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Optical phase noise and carrier-envelope offset noise of mode-locked lasers

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ABSTRACT The timing jitter, optical phase noise, and carrier-envelope offset (CEO) noise of passively mode-locked lasers are closely related. New key results concern analytical calculations of the quantum noise limits for optical phase noise and CEO noise. Earlier results for the optical phase noise of actively mode-locked lasers are generalized, particularly for application to passively mode-locked lasers. It is found, for example, that mode locking with slow absorbers can lead to optical linewidths far above the Schawlow–Townes limit. Furthermore, mode-locked lasers can at the same time have nearly quantum-limited timing jitter and a strong optical excess phase noise. A feedback timing stabilization via cavity length control can, depending on the situation, reduce or greatly increase the optical phase noise, while not affecting the CEO noise. Besides presenting such findings, the paper also tries to clarify some basic aspects of phase noise in mode-locked lasers.

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1 Introduction

The noise properties of mode-locked lasers, in particular the timing jitter [1, 2], the optical phase noise, and the carrier-envelope offset (CEO) noise [3, 4], are important for many applications, e.g. in frequency metrology and data transmission. These types of noise can have different origins, the most important of which are usually mechanical vibrations of the laser cavity, thermal effects in gain medium and/or laser cavity, and quantum fluctuations. As is well known, the latter are related mainly to spontaneous emission in the gain medium and to vacuum noise entering the laser cavity through the output coupler mirror and other elements with optical losses. Depending on the circumstances, the noise performance can be close to quantum limited or many orders of magnitude above the quantum limit. One may expect that quantum-limited performance in terms of timing jitter and optical phase noise should usually come in combination, but in the following it will be demonstrated that e.g. mirror vibrations can lead to strongly enhanced optical phase noise, while the timing jitter remains quantum limited. Also, the further reduction of quantum-limited timing jitter with a feedback control of the cavity length can even increase the optical phase noise to a very high level. These and other somewhat surprising findings are discussed in this paper.

The paper begins with a careful introduction of the definitions used in Sect. 2, which is particularly important in order to distinguish different kinds of phase noise. Section 3 discusses how timing jitter and optical phase noise are related to each other in mode-locked lasers of different types. This has consequences for the CEO noise, as discussed in Sect. 4. Finally, Sect. 5 discusses the influence of technical noise sources and possible side effects of a timing stabilization, before Sect. 6 summarizes some conclusions.

2 Definitions

In order to avoid confusion, we first carefully introduce various quantities related to the noise characteristics of mode-locked lasers. For all noise spectra, we use two-sided power spectral densities (PSDs), as is common in physics, while the engineering disciplines typically use the (two-times higher) one-sided PSDs (defined only for positive frequencies). Some more mathematical background is given in Ref. [5].

To begin with, the PSD $S_{\Delta t}(f)$ is related to the timing error Δt, defined as the deviation of the temporal pulse position from the corresponding position of a timing reference, which will always be assumed to be noiseless. We also frequently use the PSD $S_{\varphi,t}(f)$ of the timing phase, the latter being defined as

$$\varphi_t = 2\pi f_{\text{rep}} \Delta t \,, \tag{1}$$

so that

$$S_{\varphi,t}(f) = \left(2\pi f_{\text{rep}}\right)^2 S_{\Delta t}(f) \,. \tag{2}$$

The timing phase should not be confused with the optical phase, as discussed below. Without noise, the optical spectrum of a mode-locked laser consists of equidistant lines:

$$\nu_j = \nu_{\text{ceo}} + j f_{\text{rep}} \,, \tag{3}$$

with the CEO frequency ν_{ceo} [3] and an integer index j. The CEO frequency is typically chosen to be between 0 and f_{rep},

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i.e. in a frequency region where a laser cannot emit, and is thus just a measure for the common shift of all lines in the spectrum from the position where one extrapolated line would be at $\nu = 0$. If $\nu_{ceo} \neq 0$, the CEO phase systematically changes from pulse to pulse by $2\pi\nu_{ceo}/f_{rep}$ (Equation (10) in Ref. [4]). Note that although there are subtleties related to the definition of the CEO phase of single pulses [6] (particularly for non-trivial pulse shapes), the CEO frequency at least without noise influences is unambiguously defined by the optical frequencies.

The optical phase noise (sometimes called carrier phase noise) is quantified by $S_{\varphi,\mathrm{opt}}(f)$, where the definition of the optical phase φ_{opt} raises some subtle issues. In our context, we take φ_j to be the phase value of the complex amplitude corresponding to line j, which is reasonable on time scales long compared to the repetition period, as are usually relevant in the context of noise. (Note that one requires a measurement time of several pulse periods to clearly distinguish the individual lines and obtain their oscillation phases.) Precisely speaking, φ_j is the phase deviation from a sinusoidal oscillation with the average frequency ν_j. In the following, we sometimes refer to the optical phase as a continuous function $\varphi_{\mathrm{opt}}(\nu)$ of frequency, although it is actually only defined for the frequencies ν_j. φ_{opt} without frequency argument is then meant to be the phase value for a selected line at the center of the optical spectrum, corresponding to a frequency which we denote by ν_{opt}.

CEO noise can be understood as the noise of the CEO phase or (apart from a factor $1/2\pi$) its temporal derivative, the CEO frequency. However, some issues arise from the fact that there is no real line in the laser spectrum with the frequency ν_{ceo}. We discuss these further in Sect. 4.

As an instantaneous frequency is essentially the temporal derivative of a phase, one can specify a frequency noise, i.e. the noise of an instantaneous frequency, instead of phase noise. The PSDs are related to each other by

$$S_\nu(f) = f^2 S_\varphi(f)\,. \tag{4}$$

Phase noise PSDs proportional to f^{-2}, as often encountered in this paper, then correspond to (noise-)frequency-independent instantaneous frequency noise, i.e. to white frequency noise.

3 Quantum limits of optical phase noise and timing jitter

Although classical noise is often dominating in lasers, in this section we are mostly interested in the quantum limits for different kinds of noise, i.e. the noise levels obtained if only quantum noise occurs in a mode-locked laser. Note that this is not the same as the fundamental quantum noise limit, applicable to any light source generating pulses with the given properties. The noise in the output of a laser will in many cases be well above the fundamental quantum noise limit, even if the laser is subject only to quantum noise. For example, intensity noise will always be above the shot noise level around the relaxation oscillation frequency. In this paper, we always refer to the quantum limit as that of a laser with given parameters, not the fundamental one.

3.1 *Timing noise*

We begin by recalling a previous key result [2], according to which the quantum-limited PSD of the timing error is

$$S_{\Delta t}(f) \approx 0.53\,\frac{\theta\,h\nu}{E_p}\frac{l_{tot}}{T_{rt}}\tau_p^2\frac{1}{(2\pi f)^2} \tag{5}$$

and correspondingly for the timing phase

$$S_{\varphi,t}(f) \approx 0.53\,\frac{\theta\,h\nu}{E_p}\frac{l_{tot}}{T_{rt}}\tau_p^2\left(\frac{f_{rep}}{f}\right)^2\,, \tag{6}$$

where l_{tot} denotes the total cavity losses (including the loss at the output coupler), which are on average compensated by the laser gain, $h\nu$ the photon energy, E_p the intracavity pulse energy, θ the spontaneous emission factor (> 1 for quasi-three-level gain media), τ_p the pulse duration (full width at half maximum, FWHM), and T_{rt} the cavity round-trip time. We have assumed sech^2-shaped pulses and that various effects, which couple other types of noise (in particular fluctuations of the optical center frequency) to timing noise, are negligible. (See Ref. [2] for details.) Particularly for picosecond lasers, this can be quite realistic. Modified pulse shapes would lead to a slightly different constant factor, e.g. 0.36 instead of 0.53 for Gaussian pulses, while leaving the result unchanged in other respects.

Using $f_{rep} = N/T_{rt}$ with an integer number N of pulses (to allow for harmonic mode locking with $N > 1$), we obtain

$$S_{\varphi,t}(f) \approx 0.53\,\frac{\theta\,h\nu}{E_p}\frac{l_{tot}}{N}\tau_p^2 f_{rep}^3 f^{-2}\,. \tag{7}$$

For the same pulse energy, duration, and repetition rate, a harmonically mode-locked laser can have lower timing jitter, simply because the noise from the gain medium and the losses affects the pulse less frequently (only once every N pulse periods).

3.2 *Optical phase noise of actively mode-locked lasers*

For the optical phase noise, we can build on the famous Schawlow–Townes formula [7], which we use in the modified (but equivalent) form

$$\Delta\nu_{ST} = \frac{\theta\,h\nu\,l_{tot}T_{oc}}{4\pi\,T_{rt}^2\,P_{out}}\,, \tag{8}$$

where $\Delta\nu_{ST}$ is the FWHM linewidth, T_{oc} is the output coupler transmission, and P_{out} is the average output power of the laser. A later extension of this result added a factor $1+\alpha^2$, with α being Henry's linewidth enhancement factor [8], which is of interest mainly for semiconductor lasers but not further considered here. A newer derivation of both Eq. (8) and the extension with the factor $1+\alpha^2$ will appear in a recently submitted book chapter [9].

Using the facts that $S_{\varphi,\mathrm{opt}}(f) \propto f^{-2}$ in the quantum-limited case, and that the linewidth is then related to the PSD according to

$$\Delta\nu_{opt} = 2\pi\,S_{\varphi,\mathrm{opt}}(f)\,f^2\,, \tag{9}$$

we can obtain the quantum-limited optical phase noise power spectral density

$$S_{\varphi,\mathrm{opt,ST}}(f) = \frac{\theta\, h\nu\, l_{\mathrm{tot}} T_{\mathrm{oc}}}{8\pi^2\, T_{\mathrm{rt}}^2\, P_{\mathrm{out}} f^2} = \frac{\theta\, h\nu\, l_{\mathrm{tot}}}{8\pi^2 T_{\mathrm{rt}}^2\, P_{\mathrm{int}}} f^{-2}, \qquad (10)$$

with the average intracavity power P_{int}.

Originally, Eq. (8) had been derived for single-frequency continuous-wave lasers, but it has later been shown [10, 11] that the same formula applies to all lines of an actively mode-locked laser when the total average output power of the laser is used for P_{out} (rather than e.g. the power in a particular line). This result, which may seem surprising, can be explained with the fact that the mode-locking mechanism prevents the pulses from falling apart, so that the phase values φ_j can fluctuate, but not undergo independent drifts. Specifically for active mode locking, there can only be a common drift of all phase values φ_j (apparent as equally strong optical phase noise in all lines), while all non-correlated phase changes are strongly damped by the effect of the modulator and thus can only lead to bounded fluctuations of pulse parameters such as temporal position, duration, chirp, and so on. Such bounded fluctuations indeed occur (see below), and their magnitude is different for the different lines in the spectrum, but they do not contribute to the linewidth, which is solely determined by the unbounded common phase drift of all lines. (The finite linewidth is associated with the divergence of the PSD of the phase noise for $f \to 0$ and does not depend on the PSD at higher noise frequencies.)

We note that these arguments cannot be applied to the high-frequency components of the phase noise. As any mode-locking mechanism does not rigidly lock the phases of the cavity modes, but only generates some ‘force’ which acts on the cavity modes during many round trips, the modes can be considered as basically uncoupled during short time intervals, corresponding to high noise frequencies (approximately within a decade below the Nyquist frequency, which is half the pulse repetition rate). Therefore, the quantum-limited high-frequency phase noise of the lines in the output of any mode-locked laser must be expected to be at the level according to the Schawlow–Townes formula, but this time evaluated with the power in the particular line, rather than with the total average laser power. Particularly for lasers with a small duty cycle (pulse duration divided by repetition period), i.e. with many lines in the optical spectrum, this noise level is much higher (as the power per line is low), and in any case it depends on the particular mode power. Only on longer time scales, where the mode-locking mechanism is effective, are the phases strongly coupled, and the phase noise levels of all modes come down to the Schawlow–Townes limit calculated with the total laser power.

To check these findings, we performed numerical simulations with the model which has been described in detail in Ref. [5]. The assumptions were the following: we consider a solid-state laser operating at 1064 nm, repetition rate 100 MHz with a single circulating pulse, output coupler transmission 5%, average output power 1 W, gain bandwidth 1 nm, no dispersion, and no nonlinearity. The strength of a modulator for active mode locking was adjusted so that a pulse duration of 5.8 ps was achieved. Figure 1 shows that indeed

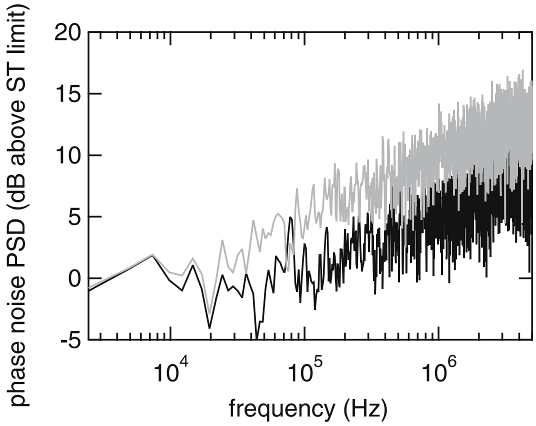

FIGURE 1 Numerically simulated optical phase noise of an actively mode-locked laser for the central line (*black*) and a line in the short-wavelength wing (*grey*), both relative to the Schawlow–Townes phase noise calculated with the total average power of the laser

the high-frequency optical phase noise is at the mentioned higher level (depending on the mode power), while the low-frequency phase noise of all lines is in agreement with the Schawlow–Townes formula evaluated with the total average power. The transition between these regimes is at noise frequencies of a few hundred kHz, corresponding to a few microseconds, i.e. to a few hundred cavity round trips. This transition would occur at higher frequencies if the mode-locking mechanism were made stronger (e.g. by increasing the modulation strength) or for lasers with shorter cavity round-trip times. In practice, the high-frequency regime with increased phase noise is probably not very relevant, because the phase noise is high only in very weak lines, where it is difficult to detect. Therefore, Eq. (10) can usually be used with P_{int} being the total intracavity average power, and the discussion above just serves to indicate the limit of validity, apart from further clarifying the role and the influence of the mode-locking mechanism. Also note that the increased high-frequency phase noise does not affect the linewidth.

For mode-locked lasers, we can rewrite Eq. (10) to obtain

$$S_{\varphi,\mathrm{opt,ST}}(f) = \frac{1}{8\pi^2}\frac{\theta\, h\nu}{E_{\mathrm{p}}}\frac{l_{\mathrm{tot}}}{N} f_{\mathrm{rep}} f^{-2} = \frac{1}{8\pi^2}\frac{\theta\, h\nu}{E_{\mathrm{p}}}\frac{l_{\mathrm{tot}}}{T_{\mathrm{rt}}} f^{-2}, \qquad (11)$$

where we have inserted $P_{\mathrm{int}} = E_{\mathrm{p}} f_{\mathrm{rep}}$ and $T_{\mathrm{rt}} = N/f_{\mathrm{rep}}$ (again allowing for harmonic mode locking), and multiplied the result with another factor N. The latter can be understood by considering that splitting a single pulse into two pulses of half the energy must double the noise PSD (just as if the pulse energy had been reduced by reducing the pump power). The consequence of this is that the linewidth of a harmonically mode-locked laser is larger than directly calculated from Eq. (10) without this correction, and is (as it should be) the same as if the laser were operated with a single circulating pulse with the same pulse energy (and correspondingly lower repetition rate).

Comparing Eqs. (6) and (11), we find that the quantum-limited timing phase noise is weaker than the quantum-limited optical phase noise, as we usually have $\tau_{\mathrm{p}} f_{\mathrm{rep}} \ll 1$. However, the opposite statement would result from comparing power spectral densities of fractional frequency noise, i.e. $S_{y,t}(f) = S_{\varphi,t}(f)\, f^2/f_{\text{rep}}^2$ and $S_{y,\text{opt}}(f) = S_{\varphi,\text{opt}}(f)\, f^2/\nu_{\text{opt}}^2$, because the timing noise refers to that of a much lower frequency (f_{rep} instead of ν_{opt}). Also, we will see in Sect. 5 that cavity length fluctuations affect $S_{\varphi,\text{opt}}(f)$ much more than $S_{\varphi,t}(f)$: quantitatively, the difference is a factor $(\nu_{\text{opt}}/f_{\text{rep}})^2$ between those effects. Therefore, it is more difficult to reach the quantum noise limit for the optical phase noise than for the timing noise. This is actually in agreement with experimental observations: a number of mode-locked lasers have exhibited close to quantum-limited timing jitter, while phase noise at the Schawlow–Townes limit is very hard to reach in practice.

3.3 *Optical phase noise of passively mode-locked lasers*

Previously, we considered actively mode-locked lasers. For passive mode locking (without timing stabilization), we have to modify the phase noise results. The most basic difference to active mode locking is that here we can also have an unbounded drift of the slope $\partial\varphi/\partial\nu$, related to timing drifts, in addition to drifts of the average optical phase. We again consider the case that only quantum noise acts in the laser. Instead of generalizing the already rather involved calculations of Ref. [10], as a first approach we calculate the resulting additional optical phase noise on the basis of the timing fluctuations from Eq. (6). First we note the similarity of Eq. (6) and Eq. (10), so that we can rewrite Eq. (6) as

$$S_{\varphi,t}(f) \approx \left(2\pi f_{\text{rep}}\tau_{\text{p}}\right)^2 S_{\varphi,\text{opt,ST}}(f) \qquad (12)$$

(where we have replaced 0.53 with 0.5), which shows that the ratio of $S_{\varphi,t}(f)$ and $S_{\varphi,\text{opt,ST}}(f)$ is determined only by the duty cycle. Of course, Eq. (12) holds only for low noise frequencies as discussed above. Next, we consider the additional phase fluctuations at a frequency $\nu = \bar{\nu} + \Delta\nu$ (e.g. in a wing of the optical spectrum) caused by the timing fluctuations:

$$\varphi(\bar{\nu} + \Delta\nu) = \varphi(\bar{\nu}) + \frac{\partial\varphi}{\partial\nu}\Delta\nu = \varphi(\bar{\nu}) + 2\pi\,\Delta t\,\Delta\nu\,, \qquad (13)$$

so that the total PSD of the phase at $\nu = \bar{\nu} + \Delta\nu$ is

$$\begin{aligned} S_{\varphi,\text{opt,wing}}(f) &= S_{\varphi,\text{opt,ST}}(f) + (2\pi\,\Delta\nu)^2\, S_{\Delta t}(f) \\ &= S_{\varphi,\text{opt,ST}}(f) + \left(\frac{\Delta\nu}{f_{\text{rep}}}\right)^2 S_{\varphi,t}(f) \\ &= S_{\varphi,\text{opt,ST}}(f)\left[1 + \left(2\pi\,\Delta\nu\,\tau_{\text{p}}\right)^2\right] \qquad (14) \end{aligned}$$

provided that phase and timing fluctuations are uncorrelated. This condition should normally be fulfilled for quantum noise, unless some coupling mechanism in the laser cavity correlates these quantities. However, in passively mode-locked lasers one would normally not expect this to occur: the temporal position of a pulse should not affect its optical phase changes (as no component of the laser 'knows the time'), and vice versa.

With a typical time–bandwidth product $\Delta\nu_{\text{p}}\tau_{\text{p}}$ ($\Delta\nu_{\text{p}}$ = pulse bandwidth) in the order of 0.3, Eq. (14) shows that the phase noise in the wings of the spectrum is a few dB stronger than at the center, and the linewidth in the wings is accordingly somewhat increased. This holds independently of the pulse duration (for a given value of $\Delta\nu_{\text{p}}\tau_{\text{p}}$) and of all laser parameters. Of course, the situation would be completely different with classical noise sources such as e.g. mirror vibrations, as discussed in Sect. 5.

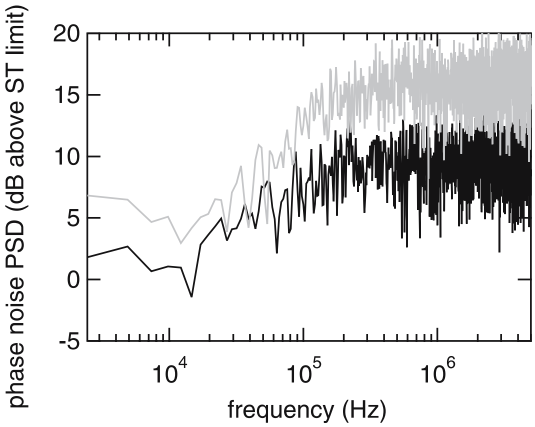

FIGURE 2 Like Fig. 1, but for a passively mode-locked laser with fast saturable absorber

We have again used the above-mentioned numerical model, this time with a fast saturable absorber instead of the modulator, to verify these findings. The absorber has a modulation depth of 2.2%, so that the pulse duration is again 5.8 ps. Figure 2 shows that the phase noise in the central line of the spectrum is basically the same as in the case of the actively mode-locked laser, while the line in the wing indeed has somewhat more noise at low frequencies, as expected. This means that the numerical simulation confirms the analytical calculation on a fairly independent basis, because e.g. it does not rely on the arguments underlying Eq. (13) and Eq. (14).

Note that in practice the driving signal of the modulator in an actively mode-locked laser also has some phase noise, which can easily lead to a timing jitter well above the quantum limit of the laser, and consequently also to increased optical phase noise. The latter then results from the additional timing jitter in the same way as discussed above for a passively mode-locked laser.

For mode locking with a slow saturable absorber (e.g. a semiconductor saturable absorber mirror, SESAM [12, 13]), one might not expect to see significantly different results. However, the simulation with a modulation depth of 3% and a saturation parameter of 4 (again chosen for a pulse duration of 5.8 ps) shows drastically increased low-frequency phase noise, as shown in Fig. 3, which also leads to correspondingly increased linewidth values. This finding can be explained as follows. Fluctuations of the optical center frequency (i.e. of the position of the envelope of the optical spectrum) correspond to fluctuations of the slope of the temporal phase. As the slow absorber attenuates the leading wing of the pulse much more than the trailing wing, it converts such changes of the slope of the temporal phase into changes of the averaged phase of the temporal envelope [14]. According to the Fourier transform, the complex amplitude of the central component of the spectrum is just an average of the temporal envelope, so that its phase is also affected by the absorber action. (In contrast, a fast absorber treats the pulses symmetrically and therefore does not introduce such a coupling.) Indeed, the simulation also confirmed (not shown

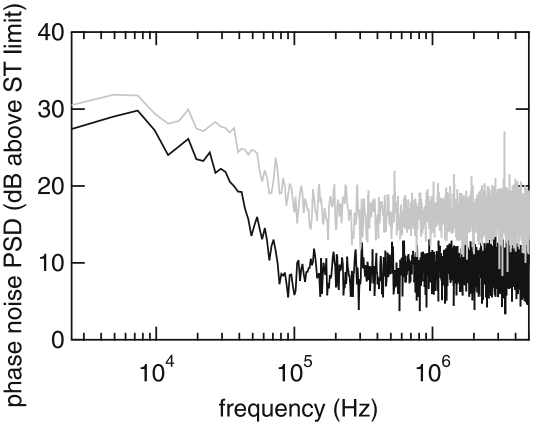

FIGURE 3 Like Fig. 1, but for a passively mode-locked laser with slow saturable absorber

in the figure) that the low-frequency optical phase noise is strongly correlated with fluctuations of the center frequency, which strongly supports this interpretation. Also, the mentioned coupling works in the other direction as well and significantly increases the low-frequency noise of the center frequency. We thus have some complicated nonlinear dynamics, introduced by the coupling of center frequency noise to optical phase noise at the slow absorber, and these dynamics can obviously raise the noise level quite significantly. Of course, this effect may be much stronger or weaker for other laser parameters. (A study of such dependences goes beyond the scope of this paper.)

Note that although the simple absorber model used for the simulations may not very accurately reflect the behavior of a real SESAM, it must be expected that a significant coupling of center frequency to phase noise exists, with the effect of increasing the linewidth by orders of magnitude. On the other hand, a fast absorber should not to be able to produce such a coupling, as explained above, so that fast absorbers turn out to be preferable when a low optical linewidth is required.

Of course, similar mechanisms which couple different kinds of noise in a mode-locked laser can easily occur, for example via optical nonlinearities. For example, a similar increase of the low-frequency phase noise is observed in the model when a Kerr nonlinearity is introduced into a laser which is mode locked with a fast absorber. In this situation, the pulses are chirped, and it is not surprising that chirps break the symmetry of the pulses and thus open the possibility for a whole range of additional coupling mechanisms. Therefore, we must expect that the linewidths can be orders of magnitude above the Schawlow–Townes linewidth even if quantum fluctuations were the only source of noise in a mode-locked laser. In other words, not only excess noise such as mirror vibrations (see Sect. 5) can prevent a mode-locked laser from operating near the Schawlow–Townes limit. This is in striking contrast with the situation of a single-frequency laser, where the optical phase is typically at most weakly coupled to the intracavity power. However, it is not surprising that a mode-locked laser with its many more degrees of freedom (not just pulse energy and phase, but also timing, pulse duration and shape, center frequency, chirp, etc.) can exhibit much richer dynamical features.

4 Carrier-envelope offset (CEO) noise

We have seen in Sect. 3 that for simple actively mode-locked lasers with a noiseless modulator signal the quantum limit of the low-frequency phase noise is at the Schawlow–Townes limit (i.e. the same as for a continuous-wave laser with the same average power), while for a passively mode-locked laser the timing jitter contributes a few dB to the PSD of the optical phase noise in the wings of the spectrum. However, this contribution from the timing jitter can be totally dominating for larger frequency offsets from the center of the spectrum. In particular, we can extrapolate to zero frequency by using Eq. (14) with $\Delta\nu = -\nu_{\text{opt}}$ and thus obtain the PSD of the quantum-limited carrier-envelope offset phase noise:

$$S_{\varphi,\text{ceo}}(f) = S_{\varphi,\text{opt,ST}}(f)\left(2\pi\nu_{\text{opt}}\tau_{\text{p}}\right)^2 = \frac{1}{2}\frac{\theta\, h\nu}{E_{\text{p}}}\frac{l_{\text{tot}}}{N}\left(\nu_{\text{opt}}\tau_{\text{p}}\right)^2 f_{\text{rep}} f^{-2}\,. \tag{15}$$

Note that $\nu_{\text{ceo}} < f_{\text{rep}} \ll \nu_{\text{opt}}$, so that $\Delta\nu = -\nu_{\text{opt}}$ is a good approximation. We have neglected the direct Schawlow–Townes phase noise term, which is much smaller provided that $\nu_{\text{opt}}\tau_{\text{p}}$, which is the number of optical cycles within the pulse duration, is large. In other words, for laser pulses with many cycles, the PSD of the quantum-limited CEO phase noise is larger than that according to the Schawlow–Townes formula by a factor in the order of the squared number of cycles of the pulses. For picosecond lasers, this factor can be huge.

It may appear to be a rather bold extrapolation to go from within the optical spectrum to frequencies far outside the spectrum, such as e.g. the CEO frequency. Note, however, that this kind of extrapolation is not an invention made in the context of our argument, but rather essentially part of the definition of the CEO frequency and phase. In practice, one will measure the CEO phase not by detecting the laser output in the radio-frequency domain (where there is basically no spectral density), but rather by comparing phases of lines within the spectrum [3] (possibly after spectral broadening in a nonlinear fiber). Thus, our argument does not really imply a physical extrapolation down to $\nu = 0$, but only relies on phase noise properties within the optical spectrum, which we have at least numerically tested in Sect. 3.3. In other words, the extrapolation is only a mathematical one, implicit in the definition of the CEO frequency.

The situation is graphically illustrated in Fig. 4a, showing the (extrapolated) spectral phase versus optical frequency. (We consider only low noise frequencies, where the unbounded drifts of optical phase and timing are much stronger than the bounded fluctuations of other parameters.) For narrow-bandwidth pulses, the spectral phase of a passively mode-locked laser with quantum noise influences always stays close to a straight line, which intersects the horizontal axis somewhere in the range of the optical spectrum. At least for frequencies well outside the optical spectrum, these fluctuations can be approximately described as rotations of the straight line around the center of the spectrum. Apparently, the quantum-limited CEO phase noise for long pulses is thus much stronger than the quantum-limited optical phase noise. On the other hand, this figure also illustrates the moderate in-

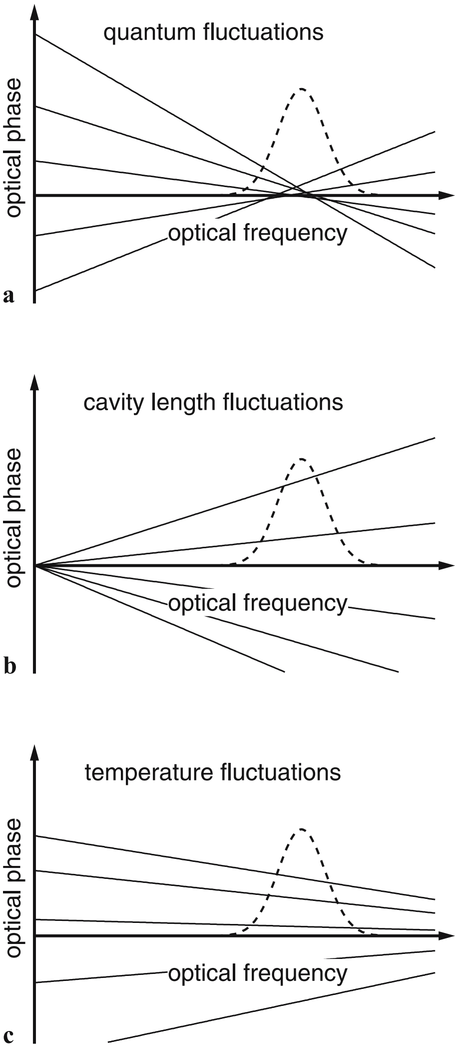

FIGURE 4 Temporal evolution of optical phase under the influence of (**a**) quantum fluctuations, (**b**) cavity length fluctuations, (**c**) temperature fluctuations. The *lines* indicate the (extrapolated) phase values at five different times

crease of optical phase noise within the optical spectrum, as discussed in Sect. 3.3.

Note that although the timing stabilization of a passively mode-locked laser via feedback on the cavity length can strongly suppress the low-frequency timing jitter, this would not affect the validity of Eq. (15), as the cavity length changes have no effect on the CEO phase noise (see Sect. 5). Graphically, we can illustrate the effect of cavity length changes as rotations around the origin of the coordinate system (see Sect. 5), which thus do not affect frequencies close to $\nu = 0$.

Other important remarks concern the measurement of CEO phase noise. The direct recording of the CEO phase values of the emitted pulses (interpreted as the optical phase at the maximum of the pulse envelope) is probably not a feasible way to achieve this, because the definition of the CEO phase of a single pulse is problematic for long pulses and non-trivial pulse shapes, and because sufficient accuracy and measurement bandwidth would probably be hard to achieve. The most common method for determination of the CEO frequency is to use an f–2f interferometer [3], where a beat signal is generated from the high-frequency part of the optical spectrum and the frequency-doubled low-frequency part. This requires an octave-spanning spectrum; similar techniques require a somewhat lower bandwidth at the expense of higher complexity [3]. Technical problems for CEO noise measurements arise from additional noise in the f–2f interferometer (e.g. drifts of arm lengths and temperature fluctuations in the frequency doubler) and from the nonlinear processes in a fiber, if a fiber is used to achieve sufficient bandwidth by nonlinear spectral broadening [4, 15, 16]. A more fundamental problem is that the noise of the generated CEO signal is actually determined by the optical phase noise at two different locations in the spectrum, which is influenced not only by common-mode phase noise and timing jitter, but also (particularly at higher frequencies) by fluctuations of pulse parameters like duration, chirp, and center frequency, the extrapolation of which to zero frequency cannot be regarded as CEO noise. A conclusion from these thoughts is that CEO noise of a laser is actually a somewhat problematic concept – in contrast e.g. to the average CEO frequency, which can be unambiguously determined from the optical frequencies. Consequences of these thoughts are discussed in Sect. 6.

5 Influence of cavity length variations and temperature changes

For comparison with the effects of quantum noise, we also consider the effects of cavity length variations, as caused e.g. by random mechanical vibrations. A length change δL of a linear cavity will introduce an optical phase change $\delta\varphi = 2\pi(\nu/c) \times 2\delta L$ per cavity round trip. For the line with index j, cavity length vibrations with a given PSD $S_L(f)$ are converted into phase fluctuations with the PSD

$$S_{\varphi j}(f) = \left(\frac{4\pi\nu_j/c}{2\pi f\, T_{\mathrm{rt}}}\right)^2 S_L(f) = \left(\frac{2\nu_j}{c\, f\, T_{\mathrm{rt}}}\right)^2 S_L(f) \qquad (16)$$

(see Section 3.2 in Ref. [2]). The situation is illustrated in Fig. 4b, where the straight line for phase versus frequency rotates around the origin of the coordinate system, as the phase deviations are proportional to the optical frequency. Using the rubber-band model of Ref. [17], this can be described as a 'fixed point' at zero frequency. (The rubber-band model has actually been introduced for optical frequencies, but can be applied to phase values in an analogous fashion.) We conclude that random cavity length changes generate a certain optical phase noise, and at the same time a much weaker timing phase noise with

$$S_{\varphi,t}(f) = \left(\frac{f_{\mathrm{rep}}}{\nu_{\mathrm{opt}}}\right)^2 S_{\varphi,\mathrm{opt}}(f) \qquad (17)$$

(according to the much lower frequency to which timing noise refers), and no CEO noise. The latter is obvious, because an air path does not change the CEO phase of a pulse, if the dispersion of air can be neglected [18]. The smaller timing phase noise PSD simply results from the fact that the timing phase is calculated with reference to the pulse period, rather than to the much shorter optical period.

This result can now be compared with that of Eq. (12), which holds for quantum noise influences. As we typically have $2\pi\tau_p \gg 1/\nu_{opt}$, the ratio of the PSDs of timing phase and optical phase is much larger if the noise of both results from quantum fluctuations, as opposed to mirror fluctuations. As a consequence, cavity length fluctuations can be strong enough to dominate the optical phase noise in a situation where they are weak enough to allow for quantum-limited timing jitter. Indeed, it has been observed for passively mode-locked Er:Yb:glass miniature lasers that the timing noise is close to quantum limited for noise frequencies of about 1–100 kHz [19], while the optical phase noise is well above the Schawlow–Townes limit [20]. However, in that case the origin of the increased optical phase noise is not clear; the absence of lines in the phase noise spectrum (which would indicate acoustical resonances) actually makes it more likely that some other effect is responsible for the excess noise. Still, basically the same analysis would apply e.g. if it were thermal fluctuations in the gain medium that modulate the refractive index.

An interesting situation arises if the timing noise is suppressed with a feedback timing stabilization which acts on the cavity length via a piezo translator below a cavity mirror [21, 22]. If the timing jitter results dominantly from cavity vibrations, the feedback obviously also reduces the resulting optical phase noise. However, if the timing jitter results from quantum noise, its compensation with cavity length changes will even introduce huge excess phase noise, as is illustrated by the following arguments. Consider a timing offset δt, induced by quantum fluctuations, with the corresponding change

$$\delta\varphi(\nu) \approx 2\pi\,(\nu - \nu_{opt})\,\delta t \tag{18}$$

of the spectral phase, neglecting the change at $\nu = \nu_{opt}$. Now, by switching on the cavity length control, one corrects the timing (the slope of the optical phase) by adding corrections which are proportional to the absolute optical frequency, rather than to the frequency offset from ν_{opt}. For perfect suppression of the timing jitter, one would obtain a frequency-independent optical phase change

$$\delta\varphi(\nu) = -2\pi\,\nu_{opt}\,\delta t \tag{19}$$

and thus

$$\delta\varphi_{opt} = -2\pi\nu_{opt}\,\delta t = -\left(\nu_{opt}/f_{rep}\right)\,\delta\varphi_t\,. \tag{20}$$

This shows that the resulting optical phase change is huge compared to the original timing phase error $\delta\varphi_t$. Graphically (see Fig. 2), the situation is obvious: cavity length control makes the straight line horizontal by rotating it around the 'wrong' fixed point, which in this case is at $\nu = 0$ rather than at $\nu = \nu_{opt}$. It is also instructive to calculate the magnitude of the induced optical phase noise PSD:

$$\begin{aligned} S_{\varphi,opt}(f) &= \left(\frac{\nu_{opt}}{f_{rep}}\right)^2 S_{\varphi,t}(f) \\ &\approx \left(\frac{\nu_{opt}}{f_{rep}}\right)^2 \left(2\pi f_{rep}\tau_p\right)^2 \times S_{\varphi,opt,ST}(f) \\ &= \left(2\pi\nu_{opt}\tau_p\right)^2 S_{\varphi,opt,ST}(f)\,. \end{aligned} \tag{21}$$

This is exactly the same formula as Eq. (15) for the CEO noise, which for long pulses is also far stronger than according to the Schawlow–Townes formula. The equality with CEO noise is actually not surprising, as the (perfect) timing stabilization enforces the same phase noise at all optical frequencies while not changing the CEO noise. Note, however, that Eq. (21) of course holds only well within the bandwidth of the timing stabilization.

One may consider other control parameters with different fixed points. For example, temperature changes in a silica-based 1535-nm fiber laser (which could e.g. be induced via changes of the pump power) can be shown to have a fixed point of roughly 2.5 times ν_{opt} (see Fig. 4c). This means that the excess phase noise generated by a timing stabilization would be even somewhat larger than with cavity length control, as the distance of the fixed point from ν_{opt} is larger. Other control parameters, related e.g. to the beam alignment in a bulk laser, or associated with a coupling of intensity and phase via optical nonlinearities, can again have totally different fixed points.

To avoid excess phase noise, one may search for a control with the correct fixed point at ν_{opt}. This would be a pure amplitude modulator with no effect on the optical phase. Alternatively, one may use a combination of two controls with different fixed points (but similar frequency response).

In a situation with mixed noise sources, it should be possible to distinguish the effect of different sources by simultaneously measuring the fluctuations of the optical phase and the timing phase. The optical phase of a single line could be monitored e.g. by recording a beat with a stable single-frequency laser, while the timing phase can be recorded e.g. with the method described in Ref. [23]. (To eliminate the need of optical and timing reference sources with superior quality, one may measure relative fluctuations between two identical lasers.) The correlation of optical and timing phases then reveals information on the noise source. For example, quantum noise would lead to uncorrelated noise, while cavity length fluctuations lead to a strong correlation (with a magnitude ratio of ν_{opt}/f_{rep}), and thermal changes with a fixed point above the optical frequency lead to a correlation with opposite sign. Similar measurements have been performed with sinusoidal modulations of parameters such as the pump power [24], but seemingly not yet with noise.

6 Conclusions and outlook

We have seen that timing noise, optical phase noise, and carrier-envelope offset noise of mode-locked lasers are strongly related to each other: on a mathematical basis, independent of any concrete laser, as well as by a number of physical mechanisms which couple different kinds of noise with each other.

From the discussion at the end of Sect. 4, we can conclude that we should consider the phase and amplitude noise in all lines of the optical spectrum as the fundamental physical phenomenon, which directly describes noise effects in optical beat measurements and also determines the fluctuations of pulse timing and CEO phase. For low noise frequencies, phase noise dominates over amplitude noise because of the unbounded phase fluctuations (at least concerning one or two degrees of freedom). The optical phase noise can then be decomposed into common-mode phase noise plus timing jitter (i.e. fluctuations of the slope of the spectral phase), plus fluctuations of second and higher order in the offset from the optical center frequency. Mathematically, we can make a Taylor expansion of the time-dependent phase values of all lines:

$$\begin{aligned}\varphi_j(t) &= \varphi_{\text{opt, com}}(t) + \left(\nu_j - \bar{\nu}\right) \frac{\partial \varphi}{\partial \nu}(t) + \ldots \\ &= \varphi_{\text{opt, com}}(t) + \frac{\nu_j - \bar{\nu}}{f_{\text{rep}}} \varphi_t(t) + \ldots ,\end{aligned} \tag{22}$$

where $\varphi_{\text{opt, com}}$ describes the common-mode phase fluctuations, and the omitted higher-order fluctuations correspond to fluctuations of pulse duration, pulse shape, chirp, etc. An alternative decomposition would be based on a Taylor expansion around $\nu = 0$, i.e. on CEO noise and timing jitter as the basic lower-order parameters:

$$\begin{aligned}\varphi_j(t) &= \varphi_{\text{ceo}}(t) + \nu_j \frac{\partial \varphi}{\partial \nu}(t) + \ldots \\ &= \varphi_{\text{ceo}}(t) + \frac{\nu_j}{f_{\text{rep}}} \varphi_t(t) + \ldots .\end{aligned} \tag{23}$$

However, the latter decomposition is affected by the above-mentioned problems related to CEO noise. Basically, it is appropriate only where the higher-order noise contributions are small, i.e. at low noise frequencies, or where a noise contribution proportional to ν (e.g. from vibrations) is dominant.

In the case of a laser with long pulses and quantum noise only, $\varphi_{\text{ceo}}(t)$ and $\varphi_t(t)$ in Eq. (23) are strongly correlated, while $\varphi_{\text{opt, com}}(t)$ and $\varphi_t(t)$ in Eq. (22) would be uncorrelated (at least in simple cases without additional coupling mechanisms). In such cases, Eq. (22) provides the more natural description. On the other hand, cavity length fluctuations lead to strong correlations between $\varphi_{\text{opt, com}}(t)$ and $\varphi_t(t)$, while they do not affect $\varphi_{\text{ceo}}(t)$. Here, one may prefer Eq. (23).

We have also seen that in mode-locked lasers there can be a large number of physical mechanisms which couple different kinds of noise with each other and thus lead to noise correlations. To name a few important examples:

- Center frequency noise is coupled to optical phase noise in a slow saturable absorber and to timing noise via intracavity dispersion, so that optical phase noise and timing noise can be correlated.
- Intensity noise can couple to optical phase noise via a Kerr nonlinearity, and also to timing noise via self-steepening.
- Intensity fluctuations are coupled to gain fluctuations, which also couple to the optical phase and the pulse timing via refractive-index changes.

Therefore, we must expect to find very different noise behavior in different mode-locked lasers, depending on the dominant noise sources but also on the coupling mechanisms within the lasers. Note also that the importance of certain coupling mechanisms can strongly depend on the laser parameters (see e.g. Section 6 in Ref. [2]). A good way to investigate such coupling effects is to monitor the effects of sinusoidally modulating a parameter such as e.g. the pump power [24], and to compare the results to theoretical expectations. The resulting knowledge could then be used for an extended theoretical model, used for calculating noise properties in lasers with certain coupling mechanisms.

The discussion of simple situations without additional coupling mechanisms has already led to interesting conclusions. For example, we explained why the quantum limit for the timing jitter is in general more easily reached than that of the optical phase noise: mirror vibrations can be strong enough to substantially increase the optical phase noise (and thus the linewidth), while still being weak enough to permit quantum-limited timing jitter, particularly if the pulse duration is long compared to an optical cycle. Another finding is of practical importance for the development of frequency comb sources: the quantum limit for CEO noise is higher for lasers generating longer pulses. This means that e.g. mode-locked Ti:sapphire lasers have the fundamentally better potential for very low CEO noise, compared to fiber lasers, because the latter cannot generate as short pulses. Our statement cannot be considered to be proven by experimental results, because it is not clear which effects limited the performance of various lasers, but it should give some valuable guidance.

In general, it will be interesting to obtain more experimental data on the noise performance of different types of lasers under carefully controlled conditions. Until now, there have been many reports of noise measurements on mode-locked lasers, but nearly always concerning only one type of noise, and hardly ever correlations between different types of noise. In particular, it should be interesting to simultaneously record the fluctuations of the optical phase, the pulse energy, the timing position, and the CEO phase as measured with an f–2f interferometer. The data become even more valuable when they are complemented with tests of the effect of sinusoidal modulations of parameters such as the pump power [24]. The results of such investigations should be of great value for identifying both the origins of noise and the limits of ultraprecise frequency metrology with different kinds of mode-locked lasers. First results with simultaneous recording of timing position, optical phase, and CEO phase are presented in Ref. [25].

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