Supplementary Material for: Classical many-body time crystals

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1 Derivation of λ_{th}

We present here the derivation of the many-body parametric driving threshold amplitude for N resonators that are equally coupled to one another. For the prupose of this calculation, it suffices to consider N coupled *linear* resonators. The slow-flow equation describing this system (cf. Eq.(2) in the paper with coupling $\beta_{ij} = \beta/\sqrt{N}$ for all $i \neq j$) is given by:

$$\dot{\boldsymbol{X}} = A\boldsymbol{X}, \tag{1}$$

where the matrix A is given by:

$$A = \begin{pmatrix} a & b & \cdots & b \\ b & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix}. \tag{2}$$

and the individual matrix entries a and b are given by:

$$a = \begin{pmatrix} -\frac{\gamma}{2} & -\frac{1}{4\omega}\left(\lambda\omega_0^2 + 2(\omega_0^2 - \omega^2)\right) \\ -\frac{1}{4\omega}\left(\lambda\omega_0^2 - 2(\omega_0^2 - \omega^2)\right) & -\frac{\gamma}{2} \end{pmatrix}, \tag{3}$$

$$b = \begin{pmatrix} 0 & \frac{\beta^2/N}{2\omega} \\ -\frac{\beta^2/N}{2\omega} & 0 \end{pmatrix}. \tag{4}$$

The dynamics of the linear system can be deduced by decomposing the initial state $\boldsymbol{X}$ into the eigenvectors of A. The time evolution of each eigenvector is then determined by $e^{\Lambda t}$, where Λ is the corresponding eigenvalue. $\text{Im}\Lambda \neq 0$ imposes an oscillatory behavior whose envelope decreases exponentially for $\text{Re}\Lambda < 0$ and increases exponentially for $\text{Re}\Lambda > 0$. To evaluate these eigenvalues and eigenvectors, it is useful to rewrite the matrix A as:

$$A = Id_N \otimes a + \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \otimes b = Id_N \otimes a + M_N \otimes b\,. \tag{5}$$

Based on the structure of A, the eigenvectors obey the ansatz $w_m = r_m \otimes s_m$, where r_m are the N dimensional eigenvectors of M_N with eigenvalue ρ_m and s_m are the 2-dimensional eigenvectors of $a + \rho_m b$ with eigenvalues σ_m. Since M_N has N eigenvectors, r_m, with 2 corresponding s_m, this ansatz describes all the $2N$ eigenvectors and eigenvalues of the matrix A. The 2-vector s_m describes the amplitude and momentum (u_m, v_m) in a particular mode's phase-space and r_m generically describes the relative amplitudes and the phase configuration, e.g., $r_m = (1, -1)$ means that the two oscillators have opposite phases. We can readily show that this ansatz is indeed an eigenvector of A:

$$\begin{aligned} Aw_m &= Id_N r_m \otimes a s_m + M_N r_m \otimes b s_m \\ &= r_m \otimes (a + \rho_m b) s_m \\ &= \sigma_m w_m\,. \end{aligned} \tag{6}$$

Since M_N has a simple structure, we see that the eigenvectors $\{r_m\}$ take the form $r_0 = (1, 1, \cdots)$ with eigenvalue $\rho_0 = N - 1$ and $r_m = (0, \cdots 0, 1, -1, 0 \cdots)^T$, where the $+1$ is the m^{th} entry, are eigenvectors of M_N with eigenvalues $\rho_m = -1$ ($m \in \mathbb{N}$, $0 < m \leq N-1$). Note that the eigenvectors r_m effectively determine the normal mode transformations of the problem. Next, we evaluate the eigenvectors s_m and eigenvalues σ_m of

$$a + \rho_m b = \begin{pmatrix} a_1 & a_2 - a_3 \\ a_2 + a_3 & a_1 \end{pmatrix}, \tag{7}$$

where $a_1 = -\frac{\gamma}{2}$, $a_2 = -\frac{\lambda \omega_0^2}{4\omega}$ and $a_3 = \frac{(\omega_0^2 - \omega^2)}{2\omega} - \rho_m \frac{\beta^2/N}{2\omega}$. These are given by,

$$\sigma_{m,\pm} = a_1 \pm \sqrt{a_2^2 - a_3^2}\,, \tag{8}$$

$$s_{m,\pm} = \begin{pmatrix} \pm\sqrt{a_2 - a_3} \\ \sqrt{a_2 + a_3} \end{pmatrix}. \tag{9}$$

To summarize, the $2N$ eigenvectors of the matrix A are given by

$$w_{m,\pm} = r_m \otimes s_{m,\pm}\,, \tag{10}$$

with corresponding eigenvalues $\sigma_{m,\pm}$ and $0 \leq m \leq N-1$.

If $\mathrm{Re}\sigma_{m,\pm} > 0$, the corresponding $w_{\pm,m}$ grows exponentially indicating a parametric instability. We obtain the parametric driving threshold $\lambda_{th,m}$ for this instability by imposing the condition:

$$\sigma_{m,+} = a_1 + \sqrt{a_2^2 - a_3^2} = 0\,. \tag{11}$$

Note that we have $a_1 < 0$, whereas $\sqrt{a_2^2 - a_3^2}$ can be either real-valued and positive or complex-valued. Solving Eq. 11, we obtain

$$\lambda_{th,m} = \frac{4\omega}{\omega_0^2}\sqrt{a1^2 + a_3^2} = \frac{4\omega}{\omega_0^2}\sqrt{\frac{\gamma^2}{4} + \left(\frac{\omega^2 - \omega_0^2}{2\omega} + \rho_m \frac{\beta^2/N}{2\omega}\right)^2}. \quad (12)$$

For identical oscillators, we see that there are primarily two instability thresholds corresponding to (i) the instability of the symmetric normal mode, $w_{0,+}$, and (ii) to the instability of all other normal modes: $w_{m,+}$ including the anti-symmetric mode.

2 Calibration measurements

In Fig. 1, we present test measurements that we have performed to ensure that the weakly coupled strings were degenerate in frequency. On timescales of hours, thermal drift sometimes caused detuning between the strings, which we balanced by adjusting the tension of the strings separately. In Fig. 2, we show the fits used to extract the nonlinear coefficients of the two weakly coupled strings. Please refer to Ref. [23] of the main text for details regarding the model of a nonlinear parametric oscillator.

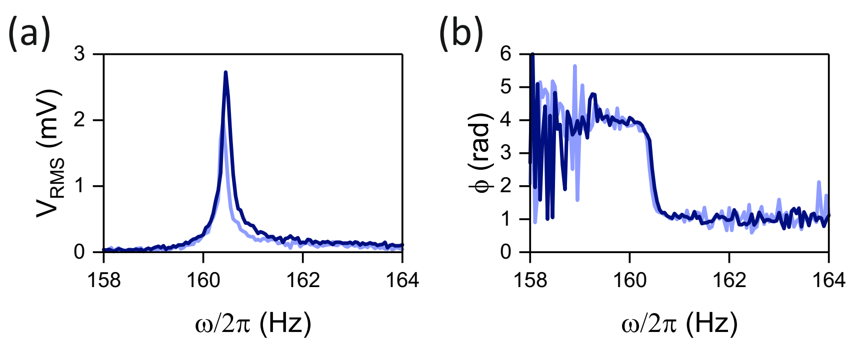

Figure 1: (a) Amplitude and (b) phase response of the two resonators in the linear regime. We use weak external driving, no parametric drive, and weak coupling to observe a Lorentzian response. Light and dark blue correspond to resonator 1 and 2, respectively. These measurements are taken immediately before the nonlinear parametric measurements shown in Fig. 3 of the main text to ensure that the two modes are degenerate in frequency.

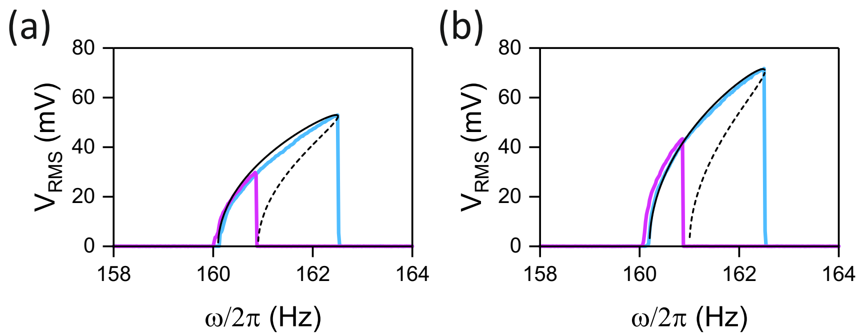

Figure 2: (a) Amplitude response of resonator 1 and (b) resonator 2 with weak coupling and strong parametric driving. Blue and magenta lines correspond to sweeps with increasing and decreasing frequency, respectively. These are the same data as shown in Fig. 3c of the main text. Solid and dashed black lines are stable and unstable theory solutions, respectively. From fitting these solutions to the measured data, we retrieve the values of $\alpha_{1,2}$ and $\eta_{1,2}$ stated in the main text. Note that in the strong coupling case we find that the nonlinear damping decreases, as determined from the frequency at which the stable and unstable solutions merge.