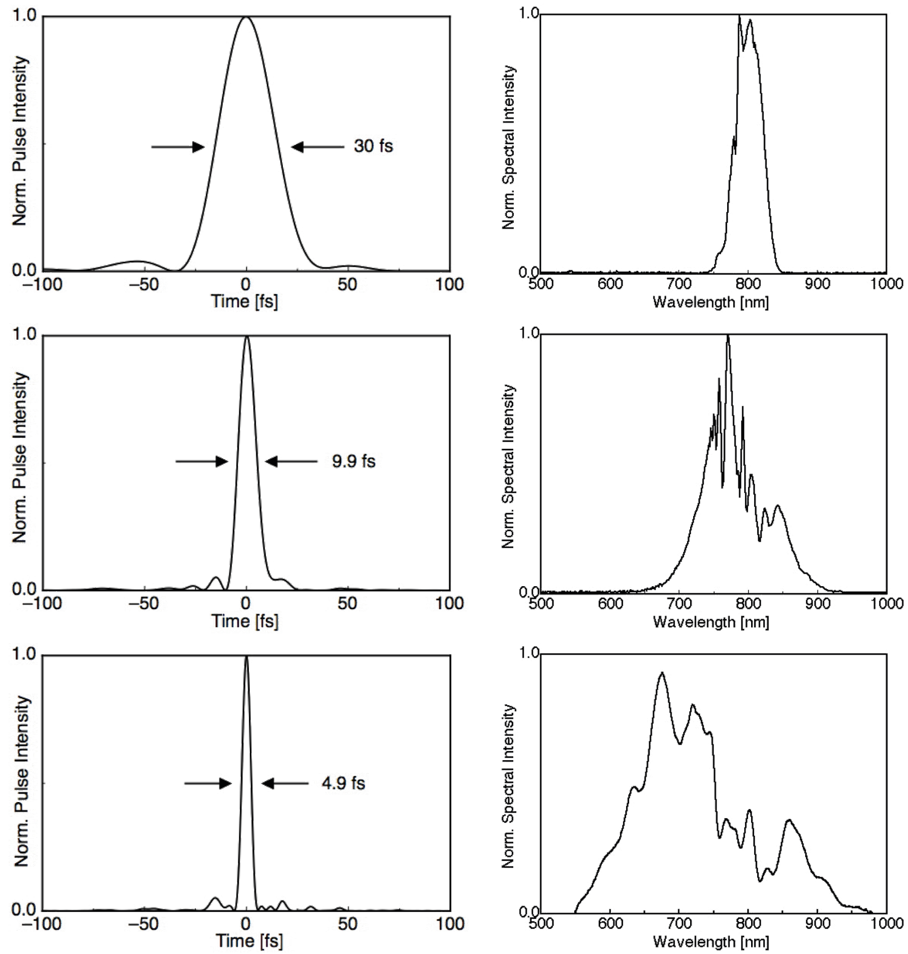
$$\phi_{tot}(t) \equiv \omega_0 t + \Gamma_2 t^2$$

$$\omega(t) \equiv \frac{d\phi_{tot}(t)}{dt} = \omega_0 + 2\Gamma_2 t$$

Time-bandwidth products

$I(t)$ $(x \equiv t/\tau)$	τ_p/τ	$\Delta\nu_p \cdot \tau_p$
1. Gaussian $I(t) = e^{-x^2}$	$2\sqrt{\ln 2}$	0.4413
2. Hyperbolic secant (soliton pulse) $I(t) = \text{sech}^2 x$	1.7627	0.3148
3. Rectangle $I(t) = \begin{cases} 1, & \lvert t\rvert \leq \tau/2 \\ 0, & \lvert t\rvert > \tau/2 \end{cases}$	1	0.8859
4. Parabolic $I(t) = \begin{cases} 1 - x^2, & \lvert t\rvert \leq \tau/2 \\ 0, & \lvert t\rvert > \tau/2 \end{cases}$	1	0.7276
5. Lorentzian $I(t) = \frac{1}{1 + x^2}$	2	0.2206
6. Symmetric two-sided exponent $I(t) = e^{-2\lvert x\rvert}$	$\ln 2$	0.1420

Spectral phase yielding shortest pulse



$\langle t \rangle$: center of gravity

$$\langle t \rangle = \frac{\int_{-\infty}^{+\infty} t \cdot I(t)dt}{\int_{-\infty}^{+\infty} I(t)dt}$$

$$\tilde{E}(\omega) = |\tilde{E}(\omega)| e^{i\varphi(\omega)}$$

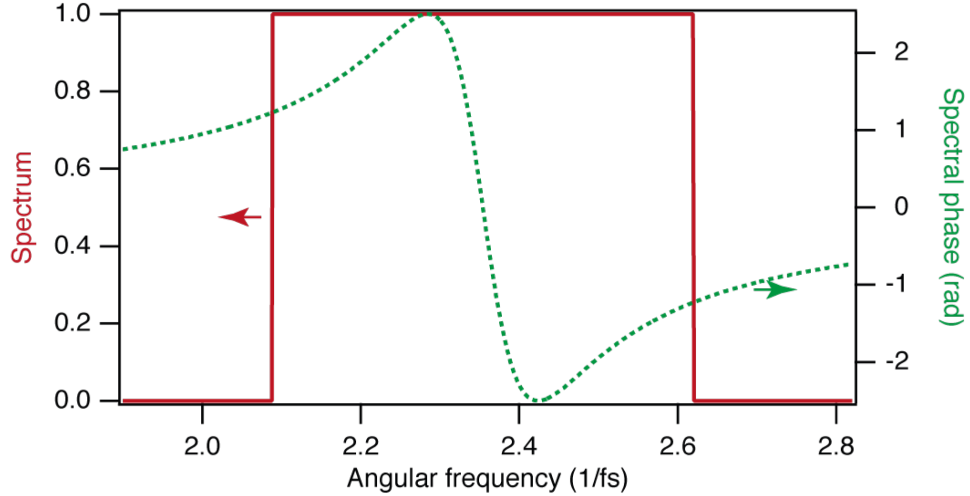
rms pulse duration Δt:

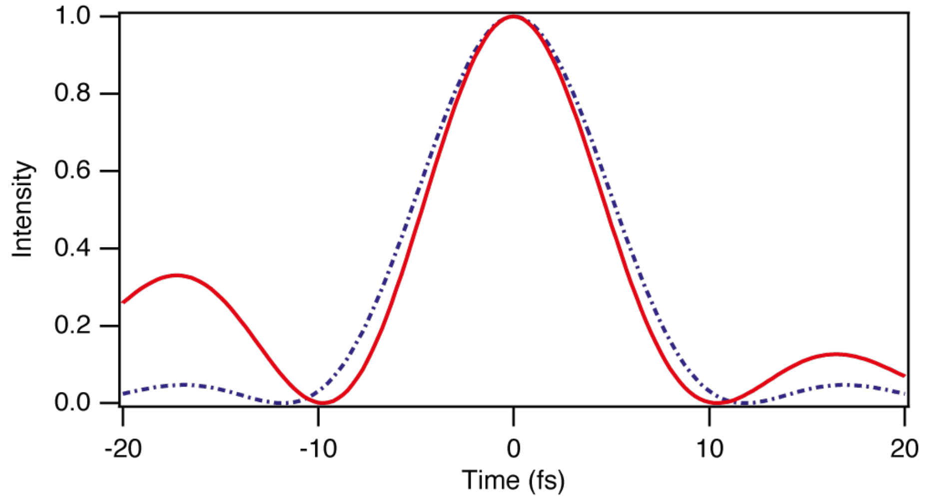
$$\Delta t^2 = \left\langle (t - \langle t \rangle)^2 \right\rangle = \int_{-\infty}^{\infty} (t - \langle t \rangle)^2 I(t)dt \Big/ \int_{-\infty}^{\infty} I(t)dt$$

shortest pulse for:

$$\frac{\partial \phi(\omega)}{\partial \omega} = 0, \quad \text{where} \quad \phi(\omega) := \varphi(\omega) - \omega \langle t \rangle$$

Inappropriateness of FWHM definition

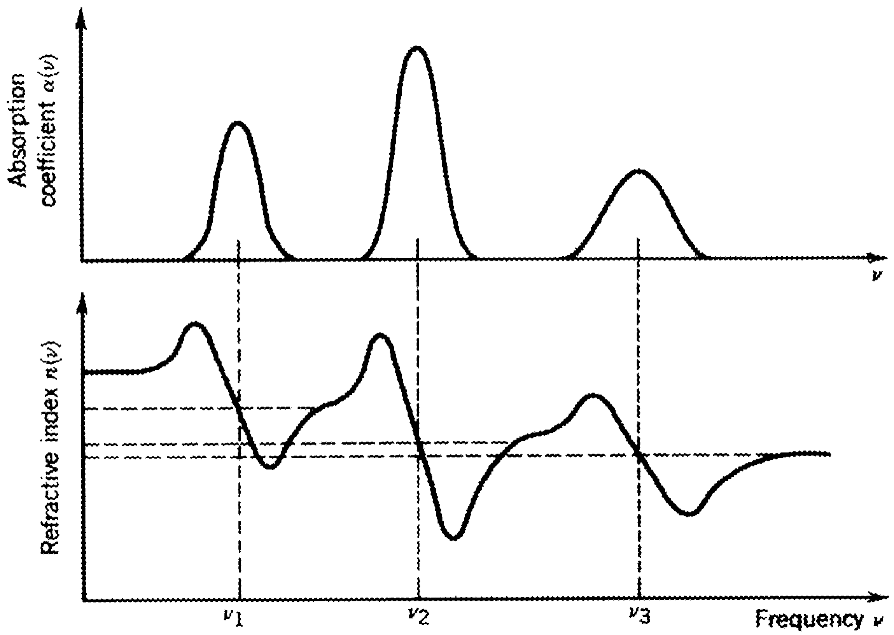


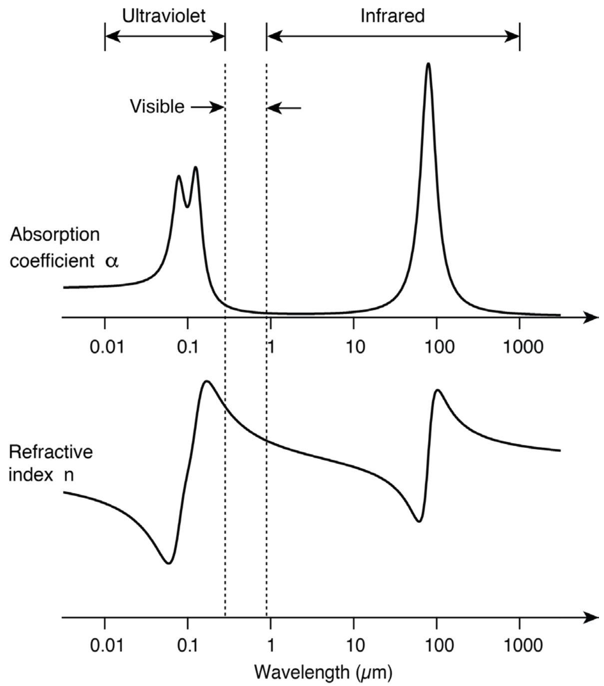


Temporal FWHM of pulse with nonlinear spectral phase is shorter than "transform-limited" pulse

FWHM is the standard in the community – but one has to be aware of its limitations

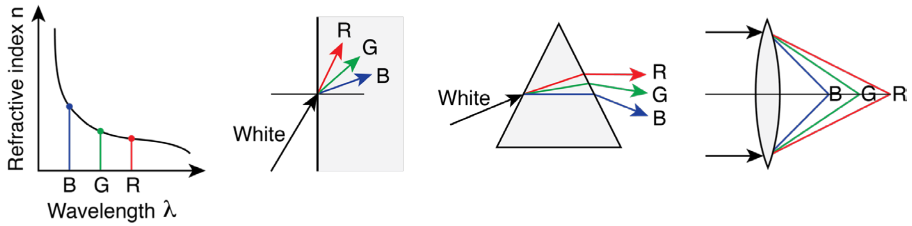
Optical dispersion





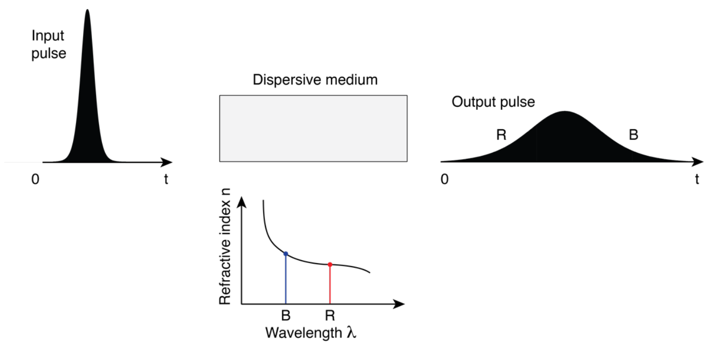
Positive dispersion:

$$\frac{\partial^2 n}{\partial \omega^2} > 0, \quad \frac{\partial^2 n}{\partial \lambda^2} > 0$$

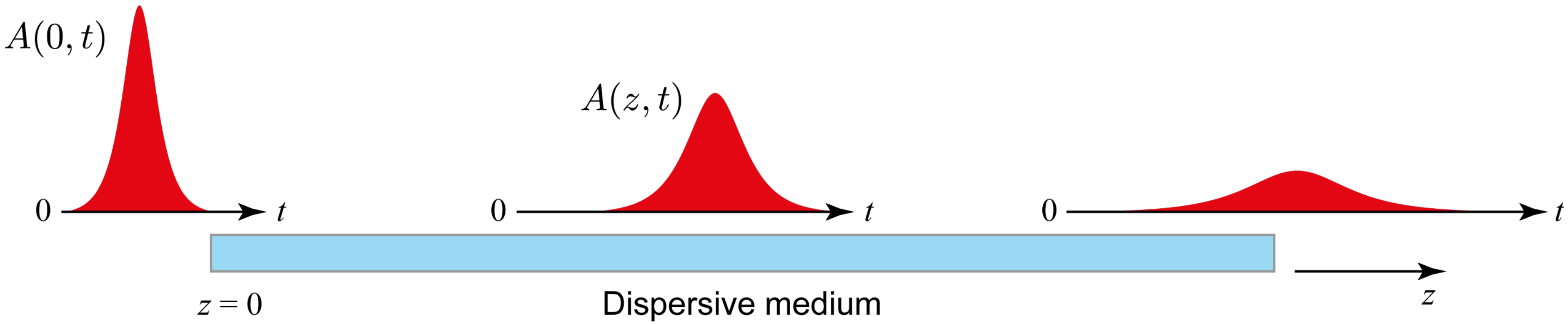


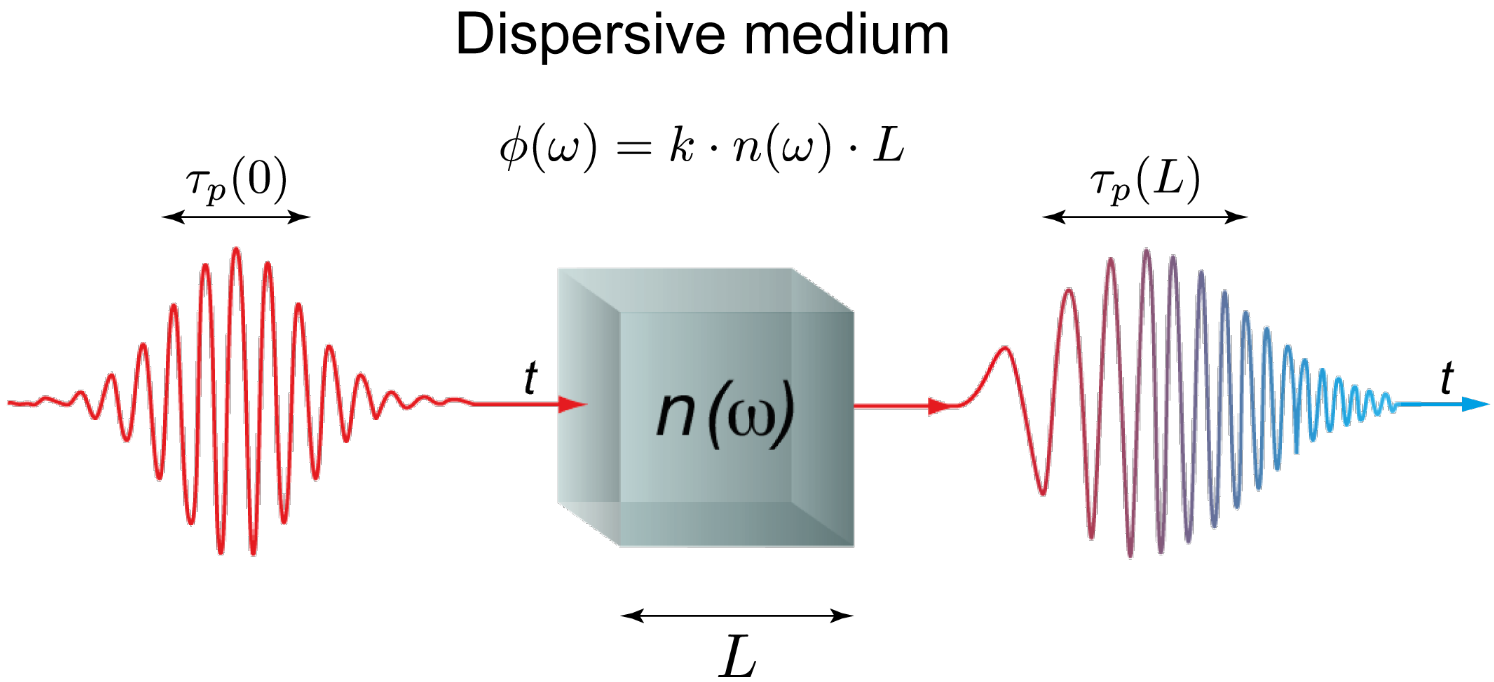
Negative dispersion:

$$\frac{\partial^2 n}{\partial \omega^2} < 0, \quad \frac{\partial^2 n}{\partial \lambda^2} < 0$$



Dispersive pulse broadening





Linear pulse propagation

Helmholtz equation:

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + [k_n(\omega)]^2\, \tilde{E}(z,\omega) = 0$$

$$\tilde{E}(z,\omega_0+\Delta\omega) = \tilde{A}(z,\Delta\omega)e^{-ik_n(\omega_0)z}$$

$$\Delta\omega \equiv \omega - \omega_0$$

$$\tilde{A}(z,\Delta\omega) = \int A(z,t)e^{-i\Delta\omega t}dt$$

$$\frac{\partial^2}{\partial z^2}\tilde{A}(z,\Delta\omega) - 2ik_n(\omega_0)\frac{\partial}{\partial z}\tilde{A}(z,\Delta\omega) - [k_n(\omega_0)]^2\,\tilde{A}(z,\Delta\omega) + [k_n(\omega_0+\Delta\omega)]^2\,\tilde{A}(z,\Delta\omega) = 0$$

"slowly varying envelope approximation"

$$\left|\frac{\partial A}{\partial z}\right| \ll |k_n(\omega_0)A|\,, \qquad \left|\frac{\partial A}{\partial t}\right| \ll |\omega_0 A|$$

$$\frac{\partial}{\partial z}\tilde{A}(z,\Delta\omega) + i\Delta k_n \tilde{A}(z,\Delta\omega) = 0$$

$$\Delta k_n \equiv k_n(\omega_0+\Delta\omega) - k_n(\omega_0)$$

Very simple equation of motion for the pulse envelope in the spectral domain with a very easy solution (i.e. a phase shift $\Delta k_n z$ for each frequency component)

$$\tilde{A}(z,\Delta\omega) = \tilde{A}(0,\Delta\omega)e^{-i\Delta k_n z} = \tilde{A}(0,\Delta\omega)e^{-i[k_n(\omega_0+\Delta\omega)-k_n(\omega_0)]z}$$

First and second order dispersion

Taylor expansion around the center frequency ω_0: $\Delta\omega = \omega - \omega_0$

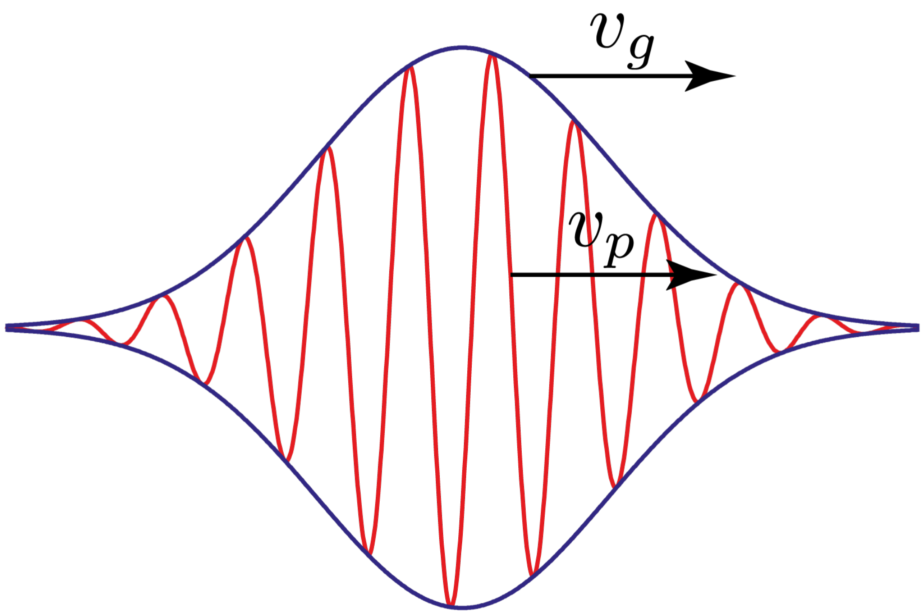
$$k_n(\omega) \approx k_n(\omega_0) + k_n' \Delta\omega + \frac{1}{2} k_n'' \Delta\omega^2 + \cdots$$

First order dispersion: $k_n' = dk_n / d\omega$

Second order dispersion: $k_n'' = d^2 k_n / d\omega^2$

Phase and group velocity

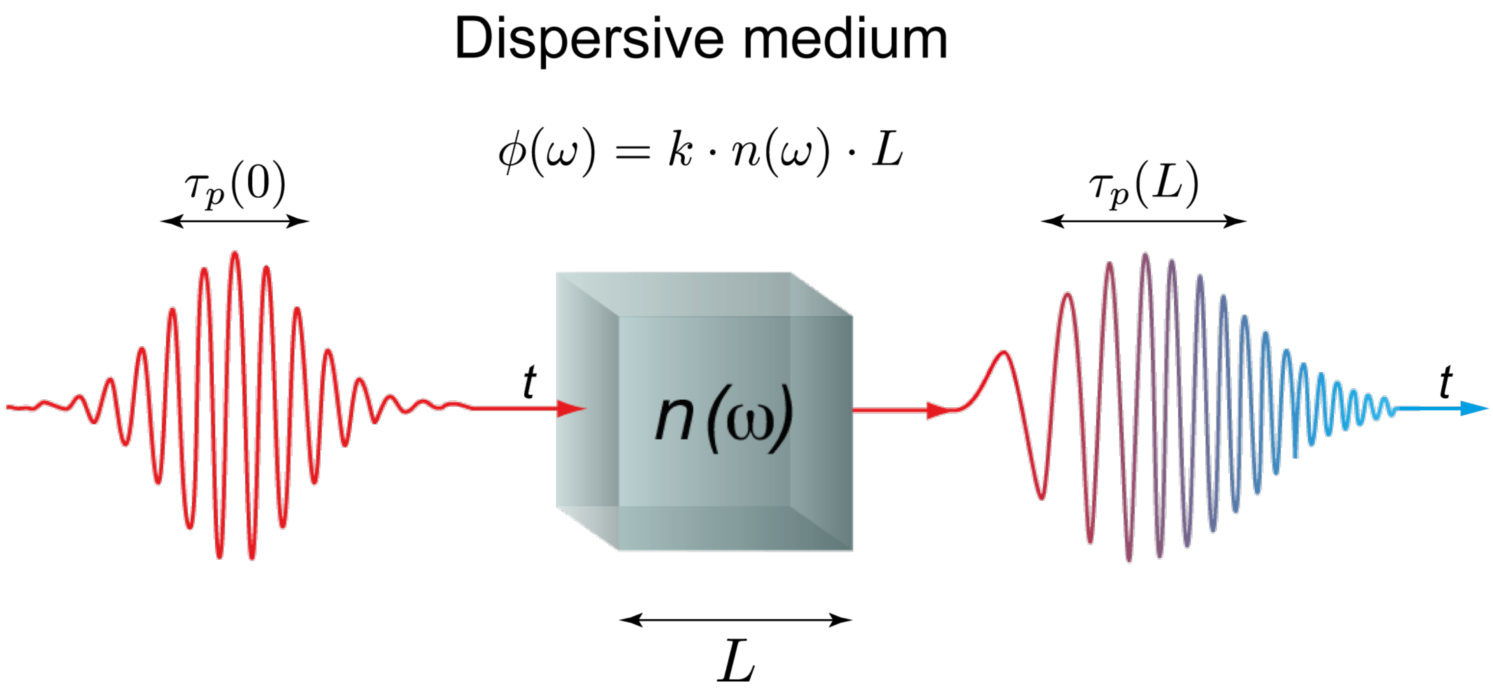
$$E(z,t) \propto \exp\left[i\omega_0\left(t-\frac{z}{v_p(\omega_0)}\right)\right]\cdot\exp\left[-\Gamma(L_d)\cdot\left(t-\frac{z}{v_g(\omega_0)}\right)^2\right]$$



$$v_p(\omega_0) \equiv c_n = \left.\frac{\omega}{k_n}\right|_{\omega=\omega_0}$$

$$v_g(\omega_0) \equiv \frac{1}{k_n'(\omega_0)} = \frac{1}{\left(\frac{dk_n}{d\omega}\right)_{\omega=\omega_0}} = \left(\frac{d\omega}{dk_n}\right)_{\omega=\omega_0}$$

Dispersive pulse broadening



Gaussian pulse:
(initially unchirped pulse)

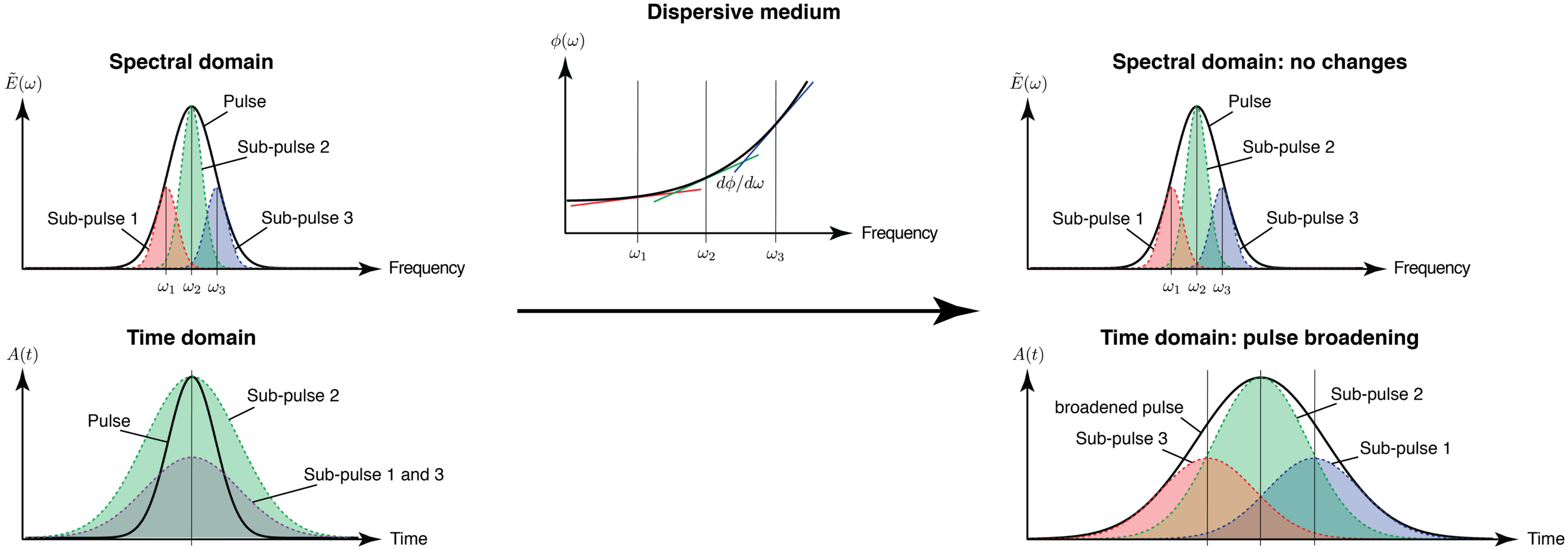
$$\frac{\tau_p(z)}{\tau_p(0)} = \sqrt{1 + \left(\frac{4 \ln 2 \; d^2\phi / d\omega^2}{\tau_p^2(0)} \right)^2}$$

Approximation for
(strong pulse broadening)

$$\frac{d^2\phi}{d\omega^2} \gg \tau_p^2(0)$$

$$\tau_p(z) \approx \frac{d^2\phi}{d\omega^2} \Delta\omega_p$$

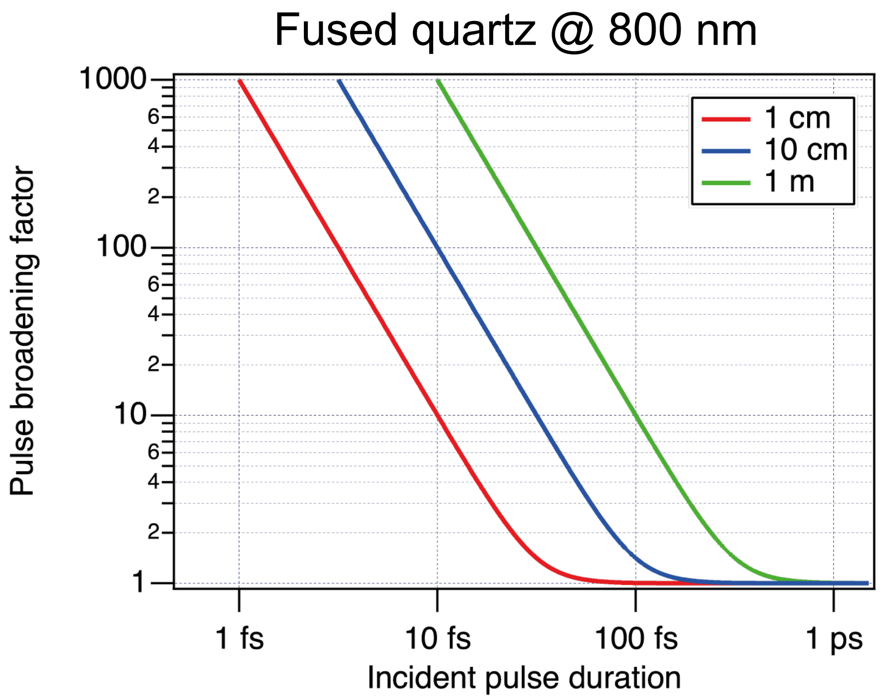
Dispersive pulse broadening



Phase Velocity v_p	$\frac{\omega}{k_n}$	$\frac{c}{n}$
Group Velocity v_g	$\frac{d\omega}{dk_n}$	$\frac{c}{n}\frac{1}{1-\frac{dn}{d\lambda}\frac{\lambda}{n}}$
Group Delay T_g	$T_g = \frac{z}{v_g} = \frac{d\phi}{d\omega}, \quad \phi \equiv k_n z$	$\frac{nz}{c}\left(1-\frac{dn}{d\lambda}\frac{\lambda}{n}\right)$
Dispersion 1. Order (Group Delay, GD)	$\frac{d\phi}{d\omega}$	$\frac{nz}{c}\left(1-\frac{dn}{d\lambda}\frac{\lambda}{n}\right)$
Dispersion 2. Order (Group Delay Dispersion, GDD)	$\frac{d^2\phi}{d\omega^2}$	$\frac{\lambda^3 z}{2\pi c^2}\frac{d^2 n}{d\lambda^2}$
Dispersion 3. Order (Third-Order Dispersion, TOD)	$\frac{d^3\phi}{d\omega^3}$	$\frac{-\lambda^4 z}{4\pi^2 c^3}\left(3\frac{d^2 n}{d\lambda^2}+\lambda\frac{d^3 n}{d\lambda^3}\right)$

Example: fused quartz

$n(800\,\text{nm}) = 1.45332$	
$\left.\frac{\partial n}{\partial \lambda}\right\|_{800\,\text{nm}} = -0.017\frac{1}{\mu\text{m}}$	$\left.\frac{\partial k_n}{\partial \omega}\right\|_{800\,\text{nm}} = 4.84\frac{\text{ns}}{\text{m}}$
$\left.\frac{\partial^2 n}{\partial \lambda^2}\right\|_{800\,\text{nm}} = 0.04\frac{1}{\mu\text{m}^2}$	$\left.\frac{\partial^2 k_n}{\partial \omega^2}\right\|_{800\,\text{nm}} = 36.1\frac{\text{fs}^2}{\text{mm}}$

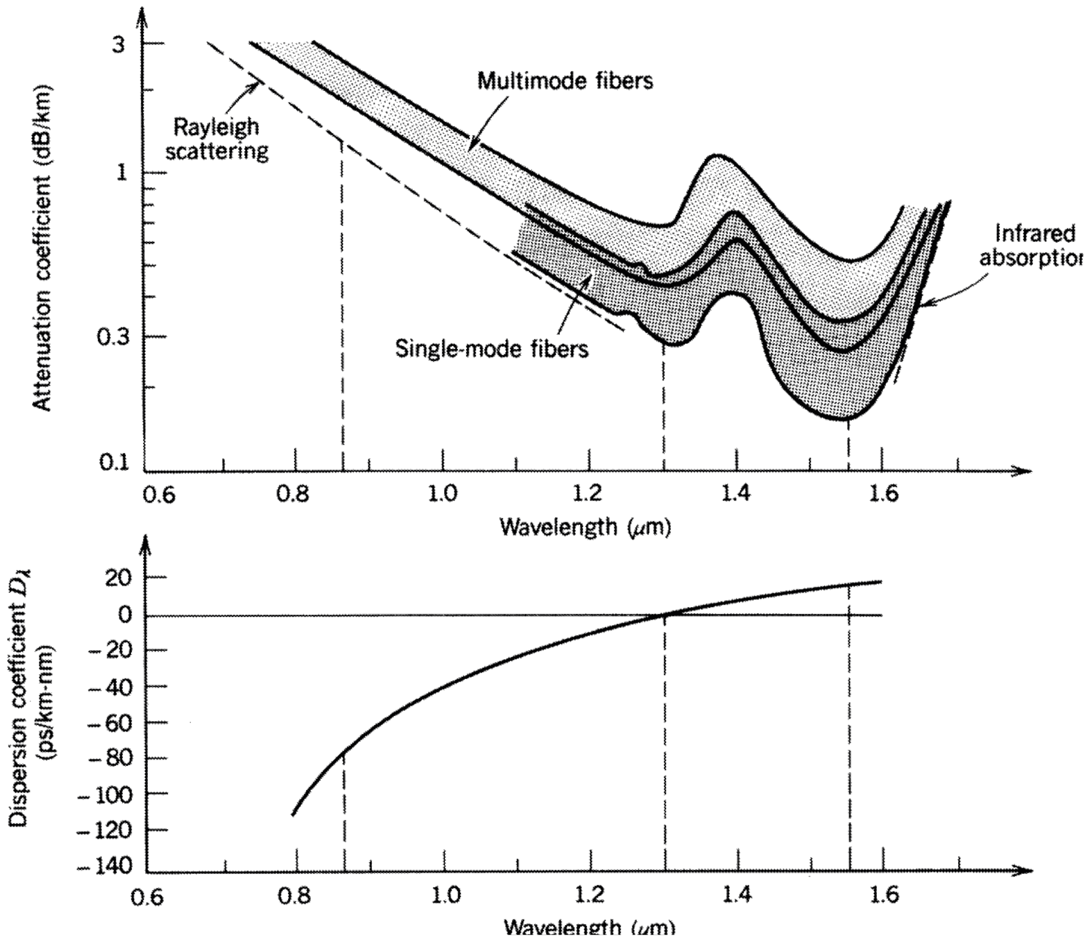


Example:
What is the final pulse duration:

10 fs pulse through 10 lenses,
each 1 cm thick?

10 fs pulse through 10 cm fused quartz?

Optical communication



λ_0 [μm]	Losses [dB/km]	2nd order dispersion [ps/km·nm]
0.87	1.5	-80
1.312	0.3	0
1.55	0.16	+17

$$\text{GVD:}\quad D_\lambda \equiv -\frac{1}{L_F}\frac{dT_g}{d\lambda} = \frac{\omega^2}{2\pi c L_F}\frac{dT_g}{d\omega} \quad\Rightarrow\quad \text{units:}\ \frac{\text{ps}}{\text{km}\cdot\text{nm}}$$

$$\tau_p(L_d) \approx \frac{d^2\phi}{d\omega^2}\Delta\omega = \frac{2\pi c L_d D_\lambda}{\omega^2}\Delta\omega = D_\lambda \cdot L_d \cdot \Delta\lambda$$

Higher order dispersion

$$\phi(\omega) = \phi_0 + \frac{\partial \phi}{\partial \omega}(\omega - \omega_0) + \frac{1}{2}\frac{\partial^2 \phi}{\partial \omega^2}(\omega - \omega_0)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial \omega^3}(\omega - \omega_0)^3 + \ldots$$

First order dispersion (group delay) $\partial \phi / \partial \omega$

Second order dispersion (group delay dispersion - GDD) $\partial^2 \phi / \partial \omega^2$

Third order dispersion (TOD) $\partial^3 \phi / \partial \omega^3$

…

Wigner representation of ultrashort pulses

Wigner distribution:

$$W(t,\omega) = \int_{-\infty}^{\infty} E\left(t + \frac{t'}{2}\right) E^*\left(t - \frac{t'}{2}\right) e^{-i\omega t'} dt'$$

"window" function

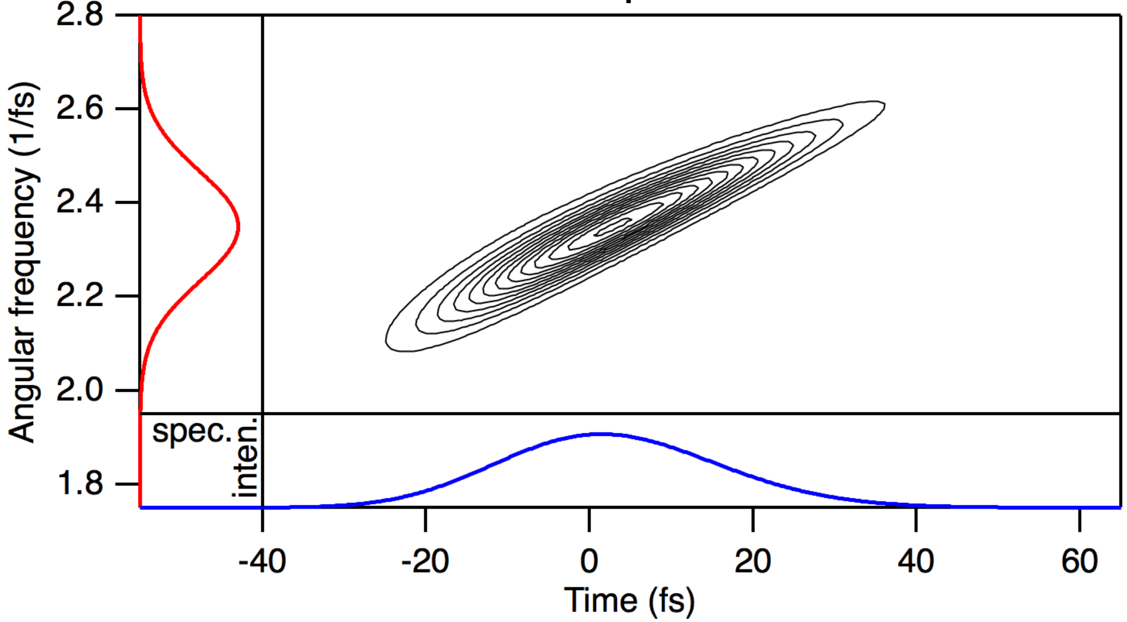
$$W(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}\left(\omega + \frac{\omega'}{2}\right) \tilde{E}^*\left(\omega - \frac{\omega'}{2}\right) e^{i\omega' t} d\omega'$$

Time-frequency representation /

windowed Fourier transform /

"instantaneous spectrum vs time"

Example:



Properties:

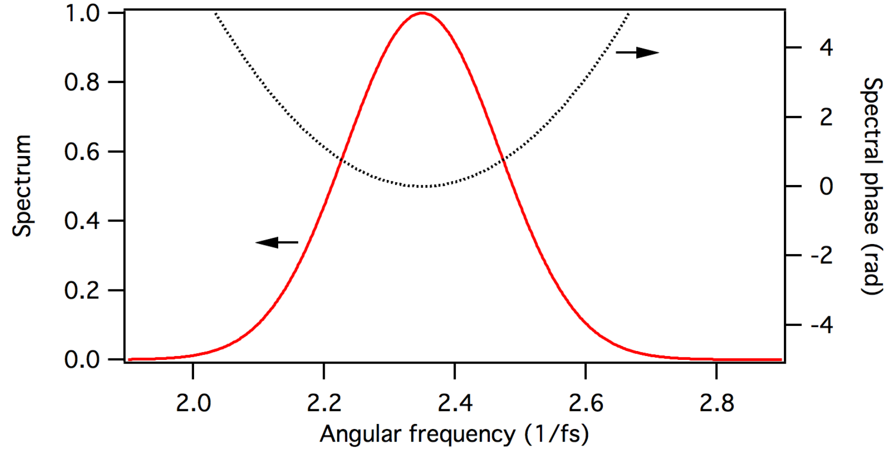
- $I(t) = |E(t)|^2 = \int_{-\infty}^{\infty} W(t,\omega) d\omega$
- $I(\omega) = |\tilde{E}(\omega)|^2 = \int_{-\infty}^{\infty} W(t,\omega) dt$

- Linear propagation in non-absorbing medium conserves area (≘ minimum time-bandwidth product)

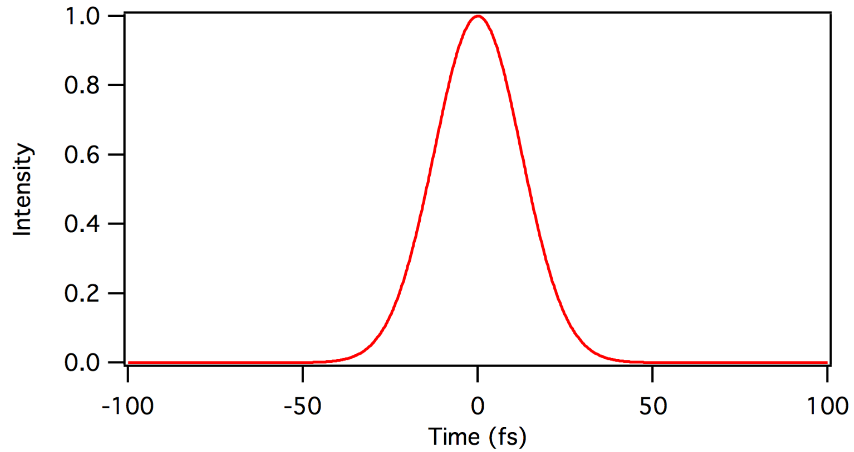
Effect of dispersion orders on Wigner trace

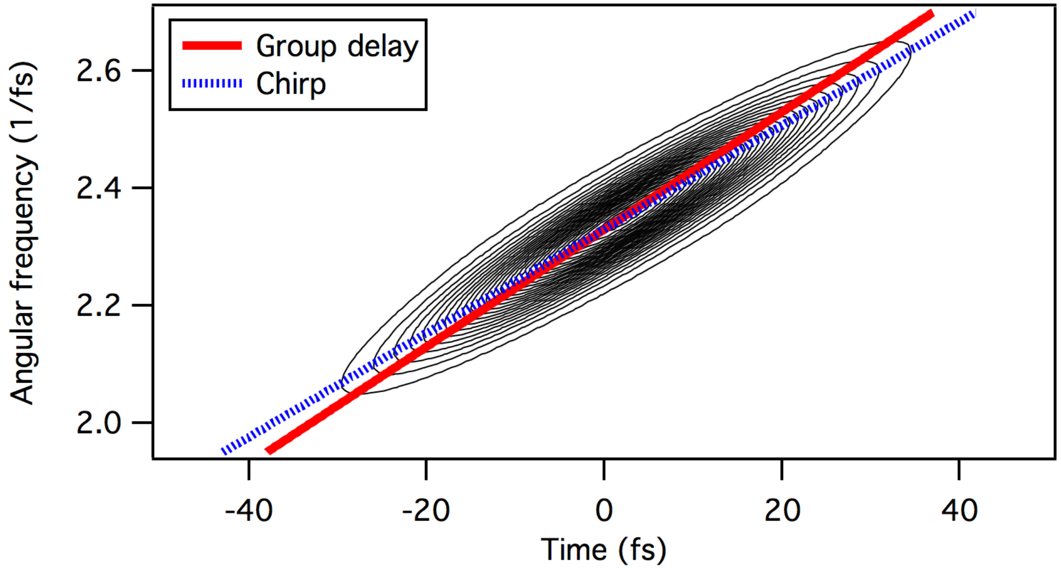
Everything calculated for an initially 10-fs long Gaussian pulse

After 100 fs^2 of GDD:



$$\phi(\omega) = \frac{1}{2} \cdot 100\,\text{fs}^2 \cdot (\omega - \omega_0)^2$$



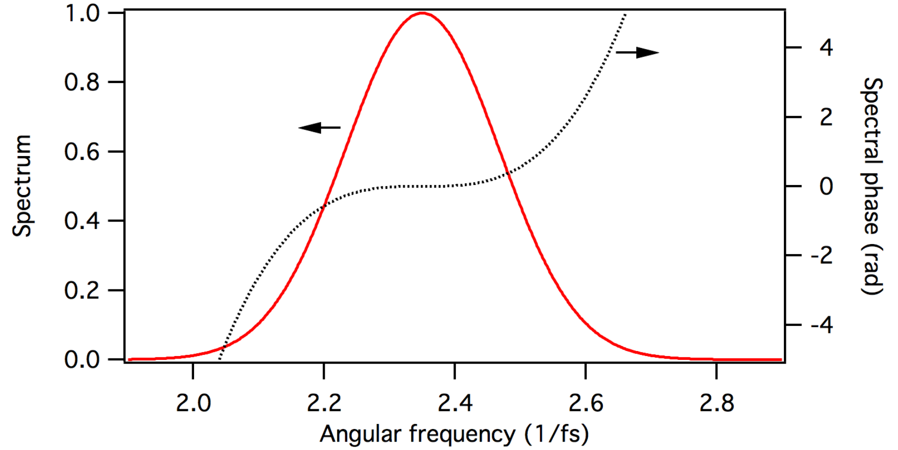


- Pulse remains Gaussian in time!
- Chirp is linear

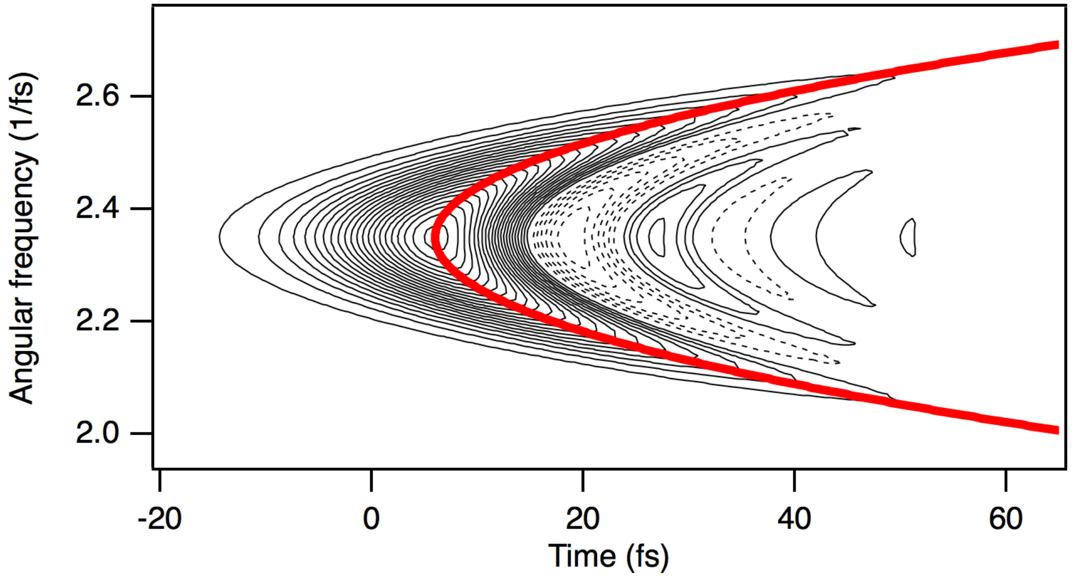
Effect of dispersion orders on Wigner trace

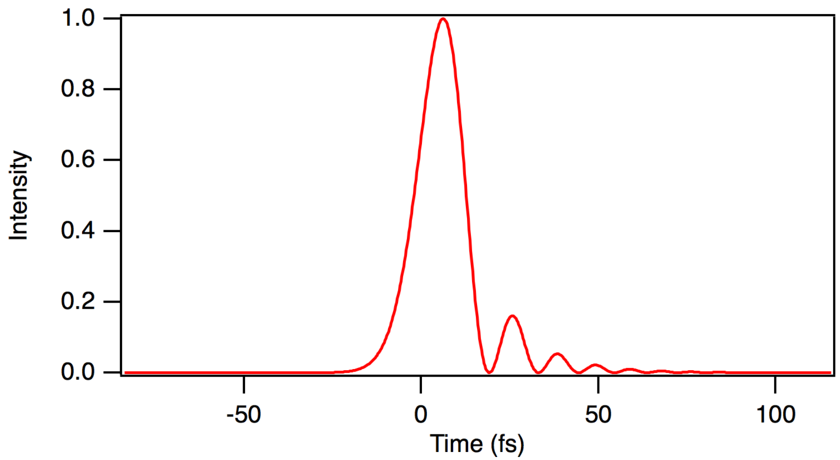
Everything calculated for an initially 10-fs long Gaussian pulse

After 1000 fs^3 of TOD:



$$\phi(\omega) = \frac{1}{6} \cdot 1000\,\text{fs}^3 \cdot (\omega - \omega_0)^3$$



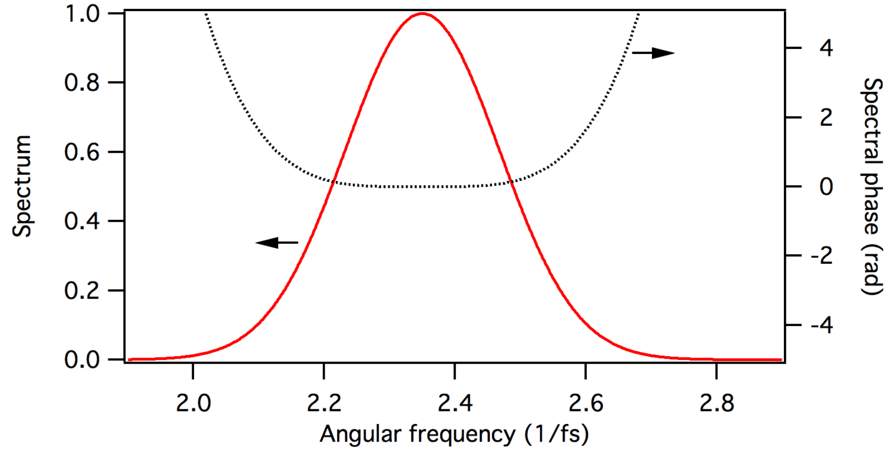


- "Beating of simultaneous frequencies" causes post-(pre-)pulses

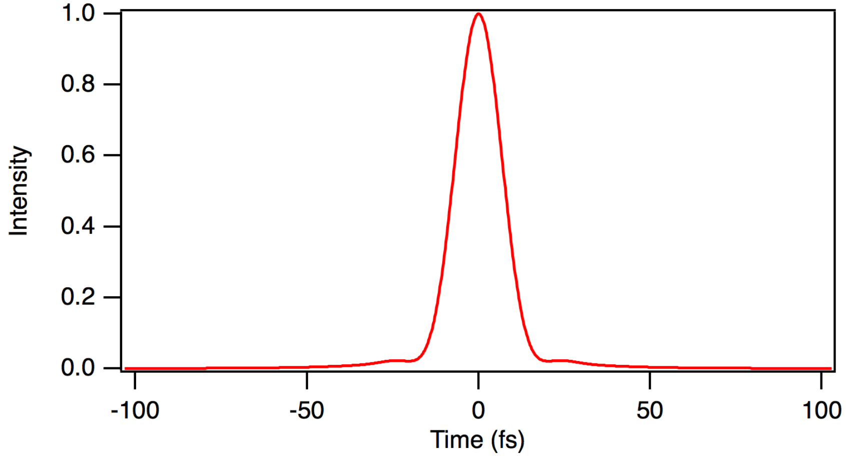
Effect of dispersion orders on Wigner trace

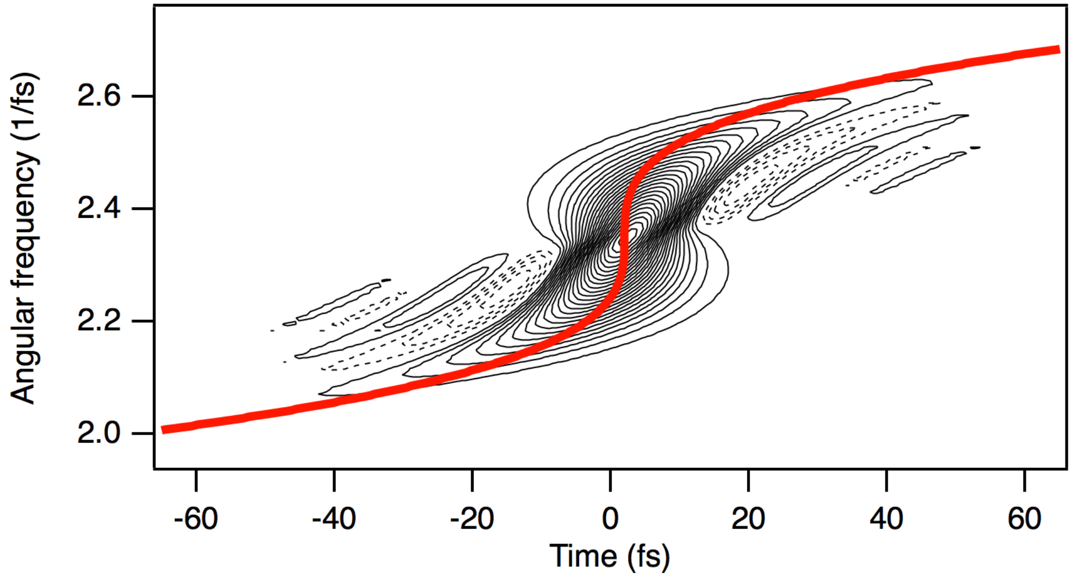
Everything calculated for an initially 10-fs long Gaussian pulse

After 10000 fs⁴ of FOD:



$$\phi(\omega) = \frac{1}{24} \cdot 10000\,\mathrm{fs}^4 \cdot (\omega - \omega_0)^4$$



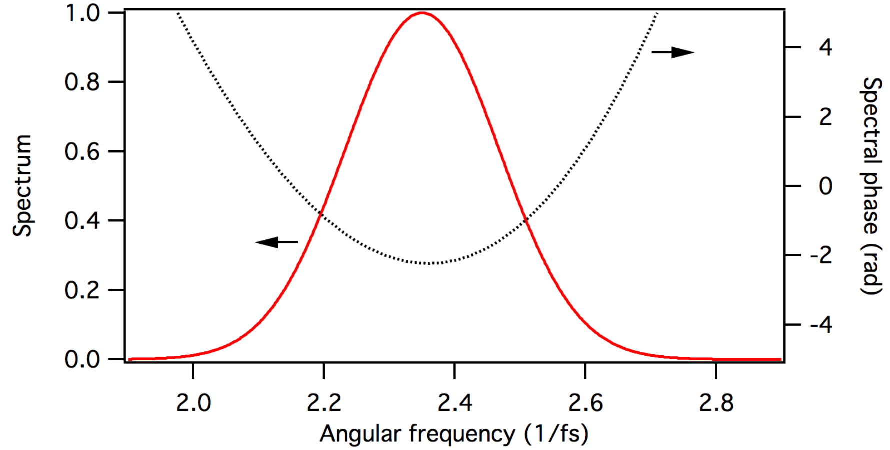


- Even dispersion orders yield symmetric pulse distortions in time (for a symmetric spectrum)

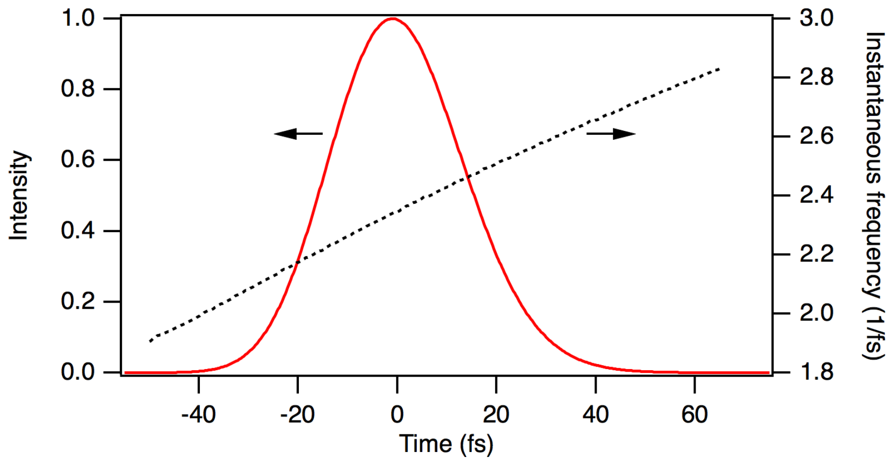
Pulse after 3 mm of fused silica

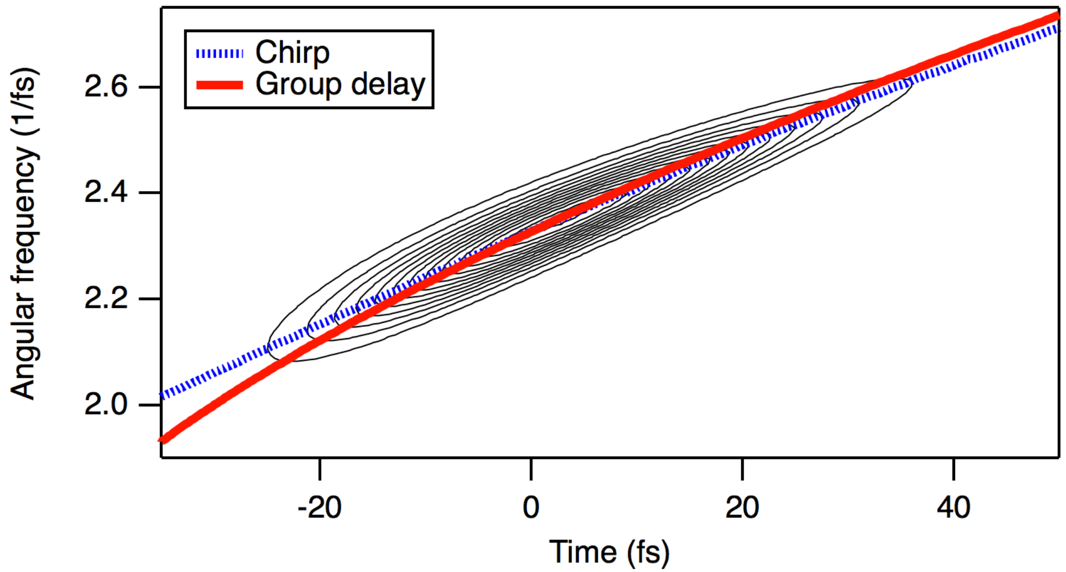
Everything calculated for an initially 10-fs long Gaussian pulse

After 3 mm of fused silica (800 nm center wavelength):



$$\phi(\omega) = 3\,\mathrm{mm} \cdot k \cdot n(\omega)$$





- Chirp is dominantly linear – however, influence of higher orders clearly visible
- Pulse is not Gaussian in time anymore!

Higher order dispersion

Material	Refractive index $n(\lambda)$ At 800 nm	Propagation constant $k_n(\omega)$ At 800 nm
Fused quartz	$n(800\,\mathrm{nm}) = 1.45332$	
	$\left.\frac{\partial n}{\partial \lambda}\right\|_{800\,\mathrm{nm}} = -0.017 \frac{1}{\mu\mathrm{m}}$	$\left.\frac{\partial k_n}{\partial \omega}\right\|_{800\,\mathrm{nm}} = 4.84 \cdot 10^{-9} \frac{\mathrm{s}}{\mathrm{m}}$
	$\left.\frac{\partial^2 n}{\partial \lambda^2}\right\|_{800\,\mathrm{nm}} = 0.04 \frac{1}{\mu\mathrm{m}^2}$	$\left.\frac{\partial^2 k_n}{\partial \omega^2}\right\|_{800\,\mathrm{nm}} = 3.61 \cdot 10^{-26} \frac{\mathrm{s}^2}{\mathrm{m}} = 36.1 \frac{\mathrm{fs}^2}{\mathrm{mm}}$
	$\left.\frac{\partial^3 n}{\partial \lambda^3}\right\|_{800\,\mathrm{nm}} = -0.24 \frac{1}{\mu\mathrm{m}^3}$	$\left.\frac{\partial^3 k_n}{\partial \omega^3}\right\|_{800\,\mathrm{nm}} = 2.74 \cdot 10^{-41} \frac{\mathrm{s}^3}{\mathrm{m}} = 27.4 \frac{\mathrm{fs}^3}{\mathrm{mm}}$
SF10-glass	$n(800\,\mathrm{nm}) = 1.71125$	
	$\left.\frac{\partial n}{\partial \lambda}\right\|_{800\,\mathrm{nm}} = -0.0496 \frac{1}{\mu\mathrm{m}}$	$\left.\frac{\partial k_n}{\partial \omega}\right\|_{800\,\mathrm{nm}} = 5.70 \cdot 10^{-9} \frac{\mathrm{s}}{\mathrm{m}}$
	$\left.\frac{\partial^2 n}{\partial \lambda^2}\right\|_{800\,\mathrm{nm}} = 0.176 \frac{1}{\mu\mathrm{m}^2}$	$\left.\frac{\partial^2 k_n}{\partial \omega^2}\right\|_{800\,\mathrm{nm}} = 1.59 \cdot 10^{-25} \frac{\mathrm{s}^2}{\mathrm{m}} = 159 \frac{\mathrm{fs}^2}{\mathrm{mm}}$
	$\left.\frac{\partial^3 n}{\partial \lambda^3}\right\|_{800\,\mathrm{nm}} = -0.997 \frac{1}{\mu\mathrm{m}^3}$	$\left.\frac{\partial^3 k_n}{\partial \omega^3}\right\|_{800\,\mathrm{nm}} = 1.04 \cdot 10^{-40} \frac{\mathrm{s}^3}{\mathrm{m}} = 104 \frac{\mathrm{fs}^3}{\mathrm{mm}}$
Sapphire	$n(800\,\mathrm{nm}) = 1.76019$	
	$\left.\frac{\partial n}{\partial \lambda}\right\|_{800\,\mathrm{nm}} = -0.0268 \frac{1}{\mu\mathrm{m}}$	$\left.\frac{\partial k_n}{\partial \omega}\right\|_{800\,\mathrm{nm}} = 5.87 \cdot 10^{-9} \frac{\mathrm{s}}{\mathrm{m}}$
	$\left.\frac{\partial^2 n}{\partial \lambda^2}\right\|_{800\,\mathrm{nm}} = 0.064 \frac{1}{\mu\mathrm{m}^2}$	$\left.\frac{\partial^2 k_n}{\partial \omega^2}\right\|_{800\,\mathrm{nm}} = 5.80 \cdot 10^{-26} \frac{\mathrm{s}^2}{\mathrm{m}} = 58 \frac{\mathrm{fs}^2}{\mathrm{mm}}$
	$\left.\frac{\partial^3 n}{\partial \lambda^3}\right\|_{800\,\mathrm{nm}} = -0.377 \frac{1}{\mu\mathrm{m}^3}$	$\left.\frac{\partial^3 k_n}{\partial \omega^3}\right\|_{800\,\mathrm{nm}} = 4.21 \cdot 10^{-41} \frac{\mathrm{s}^3}{\mathrm{m}} = 42.1 \frac{\mathrm{fs}^3}{\mathrm{mm}}$

Higher order dispersion

$$\phi(\omega) = \phi_0 + \frac{\partial \phi}{\partial \omega}(\omega - \omega_0) + \frac{1}{2}\frac{\partial^2 \phi}{\partial \omega^2}(\omega - \omega_0)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial \omega^3}(\omega - \omega_0)^3 + \ldots$$

First order dispersion (group delay) $\partial \phi / \partial \omega$

Second order dispersion (group delay dispersion - GDD) $\partial^2 \phi / \partial \omega^2$

Third order dispersion (TOD) $\partial^3 \phi / \partial \omega^3$

…

GDD becomes important, when: $\mathrm{GDD} > \tau_0^2$

TOD becomes important, when: $\mathrm{TOD} > \tau_0^3$

Dispersion lengths: $L_D \equiv \frac{\tau_0^2}{|k_n''|}, \quad L_D' \equiv \frac{\tau_0^3}{|k_n'''|}$

$$\frac{\tau_p(z)}{\tau_p(0)} \approx z\sqrt{1 + \frac{1}{L_D^2} + \frac{1}{4{L_D'}^2}}$$

Gaussian pulse $E(t) \propto \exp\left(-\frac{t^2}{2\tau^2}\right) \Rightarrow \tau_p = 2\sqrt{\ln 2}\tau = 1.665\tau$

Soliton pulse $E(t) \propto \operatorname{sech}\left(\frac{t}{\tau}\right) \Rightarrow \tau_p = 2\ln\left(1 + \sqrt{2}\right)\tau = 1.763\tau$