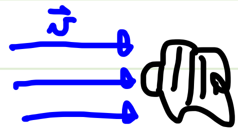
Topic: Superconductivity, BCS, superfluidity

* We know the concept of a Cooper pair.
* We appreciate the role of the Fermi surface
* We know the basics of BCS theory.

1. Superfluidity

We recall the Landau criterion for superfluidity. We assume a quasiparticle spectrum



$$H = \sum_k \varepsilon(k)\, a_k^{+} a_k .$$

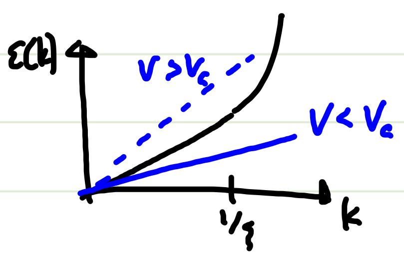
Trying to move the liquid past an obstacle with velocity $\vec{v}$ we calculate the energy in the co-moving frame

$$E = \varepsilon(k) + \vec{k}\cdot\vec{v} + \frac{M v^2}{2} .$$

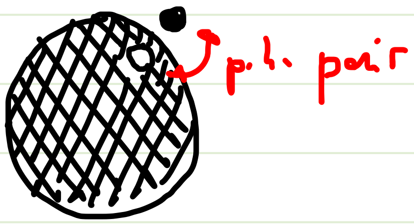
Here, we assumed that we excited a q.p. with momentum $\vec{k}$. This only happens if $\varepsilon(k) + \vec{k}\vec{v} < 0$

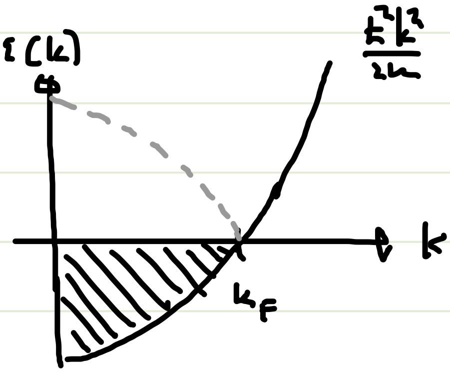
$$\Rightarrow \boxed{v_{Landau} = \frac{\varepsilon(k)}{k}}$$

How did it look for a BEC?

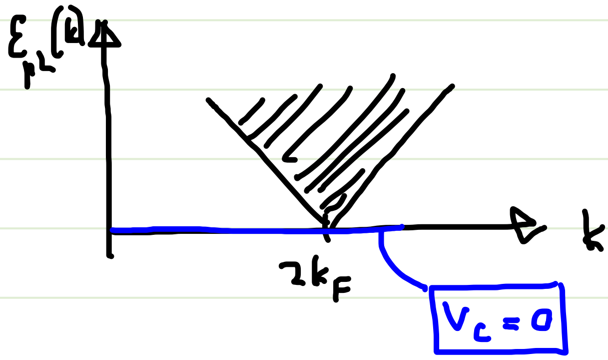


How does the excitation spectrum look like for fermions in a fermi sea?

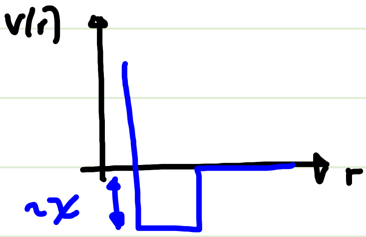




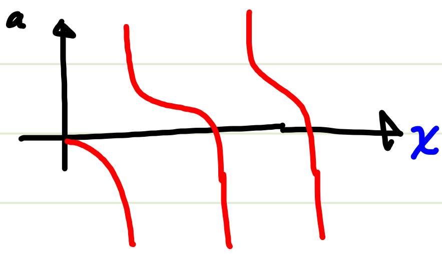
$\Rightarrow$ for a particle-hole pair:



Can fermions "pair" into "bosons" to become superfluid?
Most probably we need attractive interaction for this to happen.
Recall that we can have scattering resonances:



$\Rightarrow$

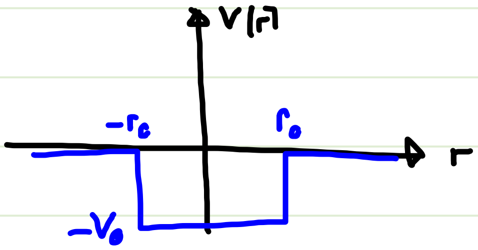


And $$H_{int} = \frac{4\pi\hbar^2 a}{m} \int d\vec{x}\, \psi_\uparrow^\dagger(\vec{x}) \psi_\uparrow(\vec{x}) \psi_\downarrow^\dagger(\vec{x}) \psi_\downarrow(\vec{x}) \qquad (1)$$

However, this is only valid up to some cutoff energy scale Λ. Let us see what we can get from (1).

2. Pairing

Let us recall the solution of two interacting particles. We can always reduce the two-particle problem to a single-particle (relative coordinate) in potential.

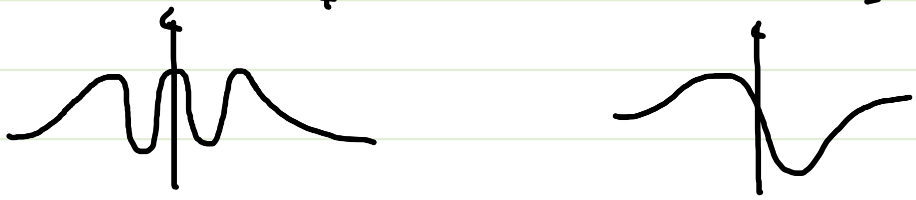


$$k_0 = \frac{\sqrt{2m(V_0-|E|)}}{\hbar} \qquad \alpha = \frac{\sqrt{2m|E|}}{\hbar}$$

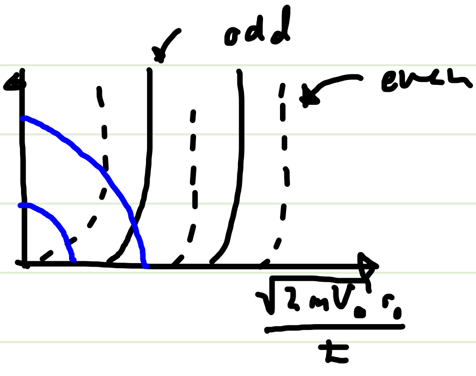
$$-V_0 < E < 0$$

1d:

$$\varphi_+(r) = \begin{cases} a_+ \cos(k_0 r) \\ b_+ e^{-\alpha|r|} \end{cases} \qquad \varphi_-(r) = \begin{cases} a_- \sin(k_0 r) \\ b_- \,\mathrm{sign}(r)\, e^{-\alpha|r|} \end{cases}$$



$\Rightarrow$ match at $r = \pm r_0$ both $\varphi_\pm$ and $\varphi'_\pm$ $\Rightarrow$



$\Rightarrow$ always an even solution; V_0^{max} for odd solution

2d:

$$H = -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\right] + V(r)$$

$$\Rightarrow \psi(r,\varphi) = e^{in\varphi}\, u_n(r) \qquad \Rightarrow u_n(r) = \begin{cases} a\, J_n(k_0 r) & r < r_0 \\ b\, K_n(k_0 r) & r > r_0 \end{cases}$$

⇒ match u_n and u_n' at r_0 ⇒

$$E = -\frac{V_0}{\xi_0^2}\, e^{-4/\xi_0^2}$$

⇒ there is always an exponentially weakly bound state.

3d: $\varphi(r,\vartheta,\varphi) = Y_{lm}(\vartheta,\varphi)\, R(r)$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{\hbar^2 l(l+1)}{2mr^2}+V(r)\right]R(r) = E\,R(r)$$

$l=0 \Rightarrow R(r) = \frac{u(r)}{r}$ with $u(0)=0$ ⇒

$$-\frac{\hbar^2}{2m}u'' + V(r)u = Eu$$

⇒ formally equivalent to 1d with $u(r) = -u(-r)$ ⇒ critical V_0 needed.

⇒ Does this mean we need a finite V_0 for superfluidity in 3d?

2.1 The cooper problem

Let us examine the two particle problem on top of a Fermi-surface.

$$\left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + V(r)\right]\psi(r) = \left(E + \frac{\hbar^2 k_F^2}{m}\right)\psi(r)$$

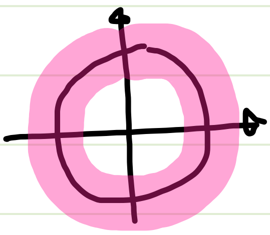
→ $\frac{1}{m}$: 2-Particles; singlet; E measured from E_F

$$\Rightarrow \; g(\vec{k}) = \int d\vec{r}^3 e^{-\vec{k}\vec{r}} \psi(\vec{r}) \quad ; \quad V(\vec{k}) = \int d\vec{r}^3 \, V(\vec{r}) e^{-i\vec{k}\vec{r}}$$

We obtain

$$\frac{\hbar^2 \vec{k}^2}{m} g(k) + \int \frac{d\vec{k}'^3}{(2\pi)^3} V(\vec{k}-\vec{k}') \, g(k') = \left(E + \frac{\hbar^2 k_F^2}{m}\right) g(k)$$

Let us now assume: $V(k-k') = \begin{cases} -V_0 & \text{for } \varepsilon_F < \frac{\hbar^2 k^2}{2m}, \frac{\hbar^2 k'^2}{2m} < \varepsilon_F + \Lambda \\ 0 & \text{otherwise} \end{cases}$



$$\Rightarrow \left(-\frac{\hbar^2 k^2}{m} + E + \frac{\hbar^2 k_F^2}{m}\right) g(k) = -V_0 \int_{\varepsilon_F < \frac{\hbar^2 k'^2}{2m} < \varepsilon_F + \Lambda} d^3k' \; g(k')$$

$\Rightarrow$ divide by and integrate over $\int d\vec{k}^3/(2\pi)^3$

$$1 = V \int (d\vec{k}^3) \frac{1}{\frac{\hbar \vec{k}^2}{m} - E - 2\varepsilon_F}$$

We now set $\xi = \frac{\hbar^2 k^2}{2m} - \varepsilon_F$; use $N(\xi) = \frac{4\pi k^2}{(2\pi)^3} \frac{dk}{d\xi}$ $\Rightarrow$

$$1 = V \int_0^{\Lambda} d\xi \, N(\xi) \frac{1}{2\xi - E} \approx V N(0) \int_0^{\Lambda} d\xi \frac{1}{2\xi - E} \approx \frac{V_0 N(0)}{2} \log\left(-\frac{2\Lambda}{E}\right)$$

$$\Rightarrow \boxed{E = -2\Lambda e^{-\frac{1}{V_0 N(0)}}}$$

* Bound state for arbitrarily weak V_0
* Binding energy exponentially small
* Looks like 2d

↳ Fermi-surface effect!

* Assumed FS → bound state ... → what about the next "layer"?
* if $k_F' \to 0$ $\quad N(\xi) \sim \xi^{1/2} \Rightarrow$ 3d $\Rightarrow$ need minimal V_0

⇒ we need more many-body-like theory

3. Bardeen-Cooper-Schrieffer theory

We saw that we need a many-body wave-function. We want to use a variational wave-function. There are several (equivalent) ways to go about this issue.

1.) Inspired by $\psi_B \propto e^{-g_k b_k^+}|0\rangle$ coherent state of bosons:

$$\psi_{BCS} \propto e^{\sum_k g(k) c^+_{k\uparrow} c^+_{-k\downarrow}}|0\rangle =$$

$$= \prod_k \left(1 - g(k)\, c^+_{k\uparrow} c_{-k\downarrow}\right)|0\rangle$$

$$= \prod_k \left(u_k + v_k\, c^+_{k\uparrow} c^+_{-k\downarrow}\right)|0\rangle$$

$\Rightarrow \langle \psi_{BCS} | H | \psi_{BCS} \rangle = E(u,v) \Rightarrow$ Minimize w.r.t. $\{u_k, v_k\}$ with $u_k^2 + v_k^2 = 1$

2.) Inspired by FS. Goal: find the best Slater determinant that minimizes the g.s.-energy. How do we generate a Slater determinant? $\Rightarrow$ Via a quadratic Hamiltonian!

$$H = H_0 + V \sum_i \psi_{i\uparrow}^\dagger \psi_{i\uparrow} \psi_{i\downarrow}^\dagger \psi_{i\downarrow} \quad ; \quad H_0 = \sum_{k\sigma} \xi(k)\, c_{k\sigma}^\dagger c_{k\sigma}$$

$$c_{k\sigma} = \frac{1}{\sqrt{N}} \sum_i e^{-i\vec{k}\vec{r}_i} c_{i\sigma}$$

$$H_{MF} = H_0 + \Delta \sum_i \left(\psi_{i\uparrow}^\dagger \psi_{i\downarrow}^\dagger + \text{H.c.} \right) - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma}$$

$\Rightarrow$ $|\psi_{MF}\rangle$ is g.s. of H_{MF}, a Slater-determinant.

$$\langle H \rangle_{MF} = \langle H_{MF} \rangle - \langle H - H_{MF} \rangle$$

$$= \langle H_{MF} \rangle - \mu \sum_{i\sigma} \langle c_{i\sigma}^\dagger c_{i\sigma} \rangle + \Delta \sum_i \left\{ \langle \psi_{i\uparrow}^\dagger \psi_{i\downarrow}^\dagger \rangle + \text{H.c.} \right\}$$

$$- V \sum_i \left\{ \langle \psi_{i\uparrow}^\dagger \psi_{i\uparrow} \rangle \langle \psi_{i\downarrow}^\dagger \psi_{i\downarrow} \rangle + \langle \psi_{\uparrow i}^\dagger \psi_{\downarrow i}^\dagger \rangle \langle \psi_{i\uparrow} \psi_{i\downarrow} \rangle \right\}$$

Now we minimize:

Feynman Hellman

$$0 \overset{!}{=} \frac{\partial}{\partial \mu} \langle H \rangle_{MF} = \cancel{\langle \frac{\partial}{\partial \mu} H_{MF} \rangle_{MF}} - \cancel{\sum_i \langle c_{i\sigma}^\dagger c_{i\sigma} \rangle} + \Delta \sum_i \left\{ \frac{\partial}{\partial \mu} \langle \psi_{\uparrow i}^\dagger \psi_{i\downarrow}^\dagger \rangle + \text{H.c.} \right\}$$

$$- V \sum_i \left\{ \left(\frac{\partial}{\partial \mu} \langle \psi_{i\uparrow}^\dagger \psi_{i\uparrow} \rangle \right) \langle \psi_{i\downarrow}^\dagger \psi_{i\downarrow} \rangle + \langle \psi_{i\uparrow}^\dagger \psi_{i\uparrow} \rangle \frac{\partial}{\partial \mu} \langle \psi_{i\downarrow}^\dagger \psi_{i\downarrow} \rangle \right.$$

$$\left. + \left[\frac{\partial}{\partial \mu} \langle \psi_{i\uparrow}^\dagger \psi_{i\downarrow}^\dagger \rangle \right] \langle \psi_{\uparrow i} \psi_{\downarrow i} \rangle + \langle \psi_{\uparrow i}^\dagger \psi_{\downarrow i}^\dagger \rangle \frac{\partial}{\partial \mu} \langle \psi_{\uparrow i} \psi_{i\downarrow} \rangle \right\}$$

generally $\frac{\partial}{\partial\mu}\langle\cdot\rangle \neq 0 \Rightarrow$ we need to nullify the prefactors

$$\Delta = \frac{V}{N}\sum_i \langle c^+_{i\downarrow} c^+_{i\uparrow}\rangle = \frac{V}{N}\sum_{k k' i} \langle c^+_{k\uparrow} c^+_{k'\downarrow}\rangle e^{i r_i (k+k')} = \frac{V}{N}\sum_k \langle c^+_{k\uparrow} c^+_{-k\downarrow}\rangle$$

$$\mu = \frac{1}{2N}\sum_{i\sigma} \langle c^+_{i\sigma} c_{i\sigma}\rangle$$

This leads us to the BCS-self-consistency equations:

$$\Delta = \int \frac{dk^3}{(2\pi)^3} \langle c^+_{k\uparrow} c^+_{-k\downarrow}\rangle_{MF}$$

$$\mu = \int \frac{dk^3}{(2\pi)^3} \langle c^+_{k\uparrow} c_{k\uparrow}\rangle_{MF} = \int \frac{dk^3}{(2\pi)^3} \langle c^+_{k\downarrow} c_{k\downarrow}\rangle_{MF}$$

We assumed $\mu \neq \mu_i$, $\Delta \neq \Delta_i$; $\mu = \mu_\sigma$, i.e., an s-wave pairing with no broken spin- or translation symmetry.

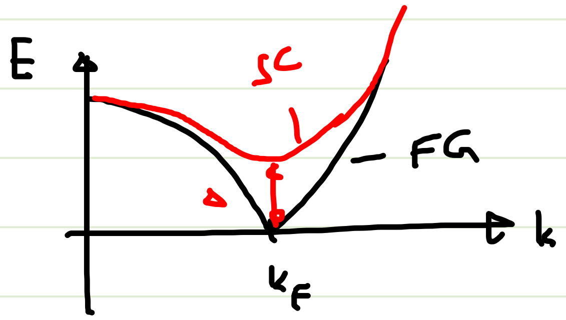
Let us solve these equations.

$$H_{MF} = \sum_k \vec{\psi}^+_k \left[\xi_k \hat{\sigma}_z + \mathrm{Re}\,\Delta\, \hat{\sigma}_x + \mathrm{Im}\,\Delta\, \hat{\sigma}_y \right] \vec{\psi}_k \quad ; \quad \vec{\psi}_k = \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix}$$

$$\sum_k \vec{\gamma}^+_k U^+ \left[\xi_k \hat{\sigma}_z + \Delta \hat{\sigma}_x \right] U \vec{\gamma}_k \qquad \vec{\gamma}_k = \begin{pmatrix} \gamma_{+k} \\ \gamma^+_{-k} \end{pmatrix}$$

$$\sum_k \gamma^+_{+k}\gamma_{+k}\sqrt{\xi_k^2 + \Delta^2} - \gamma_{-k}\gamma^+_{-k}\sqrt{\xi_k^2 + \Delta^2}$$

$$\Rightarrow \quad H_{MF} = \sum_k \underbrace{\sqrt{\xi_k^2 + \Delta^2}}_{E_k} \left\{ \gamma_{k\uparrow}^{+} \gamma_{k\uparrow} + \gamma_{k\downarrow}^{+} \gamma_{k\downarrow} \right\}$$



Using $\quad U_k = \begin{pmatrix} u_k & v_k \\ v_{-k}^{*} & u_{-k}^{*} \end{pmatrix}$ with $\quad u_k = \dfrac{E_k + \xi_k}{\sqrt{2E_k(E_k+\xi_k)}}$

$$v_k = \frac{-\Delta}{\sqrt{2E_k(E_k+\xi_k)}}$$

$$\Rightarrow \quad \langle c_{k\uparrow}^{+} c_{-k\downarrow}^{+} \rangle = \dots\dots = \qquad \Rightarrow$$

$$\Delta = -V \int d\xi \, N(\xi) \, \frac{\Delta}{2E_k} \tanh\left(\frac{E_k}{2k_B T}\right)$$

$\Rightarrow$ at $T=0$:

$$1 = -VN(0) \int_0^{\Lambda} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}} = -VN(0) \log\left(\frac{2\Lambda}{\Delta}\right)$$

$$\Delta = 2\Lambda e^{-\frac{1}{N(0)V}}$$

$\Rightarrow$ at $\Delta \to 0$:

$$1 = -VN(0) \int_0^{\Lambda} d\xi \frac{1}{2\xi} \tanh\left(\frac{\xi}{2k_B T_c}\right)$$

$$k_B T_c = 1.134 \, e^{-\frac{1}{N(0)V}}$$

What did we achieve?

* Self-consistent solution
* $\Delta(T=0)$
* T_c

And: