### **Problem 1.** *Lattice Dirac model*

Implement the Lattice Dirac model. This is the fixed spin sector or the Bernevig-Hughes-Zhang model discussed in the lecture:

$$H = \begin{pmatrix} m + \cos k_x + \cos k_y & \sin k_x - \mathrm{i} \sin k_y \\ \sin k_x + \mathrm{i} \sin k_y & -m - \cos k_x - \cos k_y \end{pmatrix} \tag{1}$$

1. Study the spectrum of the system as a function of $m$. For which coupling do you observe a closure of the gap?
2. For different values of $m$, study the character of the bands along a high symmetry line of the Brillouin zone. Do you observe band inversion?
3. Consider an cylinder geometry, where $k_y$ is still a good quantum number. Compare the spectrum of an open and periodic geometry along $x$ for different values of $m$ and sample length $L_x = 50$.

You should be able to reproduce the results and figures presented in the lecture notes.

### **Problem 2.** *Hohenberg-Kohn theorems*

Give a proof of the two Hohenberg-Kohn theorems seen in the lecture.

1. The total energy of an electronic system in an external potential $V_{ne}(\mathbf{r})$ is a unique functional of the electron density $\rho(\mathbf{r})$:

   $$E[\rho] = \int \mathrm{d}^3\mathbf{r} V_{ne}(\mathbf{r})\rho(\mathbf{r}) + F[\rho], \tag{2}$$

   where $F[\rho]$ is an unknown, but otherwise universal functional of the electron density $\rho(\mathbf{r})$ only.
   *Hint:* Proceed by *reductio ad absurdum.*

2. The density that minimises the total energy is the groundstate density $\rho_0(\mathbf{r})$. The corresponding energy is the groundstate energy $E_0$.

   $$E[\rho] \geq E[\rho_0] = E_0. \tag{3}$$

   *Hint:* Start from the variational principle for wavefunctions: $E_0 \leq \langle\psi|H|\psi\rangle$.