Quantum-enhanced sensing of displacements and rotations in continuous-variable systems

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June 24, 2023

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1 Introduction

The classical measurement of a parameter encoded in a system's state is limited in its sensitivity. To go beyond this classical limit, one can exploit quantum resources, such as entangled states. It is thanks to the strong nonclassical correlations present in entangled states that the classical limit can be overcome. The field studying this topic is called quantum metrology [1].

One could then ask the following question: What measurement should be performed on a quantum state to obtain the lowest possible uncertainty in a parameter estimation task? This question is what this report tries to address for a number of relevant states in continuous-variable systems. The best possible sensitivity for a given state is expressed by the so-called Cramér-Rao bound, which is explained in Section 2. By exploring different possible measurements, one can then determine which measurements are better than the others, and which ones are optimal by comparing how close the achievable sensitivities are to the Cramér-Rao bound.

In the following, we investigate the saturation of this Cramér-Rao bound for different probe states, parameter encoding generators and measurements. In Section 2, we summarize the key concepts needed to investigate such performances. In Section 3, we compare the quantum Fisher information (QFI) to the sensitivity χ in the case of displacement sensing measured by quadrature measurement as well as in the case of a rotation sensing also by quadrature measurement. Finally, Section 4 comes back to displacement sensing, but this time as measured by displaced parity measurements.

2 Quantum metrology

A usual way to obtain precision measurements is to map physical quantities to phase shifts that can be measured by interferometry [1]. When trying to measure a phase θ by classical means, the uncertainty $\Delta\theta$ is bounded by the standard quantum limit $\Delta\theta \geq \frac{1}{\sqrt{N}}$ which arises when considering uncorrelated or classically-correlated particles [1]. Here, N stands for the number of particles in the probe state. Theoretically, however, the ultimate bound that can be reached is the so-called Heisenberg bound $\Delta\theta \geq \frac{1}{N}$. Quantum-enhanced metrology is the field that studies how to go beyond this standard quantum limit to get as close as possible to the Heisenberg bound.

It turns out that only certain types of correlations overcome this limit, as entanglement itself is necessary but not sufficient. A way to quantify the maximal phase sensitivity for a given probe state and measurement is given by the quantum Fisher information (QFI), and by the corresponding quantum Cramér-Rao bound [1]. The quantum Fisher information can be defined as

$$F_{\mathrm{Q}}[\hat{\rho}_\theta] = \max_{\hat{E}} F(\theta), \tag{1}$$

where we maximize the (classical) Fisher information $F(\theta)$ [2] associated with the probability distribution obtained by applying the generalized measurement $\hat{E}$ (as defined in [3]) on the quantum state under consideration [1]. For the usual case of a projective measurement $\hat{M} = \sum_m m\hat{P}_m$ with projectors $\hat{P}_m$, the generalized measurement $\hat{E}$ with value m reduces to $\hat{E}_m = \hat{P}_m^\dagger \hat{P}_m = \hat{P}_m$ (see [4]). The (classical) Fisher information is defined as [1]

$$F(\theta) = \sum_m \frac{1}{P(m|\theta)} \left(\frac{\partial P(m|\theta)}{\partial \theta}^2 \right), \tag{2}$$

where $P(m|\theta) = \mathrm{Tr}[\hat{\rho}_\theta \hat{E}_m]$ represents the probability of getting the result m when measuring $\hat{M}$ (in the projective case) given a phase change θ. The Fisher information $F(\theta)$ is a measure of the amount of information about the parameter θ that is available given samples from the distribution $P(m|\theta)$ [5](*i.e.* from the observable $\hat{M}$). From the QFI, one can obtain the quantum Cramér-Rao bound

$$\Delta\theta \geq \frac{1}{\sqrt{\tau F_{\mathrm{Q}}[\hat{\rho}_\theta]}} \tag{3}$$

where τ is the number of independent measurements on the probe state $\hat{\rho}_\theta$. This gives a lower bound to our uncertainty on the parameter estimation θ. Entanglement that overcomes the standard quantum bound are uniquely specified by the condition $F_{\mathrm{Q}} > N$ [1].

It is clear that Eq. (1) is difficult to calculate in practice. However, it turns out there is a useful upper bound [1]

$$F_{\mathrm{Q}}[\hat{\rho}_0, \hat{G}] \leq 4\Delta^2 \hat{G}, \tag{4}$$

where $\hat{G}$ is the operator responsible for imprinting on the state $\hat{\rho}$ the parameter θ (*i.e.* $\hat{\rho}_\theta = e^{i\hat{G}\theta} \hat{\rho}_0 e^{-i\hat{G}\theta}$), which reduces to an equality for pure states.

On the other hand, for a given state $\hat{\rho}_\theta$ and a specific measurement $\hat{M}$, the method of moments (see Section 2.2) allows us to compute the uncertainty $\Delta\theta_{\mathrm{mom}}$ for estimating θ. To find the measurement attaining the lowest possible uncertainty, one can try to find an operator $\hat{M}$ such that that $\Delta\theta_{\mathrm{mom}}$ saturates the Cramér-Rao bound. If one defines the sensitivity χ as

$$\chi = \frac{1}{\tau \Delta^2 \theta_{\mathrm{mom}}}, \tag{5}$$

then the goal becomes to find a measurement $\hat{M}$ for which $\chi = F_{\mathrm{Q}}[\hat{\rho}_\theta]$.

The rest of this section gathers the general theoretical background needed in the following sections. We start by defining the density matrix necessary to describe mixed state, as well as the Wigner function, extremely useful for the visualization of quantum states in phase-space. Then, we explore the method of moments in more details, as it is the main building block of our work. Besides this, we also prove a relationship between the QFI and the Wigner function.

2.1 Definitions

In [6], we learn the definition of a density matrix

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i| \,, \tag{6}$$

where p_i is the probability of the system being in the pure state $|\psi_i\rangle$, with $\langle\psi_i| |\psi_j\rangle = \delta_{ij}$. The density matrix is useful to describe so-called mixed quantum states, *i.e.* states which cannot be written as a pure state (wavefunction).

From the density matrix, it is possible to define a quasi-probability distribution called the Wigner function

$$W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}q \,\langle x + \frac{1}{2}q| \,\hat{\rho}\, |x - \frac{1}{2}q\rangle \, e^{ipq}, \tag{7}$$

where $|x \pm \frac{1}{2}q\rangle$ are the eigenkets of the position operator [6]. Here and in the following we will be working with the dimensionless quantities x and p.

Similarly to the density matrix, the Wigner function gives also a complete description of the system's state. Therefore, it allows us to compute expectation values of operators, or other quantities of interest. For example, it is possible to write the quantum fidelity as an integral of Wigner functions:

$$F(\hat{\rho},\hat{\sigma}) \equiv \left(\mathrm{Tr}[\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}]\right)^2 = 2\pi \int\int \mathrm{d}x\mathrm{d}p W_{\hat{\rho}}(x,p) W_{\hat{\sigma}}(x,p). \tag{8}$$

This relation is proved in Appendix A.

In turns out that the Wigner function can also be written as the expectation value of a displaced parity measurement [6]:

$$W(\alpha) = \frac{2}{\pi}\mathrm{Tr}\left[\hat{\rho}\hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha)\right], \tag{9}$$

so

$$W(x,p) = \frac{1}{\pi}\mathrm{Tr}\left[\hat{\rho}\hat{D}(x,p)\hat{\Pi}\hat{D}^\dagger(x,p)\right], \tag{10}$$

where $\hat{D}(x,p)$ is the displacement operator and $\hat{\Pi}$ is the parity operator and $\alpha = \frac{x+ip}{\sqrt{2}}$. The factor of 2 disappears due to the need of normalization $\int \mathrm{d}x\mathrm{d}pW(x,p) = 1$ after the change of variables.

2.2 Method of moments

Let us consider the measurement of an observable $\hat{M}$ onto a state $\hat{\rho}$, performed before and after some small phase shift $\Delta\theta$ from the initial phase θ_0. We will write the expectation value of the measurement $\hat{M}$ applied on the unperturbed state and perturbed state as

$\langle \hat{M} \rangle_{\theta_0}$ and $\langle \hat{M} \rangle_{\theta_0+\Delta\theta}$, respectively. For a small perturbation $\Delta\theta$ around θ_0 we get to first order

$$\left\langle \hat{M} \right\rangle_{\theta_0+\Delta\theta} \simeq \left\langle \hat{M} \right\rangle_{\theta_0} + \frac{\partial \left\langle \hat{M} \right\rangle_\theta}{\partial\theta}\bigg|_{\theta_0} \Delta\theta \,. \tag{11}$$

This means that a perturbation $\Delta\theta$ imprinted on the state can be estimated from $\left\langle \hat{M} \right\rangle_{\theta_0+\Delta\theta}$ by inverting Eq. (11). Here, note that it is essential to know the derivative $\frac{\partial \left\langle \hat{M} \right\rangle}{\partial\theta}$.

To be able to detect the perturbation $\Delta\theta$, however, one requires this perturbation to result in a change of the distribution of measurement results that is larger than its standard deviation. Namely, we want

$$\left\langle \hat{M} \right\rangle_{\theta_0+\Delta\theta} - \left\langle \hat{M} \right\rangle_{\theta_0} \geq \frac{\Delta \hat{M}}{\sqrt{\tau}} \,. \tag{12}$$

From the two last expressions, we conclude that

$$\Delta\theta \geq \frac{\Delta \hat{M}}{\sqrt{\tau} \left| \frac{\partial \left\langle \hat{M} \right\rangle}{\partial\theta} \right|} \,. \tag{13}$$

Note that, in general, $\Delta \hat{M}$ depends on θ_0. Let us define here the sensitivity as

$$\chi = \frac{\left| \frac{\partial \langle \hat{M} \rangle}{\partial\theta} \right|^2}{\Delta^2 \hat{M}}, \tag{14}$$

such that we get

$$\Delta\theta \geq \frac{1}{\sqrt{\tau\chi}}. \tag{15}$$

It is very interesting and useful to realize that, using von Neumann's equation of motion $i\frac{\partial\rho}{\partial t} = [H, \rho]$ with Hamiltonian $\hat{H} = \hat{G}$ and parameter $t = \theta$, we find

$$\begin{aligned} \frac{\partial \langle \hat{M} \rangle}{\partial\theta} &= \mathrm{Tr}\left[\hat{M} \frac{d\rho}{d\theta} \right] \\ &= -i\mathrm{Tr}[\hat{M}[\rho, \hat{G}]] \\ &= -i\mathrm{Tr}[\hat{M}\rho\hat{G} - \hat{M}\hat{G}\rho] \\ &= -i\mathrm{Tr}[\hat{G}\hat{M}\rho - \hat{M}\hat{G}\rho] \\ &= -i\mathrm{Tr}[[\hat{G}, \hat{M}]\rho] \\ &= i\mathrm{Tr}[[\hat{M}, \hat{G}]\rho] \\ &= i\langle[\hat{M}, \hat{G}]\rangle. \end{aligned} \tag{16}$$

This tells us that the sensitivity χ can also be written as

$$\chi = \frac{\left|\langle[\hat{M},\hat{G}]\rangle\right|^2}{\Delta^2\hat{M}}, \tag{17}$$

which will be useful in sections 3.1 and 3.2. As a sanity check, we could ask ourselves what happens if $\hat{M}$ and $\hat{G}$ happened to commute. In that case, we have a measurement $\hat{M}$ commuting with the Hamiltonian $\hat{G} = \hat{H}$, or in other words, the observable $\hat{M}$ represents a symmetry of the system. It is therefore not surprising to find absolutely no sensibility to the phase of the evolution θ when measuring $\hat{M}$.

2.3 Link between QFI and Wigner

Having introduced the QFI and the Wigner function, we are tempted to ask if there exists a simple connection between the two.

In the following, we show that the QFI F_Q is bounded from above by

$$F_\mathrm{Q} \leq 4\pi \int \mathrm{d}x\mathrm{d}p(\partial_\theta W(x,p))^2, \tag{18}$$

with an equality for pure states. To do so, we first prove the two following results

$$F_Q[\hat{\rho}] \leq 2\mathrm{Tr}\left[\left(\frac{\partial}{\partial\theta}\hat{\rho}\right)^2\right], \tag{19}$$

and

$$\mathrm{Tr}\left[\left(\frac{\partial}{\partial\theta}\hat{\rho}\right)\right] = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}p\left(\frac{\partial}{\partial\theta}W(x,p)\right)^2, \tag{20}$$

where we assume $\theta = x$, so we consider a translation in the x direction.

To find the first result, we use the second order expansion of $\hat{\rho}$ around θ:

$$\hat{\rho}(\theta+\mathrm{d}\theta) = \hat{\rho}(\theta) + \partial_\theta\hat{\rho}\mathrm{d}\theta + \frac{1}{2}\partial_\theta^2\hat{\rho}(\mathrm{d}\theta)^2 , \tag{21}$$

together with the definition of Uhlmann fidelity [7]

$$F(\rho,\sigma) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]\right)^2. \tag{22}$$

and with the relation [8]

$$F_Q(\mathrm{d}\theta)^2 = 8(1-\sqrt{F[\hat{\rho}(\theta),\hat{\rho}(\theta+\mathrm{d}\theta)]}). \tag{23}$$

Lastly, we use the fact that the fidelity can be upper bounded by the superfidelity [9]

$$F(\rho,\sigma) \leq \mathrm{Tr}[\sigma\rho] + \sqrt{1-\mathrm{Tr}[\rho^2]}\sqrt{1-\mathrm{Tr}[\sigma^2]}. \tag{24}$$

Now that these relations have been introduced, we can insert $\sigma = \rho(\theta + \mathrm{d}\theta)$ and expand in second order in (24). First, we can use the following second order expansion around $x = 0$:

$$\sqrt{a}\sqrt{a-x} \approx a - \frac{x}{2} - \frac{x^2}{8a}. \tag{25}$$

Here we use $a = 1 - \mathrm{Tr}[\rho^2]$ and $x = \mathrm{Tr}[\sigma^2] - \mathrm{Tr}[\rho^2]$. This gives

$$\begin{aligned}\sqrt{1-\mathrm{Tr}[\rho^2]}\sqrt{1-\mathrm{Tr}[\sigma^2]} &\approx 1 - \mathrm{Tr}[\rho^2] - \frac{1}{2}(\mathrm{Tr}[\sigma^2] - \mathrm{Tr}[\rho^2]) - \frac{1}{8}\frac{1}{1-\mathrm{Tr}[\rho^2]}(\mathrm{Tr}[\sigma^2] - \mathrm{Tr}[\rho^2])^2 \\ &= 1 - \mathrm{Tr}[\rho^2 + \rho\partial_\theta\rho \mathrm{d}\theta + \frac{1}{2}((\partial_\theta\rho)^2 + \rho\partial_\theta^2\rho)(\mathrm{d}\theta)^2] - \frac{(\mathrm{d}\theta)^2}{2}\frac{1}{1-\mathrm{Tr}[\rho^2]}\mathrm{Tr}[\rho\partial_\theta\rho]^2\end{aligned} \tag{26}$$

On the other hand, we have

$$\mathrm{Tr}[\rho\sigma] = \mathrm{Tr}[\rho^2 + \rho\partial_\theta\rho\mathrm{d}\theta + \frac{1}{2}\rho\partial_\theta^2\rho(\mathrm{d}\theta)^2] \tag{27}$$

Taking the sum of both equations gives (inserting in (24))

$$F(\rho,\sigma) \leq 1 - \frac{(\mathrm{d}\theta)^2}{2}\left[\mathrm{Tr}[(\partial_\theta\rho)^2] + \frac{1}{1-\mathrm{Tr}[\rho^2]}\mathrm{Tr}[\rho\partial_\theta\rho]^2\right]. \tag{28}$$

One can notice that the second term in the parenthesis can be written in a more convenient way

$$\frac{1}{1-\mathrm{Tr}[\rho^2]}\mathrm{Tr}[\rho\partial_\theta\rho]^2 = \left(\frac{\partial}{\partial\theta}\sqrt{1-\mathrm{Tr}[\rho^2]}\right)^2. \tag{29}$$

So, in the situation where the purity $\mathrm{Tr}[\rho^2]$ does not depend on θ (pure states), this term vanishes. This yields

$$F(\rho,\sigma) \leq 1 - \frac{(\mathrm{d}\theta)^2}{2}\mathrm{Tr}[(\partial_\theta\rho)^2]. \tag{30}$$

Now let us focus on pure initial states ($\mathrm{Tr}[\rho^2] = 1$) and unitary evolution. In this case, (22) gives $F(\rho,\sigma) = \mathrm{Tr}[\rho\sigma]$. Taking the second order expansion gives

$$F(\rho,\sigma) = \mathrm{Tr}[\rho^2] + \mathrm{d}\theta\mathrm{Tr}[\rho\partial_\theta\rho] + \frac{(\mathrm{d}\theta)^2}{2}\mathrm{Tr}[(\partial_\theta\rho)^2]. \tag{31}$$

The second term vanishes with the assumption that the purity is independent of θ:

$$0 = \partial_\theta\mathrm{Tr}[\rho^2] = \mathrm{Tr}[2\rho\partial_\theta\rho]. \tag{32}$$

Moreover, the last term is now $-\frac{(\mathrm{d}\theta)^2}{2}\mathrm{Tr}[(\partial_\theta\rho)^2]$ since $\partial_\theta(\rho\partial_\theta) = 0 = (\partial_\theta\rho)^2 + \rho\partial_\theta^2\rho$. In that case, we get

$$F(\rho,\sigma) = 1 - \frac{(\mathrm{d}\theta)^2}{2}\mathrm{Tr}[(\partial_\theta\rho)^2]. \tag{33}$$

If we now insert these results in (23), we obtain

$$F_Q \leq 2\mathrm{Tr}[(\partial_\theta \rho)^2], \tag{34}$$

with equality for pure initial states with unitary evolution.

We now want to show the second result (20). To do so, we use a general result about Wigner-Weyl transforms (see [10])

$$\mathrm{Tr}[\hat{A}\hat{B}] = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}p\tilde{a}(x,p)\tilde{b}(x,p), \tag{35}$$

where $\tilde{g}(x,p)$ is the Wigner-Weyl transform of the operator $\hat{G}$

$$\tilde{g}(x,p) = \int \mathrm{d}y \, \langle x + \frac{y}{2}| \, G \, |x - \frac{y}{2}\rangle \, e^{-ipy}. \tag{36}$$

Inserting $\hat{A} = \hat{B} = \partial_\theta \hat{\rho}$ gives

$$\mathrm{Tr}[(\partial_\theta \rho)^2] = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}p \left(\widetilde{\partial_\theta \hat{\rho}}(x,p)\right)^2, \tag{37}$$

where

$$\begin{aligned} \widetilde{\partial_\theta \hat{\rho}}(x,p) &= \int \mathrm{d}y \, \langle x + \frac{y}{2}| \, \partial_\theta \hat{\rho} \, |x - \frac{y}{2}\rangle \, e^{-ipy} \\ &= \partial_\theta \int \mathrm{d}y \, \langle x + \frac{y}{2}| \, \hat{\rho} \, |x - \frac{y}{2}\rangle \, e^{-ipy}. \end{aligned} \tag{38}$$

Now, if we recall (7),we find

$$\widetilde{\partial_\theta \hat{\rho}}(x,p) = 2\pi \partial_\theta W(x,p). \tag{39}$$

Inserting in (37) finally gives

$$\mathrm{Tr}[(\partial_\theta \rho)^2] = 2\pi \int \mathrm{d}x\mathrm{d}p \, (\partial_\theta W(x,p))^2 \tag{40}$$

Inserting Eq. (40) into Eq. (34) gives the result

$$F_\mathrm{Q} \leq 4\pi \int \mathrm{d}x\mathrm{d}p(\partial_\theta W(x,p))^2 \tag{41}$$

with equality for pure states.

3 Sensing from quadrature measurements

In this Chapter we focus on sensing displacements and rotation by performing quandrature measurements on continuous-variable states.

3.1 Displacement sensing with quadrature measurements

The displacement operator can be written as

$$\hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{i\hat{G}(\phi)\theta} \ , \tag{42}$$

where $\alpha = \theta e^{-i\phi}/\sqrt{2}$, and $\hat{G}(\phi) = \sin(\phi)\hat{x} - \cos(\phi)\hat{p}$ is the generator of the perturbation.

Let us consider the goal of estimating the amplitude θ of a displacement. We can assume the phase ϕ of the displacement to be either known or unknown at the time of the state preparation. The task then consists of optimizing the state preparation and the final measurement, in order to maximize the sensitivity.

Assuming an unknown phase ϕ for the displacement, in the worst-case scenario the QFI is lower bounded by

$$F_Q^{\min}[\hat{\rho}] = \min_\phi F_Q[\hat{\rho}, \hat{G}(\phi)] \ . \tag{43}$$

Alternatively, we can define the average QFI as

$$F_Q^{\mathrm{avg}}[\hat{\rho}] = \frac{1}{2\pi} \int_0^{2\pi} d\phi F_Q[\hat{\rho}, \hat{G}(\phi)]. \tag{44}$$

The optimization strategy consists here in maximizing either $F_Q^{\min}$ or F_Q^{avg}. Indeed, in real case scenarios, one might not know the direction of the displacement. The best one can do is then to optimize a quantity that is independent of ϕ, so in our case, either the worst-case scenario or the average scenario.

To evaluate the QFI, we exploit the fact that for pure states it coincides with four times the variance of the generator. Then, the variance along some direction ϕ in phase space can be written in terms of the covariance matrix Γ as

$$\mathrm{Var}[\hat{G}(\phi)] = \boldsymbol{u}^T \Gamma \boldsymbol{u}. \tag{45}$$

where $\boldsymbol{u} = (\sin(\phi), -\cos(\phi))$ and

$$\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x}, \hat{p}) \\ \mathrm{Cov}(\hat{x}, \hat{p}) & \mathrm{Var}(\hat{p}) \end{pmatrix} \ . \tag{46}$$

A derivation of this statement is provided in Appendix B.

The following question will be to understand what measurement M can achieve the sensitivity predicted by the QFI. For a general measurement $\hat{M}$, remember we have

$$\chi = \frac{\left|\left\langle [\hat{M}, \hat{G}] \right\rangle\right|^2}{\mathrm{Var}[\hat{M}]} \leq F_Q[\hat{\rho}, \hat{G}] \ . \tag{47}$$

This means that for a given $\hat{M}$, one can see if the inequality is saturated and $\chi = F_Q$.

In the following, we will consider $\hat{M}$ to be a linear quadrature measurement: $\hat{M}(\varepsilon) = \sin(\varepsilon)\hat{x} - \cos(\varepsilon)\hat{p}$. We compute the associated sensitivity χ for Fock, coherent, Gaussian and cat states, and compare it to the QFI. We also make the distinction between cases where we know the direction ϕ of the displacement, and where we do not.

3.1.1 Fock states

Fock states are defined as $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. We use the following definitions for $\hat{x}$ and $\hat{p}$:

$$\hat{x} = \frac{1}{\sqrt{2}}(a + a^\dagger) \ , \qquad \hat{p} = \frac{1}{i\sqrt{2}}(a - a^\dagger) \ . \tag{48}$$

By computing expectation values of $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$, $\hat{x}\hat{p}$ and $\hat{p}\hat{x}$ as in Appendix C.1, we find the covariance matrix

$$\Gamma = \begin{pmatrix} n + \frac{1}{2} & 0 \\ 0 & n + \frac{1}{2} \end{pmatrix} . \tag{49}$$

So finally,

$$\begin{aligned} \mathrm{Var}[\hat{G}(\phi)] &= \boldsymbol{u}^T \Gamma \boldsymbol{u} \\ &= \begin{pmatrix} \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} (n + \frac{1}{2})\sin\phi \\ -(n + \frac{1}{2})\cos\phi \end{pmatrix} \\ &= n + \frac{1}{2}, \end{aligned} \tag{50}$$

which does not depend on ϕ, as we might expect from the rotational symmetry of Fock states. For pure states, $F_Q[\hat{\rho}, \hat{G}] = 4\,\mathrm{Var}[\hat{G}]$, so here

$$F_Q = 4n + 2. \tag{51}$$

The minimum and average QFI are trivially

$$F_Q^{\mathrm{min}} = F_Q^{\mathrm{avg}} = 4n + 2. \tag{52}$$

Now we will compute χ. To do this, we need to know $\mathrm{Var}[\hat{M}]$ for quadrature measurements. We can directly use (50) to obtain again

$$\mathrm{Var}[\hat{M}] = n + \frac{1}{2}. \tag{53}$$

We also need the commutator $[\hat{M}, \hat{G}]$:

$$
\begin{aligned}
[\hat{M}, \hat{G}] &= [\sin\varepsilon\hat{x} - \cos\varepsilon\hat{p}, \sin\phi\hat{x} - \cos\phi\hat{p}] \\
&= \sin\varepsilon\sin\phi\hat{x}^2 - \cos\varepsilon\sin\phi\hat{p}\hat{x} - \sin\varepsilon\cos\phi\hat{x}\hat{p} + \cos\varepsilon\cos\phi\hat{p}^2 - \sin\varepsilon\sin\phi\hat{x}^2 \\
&\quad + \sin\varepsilon\cos\phi\hat{p}\hat{x} + \cos\varepsilon\sin\phi\hat{x}\hat{p} - \cos\varepsilon\cos\phi\hat{p}^2 \\
&= (\cos\varepsilon\sin\phi - \sin\varepsilon\cos\phi)[\hat{x}, \hat{p}] \\
&= i(\cos\varepsilon\sin\phi - \sin\varepsilon\cos\phi).
\end{aligned}
\tag{54}
$$

The norm squared of the expectation value is then

$$
\left|\langle[\hat{M}, \hat{G}]\rangle\right|^2 = (\cos\varepsilon\sin\phi - \sin\varepsilon\cos\phi)^2. \tag{55}
$$

Putting together these results gives the sensitivity

$$
\chi = \frac{\sin^2(\varepsilon - \phi)}{n + \frac{1}{2}} \leq 4n + 2. \tag{56}
$$

Let us assume first that we know the displacement direction ϕ. In this case, we are able to compute $\max_\varepsilon \chi$ by solving the first derivative for 0 and checking the sign of the second derivative. Doing so yields a maximum at $\varepsilon = \phi + \frac{\pi}{2}$, so

$$
\max_\varepsilon \chi = \frac{1}{n + \frac{1}{2}}. \tag{57}
$$

We see here that the upper bound is only achieved for $n = 0$, and that the maximum gets further away from the QFI as n grows larger. This implies that (linear) quadrature measurements are in general not optimal for detecting displacements of Fock states.

The fact that the sensitivity is maximal when ε and ϕ are perpendicular makes sense intuitively: if we take for example $\phi = 0$, then $\hat{G} = -\hat{p}$ generates a displacement along x. Since $\mathrm{Var}[\hat{M}]$ does not depend on ε, one can maximize χ by maximizing the commutator $[\hat{M}, \hat{G}]$, which means taking $\hat{M} = \hat{x}$, as expected.

Now, if we consider ϕ to be unknown, then we cannot choose the optimal measurement direction ε. So, to estimate the sensitivity of a measurement protocol, we can fix the measurement direction to be $\hat{M} = \hat{x}$, and take the average of χ over all possible displacement directions ϕ:

$$
\begin{aligned}
\chi^{\text{avg}} &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \frac{\sin^2(\varepsilon - \phi)}{n + \frac{1}{2}} \\
&= \frac{1}{4\pi}\frac{1}{n + \frac{1}{2}}[\varepsilon - \phi - \cos(\varepsilon - \phi)\sin(\varepsilon - \phi)]_0^{2\pi} \\
&= \frac{1}{2n + 1}.
\end{aligned}
\tag{58}
$$

This is only half of the best-case scenario where we know the displacement direction ϕ.

3.1.2 Coherent states

We can write coherent states in the Fock basis as [11]:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,. \tag{59}$$

We can compute the expectation values of $\hat{x}$, $\hat{p}$, $\hat{x}^2$, etc. just as for the Fock states (see Appendix C.2), to find the covariance matrix

$$\Gamma = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} . \tag{60}$$

So finally,

$$\begin{aligned} \mathrm{Var}[\hat{G}(\phi)] &= \boldsymbol{u}^T \Gamma \boldsymbol{u} \\ &= \begin{pmatrix} \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sin\phi \\ -\frac{1}{2}\cos\phi \end{pmatrix} \\ &= \frac{1}{2}, \end{aligned} \tag{61}$$

which again does not depend on ϕ. This means that, also for coherent states, the minimum and average QFI are

$$F_Q = F_Q^{\mathrm{min}} = F_Q^{\mathrm{avg}} = 2. \tag{62}$$

The variance of $\hat{M}$ is here $\mathrm{Var}[\hat{M}] = \frac{1}{2}$, and the commutator is the same as before, namely $[\hat{M}, \hat{G}] = i(\cos\varepsilon \sin\phi - \sin\varepsilon\cos\phi)$. The sensitivity χ is thus

$$\chi = 2\sin^2(\varepsilon - \phi) \leq 2. \tag{63}$$

We then get a maximum at $\varepsilon = \phi + \frac{\pi}{2}$, so

$$\max_{\varepsilon} \chi = 2, \tag{64}$$

which corresponds to the upper bound given by the QFI. If we do not know the displacement direction ϕ, we find the average sensitivity

$$\begin{aligned} \chi^{\mathrm{avg}} &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi 2\sin^2(\varepsilon - \phi) \\ &= \frac{1}{2\pi} [\varepsilon - \phi - \sin(\varepsilon - \phi)\cos(\varepsilon - \phi)]_0^{2\pi} \\ &= 1, \end{aligned} \tag{65}$$

which is again half as large as the maximum sensitivity.

Note that the results in this Section coincide with the case of Fock state $|0\rangle$. This is expected, since coherent states can be written as displaced vacuum states, and that for displacement sensing there is not preferred definition of a phase-space origin.

3.1.3 Gaussian states

We can define Gaussian states as those states that have a Gaussian Wigner function [12]. In the case of a single mode, the density matrix corresponding to the vacuum state with density matrix $\rho_0 = |0\rangle\langle 0|$ is given by (Eq. 2.15 in [12])

$$\rho = D(\alpha)S(\xi)\rho_0 S^\dagger(\xi)D^\dagger(\alpha), \tag{66}$$

where $S(\xi)$ is the squeezing operator

$$S(\xi) = e^{\frac{1}{2}\xi(\hat{a}^\dagger)^2 - \frac{1}{2}\xi^*\hat{a}^2} \tag{67}$$

and $\xi = re^{i\gamma}$. In other words, any Gaussian state can be written as a displaced squeezed vacuum state.

It turns out that Gaussian states can be fully described by its associated covariance matrix Γ. From [12], we know that it can be written as $\Gamma = \Sigma_\xi^T \Gamma_{\rho_0} \Sigma_\xi$, where $\Sigma_\xi = \mu \mathbb{1}_2 + R_\xi$, and $R_\xi = \begin{pmatrix} \Re[\nu] & \Im[\nu] \\ \Im[\nu] & -\Re[\nu] \end{pmatrix}$. The parameters μ and ν are related to the amplitude of the squeezing r and its direction γ by $\mu = \cosh r$ and $\nu = e^{i\gamma}\sinh r$. The vacuum covariance matrix is known to be $\frac{1}{2}\mathbb{1}_2$ from Subsection 3.1.1. We then find

$$\begin{aligned} \Gamma &= \frac{1}{2}\begin{pmatrix} \Re[\nu] + \mu & \Im[\nu] \\ \Im[\nu] & -\Re[\nu] + \mu \end{pmatrix}^2 \\ &= \frac{1}{2}\begin{pmatrix} \cosh(2r) + \sinh(2r)\cos\gamma & \sinh(2r)\sin\gamma \\ \sinh(2r)\sin\gamma & \cosh(2r) - \sinh(2r)\cos\gamma \end{pmatrix}, \end{aligned} \tag{68}$$

which does not depend on α. This means that we can assume $\alpha = 0$, leaving us a squeezed vacuum state. From [12], we also know that $n = \langle\xi|\,\hat{n}\,|\xi\rangle = |\nu| = \sinh^2 r$. This gives

$$2n + 1 = 2\sinh^2 r + 1 = \sinh^2 r + \cosh^2 r = \cosh(2r), \tag{69}$$

and therefore

$$\sinh(2r) = 2\sqrt{n(n+1)}, \tag{70}$$

since here we consider $r \geq 0$.

The covariance matrix gives

$$\begin{aligned}
\mathrm{Var}[\hat{G}(\phi)] &= \boldsymbol{u}^T \Gamma \boldsymbol{u} \\
&= \frac{1}{2}\begin{pmatrix}\sin\phi & -\cos\phi\end{pmatrix}\begin{pmatrix}[\cosh(2r)+\sinh(2r)\cos\gamma]\sin\phi - [\sinh(2r)\sin\gamma]\cos\phi \\ [\sinh(2r)\sin\gamma\sin\phi]\sin\phi - [\cosh(2r)-\sinh(2r)\cos\gamma]\cos\phi\end{pmatrix} \\
&= \frac{1}{2}\big([\cosh(2r)+\sinh(2r)\cos\gamma]\sin^2\phi - 2\sinh(2r)\sin\gamma\cos\phi\sin\phi \\
&\quad +[\cosh(2r)-\sinh(2r)\cos\gamma]\cos^2\phi\big) \\
&= \frac{1}{2}\left(\cosh(2r) - \sinh(2r)\cos\gamma\cos(2\phi) - \sinh(2r)\sin\gamma\sin(2\phi))\right) \\
&= \frac{1}{2}[\cosh(2r) - \sinh(2r)(\cos\gamma\cos(2\phi) + \sin\gamma\sin(2\phi))] \\
&= \frac{1}{2}(\cosh(2r) - \sinh(2r)\cos(\gamma - 2\phi)),
\end{aligned} \tag{71}$$

From which we find

$$F_Q[\hat{\rho}, \hat{G}] = 2(\cosh(2r) - \sinh(2r)\cos(\gamma - 2\phi)). \tag{72}$$

To evaluate the minimum QFI we take the derivative of F_Q with respect to ϕ

$$\frac{\mathrm{d}F_Q[\hat{G}]}{\mathrm{d}\phi} = -4\sinh(2r)\sin(\gamma - 2\phi) \tag{73}$$

which is zero if $\phi = \frac{\gamma}{2}$ (or if $r = 0$, but in that case we just have a vacuum state). We then get

$$\begin{aligned}
F_Q^{\min} &= 2[\cosh(2r) - \sinh(2r)] \\
&= 2e^{-2r}.
\end{aligned} \tag{74}$$

We can convince ourselves that this does indeed represent a minimum by seeing that the second derivative is given by

$$\frac{\mathrm{d}^2F_Q[\hat{G}]}{\mathrm{d}\phi^2} = 8\sinh(2r)\cos(\gamma - 2\phi) \geq 0 \tag{75}$$

in the neighbourhood of $\phi = \frac{\gamma}{2}$ since $r \geq 0$. Integrating ϕ over a period and dividing by 2π gives

$$\begin{aligned}
F_Q^{\mathrm{avg}} &= 2\cosh(2r) \\
&= 4n + 2.
\end{aligned} \tag{76}$$

For calculating the sensitivity, $[\hat{M}, \hat{G}]$ is the same as before, but the variance of $\hat{M}$ is now given by (see (71))

$$\mathrm{Var}[\hat{M}] = \frac{1}{2}[\cosh(2r) - \cos(\gamma - 2\varepsilon)\sinh(2r)]. \tag{77}$$

Finally, we find

$$\chi = \frac{\sin^2(\varepsilon - \phi)}{n + \frac{1}{2} - \sqrt{n(n+1)}\cos(\gamma - 2\varepsilon)} \leq F_{\mathrm{Q}} = 2(\cosh(2r) - \sinh(2r)\cos(\gamma - 2\phi)) \tag{78}$$

We notice here that this is similar to the Fock state sensitivity (56), but with an additional term in the denominator. If we consider ϕ to be known, then the maximal sensitivity is found by taking derivatives of χ with respect to ε. We can also set $\phi = 0$ without loss of generality. For the sake of the calculations, we call $a = n + \frac{1}{2}$ and $b = \sqrt{n(n+1)}$. The derivative is then given by

$$\partial_\varepsilon \chi = \frac{2\cos(\varepsilon)\sin(\varepsilon)}{a - b\cos(\gamma - 2\varepsilon)} + \frac{2b\sin(\gamma - 2\varepsilon)\sin^2(\varepsilon)}{(a - b\cos(\gamma - 2\varepsilon))^2}. \tag{79}$$

Solving for the zeros and checking which zeros give a maximum, we find that χ is maximal at

$$\varepsilon = -\cos^{-1}\left(-\frac{\sqrt{b^2 - b^2\cos(2\gamma)}}{\sqrt{2}\sqrt{a^2 + b^2 - 2ab\cos(\gamma)}}\right) \tag{80}$$

with value

$$\chi^{\max} = \frac{1 - \frac{4n(1+n)\sin^2(\gamma)}{1+8n+8n^2-4\sqrt{n(1+n)}(1+2n)\cos(\gamma)}}{\frac{1}{2} + n - \sqrt{n(1+n)}\cos\left(\gamma + 2\cos^{-1}\left(-\frac{2\sqrt{n(1+n)}\sin^2(\gamma)}{\sqrt{1+8n+8n^2-4\sqrt{n(1+n)}(1+2n)\cos(\gamma)}}\right)\right)} \tag{81}$$

If we do not know ϕ, we take the average sensitivity

$$\chi^{\mathrm{avg}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \frac{\sin^2(\varepsilon - \phi)}{n + \frac{1}{2} - \sqrt{n(n+1)}\cos(\gamma - 2\varepsilon)} \tag{82}$$

which can be integrated numerically.

3.1.4 Cat states

Cat states are defined as quantum superposition of coherent states, such as [11]

$$|\Psi\rangle = \mathcal{N}[|\alpha\rangle + e^{i\theta}|-\alpha\rangle], \tag{83}$$

where $\mathcal{N}^{-1} = 2 + 2\cos\theta e^{-2|\alpha|}$ is the normalization constant. In this subsubsection we will consider for convenience α to be real. We can take specific states such as $\theta = 0$ (even cat state), or $\theta = \pi$ (odd cat state). In the following, we will consider the even cat state

$$|\Psi_e\rangle = \mathcal{N}_e[|\alpha\rangle + |-\alpha\rangle]. \tag{84}$$

The covariance matrix is given by (see Appendix C.3)

$$\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x},\hat{p}) \\ \mathrm{Cov}(\hat{p},\hat{x}) & \mathrm{Var}(\hat{p}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}} & 0 \\ 0 & \frac{1}{2} - \frac{2\alpha^2 e^{-2\alpha^2}}{1+e^{-2\alpha^2}} \end{pmatrix}. \tag{85}$$

So finally,

$$\begin{aligned} \mathrm{Var}[\hat{G}(\phi)] &= \boldsymbol{u}^T \Gamma \boldsymbol{u} \\ &= \begin{pmatrix} \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \left[\frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}}\right] \sin\phi \\ -\left[\frac{1}{2} - \frac{2\alpha^2 e^{-2\alpha^2}}{1+e^{-2\alpha^2}}\right] \cos\phi \end{pmatrix} \\ &= \left[\frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}}\right] \sin^2\phi + \left[\frac{1}{2} - \frac{2\alpha^2 e^{-2\alpha^2}}{1+e^{-2\alpha^2}}\right] \cos^2\phi \\ &= \frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}} (\sin^2\phi + e^{-2\alpha^2}\cos^2\phi) \\ &= \frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}} - 2\alpha^2 \frac{1-e^{-2\alpha^2}}{1+e^{-2\alpha^2}} \cos^2\phi, \end{aligned} \tag{86}$$

which depends on both ϕ and α. We show in Appendix C.3 that $\langle n \rangle = |\alpha|^2$. The QFI is then given by

$$\begin{aligned} F_Q[\hat{\rho},\hat{G}] &= 2 + \frac{8\alpha^2}{1+e^{-2\alpha^2}} - 8\alpha^2 \frac{1-e^{-2\alpha^2}}{1+e^{-2\alpha^2}} \cos^2\phi \\ &= 2 + \frac{8n}{1+e^{-2n}} - 8n\frac{1-e^{-2n}}{1+e^{-2n}} \cos^2\phi. \end{aligned} \tag{87}$$

The minimum QFI happens quite obviously for $\phi = 0$, so

$$\begin{aligned} F_Q^{\min} &= 2 + \frac{8n}{1+e^{-2n}} - 8n\frac{1-e^{-2n}}{1+e^{-2n}} \\ &= 2 + 8n\frac{e^{-2n}}{1+e^{-2n}}. \end{aligned} \tag{88}$$

The average QFI can easily be found to be

$$\begin{aligned} F_Q^{\mathrm{avg}} &= 2 + \frac{8n}{1+e^{-2n}} - 4n\frac{1-e^{-2n}}{1+e^{-2n}} \\ &= 2 + 4n\frac{1+e^{-2n}}{1+e^{-2n}} \\ &= 2 + 4n. \end{aligned} \tag{89}$$

To evaluate the sensitivity, we compute the variance of $\hat{M}$ to be

$$\mathrm{Var}[\hat{M}] = \frac{1}{2} + \frac{2n}{1+e^{-2n}} - 2n\frac{1-e^{-2n}}{1+e^{-2n}} \cos^2\varepsilon. \tag{90}$$

This then gives the following result

$$\chi = \frac{\sin^2(\varepsilon - \phi)}{\frac{1}{2} + \frac{2n}{1+e^{-2n}} - 2n\frac{1-e^{-2n}}{1+e^{-2n}}\cos^2\varepsilon} \leq F_{\mathrm{Q}} = 2 + \frac{8n}{1+e^{-2n}} - 8n\frac{1-e^{-2n}}{1+e^{-2n}}\cos^2\phi \tag{91}$$

We can maximize ε by taking derivatives with respect to ε. Let $r = 2n\frac{1}{1+e^{-2n}}$ and $s = 1 - e^{-2n}$. We have

$$\partial_\varepsilon \chi = \frac{2\sin(\epsilon - \phi)\cos(\epsilon - \phi)}{\frac{1}{2} + r - rs\cos^2(\epsilon)} - \frac{2rs\sin(\epsilon)\cos(\epsilon)\sin^2(\epsilon - \phi)}{\left(\frac{1}{2} + r - rs\cos^2(\epsilon)\right)^2}. \tag{92}$$

If we consider the specific case where $\phi = 0$, the derivative is equal to 0 in the interval $[0, 2\pi)$ if $\varepsilon = 0$, $\varepsilon = \frac{\pi}{2}$, $\varepsilon = \pi$, or $\varepsilon = \frac{3\pi}{2}$. The maximum is reached at $\varepsilon = \frac{\pi}{2}$ or $\varepsilon = \frac{3\pi}{2}$, and is

$$\chi^{\max}(\phi = 0) = \frac{1}{\frac{1}{2} + \frac{2n}{1+e^{-2n}}}. \tag{93}$$

In the general case, the algebra becomes rather unpractical to work with, which is why it might be preferable to rely on numerical maximization on a case-by-case basis. The average sensitivity is

$$\chi^{\mathrm{avg}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \frac{\sin^2(\varepsilon - \phi)}{\frac{1}{2} + \frac{2n}{1+e^{-2n}} - 2n\frac{1-e^{-2n}}{1+e^{-2n}}\cos^2\varepsilon}. \tag{94}$$

which can be integrated numerically.

3.1.5 Fock state superposition

Let us consider here also the superposition of two Fock states $|m\rangle$ and $|n\rangle$ with $m \neq n$. The state is given by $|\psi\rangle = \frac{|m\rangle + |n\rangle}{\sqrt{2}}$. The corresponding density matrix is then

$$\begin{aligned} \rho_{mn} &= \frac{1}{2}(|m\rangle + |n\rangle)(\langle m| + \langle n|) \\ &= \frac{1}{2}(|m\rangle\langle m| + |m\rangle\langle n| + |n\rangle\langle m| + |n\rangle\langle n|). \end{aligned} \tag{95}$$

In the following we will consider the case $m = 0$. The relevant expectation values are computed in Appendix C.4. There we find

$$\Gamma = \begin{pmatrix} \frac{1}{2}(n + 1 + \delta_{0,n} + \sqrt{2}\delta_{2,n} - \delta_{n,1}) & 0 \\ 0 & \frac{1}{2}(n + 1 + \delta_{0,n} - \sqrt{2}\delta_{2,n}) \end{pmatrix}, \tag{96}$$

so using $n \neq m = 0$

$$\mathrm{Var}[\hat{G}] = \frac{1}{2}(n + 1 - \sqrt{2}\delta_{2,n}\cos(2\phi) - \delta_{n,1}\sin^2\phi), \tag{97}$$

and

$$F_{\mathrm{Q}} = 2(n + 1 - \sqrt{2}\delta_{2,n}\cos(2\phi) - \delta_{n,1}\sin^2\phi). \tag{98}$$

The minimum is given by

$$F_{\mathrm{Q}}^{\min} = \begin{cases} 2 & n = 1 \\ 2(3 - \sqrt{2}) & n = 2 \\ 2(n+1) & \text{otherwise} \end{cases}, \tag{99}$$

while the average can be written as

$$F_{\mathrm{Q}}^{\mathrm{avg}} = 2(n + 1 - \frac{1}{2}\delta_{n,1}). \tag{100}$$

We can use our results from the preceding sections again to compute the sensitivity

$$\chi = \frac{2\sin^2(\varepsilon - \phi)}{n + 1 - \sqrt{2}\delta_{2,n}\cos(2\phi) - \delta_{n,1}\sin^2\phi} \tag{101}$$

as well as the maximum over ε

$$\chi^{\max} = \frac{2}{n + 1 - \sqrt{2}\delta_{2,n}\cos(2\phi) - \delta_{n,1}\sin^2\phi}, \tag{102}$$

and the average sensitivity

$$\chi^{\mathrm{avg}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \frac{2\sin^2(\varepsilon - \phi)}{n + 1 - \sqrt{2}\delta_{2,n}\cos(2\phi) - \delta_{n,1}\sin^2\phi}. \tag{103}$$

In the case of a displacement in the x-direction, so $\phi = 0$, we find

$$\chi^{\max} = \frac{2}{n + 1 - \sqrt{2}\delta_{2,n}}. \tag{104}$$

3.1.6 Summary of the results

Let us summarize here the results we have got so far. For Fock, coherent, Gaussian and cat states, we considered two cases: ϕ is either known or unknown. In Table 1 and Table 2, we show expressions corresponding to the first case which are the sensitivity χ, its maximum $\chi^{\max}$, and the upper bound F_Q. Table 3 contains the results relative to the second situation, so the average sensitivity χ, as well as the minimum and average QFI.

	Fock	Coherent	Gaussian
χ	$\frac{\sin^2(\varepsilon-\phi)}{n+\frac{1}{2}}$	$2\sin^2(\varepsilon-\phi)$	$\frac{\sin^2(\varepsilon-\phi)}{n+\frac{1}{2}-\sqrt{n(n+1)}\cos(\gamma-2\varepsilon)}$
$\chi^{\max}$	$\frac{1}{n+\frac{1}{2}}$	2	See (81)
F_Q	$4n+2$	2	$4n+2-4\sqrt{n(n+1)}\cos(\gamma-2\phi))$

Table 1: Sensitivity χ, maximum sensitivity $\chi^{\max}$, and QFI F_Q for Fock, coherent and Gaussian states.

	Even cat	$\vert 0\rangle+\vert n\rangle$
χ	$\frac{\sin^2(\varepsilon-\phi)}{\frac{1}{2}-2n\frac{1-e^{-2n}}{1+e^{-2n}}\cos^2\varepsilon}$	$\frac{2\sin^2(\varepsilon-\phi)}{n+1-\sqrt{2}\delta_{2,n}\cos(2\phi)-\delta_{n,1}\sin^2\phi}$
$\chi^{\max}$	See 3.1.4	$\frac{2}{n+1-\sqrt{2}\delta_{2,n}\cos(2\phi)-\delta_{n,1}\sin^2\phi}$
F_Q	$2+\frac{8n}{1+e^{-2n}}-8n\frac{1-e^{-2n}}{1+e^{-2n}}\cos^2\phi$	$2(n+1-\sqrt{2}\delta_{2,n}\cos(2\phi)-\delta_{n,1}\sin^2\phi)$

Table 2: Sensitivity χ, maximum sensitivity $\chi^{\max}$, and QFI F_Q for even cat states and $|0\rangle+|n\rangle$ superpositions ($n\neq 0$).

	Fock	Coherent	Gaussian	Even cat	$\vert 0\rangle+\vert n\rangle$
χ^{avg}	$\frac{1}{2n+1}$	1	See (82)	See (94)	See (103)
$F_Q^{\min}$	$4n+2$	2	$4n+2-\sqrt{n(n+1)}$	$2+8n\frac{e^{-2n}}{1+e^{-2n}}$	See (99)
F_Q^{avg}	$4n+2$	2	$4n+2$	$4n+2$	$2(n+1-\frac{1}{2}\delta_{n,1})$

Table 3: Average sensitivity χ^{avg}, minimum QFI $F_Q^{\min}$ and average QFI F_Q^{avg} for Fock, coherent, Gaussian, even cat states and $|0\rangle+|n\rangle$ superpositions ($n\neq 0$).

We can observe the following. The bound $\chi \leq F_{\mathrm{Q}}$ (47) is saturated by no states, with the best case scenario being the vacuum state for which $\chi^{\max}=\frac{F_{\mathrm{Q}}}{2}$. For cat states with $|\alpha|^2$ large, we obtain $F_Q^{\min}[\hat{\rho}]\simeq 2$ but, if the phase of the displacement is known, one can achieve $\max_\phi F_Q[\hat{\rho},\hat{G}(\phi)]\simeq 8n+2$ (see (87)). For a Gaussian state (displaced squeezed vacuum) we observed that the covariance matrix does not depend on the displacement, and therefore without loss of generality we considered a squeezed vacuum state, which gives $F_Q^{\min}[\hat{\rho}]=4n+2-\sqrt{n(n+1)}$. This is again lower than the one of a Fock state, even if for a known phase of the displacement to be estimated the sensitivity can be higher.

3.2 Rotation sensing with quadrature measurements

We consider here the goal of estimating the angle of a rotation. The task consists again of optimizing the state preparation and the final measurement, in order to maximize the sensitivity.

The rotation operator is given by

$$\hat{R}(\theta) = e^{i\theta \hat{a}^\dagger \hat{a}} = e^{i\theta \hat{G}} \ , \tag{105}$$

where $\hat{G} = \hat{a}^\dagger \hat{a}$ is the generator of the perturbation.

In this case, the variance of $\hat{G}$ is a constant, and the QFI of a pure state is given by

$$F_Q[\hat{\rho}, \hat{G}] = 4\,\mathrm{Var}[\hat{n}] \ . \tag{106}$$

Interestingly, this result implies that states with definite particle number n are insensitive to rotations.

In the following, we compute for different states of interest the sensitivity χ, as well as the quantum Fisher information F_Q, as we did for the case of displacement sensing. The measurement chosen to be performed on the perturbed state is always a linear quadrature measurement.

3.2.1 Fock states

To compute the variance of $\hat{n}$, we clearly have

$$\langle n| \hat{G} |n\rangle = \langle n| \hat{a}^\dagger \hat{a} |n\rangle = n, \tag{107}$$

$$\langle n| (\hat{G})^2 |n\rangle = \langle n| (\hat{a}^\dagger \hat{a})^2 |n\rangle = n^2, \tag{108}$$

which gives the expected

$$\mathrm{Var}[\hat{G}] = 0. \tag{109}$$

For the senitivity, we can reuse the variance of $\hat{M}$ computed for the displacement sensing case: $\mathrm{Var}[\hat{M}] = \frac{1}{2}$. However, the commutator $[\hat{M}, \hat{G}]$ is now (see Appendix D.1)

$$[\hat{M}, \hat{G}] = i(\cos\varepsilon \hat{x} + \sin\varepsilon \hat{p}). \tag{110}$$

Recalling (176) and (177), we find that the expectation value of the commutator is 0, so

$$\chi = 0 \leq F_Q = 0. \tag{111}$$

This means that no matter what ε we choose, we will not be able to get any information on the perturbation if we consider a measurement $\hat{M}$ that is a linear combination of $\hat{x}$ and $\hat{p}$.

3.2.2 Coherent states

For a coherent state we have

$$\langle\alpha|\hat{G}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\,, \tag{112}$$

$$\langle\alpha|(\hat{G})^2|\alpha\rangle = \langle\alpha|(\hat{a}^\dagger\hat{a})^2|\alpha\rangle = |\alpha|^2 + \langle\alpha|(\hat{a}^\dagger)^2\hat{a}^2|\alpha\rangle = |\alpha|^2(1+|\alpha|^2), \tag{113}$$

which gives

$$\mathrm{Var}[\hat{G}] = |\alpha|^2\,, \tag{114}$$

and thus

$$F_{\mathrm{Q}} = 4\,|\alpha|^2\,. \tag{115}$$

Using (61) directly gives

$$\mathrm{Var}[\hat{M}] = \frac{1}{2}. \tag{116}$$

The commutator was already computed in (215), so we can compute the expectation value directly using the expectation values of $\hat{x}$ and $\hat{p}$ with respect to the coherent states from (183) and (184):

$$\langle[\hat{M},\hat{G}]\rangle = \frac{i}{\sqrt{2}}(\cos\varepsilon(\alpha+\alpha^*) + i\sin\varepsilon(\alpha^*-\alpha)). \tag{117}$$

Taking the norm squared gives

$$\left|\langle[\hat{M},\hat{G}]\rangle\right|^2 = 2(\cos\varepsilon\Re[\alpha] + \sin\varepsilon\Im[\alpha])^2. \tag{118}$$

With this we are now able to find the sensitivity

$$\chi = 4(\cos\varepsilon\Re[\alpha] + \sin\varepsilon\Im[\alpha])^2 \leq 4\,|\alpha|^2\,. \tag{119}$$

If we let $\alpha = |\alpha|\,e^{i\beta}$, we find by solving $\partial_\varepsilon\chi = 0$

$$\varepsilon = \beta + k\pi, \tag{120}$$

where k is an integer. The only solution in $[0,2\pi]$ where the second derivative is negative is when $k=0$. In that case we get $\varepsilon = \beta$ and

$$\chi^{\max} = 4\,|\alpha|^2(\cos^2(\beta)+\sin^2(\beta))^2 = 4\,|\alpha|^2\,, \tag{121}$$

which corresponds exactly to the upper bound given by the QFI. This means that we have to measure along $\varepsilon = \arg(\alpha)$ to get maximal sensitivity. This can also be understood intuitively: imagine a coherent state on the x axis (*i.e.* with real alpha), that is subject to a small rotation. We find $\epsilon = \arg(\alpha) = 0$, meaning that, as expected, the best measurement to be performed is along p.

3.2.3 Gaussian states

The following results will be useful for computing expectation values [12]:

$$D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha, \tag{122}$$

$$D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \alpha^*, \tag{123}$$

$$S^\dagger(\xi)\hat{a}S(\xi) = \mu\hat{a} + \nu\hat{a}^\dagger, \tag{124}$$

$$S^\dagger(\xi)\hat{a}^\dagger S(\xi) = \mu\hat{a}^\dagger + \nu^*\hat{a}. \tag{125}$$

Using these, we then find (see Appendix D.2)

$$\langle\hat{n}\rangle = |\alpha|^2 + |\nu|^2\,, \tag{126}$$

$$\langle\hat{n}\rangle^2 = |\alpha|^4 + 2\,|\alpha|^2\,|\nu|^2 + |\nu|^4\,, \tag{127}$$

and similarly

$$\langle\hat{n}^2\rangle = |\alpha|^2 + |\nu|^2 + 2\,|\nu|^4 + \mu^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + 4\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^* + |\alpha|^4\,, \tag{128}$$

which gives

$$\mathrm{Var}[\hat{G}] = |\alpha|^2 + |\nu|^2 + |\nu|^4 + \mu^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + 2\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^*. \tag{129}$$

The QFI is then

$$F_Q[\rho,\hat{G}] = 4\left[|\alpha|^2 + |\nu|^2 + |\nu|^4 + \mu^2\,|\nu|^2 + 2\,|\alpha|^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + \alpha^2\mu\nu^*\right]. \tag{130}$$

We can recall from (77) that $\mathrm{Var}[\hat{M}] = \frac{1}{2}[\cosh(2r) - \cos(\gamma-2\varepsilon)\sinh(2r)]$. The expectation values of $\hat{x}$ and $\hat{p}$ are found to be

$$\langle\hat{x}\rangle = \frac{\alpha+\alpha^*}{\sqrt{2}}, \text{ and} \tag{131}$$

$$\langle\hat{p}\rangle = \frac{\alpha-\alpha^*}{i\sqrt{2}}. \tag{132}$$

This gives

$$\langle[\hat{M},\hat{G}]\rangle = i\sqrt{2}(\cos\varepsilon\,\Re[\alpha] + \sin\varepsilon\,\Im[\alpha]) \tag{133}$$

and therefore the sensitivity

$$\chi = \frac{4(\Re[\alpha]^2\cos^2\varepsilon + \Im[\alpha]^2\sin^2\varepsilon)}{\cosh(2r) - \cos(\gamma-2\varepsilon)\sinh(2r)}. \tag{134}$$

The maximum is here given by

$$\chi^{\max} = \frac{4\max\{\Re[\alpha]^2, \Im[\alpha]^2\}}{\cosh(2r) - \cos(\gamma-2\varepsilon)\sinh(2r)}, \tag{135}$$

and it is achieved for $\varepsilon = 0$ if $|\Im[\alpha]| < |\Re[\alpha]|$, and for $\varepsilon = \frac{\pi}{2}$ otherwise.

3.2.4 Cat states

We consider here the general case of a cat state (83) with α complex and θ in $[0, 2\pi]$. The expectation values are given by (see Appendix D.3)

$$\langle\Psi| \hat{a}^\dagger \hat{a} |\Psi\rangle = |\alpha|^2 , \tag{136}$$

$$\langle\Psi| (\hat{a}^\dagger \hat{a})^2 |\Psi\rangle = \frac{|\alpha|^2}{2 + 2\cos\theta e^{-2|\alpha|^2}} (2 + 2|\alpha|^2 + 2\cos\theta e^{-2|\alpha|^2}(-1 + |\alpha|^2)). \tag{137}$$

For α large, the last equality reduces to

$$\langle\Psi| (\hat{a}^\dagger \hat{a})^2 |\Psi\rangle \simeq |\alpha|^2 (1 + |\alpha|^2). \tag{138}$$

This finally gives for large α

$$\mathrm{Var}[\hat{G}] \simeq |\alpha|^2 . \tag{139}$$

If we now still consider α to be large, and if we restrict ourselves to the even cat state considered before, $\mathrm{Var}[\hat{M}]$ is given by (90). The commutator is given by (215). If we recall (193) and (194), we immediately see that

$$\chi = 0 \leq 4|\alpha|^2 . \tag{140}$$

This means again that choosing $\hat{M}$ to be linear cannot give us any information about the rotation of an even cat state.

3.2.5 Summary of the results

Let us summarize here the results for rotation sensing we got so far. In Table 4, we show the sensitivity χ, its maximum $\chi^{\max}$, and the upper bound F_Q for the different states considered.

	Fock $\lvert n\rangle$	Coherent	Gaussian	Even cat
χ	0	$4(\cos\varepsilon\Re[\alpha] + \sin\varepsilon\Im[\alpha])^2$	$\frac{4(\Re[\alpha]^2\cos^2\varepsilon + \Im[\alpha]^2\sin^2\varepsilon)}{\cosh(2r) - \cos(\gamma-2\varepsilon)\sinh(2r)}$	0
$\chi^{\max}$	0	$4\lvert\alpha\rvert^2$	$\frac{4\max\{\Re[\alpha]^2, \Im[\alpha]^2\}}{\cosh(2r) - \cos(\gamma-2\varepsilon)\sinh(2r)}$	0
F_Q	0	$4\lvert\alpha\rvert^2$	See (130)	$4\lvert\alpha\rvert^2$

Table 4: Sensitivity χ, maximum sensitivity $\chi^{\max}$, and QFI F_Q for Fock, coherent, Gaussian and cat states. For the cat state we assumed $|\alpha| \gg 1$.

This means that we cannot get any information about rotations by performing quadrature measurements on Fock states. This is no surprise, since Fock states are rotationally symmetric. Coherent states with $|\alpha| > 0$, however, have a nonzero sensitivity which is maximal when the measurement direction is $\varepsilon = \arg(\alpha)$. Gaussian states do not saturate the QFI bound, and cat states have zero sensitivity under rotations with our choice of $\hat{M}$.

4 Sensing from displaced parity measurements

In this Chapter we focus on sensing displacements by performing displaced parity measurements on continuous-variable states. The same idea can be applied to sensing rotations, which we leave for future investigations.

4.1 Displacement sensing with displaced parity measurements

Parity measurements refer to measuring whether the number of excitations are even or odd, and they are defined as

$$\hat{\Pi} = (-1)^{\hat{n}} = \sum_n (-1)^n |n\rangle \langle n| . \tag{141}$$

With this, we can define a new measurement operator $\hat{M}$ for our sensing applications, based on displaced parity measurements instead of linear quadratures:

$$\hat{M}(\beta) = \hat{D}(\beta)\hat{\Pi}\hat{D}(\beta)^\dagger \tag{142}$$

In the following, we compute χ for this $\hat{M}$ and $\hat{G} = \sin\phi\hat{x} - \cos\phi\hat{p}$ (the generator of displacements). Given the link between the Wigner function and parity measurement (10), we have $\left\langle \hat{M} \right\rangle = \pi W(x,p)$, while for the variance we use the fact that $\left\langle \hat{M}^2 \right\rangle = 1$ to write $\mathrm{Var}[\hat{M}] = \Delta^2\hat{M} = 1 - (\pi W(x,p))^2$. For simplicity, we will restrict in this Section to the case of known displacement directions, meaning $\phi = 0$.

4.1.1 Fock states

The Wigner function of a Fock state $|n\rangle$ is given by [6]:

$$W_n(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2), \tag{143}$$

where $L_n(z)$ is the n-th degree Laguerre polynomial, and $\alpha = \frac{x+ip}{\sqrt{2}}$. Expressed in terms of x and p, we get

$$W_n(x,p) = \frac{1}{\pi}(-1)^n e^{-(x^2+p^2)} L_n(2(x^2+p^2)) \tag{144}$$

We can use this formula for the Wigner function to find the sensitivity of Fock states for displacements and parity measurements. For simplicity, we choose here a displacement along the x direction. As Fock states are rotationally symmetric, this implies also no loss of generality. The derivative is

$$\begin{aligned} \frac{\partial\langle M\rangle}{\partial\theta} &= \pi\frac{\partial W_n(x-\theta,p)}{\partial\theta} \\ &= 2(-1)^n(x-\theta)e^{-p^2-(x-\theta)^2}\left(L_n\left(2\left(p^2+(x-\theta)^2\right)\right) + 2L^1_{n-1}\left(2\left(p^2+(x-\theta)^2\right)\right)\right). \end{aligned} \tag{145}$$

Taking the reference displacement $\theta_0 = 0$, we find

$$\left| \frac{\partial \langle M \rangle}{\partial \theta} \bigg|_{\theta=0} \right|^2 = 4x^2 e^{-2(x^2+p^2)} \left[L_n \left(2 \left(x^2 + p^2 \right) \right) + 2L^1_{n-1} \left(2 \left(x^2 + p^2 \right) \right) \right]^2 . \tag{146}$$

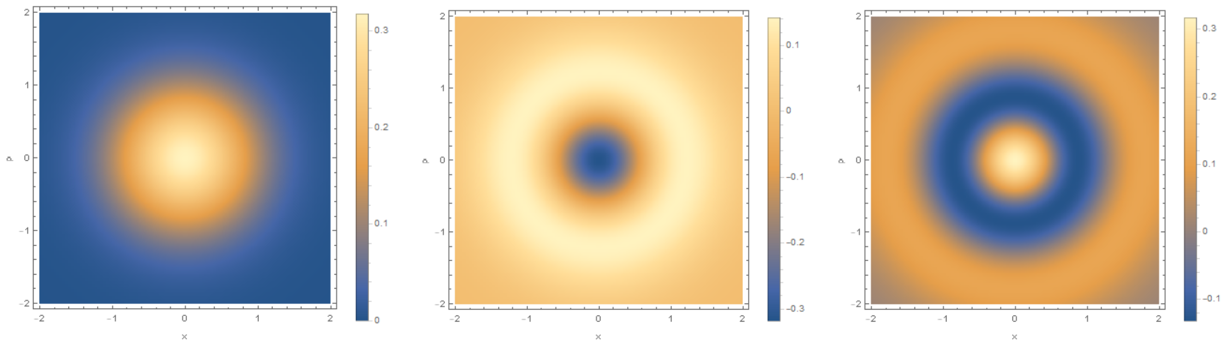
This gives the sensitivity

$$\begin{aligned} \chi &= \frac{\left| \frac{\partial \langle M \rangle}{\partial \theta} \right|^2}{1 - \pi^2 W^2(x,p)} \\ &= \frac{4x^2 \left[L_n \left(2 \left(x^2 + p^2 \right) \right) + 2L^1_{n-1} \left(2 \left(x^2 + p^2 \right) \right) \right]^2}{(-1)^{2n+1} L_n \left(2 \left(p^2 + x^2 \right) \right)^2 + e^{2(p^2+x^2)}} . \end{aligned} \tag{147}$$

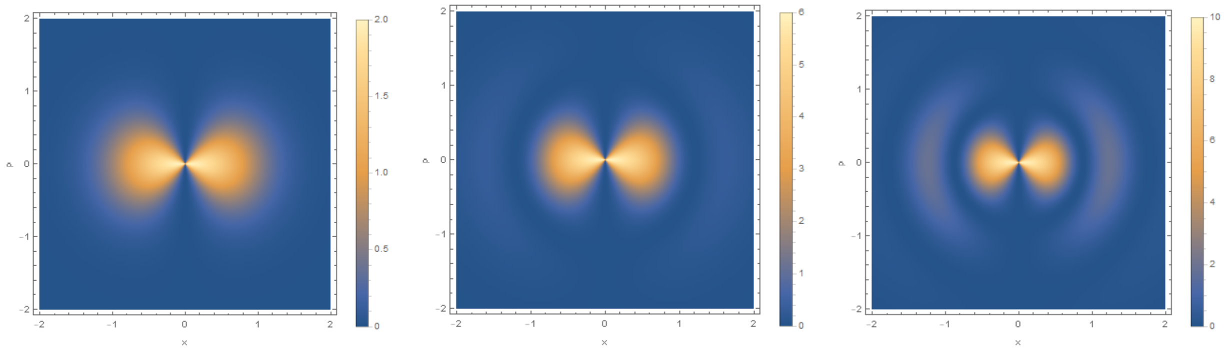
As this expression is relatively complex, we plot it for $n = 0, 1, 2$ in Fig. 1.

We can see that for $x = 0$ the sensitivity reaches the upper bound given by the QFI, implying that the measurement is optimal. For different points in phase space, however, the sensitivity quickly goes to 0. This is to be expected since the Wigner function also vanishes quickly away from the origin.

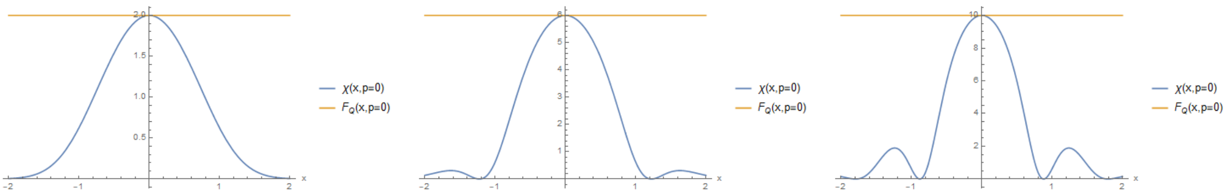
It might seem counterintuitive to see the highest sensitivity at $x = 0$, where the Wigner function has a maximum/minimum, and thus where it is to first order insensitive to displacements. In fact, this point is ill-defined, as both numerator and denominator of the expression for the sensitivity vanish at $x = 0$.This is not an uncommon situation, as pointed out in [1]. In a realistic situation, where technical noise is present (so when the denominator is not exactly 0 at the origin), the sensitivity must vanish at the origin and has a maximum at a position that depends on the amount of noise.



(a) Wigner $n = 0$. (b) Wigner $n = 1$. (c) Wigner $n = 2$.



(d) Sensitivity $n = 0$. (e) Sensitivity $n = 1$. (f) Sensitivity $n = 2$.



(g) Sensitivity and QFI $n = 0$ at $p = 0$. (h) Sensitivity and QFI $n = 1$ at $p = 0$. (i) Sensitivity and QFI $n = 2$ at $p = 0$.

Figure 1: Wigner functions and corresponding sensitivities χ for the $n = 0, 1, 2$ Fock states in the case of displaced parity measurement of a translation in the x-direction.

4.1.2 Coherent states

For a coherent state $|\beta\rangle$ the Wigner function is given by [6]

$$W_\beta(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\beta|^2}, \tag{148}$$

or in terms of x and p

$$W_\beta(\alpha) = \frac{1}{\pi} e^{-\left|x+ip-\sqrt{2}\beta\right|^2}. \tag{149}$$

We can notice that this is the same Wigner as the $|0\rangle$ Fock state with x and p replaced by $x - \sqrt{2}\Re[\beta]$ and $p - \sqrt{2}\Im[\beta]$, respectively. So, the sensitivity will for coherent state is simply the vacuum Fock state sensitivity shifted in phase space.

4.1.3 Gaussian state

By definition, the Wigner function of a Gaussian state is a Gaussian. Defining $\boldsymbol{X} = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}$, the Wigner function reads [12]

$$W(\alpha) = \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\langle\boldsymbol{X}\rangle)^T\Gamma^{-1}(\boldsymbol{X}-\langle\boldsymbol{X}\rangle)}}{(2\pi)^n\kappa_2^{2n}\sqrt{\det[\Gamma]}}, \tag{150}$$

where $\kappa_2 = 2^{-1/2}$, $n = 1$, and Γ is the covariance matrix given by (68). This then gives

$$W(\alpha) = \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\langle\boldsymbol{X}\rangle)^T\Gamma^{-1}(\boldsymbol{X}-\langle\boldsymbol{X}\rangle)}}{\pi\sqrt{\det[\Gamma]}}. \tag{151}$$

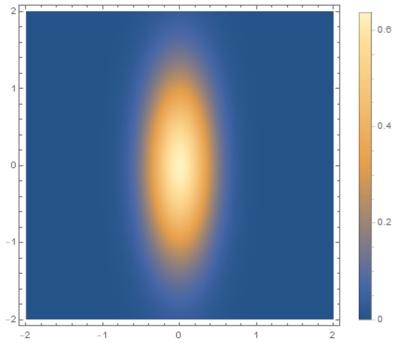
Here we consider a squeezing of a vacuum state in the p direction without displacement. This means that $\langle\boldsymbol{X}\rangle = 0$ and $\gamma = \pi$. So, in this case, the covariance matrix becomes

$$\Gamma = \frac{1}{2}\begin{pmatrix} \cosh(2r) & \sinh(2r) \\ \sinh(2r) & \cosh(2r) \end{pmatrix}. \tag{152}$$

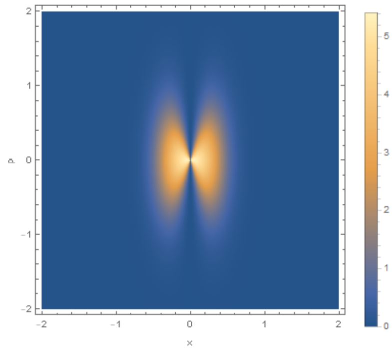
The Wigner function, as well as sensitivity χ, is plotted for $r = \frac{1}{2}$ in Fig. 2. The analytical expression for $\gamma = \pi$ is given by

$$\chi(x, p) = \frac{8e^{4r}x^2}{e^{4e^{-2r}(p^2+e^{4r}x^2)} - 1}. \tag{153}$$

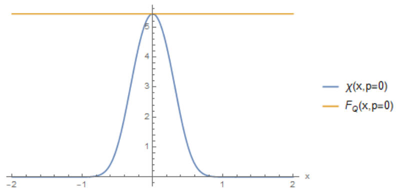
Again, we see a saturation of the bound given by the QFI at $x = 0$. However, in the presence of noise, the same conclusion as for Fock states holds.



(a) Wigner $r = \frac{1}{2}$.



(b) Sensitivity $r = \frac{1}{2}$.



(c) Sensitivity and QFI $r = \frac{1}{2}$ at $p = 0$.

Figure 2: Wigner functions and corresponding sensitivities χ for the $r = \frac{1}{2}$ squeezed vacuum state in the case of displaced parity measurement of a translation in the x-direction.

4.1.4 Cat states

For an even cat state $|\Psi\rangle = \mathscr{N}[|\beta\rangle + |-\beta\rangle]$ with β real, the Wigner function is given by [11]

$$W_\beta(x,p) = \frac{1}{2\pi[1+e^{-2\beta^2}]}\left(e^{-((x-\sqrt{2}\beta)^2+p^2)} + e^{-((x+\sqrt{2}\beta)^2+p^2)} + 2e^{-(x^2+p^2)}\cos(2\sqrt{2}\beta p)\right). \tag{154}$$

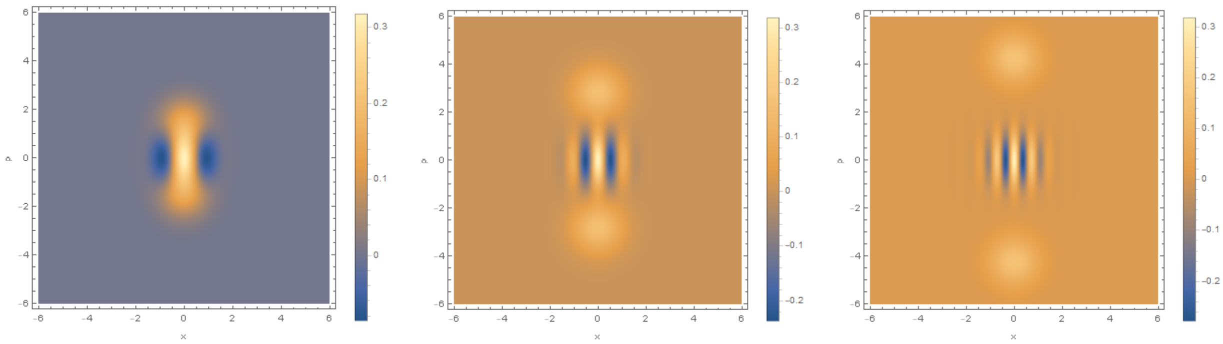
Now, we will consider $\beta \in i\mathbb{R}$ to make the cat state more sensitive to a displacement along $\phi = 0$ than with $\beta \in \mathbb{R}$. This gives us

$$W_\beta(x,p) = \frac{1}{2\pi[1+e^{-2\beta^2}]}\left(e^{-(x^2+(p-\sqrt{2}\beta)^2)} + e^{-(x^2+(p-\sqrt{2}\beta)^2)} + 2e^{-(x^2+p^2)}\cos(2\sqrt{2}\beta x)\right). \tag{155}$$

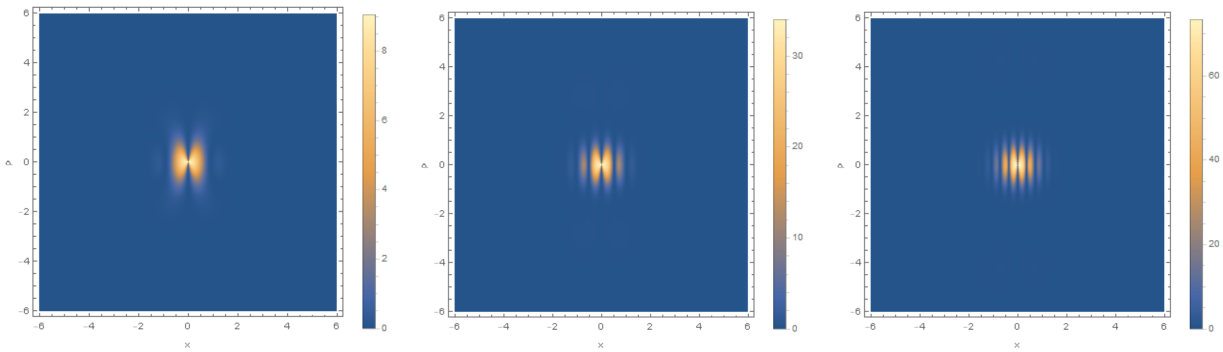
From this expression, we evaluate the sensitivity χ as before. The analytical expression is given by

$$\chi = \frac{4\left(2e^{4\beta^2+p^2}\left(\sqrt{2}\beta\sin\left(2\sqrt{2}\beta x\right) + x\cos\left(2\sqrt{2}\beta x\right)\right) + x\left(e^{\left(p-\sqrt{2}\beta\right)^2} + e^{\left(\sqrt{2}\beta+p\right)^2}\right)\right)^2}{4\left(e^{2\beta^2}+1\right)^2 e^{4\beta^2+4p^2+2x^2} - \left(2e^{4\beta^2+p^2}\cos\left(2\sqrt{2}\beta x\right) + e^{\left(p-\sqrt{2}\beta\right)^2} + e^{\left(\sqrt{2}\beta+p\right)^2}\right)^2}. \tag{156}$$

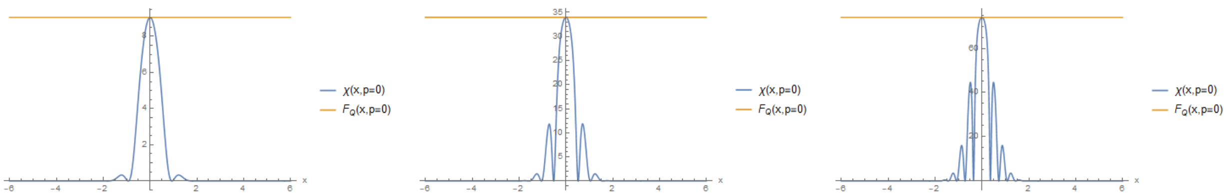
In Fig. 3 we plot this expression for some selected values of β. At $x = 0$ we observe a saturation of the bound given by the QFI. This, however, is again an ill-defined point, whose sensitivity goes to zero even for an infinitesimal amount of noise.



(a) Wigner $\beta = i$. (b) Wigner $\beta = 2i$. (c) Wigner $\beta = 3i$.



(d) Sensitivity $\beta = i$. (e) Sensitivity $\beta = 2i$. (f) Sensitivity $\beta = 3i$.



(g) Sensitivity and QFI $\beta = i$ at $p = 0$. (h) Sensitivity and QFI $\beta = 2i$ at $p = 0$. (i) Sensitivity and QFI $\beta = 3i$ at $p = 0$.

Figure 3: Wigner functions and corresponding sensitivities χ for the $\beta = 1, 2, 3$ even cat states in the case of displaced parity measurement of a translation in the x-direction.

4.1.5 Superposition of Fock states

Let us consider the superposition of two Fock states again. We can recall the density matrix

$$\rho_{mn} = \frac{1}{2}(|m\rangle\langle m| + |m\rangle\langle n| + |n\rangle\langle m| + |n\rangle\langle n|) \tag{157}$$

for $m \neq n$.

To find the corresponding Wigner function, we rely on known results for the Wigner function of $|m\rangle\langle n|$. Using the formulation from [13], after adjusting the normalization according to our definitions, we obtain

$$W_{mn}(x,p) = \begin{cases} \frac{1}{\pi}(-1)^m \left(\frac{n!}{m!}\right)^{1/2} e^{-(x^2+p^2)} \left(-\sqrt{2}(x-ip)\right)^{m-n} L_n^{m-n}\left(2(x^2+p^2)\right) & \text{if } m \geq n \\ \frac{1}{\pi}(-1)^n \left(\frac{m!}{n!}\right)^{1/2} e^{-(x^2+p^2)} \left(-\sqrt{2}(x+ip)\right)^{n-m} L_m^{n-m}\left(2(x^2+p^2)\right) & \text{if } m < n \end{cases}. \tag{158}$$

Notice that for $n = m$, we get

$$W_n(x,p) = \frac{1}{\pi}(-1)^n L_n(2(x^2+p^2))e^{-(x^2+p^2)}, \tag{159}$$

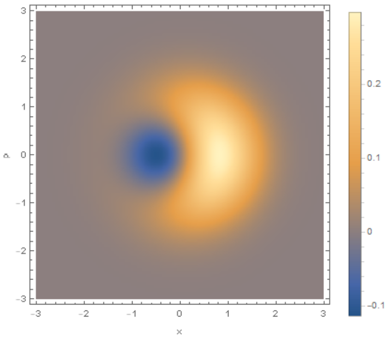
as expected.

For $m = 0$, we get the Wigner function for the superposition $|0\rangle + |n\rangle$

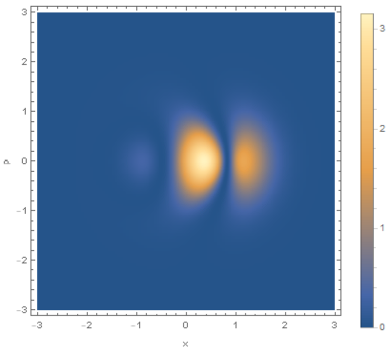
$$W(x,p) = \frac{e^{-(x^2+p^2)}}{2\pi}\left(\frac{2^{n/2}}{\sqrt{n!}}((x+ip)^n + (x-ip)^n) + (-1)^n L_n\left(2\left(p^2+x^2\right)\right) + 1\right) \tag{160}$$

We can compute χ again by taking the derivative. The results for $n = 1, 2, 3$ are plotted in Fig. 4. We see that in this case, there is saturation of the QFI bound only for $n = 2$. This might hint toward a saturation for even n, even if still for an ill-defined point. For the cases $n = 1, 3$, however, the QFI bound is not reached. This means that measuring displaced parity measurements cannot give us the best possible sensitivity for such a superposition of Fock states. If one wants to get higher sensitivities, one would probably need to perform higher order measurements of $\hat{x}$ and $\hat{p}$.

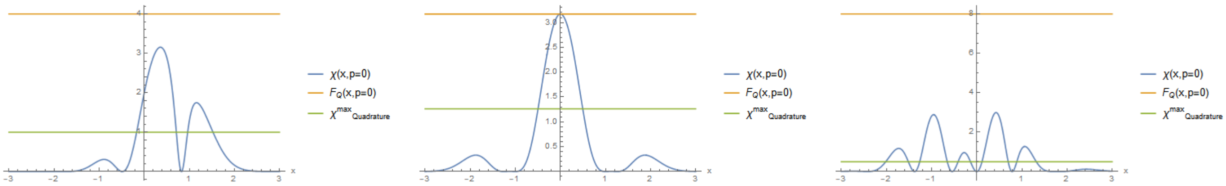
In Fig. 5, we compare the sensitivity for the $n = 2$ state, with the one for the same state after a rotation by $\frac{\pi}{2}$, which we expect to be more sensitive to displacements along the x direction. The slice view at $p = 0$ seems to indicate that it is better to use displaced parity measurements if one makes measurements where the sensitivity is maximal, rather than using quadratures.



(a) Wigner $n = 1$. (b) Wigner $n = 2$. (c) Wigner $n = 3$.

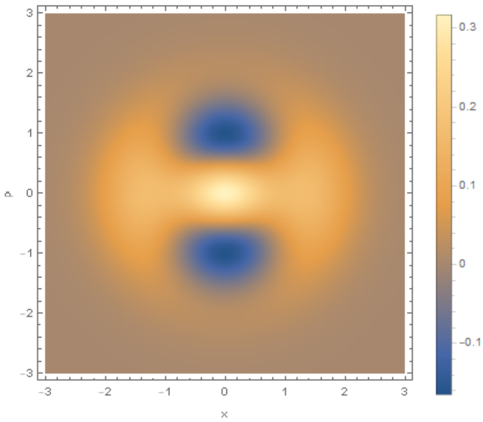


(d) Sensitivity $n = 1$. (e) Sensitivity $n = 2$. (f) Sensitivity $n = 3$.

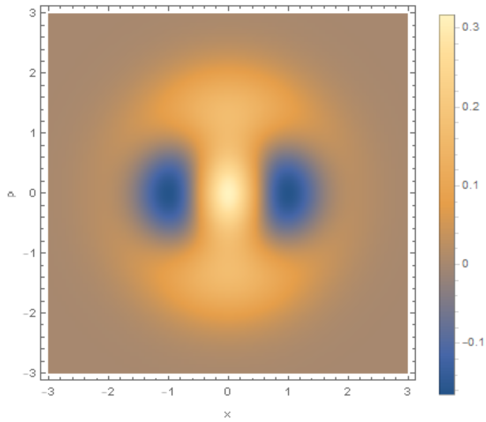


(g) Sensitivity and QFI $n = 1$ at $p = 0$. (h) Sensitivity and QFI $n = 2$ at $p = 0$. (i) Sensitivity and QFI $n = 3$ at $p = 0$.

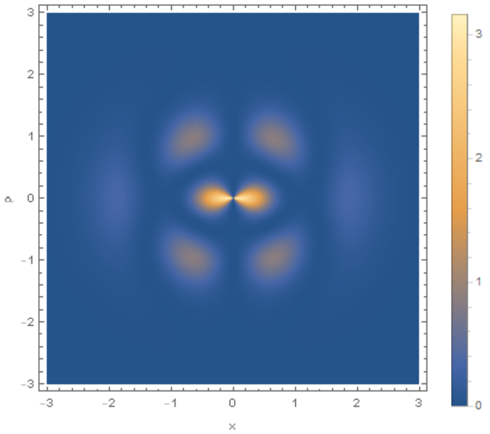
Figure 4: Wigner functions and corresponding sensitivities χ for the $n = 0, 1, 2$ Fock state superposition $|0\rangle + |n\rangle$ in the case of displaced parity measurement of a translation in the x-direction. The QFI is computed in Appendix C.4. The maximum sensitivity $\chi^{\max}$ for quadrature measurements (104) is also plotted for comparison.



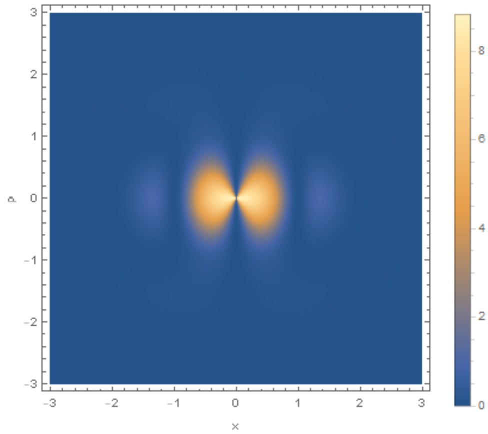
(a) Wigner $n = 2$, 0 phase shift.



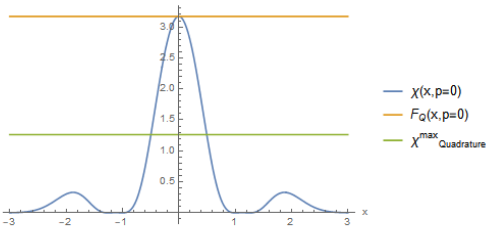
(b) Wigner $n = 2$, $\frac{\pi}{2}$ phase shift.



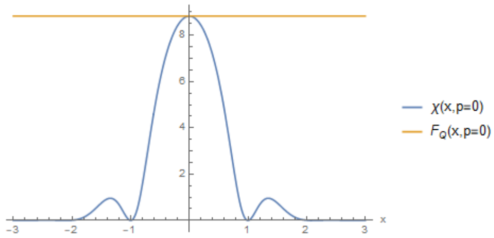
(c) Sensitivity $n = 2$, 0 phase shift.



(d) Sensitivity $n = 2$, $\frac{\pi}{2}$ phase shift.



(e) Sensitivity and QFI $n = 2$ at $p = 0$, 0 phase shift.



(f) Sensitivity and QFI $n = 2$ at $p = 0$, $\frac{\pi}{2}$ phase shift.

Figure 5: Wigner function and sensitivities for the $n = 2$ superposition with additional phase 0 and $\frac{\pi}{2}$ respectively.

5 Conclusion

In conclusion, we studied the sensitivities for various probe states, perturbations, and measurements using the framework of the quantum metrology. We started with the study of sensitivity to displacements for Fock, coherent, Gaussian, cat states, and Fock state superposition as detected by linear quadrature measurements. For the same states (except the vacuum Fock superposition) and measurement process, we also analyzed the sensitivity to rotations. Finally, we came back to the study of displaced probe states, as seen from the measurement of displaced parity operators. What we learned in this process are the conditions for which the Cramér-Rao bound is saturated. Investigating other types of measurements, such as high-order quadrature measurements, might lead to better results. In the near future, it would be interesting to text experimentally our predictions, for example with optical or mechanical systems. This could lead to the development of quantum-enhanced sensors, which is of key interest for quantum technologies.

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A Link between quantum fidelity and Wigner

It is possible to relate the fidelity to the Wigner function directly. Let us consider a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\begin{aligned}
F(\hat{\rho},\hat{\sigma}) &= \left(\mathrm{Tr}[\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}]\right)^2 \\
&= \left(\mathrm{Tr}[\sqrt{\sqrt{|\psi\rangle\langle\psi|}\hat{\sigma}\sqrt{|\psi\rangle\langle\psi|}}]\right)^2 \\
&\overset{\text{(i)}}{=} \left(\mathrm{Tr}[\sqrt{|\psi\rangle\langle\psi|\,\hat{\sigma}\,|\psi\rangle\langle\psi|}]\right)^2 \\
&= (\sqrt{\langle\psi|\,\hat{\sigma}\,|\psi\rangle}\mathrm{Tr}[|\psi\rangle\langle\psi|])^2 \\
&= \langle\psi|\,\hat{\sigma}\,|\psi\rangle \cdot 1^2 \\
&= \sum_i \langle\psi|\phi_i\rangle\langle\phi_i|\psi\rangle \\
&= \sum_i c_i\,|\langle\psi|\phi_i\rangle|^2 \\
&\overset{\text{(ii)}}{=} 2\pi\sum_i c_i \int\int \mathrm{d}x\mathrm{d}p W_{\hat{\rho}}(x,p) W_{|\phi_i\rangle\langle\phi_i|}(x,p) \\
&= \int\int \mathrm{d}x\mathrm{d}p W_{\hat{\rho}}(x,p)\int \mathrm{d}y\,\langle x+\frac{1}{2}y|\sum_i c_i\,|\phi_i\rangle\langle\phi_i|\,|x-\frac{1}{2}y\rangle\, e^{ipy} \\
&= 2\pi\int\int \mathrm{d}x\mathrm{d}p W_{\hat{\rho}}(x,p) W_{\hat{\sigma}}(x,p),
\end{aligned} \tag{161}$$

where in (i) we used that the square root of a pure state density matrix is itself, and in (ii) we used Eq. (19) of Ref. [10].

B Proof of the relation $\mathbf{Var}[\hat{G}(\phi)] = \mathbf{u}^T\Gamma\mathbf{u}$

The covariance of two operators $\hat{A}$ and $\hat{B}$ is given by $\mathrm{Cov}[\hat{A},\hat{B}] = \langle\frac{\hat{A}\hat{B}+\hat{B}\hat{A}}{2}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$. We have by definition:

$$\begin{aligned}
\mathrm{Var}[\hat{G}(\phi)] &= \langle\psi|\,\hat{G}^2(\phi)\,|\psi\rangle - (\langle\psi|\,\hat{G}(\phi)\,|\psi\rangle)^2 \\
&= \langle\psi|\sin^2\phi\hat{x}^2 - \sin\phi\cos\phi(\hat{x}\hat{p}+\hat{p}\hat{x}) + \cos^2\phi\hat{p}^2\,|\psi\rangle - (\sin\phi\,\langle\psi|\,\hat{x}\,|\psi\rangle - \cos\phi\,\langle\psi|\,\hat{p}\,|\psi\rangle)^2 \\
&= \sin^2\phi\,\langle\psi|\,\hat{x}^2\,|\psi\rangle + \cos^2\phi\,\langle\psi|\,\hat{p}^2\,|\psi\rangle - \sin\phi\cos\phi\,\langle\psi|\,\hat{x}\hat{p}+\hat{p}\hat{x}\,|\psi\rangle \\
&\quad - \sin^2\phi(\langle\psi|\,\hat{x}\,|\psi\rangle)^2 - \cos^2\phi(\langle\psi|\,\hat{p}\,|\psi\rangle)^2 + 2\sin\phi\cos\phi\,\langle\psi|\,\hat{x}\,|\psi\rangle\langle\psi|\,\hat{p}\,|\psi\rangle \\
&= \sin^2\phi\,\mathrm{Var}[\hat{x}] + \cos^2(\phi)\,\mathrm{Var}[\hat{p}] - \sin\phi\cos\phi\,\langle\psi|\,\hat{x}\hat{p}+\hat{p}\hat{x}\,|\psi\rangle + 2\sin\phi\cos\phi\,\langle\psi|\,\hat{x}\,|\psi\rangle\langle\psi|\,\hat{p}\,|\psi\rangle \\
&= \sin^2\phi\,\mathrm{Var}[\hat{x}] + \cos^2(\phi)\,\mathrm{Var}[\hat{p}] - 2\sin\phi\cos\phi\,\mathrm{Cov}[\hat{x},\hat{p}]
\end{aligned} \tag{162}$$

On the other hand,

$$\begin{aligned}\boldsymbol{u}^T\Gamma\boldsymbol{u} &= \sin^2\phi\,\mathrm{Var}(\hat{x}) - \ \mathrm{Cov}(\hat{x},\hat{p})\sin\phi\cos\phi - \ \mathrm{Cov}(\hat{p},\hat{x})\sin\phi\cos\phi + \cos^2\phi\,\mathrm{Var}(\hat{p}) \\ &= \sin^2\phi\,\mathrm{Var}(\hat{x}) + \cos^2\phi\,\mathrm{Var}(\hat{p}) - 2\sin\phi\cos\phi\,\mathrm{Cov}[\hat{x},\hat{p}].\end{aligned} \tag{163}$$

It is then clear that both expressions are the same, so indeed

$$\mathrm{Var}[\hat{G}(\phi)] = \boldsymbol{u}^T\Gamma\boldsymbol{u}. \tag{164}$$

C Calculation of the covariance matrix for translations

C.1 Fock states

The following basic properties will be useful (see [11]):

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{165}$$

$$\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|\,, \tag{166}$$

$$\langle\alpha|\,\hat{a}\hat{a}^\dagger\,|\alpha\rangle = |\alpha|^2\,. \tag{167}$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{168}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{169}$$

$$\hat{a}\hat{a}^\dagger\,|n\rangle = n\,|n\rangle\,, \tag{170}$$

$$[\hat{a},\hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}. \tag{171}$$

To find the variance of $\hat{G}(\phi)$, we just have to find Γ. We have

$$\hat{x}^2 = \frac{1}{2}(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2) \tag{172}$$

$$\hat{p}^2 = -\frac{1}{2}((\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2), \tag{173}$$

$$\hat{x}\hat{p} = \frac{i}{2}(\hat{a}\hat{a}^\dagger - \hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a}), \tag{174}$$

$$\hat{p}\hat{x} = \frac{i}{2}(\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2 - \hat{a}^2 - \hat{a}\hat{a}^\dagger). \tag{175}$$

This gives

$$\begin{aligned}\langle n|\,\hat{x}\,|n\rangle &= \sqrt{\frac{1}{2}}(\langle n|\,\hat{a}\,|n\rangle + \langle n|\,\hat{a}^\dagger\,|n\rangle) \\ &= 0,\end{aligned} \tag{176}$$

$$\begin{aligned} \langle n|\,\hat{p}\,|n\rangle &= i\sqrt{\frac{1}{2}}(\langle n|\,\hat{a}^\dagger\,|n\rangle - \langle n|\,\hat{a}\,|n\rangle) \\ &= 0, \end{aligned} \tag{177}$$

$$\begin{aligned} \langle n|\,\hat{x}^2\,|n\rangle &= \frac{1}{2}\,\langle n|\,1 + 2\hat{a}^\dagger\hat{a}\,|n\rangle \\ &= \frac{1}{2}(2n+1), \end{aligned} \tag{178}$$

$$\begin{aligned} \langle n|\,\hat{p}^2\,|n\rangle &= -\frac{1}{2}\,\langle n| -1 - 2\hat{a}^\dagger\hat{a}\,|n\rangle \\ &= \frac{1}{2}(2n+1), \end{aligned} \tag{179}$$

$$\begin{aligned} \langle n|\,\hat{x}\hat{p}\,|n\rangle &= i\frac{1}{2}\,\langle n|\,\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\,|n\rangle \\ &= \frac{i}{2}, \end{aligned} \tag{180}$$

$$\begin{aligned} \langle n|\,\hat{p}\hat{x}\,|n\rangle &= i\frac{1}{2}\,\langle n|\,\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\,|n\rangle \\ &= -\frac{i}{2}, \end{aligned} \tag{181}$$

We are now able to find Γ:

$$\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x},\hat{p}) \\ \mathrm{Cov}(\hat{p},\hat{x}) & \mathrm{Var}(\hat{p}) \end{pmatrix} = \begin{pmatrix} n+\frac{1}{2} & 0 \\ 0 & n+\frac{1}{2} \end{pmatrix}. \tag{182}$$

C.2 Coherent states

The expectation values for coherent states are given by

$$\begin{aligned} \langle \alpha|\,\hat{x}\,|\alpha\rangle &= \frac{1}{\sqrt{2}}\left(\langle \alpha|\,\hat{a} + \hat{a}^\dagger\,|\alpha\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\alpha + \alpha^*\right), \end{aligned} \tag{183}$$

$$\begin{aligned} \langle \alpha|\,\hat{p}\,|\alpha\rangle &= \frac{i}{\sqrt{2}}\left(\langle \alpha|\,\hat{a}^\dagger - \hat{a}\,|\alpha\rangle\right) \\ &= \frac{i}{\sqrt{2}}\left(\alpha^* - \alpha\right), \end{aligned} \tag{184}$$

$$\begin{aligned} \langle \alpha|\,\hat{x}^2\,|\alpha\rangle &= \frac{1}{2}\left(\langle \alpha|\,\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|\alpha\rangle\right) \\ &= \frac{1}{2}\left(\langle \alpha|\,\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|\alpha\rangle\right) \\ &= \frac{1}{2}\left(\alpha^2 + 1 + 2\,|\alpha|^2 + (\alpha^*)^2\right), \end{aligned} \tag{185}$$

$$\begin{aligned}\langle\alpha|\,\hat{p}^2\,|\alpha\rangle &= -\frac{1}{2}\left(\langle\alpha|\,(\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2\,|\alpha\rangle\right)\\ &= -\frac{1}{2}\left(\langle\alpha|\,(\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\,|\alpha\rangle\right)\\ &= -\frac{1}{2}\left((\alpha^*)^2 - 1 - 2\,|\alpha|^2 + \alpha^2\right),\end{aligned} \tag{186}$$

$$\begin{aligned}\langle\alpha|\,\hat{x}\hat{p}\,|\alpha\rangle &= \frac{i}{2}\left(\langle\alpha|\,\hat{a}\hat{a}^\dagger - \hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a}\,|\alpha\rangle\right)\\ &= \frac{i}{2}\left(\langle\alpha|\,1 - \hat{a}^2 + (\hat{a}^\dagger)^2\,|\alpha\rangle\right)\\ &= \frac{i}{2}\left(1 - \alpha^2 + (\alpha^*)^2\right),\end{aligned} \tag{187}$$

$$\begin{aligned}\langle\alpha|\,\hat{p}\hat{x}\,|\alpha\rangle &= \frac{i}{2}\left(\langle\alpha|\,\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2 - \hat{a}^2 - \hat{a}\hat{a}^\dagger\,|\alpha\rangle\right)\\ &= \frac{i}{2}\left(\langle\alpha| - 1 + (\hat{a}^\dagger)^2 - \hat{a}^2\,|\alpha\rangle\right)\\ &= \frac{i}{2}\left(-1 + (\alpha^*)^2 - \alpha^2\right).\end{aligned} \tag{188}$$

This lets us find the variances and covariances:

$$\begin{aligned}\mathrm{Var}(\hat{x}) &= \frac{1}{2}\left[(\alpha^*)^2 + 1 + 2\,|\alpha|^2 + \alpha^2\right] - \frac{1}{2}\left[\alpha^2 + 2\,|\alpha|^2 + (\alpha^*)^2\right]\\ &=\frac{1}{2},\end{aligned} \tag{189}$$

$$\begin{aligned}\mathrm{Var}(\hat{p}) &= -\frac{1}{2}\left[(\alpha^*)^2 - 1 - 2\,|\alpha|^2 + \alpha^2\right] + \frac{1}{2}\left[\alpha^2 - 2\,|\alpha|^2 + (\alpha^*)^2\right]\\ &=\frac{1}{2},\end{aligned} \tag{190}$$

$$\begin{aligned}\mathrm{Cov}(\hat{x},\hat{p}) &= \frac{i}{2}((\alpha^*)^2 - \alpha^2) - \frac{i}{2}(\alpha + \alpha^*)(\alpha^* - \alpha)\\ &= 0,\end{aligned} \tag{191}$$

We can summarize this into Γ:

$$\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x},\hat{p}) \\ \mathrm{Cov}(\hat{p},\hat{x}) & \mathrm{Var}(\hat{p}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{192}$$

C.3 Even cat states

The expectation values are here

$$
\begin{aligned}
\langle\Psi_e|\,\hat{x}\,|\Psi_e\rangle &= \frac{1}{\sqrt{2}}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)(\hat{a}+\hat{a}^\dagger)(|\alpha\rangle + |-\alpha\rangle)\\
&= \frac{1}{\sqrt{2}}\mathcal{N}_e^2(\langle\alpha|\,\hat{a}\,|\alpha\rangle + \langle\alpha|\,\hat{a}\,|-\alpha\rangle) + \langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle + \langle\alpha|\,\hat{a}^\dagger\,|-\alpha\rangle\\
&\quad + \langle-\alpha|\,\hat{a}\,|\alpha\rangle + \langle-\alpha|\,\hat{a}\,|-\alpha\rangle + \langle-\alpha|\,\hat{a}^\dagger\,|\alpha\rangle + \langle-\alpha|\,\hat{a}^\dagger\,|-\alpha\rangle\\
&= \frac{1}{\sqrt{2}}\mathcal{N}_e^2(\alpha - \alpha e^{-2|\alpha|^2} + \alpha^* - \alpha^* e^{-2|\alpha|^2} + \alpha e^{-2|\alpha|^2} - \alpha + \alpha^* e^{-2|\alpha|^2} - \alpha^*)\\
&= 0,
\end{aligned} \tag{193}
$$

$$
\begin{aligned}
\langle\Psi_e|\,\hat{p}\,|\Psi_e\rangle &= \frac{i}{\sqrt{2}}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)(\hat{a}^\dagger-\hat{a})(|\alpha\rangle + |\alpha\rangle)\\
&= \frac{i}{\sqrt{2}}\mathcal{N}_e^2(\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle + \langle\alpha|\,\hat{a}^\dagger\,|-\alpha\rangle) - \langle\alpha|\,\hat{a}\,|\alpha\rangle - \langle\alpha|\,\hat{a}\,|-\alpha\rangle\\
&\quad + \langle-\alpha|\,\hat{a}^\dagger\,|\alpha\rangle + \langle-\alpha|\,\hat{a}^\dagger\,|-\alpha\rangle - \langle-\alpha|\,\hat{a}\,|\alpha\rangle - \langle-\alpha|\,\hat{a}\,|-\alpha\rangle\\
&= \frac{i}{\sqrt{2}}\mathcal{N}_e^2(\alpha^* + \alpha^* e^{-2|\alpha|^2} - \alpha - \alpha e^{-2|\alpha|^2} - \alpha^* e^{-2|\alpha|^2} - \alpha^* - \alpha e^{-2|\alpha|^2} + \alpha)\\
&= 0,
\end{aligned} \tag{194}
$$

$$
\begin{aligned}
\langle\Psi_e|\,\hat{x}^2\,|\Psi_e\rangle &= \frac{1}{2}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)(\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2)(|\alpha\rangle + |\alpha\rangle)\\
&= \frac{1}{2}\mathcal{N}_e^2\left[\langle\alpha|\,\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|\alpha\rangle + \langle\alpha|\,\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|-\alpha\rangle\right.\\
&\quad \left. + \langle-\alpha|\,\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|\alpha\rangle + \langle-\alpha|\,\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\,|-\alpha\rangle\right]\\
&= \frac{1}{2}\mathcal{N}_e^2\Big[\alpha^2 + 1 + 2\,|\alpha| + (\alpha^*)^2 + e^{-2|\alpha|^2}\left((-\alpha)^2 + 1 - 2\,|\alpha|^2 + (\alpha^*)^2\right)\\
&\quad + e^{-2|\alpha|^2}\left(\alpha^2 + 1 - 2\,|\alpha|^2 + (-\alpha^*)^2\right) + (-\alpha)^2 + 1 + 2\,|\alpha|^2 + (-\alpha^*)^2\Big]\\
&= \frac{1}{2 + 2e^{-2|\alpha|^2}}\left[\alpha^2 + 1 + 2\,|\alpha|^2 + (\alpha^*)^2 + e^{-2|\alpha|^2}\left(\alpha^2 + 1 - 2\,|\alpha|^2 + (\alpha^*)^2\right)\right]\\
&= \frac{1}{2} + \frac{2\alpha^2}{1 + e^{-2\alpha^2}},
\end{aligned} \tag{195}
$$

$$
\begin{aligned}
\langle\Psi_e|\,\hat{p}^2\,|\Psi_e\rangle &= -\frac{1}{2}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)\left((\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\right)(|\alpha\rangle + |\alpha\rangle) \\
&= -\frac{1}{2}\mathcal{N}_e^2\left[\langle\alpha|\,\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\,|\alpha\rangle + \langle\alpha|\,\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\,|-\alpha\rangle\right. \\
&\quad \left. + \langle-\alpha|\,\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\,|\alpha\rangle + \langle-\alpha|\,\hat{a}^\dagger)^2 - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\,|-\alpha\rangle\right] \\
&= -\frac{1}{2}\mathcal{N}_e^2\Big[(\alpha^*)^2 - 1 - 2|\alpha|^2 + \alpha^2 + e^{-2|\alpha|^2}\left((\alpha^*)^2 - 1 + 2|\alpha|^2 + (-\alpha)^2\right) \\
&\quad + e^{-2|\alpha|^2}\left((-\alpha^*)^2 - 1 + 2|\alpha|^2 + \alpha^2\right) + (-\alpha^*)^2 - 1 - 2|\alpha|^2 + \alpha^2\Big] \\
&= -\frac{1}{2 + 2e^{-2|\alpha|^2}}\left[(\alpha^*)^2 - 1 - 2|\alpha|^2 + \alpha^2 + e^{-2|\alpha|^2}\left((\alpha^*)^2 - 1 + 2|\alpha|^2 + (-\alpha)^2\right)\right] \\
&= -\frac{1}{2 + 2e^{-2|\alpha|^2}}\left[(-1 + e^{-2\alpha^2}) + 4e^{-2\alpha^2}\alpha^2\right] \\
&= \frac{1}{2} - \frac{2\alpha^2 e^{-2\alpha^2}}{1 + e^{-2\alpha^2}},
\end{aligned}
\tag{196}
$$

$$
\begin{aligned}
\langle\Psi_e|\,\hat{x}\hat{p}\,|\Psi_e\rangle &= \frac{i}{2}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)\left(1 - \hat{a}^2 + (\hat{a}^\dagger)^2\right)(|\alpha\rangle + |\alpha\rangle) \\
&= \frac{i}{2}\frac{1}{1 + e^{-2|\alpha|^2}}(1 - \alpha^2 + (\alpha^*)^2)(1 + e^{-2|\alpha|^2}) \\
&= \frac{i}{2}(1 - \alpha^2 + (\alpha^*)^2) \\
&= \frac{i}{2},
\end{aligned}
\tag{197}
$$

$$
\begin{aligned}
\langle\Psi_e|\,\hat{p}\hat{x}\,|\Psi_e\rangle &= \frac{i}{2}\mathcal{N}_e^2(\langle\alpha| + \langle-\alpha|)\left(-1 - \hat{a}^2 + (\hat{a}^\dagger)^2\right)(|\alpha\rangle + |\alpha\rangle) \\
&= \frac{i}{2}\frac{1}{1 + e^{-2|\alpha|^2}}(-1 - \alpha^2 + (\alpha^*)^2)(1 + e^{-2|\alpha|^2}) \\
&= \frac{i}{2}(-1 - \alpha^2 + (\alpha^*)^2) \\
&= -\frac{i}{2}.
\end{aligned}
\tag{198}
$$

Notice that here we used at the end of each calculation that we consider α to be real. We then find

$$
\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x},\hat{p}) \\ \mathrm{Cov}(\hat{p},\hat{x}) & \mathrm{Var}(\hat{p}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{2\alpha^2}{1+e^{-2\alpha^2}} & 0 \\ 0 & \frac{1}{2} - \frac{2\alpha^2 e^{-2\alpha^2}}{1+e^{-2\alpha^2}} \end{pmatrix}. \tag{199}
$$

C.4 Vacuum state and Fock state superposition

The expectation values needed to compute the QFI are given by

$$\begin{aligned}(\langle 0| + \langle n|)\hat{x}(|0\rangle + |n\rangle) &= \langle 0|\,\hat{x}\,|0\rangle + \langle n|\,\hat{x}\,|0\rangle + \langle 0|\,\hat{x}\,|n\rangle + \langle n|\,\hat{x}\,|n\rangle \\ &= 0 + \frac{1}{\sqrt{2}}\langle n|\,\hat{a} + \hat{a}^\dagger\,|0\rangle + \frac{1}{\sqrt{2}}\langle 0|\,\hat{a} + \hat{a}^\dagger\,|n\rangle + 0 \\ &= \frac{1}{\sqrt{2}}(\langle n|\,|1\rangle + \langle 1|\,|n\rangle) \\ &= \sqrt{2}\delta_{n,1},\end{aligned} \tag{200}$$

$$\begin{aligned}(\langle 0| + \langle n|)\hat{x}^2(|0\rangle + |n\rangle) &= \frac{1}{2}(\langle 0| + \langle n|)(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2)(|0\rangle + |n\rangle) \\ &= \frac{1}{2}(\langle 0| + \langle n|)(\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2)(|0\rangle + |n\rangle) \\ &= n + 1 + \delta_{0,n} + \sqrt{2}\delta_{2,n},\end{aligned} \tag{201}$$

$$\begin{aligned}(\langle 0| + \langle n|)\hat{p}(|0\rangle + |n\rangle) &= \langle 0|\,\hat{p}\,|0\rangle + \langle n|\,\hat{p}\,|0\rangle + \langle 0|\,\hat{p}\,|n\rangle + \langle n|\,\hat{p}\,|n\rangle \\ &= 0 + \frac{1}{\sqrt{2}}\langle n|\,\hat{a} - \hat{a}^\dagger\,|0\rangle + \frac{1}{\sqrt{2}}\langle 0|\,\hat{a} - \hat{a}^\dagger\,|n\rangle + 0 \\ &= \frac{1}{\sqrt{2}}(-\langle n|\,|1\rangle + \langle 1|\,|n\rangle) \\ &= 0,\end{aligned} \tag{202}$$

$$\begin{aligned}(\langle 0| + \langle n|)\hat{p}^2(|0\rangle + |n\rangle) &= \frac{1}{2}(\langle 0| + \langle n|)(-\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - (\hat{a}^\dagger)^2)(|0\rangle + |n\rangle) \\ &= n + 1 + \delta_{0,n} - \sqrt{2}\delta_{2,n},\end{aligned} \tag{203}$$

$$\begin{aligned}(\langle 0| + \langle n|)\hat{x}\hat{p}(|0\rangle + |n\rangle) &= \frac{i}{2}(\langle 0| + \langle n|)(-\hat{a}^2 + \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2)(|0\rangle + |n\rangle) \\ &= i(1 + \delta_{0,n}),\end{aligned} \tag{204}$$

$$\begin{aligned}(\langle 0| + \langle n|)\hat{p}\hat{x}(|0\rangle + |n\rangle) &= \frac{i}{2}(\langle 0| + \langle n|)(-\hat{a}^2 - \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2)(|0\rangle + |n\rangle) \\ &= -i(1 + \delta_{0,n}).\end{aligned} \tag{205}$$

So,

$$\mathrm{Var}[\hat{x}] = \frac{1}{2}(n + 1 + \delta_{0,n} + \sqrt{2}\delta_{2,n} - \delta_{n,1}), \tag{206}$$

$$\mathrm{Var}[\hat{p}] = \frac{1}{2}(n + 1 + \delta_{0,n} - \sqrt{2}\delta_{2,n}), \tag{207}$$

$$\mathrm{Cov}[\hat{x}, \hat{p}] = 0, \tag{208}$$

which gives

$$\Gamma = \begin{pmatrix} \mathrm{Var}(\hat{x}) & \mathrm{Cov}(\hat{x}, \hat{p}) \\ \mathrm{Cov}(\hat{p}, \hat{x}) & \mathrm{Var}(\hat{p}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(n + 1 + \delta_{0,n} + \sqrt{2}\delta_{2,n} - \delta_{n,1}) & 0 \\ 0 & \frac{1}{2}(n + 1 + \delta_{0,n} - \sqrt{2}\delta_{2,n}) \end{pmatrix}. \tag{209}$$

This then gives

$$\begin{aligned}
\mathrm{Var}[\hat{G}(\phi)] &= \boldsymbol{u}^T \Gamma \boldsymbol{u} \\
&= \frac{1}{2} \begin{pmatrix} \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} (n+1+\delta_{0,n}+\sqrt{2}\delta_{2,n}-\delta_{n,1})\sin\phi \\ -(n+1+\delta_{0,n}-\sqrt{2}\delta_{2,n})\cos\phi \end{pmatrix} \\
&= \frac{1}{2}(n+1+\delta_{0,n}-\sqrt{2}\delta_{2,n}(\cos^2\phi-\sin^2\phi)-\delta_{n,1}\sin^2\phi) \\
&= \frac{1}{2}(n+1+\delta_{0,n}-\sqrt{2}\delta_{2,n}\cos(2\phi)-\delta_{n,1}\sin^2\phi),
\end{aligned} \tag{210}$$

D Rotations

D.1 Commutator $[\hat{M}, \hat{G}]$

We have

$$\begin{aligned}
\hat{G} &= \hat{a}^\dagger \hat{a} \\
&= \frac{1}{2}(\hat{x}-i\hat{p})(\hat{x}+i\hat{p}) \\
&= \frac{1}{2}(\hat{x}^2-i\hat{p}\hat{x}+i\hat{x}\hat{p}+\hat{p}^2) \\
&= \frac{1}{2}(-1+\hat{x}^2+\hat{p}^2),
\end{aligned} \tag{211}$$

so

$$\begin{aligned}
[\hat{M}, \hat{G}] &= \frac{1}{2}[\sin\varepsilon\hat{x}-\cos\varepsilon\hat{p}, -1+\hat{x}^2+\hat{p}^2] \\
&= \frac{1}{2}[\sin\varepsilon\hat{x}-\cos\varepsilon\hat{p}, \hat{x}^2+\hat{p}^2] \\
&= \frac{1}{2}[\sin\varepsilon\hat{x}^3-\cos\varepsilon\hat{p}\hat{x}^2+\sin\varepsilon\hat{x}\hat{p}^2-\cos\varepsilon\hat{p}^3-\sin\varepsilon\hat{x}^3-\sin\varepsilon\hat{p}^2\hat{x}+\cos\varepsilon\hat{x}^2\hat{p}+\cos\varepsilon\hat{p}^3] \\
&= \frac{1}{2}(\cos\varepsilon[\hat{x}^2,\hat{p}]+\sin\varepsilon[\hat{x},\hat{p}^2]).
\end{aligned} \tag{212}$$

Now, we have

$$\begin{aligned}
[\hat{x}^2, \hat{p}] &= \hat{x}^2\hat{p}-\hat{p}\hat{x}^2 \\
&= i\hat{x}+\hat{x}\hat{p}\hat{x}+i\hat{x}-\hat{x}\hat{p}\hat{x} \\
&= 2i\hat{x},
\end{aligned} \tag{213}$$

and

$$\begin{aligned}
[\hat{x}, \hat{p}^2] &= \hat{x}\hat{p}^2-\hat{p}^2\hat{x} \\
&= i\hat{p}+\hat{p}\hat{x}\hat{p}+i\hat{p}-\hat{p}\hat{x}\hat{p} \\
&= 2i\hat{p}.
\end{aligned} \tag{214}$$

Inserting in (212) gives

$$[\hat{M}, \hat{G}] = i(\cos\varepsilon\hat{x} + \sin\varepsilon\hat{p}). \tag{215}$$

D.2 Gaussian states

We can compute

$$\begin{aligned}
\langle\hat{n}\rangle &= \langle 0|\, \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\, \hat{S}^\dagger(\xi)(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\, \hat{S}^\dagger(\xi)(\hat{a}^\dagger\hat{a} + \alpha^*\hat{a} + \alpha\hat{a}^\dagger + |\alpha|^2)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\, [(\mu\hat{a}^\dagger + \nu^*\hat{a})(\mu\hat{a} + \nu\hat{a}^\dagger) + \alpha^*(\mu\hat{a} + \nu\hat{a}^\dagger) + \alpha(\mu\hat{a}^\dagger + \nu^*\hat{a}^\dagger) + |\alpha|^2]\,|0\rangle \\
&= \langle 0|\, [|\nu|^2\,\hat{a}\hat{a}^\dagger + |\alpha|^2\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2\,,
\end{aligned} \tag{216}$$

which gives

$$\langle\hat{n}\rangle^2 = |\alpha|^4 + 2\,|\alpha|^2\,|\nu|^2 + |\nu|^4\,. \tag{217}$$

Similarly, we get

$$\begin{aligned}
\langle\hat{n}^2\rangle &= \langle 0|\, \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\, \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)(\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2)\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2 + \langle 0|\, \hat{S}^\dagger(\xi)(\hat{a}^\dagger + \alpha^*)^2(\hat{a} + \alpha)^2\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2 + \langle 0|\, \hat{S}^\dagger(\xi)((\hat{a}^\dagger)^2 + 2\alpha^*\hat{a}^\dagger + (\alpha^*)^2)(\hat{a}^2 + 2\alpha\hat{a} + \alpha^2)\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2 + \langle 0|\, \hat{S}^\dagger(\xi)[(\hat{a}^\dagger)^2\hat{a}^2 + 2\alpha^*\hat{a}^\dagger\hat{a}^2 + (\alpha^*)^2\hat{a}^2 + 2\alpha(\hat{a}^\dagger)^2\hat{a} + 4\,|\alpha|^2\,\hat{a}^\dagger\hat{a} \\
&\quad + 2\alpha^*\,|\alpha|^2\,\hat{a} + \alpha^2(\hat{a}^\dagger)^2 + 2\alpha\,|\alpha|^2\,\hat{a}^\dagger + |\alpha|^4]\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2 + \langle 0|\, [(\mu\hat{a}^\dagger + \nu^*\hat{a})^2(\mu\hat{a} + \nu\hat{a}^\dagger)^2 + 0 + (\alpha^*)^2(\mu\hat{a} + \nu\hat{a}^\dagger)^2 + 0 \\
&\quad + 4\,|\alpha|^2\,|\nu|^2 + 0 + \alpha^2(\mu\hat{a}^\dagger + \nu^*\hat{a})^2 + 0 + |\alpha|^4]\,|0\rangle \\
&= |\alpha|^2 + |\nu|^2 + |\nu|^4\,\langle 0|\,\hat{a}^2(\hat{a}^\dagger)^2\,|0\rangle + \mu^2\,|\nu|^2\,\langle 0|\,\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\,|0\rangle + (\alpha^*)^2\mu\nu + 4\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^* + |\alpha|^4 \\
&= |\alpha|^2 + |\nu|^2 + |\nu|^4\,\langle 0|\,[(1 + \hat{a}^\dagger\hat{a})^2 + 1 + \hat{a}^\dagger\hat{a}]\,|0\rangle + \mu^2\,|\nu|^2\,\langle 0|\,[1 + 2\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}]\,|0\rangle \\
&\quad + (\alpha^*)^2\mu\nu + 4\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^* + |\alpha|^4 \\
&= |\alpha|^2 + |\nu|^2 + 2\,|\nu|^4 + \mu^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + 4\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^* + |\alpha|^4 \\
&= \langle\hat{n}\rangle^2 + |\alpha|^2 + |\nu|^2 + |\nu|^4 + \mu^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + 2\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^*,
\end{aligned} \tag{218}$$

which gives us

$$\mathrm{Var}[\hat{G}] = |\alpha|^2 + |\nu|^2 + |\nu|^4 + \mu^2\,|\nu|^2 + (\alpha^*)^2\mu\nu + 2\,|\alpha|^2\,|\nu|^2 + \alpha^2\mu\nu^*. \tag{219}$$

D.3 Even cat states

The expectation values are easily computed to be

$$\begin{aligned}
\langle\Psi| \hat{a}^\dagger\hat{a} |\Psi\rangle &= \mathscr{N}(\langle\alpha| + e^{-i\theta}\langle-\alpha|)(\hat{a}^\dagger\hat{a})(|\alpha\rangle + e^{i\theta}|-\alpha\rangle) \\
&= \mathscr{N}\left[\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle + e^{i\theta}\langle\alpha|\hat{a}^\dagger\hat{a}|-\alpha\rangle\right. \\
&\quad \left.+e^{-i\theta}\langle-\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle + \langle-\alpha|\hat{a}^\dagger\hat{a}|-\alpha\rangle\right] \\
&= \mathscr{N}(|\alpha|^2 - e^{i\theta}e^{-2|\alpha|^2}|\alpha|^2 - e^{-i\theta}e^{-2|\alpha|^2}|\alpha|^2 + |\alpha|^2) \\
&= \frac{|\alpha|^2}{2+2\cos\theta e^{-2|\alpha|^2}}(2 - 2\cos\theta e^{-2|\alpha|^2}) \\
&= |\alpha|^2,
\end{aligned} \tag{220}$$

$$\begin{aligned}
\langle\Psi| (\hat{a}^\dagger\hat{a})^2 |\Psi\rangle &= \langle\Psi| \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2 |\Psi\rangle \\
&= \mathscr{N}(\langle\alpha| + e^{-i\theta}\langle-\alpha|)(\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2)(|\alpha\rangle + e^{i\theta}|-\alpha\rangle) \\
&= \mathscr{N}\left[\langle\alpha|\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2|\alpha\rangle + e^{i\theta}\langle\alpha|\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2|-\alpha\rangle\right. \\
&\quad \left.+e^{-i\theta}\langle-\alpha|\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2|\alpha\rangle + \langle-\alpha|\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\hat{a}^2|-\alpha\rangle\right] \\
&= \mathscr{N}(|\alpha|^2 + |\alpha|^4 + e^{i\theta}e^{-2|\alpha|^2}(-|\alpha|^2 + |\alpha|^4) \\
&\quad - e^{-i\theta}e^{-2|\alpha|^2}(-|\alpha|^2 + |\alpha|^4) + |\alpha|^2 + |\alpha|^4) \\
&= \frac{|\alpha|^2}{2+2\cos\theta e^{-2|\alpha|^2}}(2 + 2|\alpha|^2 + 2\cos\theta e^{-2|\alpha|^2}(-1+|\alpha|^2)).
\end{aligned} \tag{221}$$